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Hilbert Coefficients of Parameters

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HILBERT COEFFICIENTS OF PARAMETERS

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1. When $e_Q^1(A) < 0$?

The purpose of my lecture is to report the recent progress in the analysis of Hilbert coefficients of parameters. My research is based discussions [11], [12], and [18] with L. Ghezzi, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos. Especially, the very recent progress is strongly inspired by Vasconcelos [40]; so the results of my lecture are joint works with them.

My lecture consists of 8 sections and the table of contents is the following.

(1) When $e_Q^1(A) < 0$?

(2) Homological degrees.

(3) When is the set

$$\Lambda_1(A) = \{e_Q^1(A) \mid Q \text{ is a parameter ideal in } A\}$$

finite?

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(4) How about $\mathrm{e}_Q^2(A)$ – uniform bounds for the sets

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

with $1 \le i \le \dim A$.

(5) A method to compute $\mathrm{e}_Q^1(A)$.

(6) Constancy of $\mathrm{e}_Q^1(A)$ with the same integral closure $\overline{Q}$.

(7) The case where $\overline{Q} = \mathfrak{m}$.

(8) A structure theorem of local rings with $\mathrm{e}_Q^1(A) = -1$.

(9) Appendix: when $\bar{\mathrm{e}}_I^1(R) \ge 0$?

In what follows, let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all $n \gg 0$. We call these integers $\mathrm{e}_I^i(A)$ the Hilbert coefficients of A with respect to I. In particular, the leading coefficient $\mathrm{e}_I^0(A)$ is called the multiplicity of A with respect to I and plays an important role in the analysis of singularity of A and I.

For example, let me consider the case where $I = Q$ is a parameter ideal in A. So, we assume that $Q = (a_1, a_2, \cdots, a_d)$ is an ideal of A generated by a system $a_1, a_2, \cdots, a_d$ of parameters. Then as is well-known,

$$\ell_A(A/Q) \ge \mathrm{e}_Q^0(A)$$

and we have definitions and characterizations of several kinds of local rings in terms of multiplicity of parameters. Let me remind some of them, which I maintain throughout this lecture. Let $\mathrm{H}_\mathfrak{m}^i(*)$ $(i \in \mathbb{Z})$ denote the i–th local cohomology functor of A with respect to $\mathfrak{m}$. We put $\mathrm{h}^i(A) = \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$ for all $i \in \mathbb{Z}$.

Definitions and characterizations 1.1. (1) *A is a Cohen-Macaulay ring if and only if $\ell_A(A/Q) = \mathrm{e}_Q^0(A)$ for some (and hence for any) parameter ideal Q in A. When this is the case, $\mathrm{H}_\mathfrak{m}^i(A) = (0)$ for all $i \ne d$.*

(2) ([37]) *We say that A is a Buchsbaum ring, if $\ell_A(A/Q) - \mathrm{e}_Q^0(A)$ is constant and independent of the choice of parameter ideals Q in A. When this is the case,*

$$\mathfrak{m}\mathrm{H}_\mathfrak{m}^i(A) = (0)$$

for all $i \neq d$. Therefore, the local cohomology modules $\mathrm{H}_\mathfrak{m}^i(A)$ are finite-dimensional vector spaces over $A/\mathfrak{m}$. (The converse is not true in general, that is, A is not necessarily a Buchsbaum ring, even if $\mathfrak{m}\mathrm{H}_\mathfrak{m}^i(A) = (0)$ for all $i \neq d$.)

(3) ([35, 36]) *We say that A is a generalized Cohen-Macaulay ring, if*

$$\sup_Q [\ell_A(A/Q) - \mathrm{e}_Q^0(A)] < \infty,$$

where Q runs over parameter ideals in A. This condition is equivalent to saying that $\mathrm{H}_\mathfrak{m}^i(A)$ are finitely generated A-modules for all $i \neq d$. (So, sometimes I call these local rings to have FLC; finite local cohomology modules.) *When this is the case, we have*

$$\sup_Q [\ell_A(A/Q) - \mathrm{e}_Q^0(A)] = \sum_{j=0}^{d-1} \binom{d-1}{j} \mathrm{h}^j(A) := \mathbb{I}(A),$$

which we call the Buchsbaum invariant (or the Stückrad-Vogel invariant) of A.

So, every Cohen-Macaulay local ring is Buchsbaum and Buchsbaum local rings are generalized Cohen-Macaulay. These definitions and characterizations are given in terms of multiplicity of parameters.

Question 1.2. How about $\mathrm{e}_Q^1(A)$? Namely, can we say anything about the structure of local rings in terms of vanishing or non-vanishing of $\mathrm{e}_Q^1(A)$ for parameters? In general we have $\mathrm{e}_Q^1(A) \leq 0$ ([28]).

As for Question 1.2, Wolmer V. Vasconcelos firstly posed the following conjecture at the conference in Yokohama 2008.

Conjecture 1.3 ([10, 39])**.** Assume that A is unmixed, that is $\dim \widehat{A}/P = d$ for all $P \in \operatorname{Ass} \widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.

Later I shall affirmatively settle this conjecture (Theorem 1.8). Before that, let me prove the inequality $\mathrm{e}_Q^1(A) \leq 0$ ([28]). This result was firstly discovered by Mandal and Verma [28] and it is also one of consequences of Theorem 1.8. After a proof of [11, Corollary 2.11] (Corollary 1.14 in this lecture) was reported in my seminar, F. Hayasaka discovered an alternate proof of it based on the following Theorem 1.4. He proved Theorem 1.4 in a more general setting, that is the case where Q is a parameter module and the multiplicity is the Buchsbaum-Rim multiplicity. Let me include a brief proof in the case of ideals.

Theorem 1.4 ([21, Theorem 1.1])**.** *Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Then*

$$\ell_A(A/Q^{n+1}) \geq \mathrm{e}_Q^0(A)\binom{n+d}{d}$$

for all $n \geq 0$. If $\ell_A(A/Q^{n+1}) = \mathrm{e}_Q^0(A)\binom{n+d}{d}$ for some $n \geq 0$, then A is a Cohen-Macaulay ring, so that

$$\ell_A(A/Q^{n+1}) = \mathrm{e}_Q^0(A)\binom{n+d}{d}$$

for all $n \geq 0$.

Proof. Let $B = A[X_1, X_2, \cdots, X_d]$ be the polynomial ring and let $M = \mathfrak{m}B + (X_1, X_2, \cdots, X_d)$ in B. Let $f_i = X_i - a_i$ $(1 \leq i \leq d)$ and put $\mathfrak{q} = (f_1, f_2, \cdots, f_d)B$. Then $f_1, f_2, \cdots, f_d$ is a regular sequence in B, as $B = A[f_1, f_2, \cdots, f_d]$. We look at the A-algebra map

$$\varphi : B \to A$$

defined by $\varphi(X_i) = a_i$ for all $1 \leq i \leq d$. Then $\mathfrak{q} = \mathrm{Ker}\varphi$. We put $C = B_M$ and extend φ to the homomorphism $\psi : C \to A$

$$\begin{array}{ccc} C & \xrightarrow{\ \psi\ } & A \\ & \nwarrow \ \circlearrowright \ \nearrow_{\varphi} & \\ & B & \end{array}$$

Then $\mathrm{Ker}\psi = \mathfrak{q}C$ and we have the identifications

$$A/Q^{n+1} = B/[\mathfrak{q}^{n+1} + (X_1, X_2, \cdots, X_d)] = C/[\mathfrak{q}^{n+1}C + (X_1, X_2, \cdots, X_d)C]$$

for all $n \geq 0$, whence $X_1, X_2, \cdots, X_d$ is a system of parameters for $C/\mathfrak{q}^{n+1}C$. Let

$$\operatorname{Assh} C/\mathfrak{q}C = \{\mathfrak{p} \in \operatorname{Supp}_C C/\mathfrak{q}C \mid \dim C/\mathfrak{p} = \dim C/\mathfrak{q}C\}.$$

Then, thanks to the associative formula of multiplicity together with the fact that $f_1, f_2, \cdots, f_d$ is a regular sequence in C, we get

$$\begin{aligned}
\ell_A(A/Q^{n+1}) &= \ell_C(C/[\mathfrak{q}^{n+1}C + (X_1, X_2, \cdots, X_d)C]) \\
&\geq \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{q}^{n+1}C) \\
&= \sum_{\mathfrak{p} \in \operatorname{Assh}_C C/\mathfrak{q}C} \ell_{C_\mathfrak{p}}(C_\mathfrak{p}/\mathfrak{q}^{n+1}C_\mathfrak{p}) \cdot \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{p}) \\
&= \sum_{\mathfrak{p} \in \operatorname{Assh}_C C/\mathfrak{q}C} \binom{n+d}{d} \ell_{C_\mathfrak{p}}(C_\mathfrak{p}/\mathfrak{q}C_\mathfrak{p}) \cdot \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{p}) \\
&= \binom{n+d}{d} \sum_{\mathfrak{p} \in \operatorname{Assh}_C C/\mathfrak{q}C} \ell_{C_\mathfrak{p}}(C_\mathfrak{p}/\mathfrak{q}C_\mathfrak{p}) \cdot \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{p}) \\
&= \binom{n+d}{d} \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{q}C) \text{ (by the associative formula)} \\
&= \binom{n+d}{d} \mathrm{e}^0_Q(A)
\end{aligned}$$

for all $n \geq 0$. Let $n \geq 0$ be now a fixed integer. We then have

$$\ell_A(A/Q^{n+1}) = \binom{n+d}{d} \mathrm{e}^0_Q(A)$$

if and only if

$$\ell_C(C/[\mathfrak{q}^{n+1}C + (X_1, X_2, \cdots, X_d)C]) = \mathrm{e}^0_{(X_1,X_2,\cdots,X_d)C}(C/\mathfrak{q}^{n+1}C),$$

which is equivalent to saying that $C/\mathfrak{q}^{n+1}C$ is a Cohen-Macaulay local ring. Because $\mathfrak{q}^{n+1}C$ is a perfect ideal of C (recall that $\mathfrak{q} = (f_1, f_2, \cdots, f_d)$ is generated by a B–regular sequence $f_1, f_2, \cdots, f_d$), this condition is equivalent to saying thatthe local ring C is Cohen-Macaulay, that is our base ring A is Cohen-Macaulay. $\square$

As consequences of Theorem 1.4 we get the following.

Corollary 1.5. *Let Q be a parameter ideal in A. Then the following assertions hold true.*

(1) ([28]) $\mathrm{e}^1_Q(A) \leq 0$.

(2) (cf. Corollary1.14) *A is a Cohen-Macaulay ring if and only if* $\mathrm{e}_Q^i(A) = 0$ *for all* $1 \le i \le d$.

Proof. We have

$$0 \le \ell_A(A/Q^{n+1}) - \mathrm{e}_Q^0(A)\binom{n+d}{d} = -\mathrm{e}_Q^1(A)\binom{n+d-1}{d-1} + (\text{terms of lower degree})$$

for all $n \gg 0$, whence $\mathrm{e}_Q^1(A) \le 0$. The second assertion is clear. $\square$

Later I will prove Corollary 1.5 in our own context.

Let me now be back to Vasconcelos' conjecture. To prove it, we need the following.

Lemma 1.6 ([15]). *If $d = 1$, then* $\mathrm{e}_Q^1(A) = -\mathrm{h}^0(A)$.

This easily follows from the fact that $A/\mathrm{H}_\mathfrak{m}^0(A)$ is a Cohen-Macaulay ring but the result itself plays a very important role in the analysis of $\mathrm{e}_Q^1(A)$.

Lemma 1.7 ([14]). *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and* $d = \dim A \ge 2$, *possessing the canonical module* K_A. *Suppose that*

$$\operatorname{Ass} A \setminus \{\mathfrak{m}\} = \operatorname{Assh} A,$$

that is $\dim A/\mathfrak{p} = d$ *for every* $\mathfrak{p} \in \operatorname{Ass} A \setminus \{\mathfrak{m}\}$. *Then the following assertions hold true.*

(1) *The local cohomology module* $\mathrm{H}_\mathfrak{m}^1(A)$ *is finitely generated.*

(2) *The set* $\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \dim A_\mathfrak{p} > \operatorname{depth} A_\mathfrak{p} = 1\}$ *is finite.*

(3) *Let $a \in \mathfrak{m}$ and assume that* $a \notin \bigcup_{\mathfrak{p} \in \operatorname{Assh} A} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}, \mathfrak{p} \ne \mathfrak{m}} \mathfrak{p}$. *Then*

$$\operatorname{Ass}_A A/(a) \setminus \{\mathfrak{m}\} = \operatorname{Assh}_A A/(a).$$

(4) *Suppose that the residue class field $A/\mathfrak{m}$ of A is infinite and let I be an $\mathfrak{m}$-primary ideal in A. Then one can choose an element $a \in I \setminus \mathfrak{m}I$ so that a is superficial with respect to I and*

$$\operatorname{Ass}_A A/(a) \setminus \{\mathfrak{m}\} = \operatorname{Assh}_A A/(a).$$

Let me talk a little bit about the proof of Lemma 1.7. In Section 2 I will provide a smarter approach to Vasconcelos' conjecture, where we will need Lemma 1.7 no more.

Proof. Let $U = \mathrm{U}_A(0)$ denote the unmixed component of (0) in A. Then $\ell_A(U) < \infty$. As $U = (0) :_A K_A$, we have the exact sequence

$$0 \to U \to A \xrightarrow{\varphi} \mathrm{Hom}_A(K_A, K_A) \to C \to 0$$

of A-modules, where $\varphi(a) = a{\cdot}1_{K_A}$ for all $a \in A$. Notice that $\mathrm{depth}_A \mathrm{Hom}_A(K_A, K_A) \geq 2$, because $d \geq 2$ and we get

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^0_{\mathfrak{m}}(C) \subseteq C,$$

so that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module.

Let $\mathfrak{p} \in \mathrm{Spec}\, A$ and suppose that $\mathfrak{p} \neq \mathfrak{m}$, $\mathrm{ht}_A \mathfrak{p} > 1$, but $\mathrm{depth}\, A_{\mathfrak{p}} = 1$. Then, localizing at $\mathfrak{p}$, we get the exact sequence

$$0 \to A_{\mathfrak{p}} \to \mathrm{Hom}_{A_{\mathfrak{p}}}([K_A]_{\mathfrak{p}}, [K_A]_{\mathfrak{p}}) \to C_{\mathfrak{p}} \to 0$$

of $A_{\mathfrak{p}}$-modules. Recall that $[K_A]_{\mathfrak{p}} = K_{A_{\mathfrak{p}}}$, because $[K_A]_{\mathfrak{p}} \neq (0)$ and we have $\mathrm{depth}_A C_{\mathfrak{p}} = 0$, since $\mathrm{depth}_{A_{\mathfrak{p}}} \mathrm{Hom}_{A_{\mathfrak{p}}}([K_A]_{\mathfrak{p}}, [K_A]_{\mathfrak{p}}) \geq 2$ and $\mathrm{depth}\, A_{\mathfrak{p}} = 1$. Thus $\mathfrak{p} \in \mathrm{Ass}_A C$, so that

$$\mathcal{F} \subseteq \mathrm{Ass}_A C \cup \{\mathfrak{m}\},$$

whence $\mathcal{F}$ is a finite set.

Let $a \in \mathfrak{m}$ such that $a \notin \bigcup_{\mathfrak{p} \in \mathrm{Assh}\, A} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}, \mathfrak{p} \neq \mathfrak{m}} \mathfrak{p}$. Let $\mathfrak{p} \in \mathrm{Ass}_A A/(a) \setminus \{\mathfrak{m}\}$. Then $(0) :_A a \subseteq U$, because a is regular in A/U. Hence the element a is $A_{\mathfrak{p}}$-regular, so that $\mathrm{depth}\, A_{\mathfrak{p}} = 1$. Because $a \in \mathfrak{p}$, we get $\mathfrak{p} \notin \mathcal{F}$ and so $\mathrm{ht}_A \mathfrak{p} = 1$. Hence $\dim A/\mathfrak{p} = d - 1$, because our local ring A is catenary; in fact, we have

$$\mathrm{Spec}\, A = \mathrm{Supp}_A K_A = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = d\}.$$

Thus

$$\mathrm{Ass}_A A/(a) \setminus \{\mathfrak{m}\} \subseteq \mathrm{Assh}_A A/(a)$$

as claimed. □

Let me now prove Vasconcelos' conjecture with the following formulation, where the implication (1) ⇒ (2) is a result of Narita ([30, Corollary 1]).

Theorem 1.8. *Suppose that A is unmixed. Then the following conditions are equivalent.*

(1) *A is a Cohen-Macaulay ring.*

(2) $\mathrm{e}_I^1(A) \geq 0$ *for every* $\mathfrak{m}$*-primary ideal* I *in* A.

(3) $\mathrm{e}_Q^1(A) \geq 0$ *for some parameter ideal* Q *in* A.

(4) $\mathrm{e}_Q^1(A) = 0$ *for some parameter ideal* Q *in* A.

Proof. Let me prove (3) $\Rightarrow$ (1). We may assume that $d > 1$, A is complete, and the residue class field $A/\mathfrak{m}$ of A is infinite. Let me choose $a \in Q \setminus \mathfrak{m}Q$ so that a is superficial with respect to Q and

$$\mathrm{Ass}_A A/(a) \setminus \{\mathfrak{m}\} = \mathrm{Assh}_A A/(a).$$

(this choice is possible; see Lemma 1.7). We put $\overline{A} = A/(a)$. Then, since a is A-regular, we have

$$\mathrm{e}_Q^1(\overline{A}) = \mathrm{e}_Q^1(A) \geq 0.$$

Therefore, if $d = 2$, then by Lemma 1.6 we see

$$\mathrm{e}_Q^1(\overline{A}) = -\mathrm{h}^0(\overline{A}) \geq 0,$$

since $\dim \overline{A} = 1$. Hence $\overline{A}$ is a Cohen-Macaulay ring, so that A is Cohen-Macaulay, because a is A-regular.

Suppose now that $d > 2$ and that our assertion holds true for $d - 1$. Let $U = \mathrm{U}_{\overline{A}}(0)$ be the unmixed component of (0) in $\overline{A}$ and put $B = \overline{A}/U$. Then, since $\ell_A(U) < \infty$ (we actually have $U = \mathrm{H}_{\mathfrak{m}}^0(\overline{A})$), we have

$$\mathrm{e}_Q^1(B) = \mathrm{e}_Q^1(\overline{A}) = \mathrm{e}_Q^1(A) \geq 0.$$

Consequently, since B is unmixed, B is a Cohen-Macaulay ring by the hypothesis of induction. Hence

$$\mathrm{H}_{\mathfrak{m}}^i(\overline{A}) = \mathrm{H}_{\mathfrak{m}}^i(B) = (0) \text{ for all } 0 < i < d - 1.$$

We now look at the exact sequence

$$(\#) \quad 0 \to \mathrm{H}_{\mathfrak{m}}^0(\overline{A}) \to \mathrm{H}_{\mathfrak{m}}^1(A) \xrightarrow{a} \mathrm{H}_{\mathfrak{m}}^1(A) \to \mathrm{H}_{\mathfrak{m}}^1(\overline{A}) \to \cdots$$

of local cohomology modules, induced from the exact sequence

$$0 \to A \xrightarrow{a} A \to \overline{A} \to 0.$$

Then because $\mathrm{H}^1_{\mathfrak{m}}(\overline{A}) = (0)$, we have $\mathrm{H}^1_{\mathfrak{m}}(A) = a\mathrm{H}^1_{\mathfrak{m}}(A)$, so that $\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, as $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module. Hence $\mathrm{H}^0_{\mathfrak{m}}(\overline{A}) = (0)$ by exact sequence (#), so that

$$\mathrm{H}^i_{\mathfrak{m}}(\overline{A}) = (0) \text{ for all } i \neq d-1.$$

Thus A is a Cohen-Macaulay ring, because so is $\overline{A}$ and a is A-regular. $\square$

In the above proof we do not use Corollary 1.5 (1).

Let Q be a parameter ideal in A and let

$$\mathrm{gr}_Q(A) = \bigoplus_{n \geq 0} Q^n/Q^{n+1}$$

denote the associated graded ring of Q. Let

$$\mathrm{H}(\mathrm{gr}_Q(A), \lambda) = \sum_{n=0}^{\infty} \ell_A(Q^n/Q^{n+1})\lambda^n$$

be the Hilbert series of $\mathrm{gr}_Q(A)$. Then we have $f(\lambda) \in Z[\lambda]$ such that

$$\mathrm{H}(\mathrm{gr}_Q(A), \lambda) = \frac{f(\lambda)}{(1-\lambda)^d}.$$

With this notaton, since $f'(1) = \mathrm{e}^1_Q(A)$, we have the following.

Corollary 1.9. *Let Q be a parameter ideal in A and let*

$$\mathrm{H}(\mathrm{gr}_Q(A), \lambda) = \frac{f(\lambda)}{(1-\lambda)^d}$$

be the Hilbert series of $\mathrm{gr}_Q(A)$, where $f(\lambda) \in \mathbb{Z}[\lambda]$. The A is a Cohen-Macaulay ring if and only if A is unmixed and $f'(1) = 0$.

We now ask what happens in the case where A is mixed. To answer this question, we need the following.

Observation 1.10. Let $U = \mathrm{U}_A(0)$ be the unmixed component of (0) in A and assume that $U \neq (0)$. We put

$$t = \dim_A U \ (< d) \quad \text{and} \quad B = A/U.$$

Let Q be a parameter ideal in A. Then for every $n \geq 0$ we have the exact sequence

$$0 \to U/[Q^{n+1} \cap U] \to A/Q^{n+1} \to B/Q^{n+1}B \to 0,$$

whence $\ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(U/[Q^{n+1} \cap U])$. Consequently, we have integers $\{\mathrm{s}_Q^i(U)\}_{0 \le i \le t}$ such that

$$\ell_A(U/[Q^{n+1} \cap U]) = \sum_{i=0}^{t} (-1)^i \cdot \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}$$

for all $n \gg 0$. Notice that $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U)$ (> 0). Hence

$$\sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(A) \binom{n+d-i}{d-i} = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(B) \binom{n+d-i}{d-i} + \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}$$

for all $n \gg 0$. Therefore, comparing the coefficients of $\binom{n+i}{i}$ in both sides, we get the following.

Fact 1.11.

$$(-1)^{d-i} \mathrm{e}_Q^{d-i}(A) = \begin{cases} (-1)^{d-i} \mathrm{e}_Q^{d-i}(B) + (-1)^{t-i} \mathrm{s}_Q^{t-i}(U) & (0 \le i \le t), \\ (-1)^{d-i} \mathrm{e}_Q^{d-i}(B) & (t < i \le d) \end{cases}$$

for $0 \le i \le d$.

Let me give an alternate proof of Corollary 1.5 (1).

Alternate proof of Corollary 1.5 (1). Let me use the same notation as in Observation 1.10. We may assume $d > 1$ and A is complete. Suppose that $\mathrm{e}_Q^1(A) > 0$. Then A is mixed by Theorem 1.8. Hence $U \ne (0)$ but $\mathrm{e}_Q^1(B) \le 0$ by Theorem 1.8, since B is unmixed. Therefore if $t < d-1$, by Observation 1.11 we get

$$0 > -\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) \ge 0,$$

which is absurd. Hence $t = d-1$, so that

$$\begin{aligned} 0 > -\mathrm{e}_Q^1(A) &= -\mathrm{e}_Q^1(B) + \mathrm{s}_Q^0(U) \\ &= -\mathrm{e}_Q^1(B) + \mathrm{e}_Q^0(U) \\ &\ge \mathrm{e}_Q^0(U) > 0 \end{aligned}$$

which is impossible. Thus $\mathrm{e}_Q^1(A) \le 0$. $\square$

Definition 1.12. A given Noetherian local ring A of dimension $d \geq 0$ is called a *Vasconcelos* ring, either if $d = 0$ or if $d > 0$ and $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q in A.

Every Vasconcelos ring of dimension at most 1 is Cohen-Macaulay.

Here is a characterization of Vasconcelos rings.

Theorem 1.13. *Suppose that* $d = \dim A \geq 2$. *Then the following conditions are equivalent.*

(1) A *is a Vasconcelos ring.*

(2) $\mathrm{e}_Q^1(A) = 0$ *for every parameter ideal* Q *in* A.

(3) $\widehat{A}/\mathrm{U}_{\widehat{A}}(0)$ *is a Cohen-Macaulay ring and* $\dim_{\widehat{A}} \mathrm{U}_{\widehat{A}}(0) \leq d-2$, *where* $\mathrm{U}_{\widehat{A}}(0)$ *denotes the unmixed component of* (0) *in the* $\mathfrak{m}$*-adic completion* $\widehat{A}$ *of* A.

(4) *The* $\mathfrak{m}$*-adic completion* $\widehat{A}$ *of* A *contains an ideal* I *such that* $\widehat{A}/I$ *is a Cohen-Macaulay ring and* $\dim_{\widehat{A}} I \leq d-2$.

When this is the case, $\widehat{A}$ *is a Vasconcelos ring,* $\mathrm{H}_{\mathfrak{m}}^{d-1}(A) = (0)$, *and the canonical module* $\mathrm{K}_{\widehat{A}}$ *of* $\widehat{A}$ *is a Cohen-Macaulay* $\widehat{A}$*-module.*

In Theorem 1.13 condition (3) is free from parameter ideals. Hence $\mathrm{e}_Q^1(A) = 0$ for every parameter ideal Q in A, once $\mathrm{e}_Q^1(A) = 0$ for some parameter. This is what the theorem says.

Proof of Theorem 1.13. Let me maintain the notation in Observation 1.10.

(1) $\Rightarrow$ (3) We may assume A is complete and $U \neq (0)$. If $t = d-1$, then by Observation 1.11 we get

$$0 = -\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) + \mathrm{e}_Q^0(U) > 0.$$

Hence $t < d-1$, so that by Observation 1.11 we get

$$0 = -\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B),$$

whence B is a Cohen-Macaulay ring.

(3) $\Rightarrow$ (2) We may assume A is complete. By Observation 1.11 we have $-\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) = 0$ for every parameter ideal Q in A.

(4) $\Rightarrow$ (3) We will show $I = U$. Notice that $\dim \widehat{A}/I = d$, since $\dim_{\widehat{A}} I < d$. Similarly, because $\dim_{\widehat{A}} I < d$, we have $I\widehat{A}_{\mathfrak{p}} = (0)$ for every $\mathfrak{p} \in \operatorname{Assh} \widehat{A}$, whence $I \subseteq U$. Suppose that $U/I \neq (0)$ and choose $\mathfrak{p} \in \operatorname{Ass}_{\widehat{A}} U/I$. Then since $\mathfrak{p} \in \operatorname{Ass}_{\widehat{A}} \widehat{A}/I$, we get $\mathfrak{p} \in \operatorname{Assh} \widehat{A}$, so that

$$U\widehat{A}_{\mathfrak{p}} = I\widehat{A}_{\mathfrak{p}} = (0),$$

which is impossible. Thus $I = U$. $\square$

The original proof of Corollary 1.5 (2) is as follows.

Corollary 1.14. *Let Q be a parameter ideal in A and assume that $\mathrm{e}_Q^i(A) = 0$ for all $1 \le i \le d$. Then A is a Cohen-Macaulay ring.*

Proof. We may assume A is complete. Since A is a Vasconcelos ring, by Theorem 1.13 $B = A/U$ is a Cohen-Macaulay ring. We must show $U = (0)$. If $U \neq (0)$, then by Observation 1.11 we get

$$0 = (-1)^{d-t}\mathrm{e}_Q^{d-t}(A) = (-1)^{d-t}\mathrm{e}_Q^{d-t}(B) + \mathrm{e}_Q^0(U) = \mathrm{e}_Q^0(U) > 0,$$

which is impossible. $\square$

We note an example of Vasconcelos rings which is not Cohen-Macaulay.

Example 1.15. Let R be a regular local ring with maximal ideal $\mathfrak{n}$ and $d = \dim R \ge 2$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. Let $D = R/(X_2, X_3, \cdots, X_d)$ and look at the idealization

$$A = R \ltimes D$$

of D over R. Then A is a Noetherian local ring with maximal ideal $\mathfrak{m} = \mathfrak{n} \times D$, $\dim A = d$, and $\operatorname{depth} A = 1$. We have $\mathrm{H}_{\mathfrak{m}}^i(A) = (0)$ for all $i \neq 1, d$ but

$$\mathrm{H}_{\mathfrak{m}}^1(A) \cong \mathrm{H}_{\mathfrak{n}}^1(D).$$

Hence A is not unmixed (and not a generalized Cohen-Macaulay ring, since $\mathrm{H}_{\mathfrak{m}}^1(A)$ is not a finitely generated A-module). For each $0 \le i \le d$ we put

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}.$$

Then the following assertions hold true.

(1) $\Lambda_i(A) = \{0\}$ for all $1 \le i \le d$ such that $i \ne d-1$.

(2) $\Lambda_0(A) = \{n \mid 0 < n \in \mathbb{Z}\}$ and $\Lambda_{d-1}(A) = \{(-1)^{d-1}n \mid 0 < n \in \mathbb{Z}\}$.

Hence A is a Vasconcelos ring, if $d > 2$. We furthermore have the following.

(3) Every parameter ideal of A is generated by a system of parameters which forms a d-sequence in A.

(4) $\mathrm{Proj}(\bigoplus_{n \ge 0} Q^n)$ is not a locally Cohen-Macaulay scheme for any parameter ideal Q in A.

Proof. Let $p : A \to R,\ p(a,x) = a$ be the projection. For each R–module M, let us denote by ${}_pM$ the A–module M which is considered to be an A–module via p. We look at the exact sequence

$$0 \to {}_pD \xrightarrow{\iota} A \xrightarrow{p} R \to 0,$$

where $\iota(x) = (0,x)$ for each $x \in D$. Let Q be a parameter ideal in A and put $\mathfrak{q} = QR$. Then we get the exact sequence

$$0 \to {}_p[D/\mathfrak{q}^{n+1}D] \to A/Q^{n+1} \to R/\mathfrak{q}^{n+1} \to 0.$$

Therefore since D is a DVR, we have

$$\begin{aligned} \ell_A(A/Q^{n+1}) &= \ell_R(R/\mathfrak{q}^{n+1}) + \ell_R(D/\mathfrak{q}^{n+1}D) \\ &= \mathrm{e}_{\mathfrak{q}}^0(R)\binom{n+d}{d} + \mathrm{e}_{\mathfrak{q}}^0(D)\binom{n+1}{1} \end{aligned}$$

for all $n \ge 0$. Hence

$$(-1)^i\mathrm{e}_Q^i(A) = \begin{cases} \mathrm{e}_{\mathfrak{q}}^0(R) & (i = 0), \\ \mathrm{e}_{\mathfrak{q}}^0(D) & (i = d-1), \\ 0 & (i \ne 0, d-1) \end{cases}$$

for $0 \le i \le d$. We now take $Q = (X_1^n, X_2, \cdots, X_d)$ $(n > 0)$. Then $\mathrm{e}_Q^0(A) = n$ and $(-1)^{d-1}\mathrm{e}_Q^{d-1}(A) = n$. Hence assertions (1) and (2) follow.

(3) Let $f_1, f_2, \cdots, f_d$ be a system of parameters in A and write $f_i = (a_i, x_i)$ with $a_i \in R$ and $x_i \in D$. After renumbering, we may assume that

$$(a_1, a_2, \cdots, a_d)D = a_1D \quad (\ne (0)).$$

Then $f_1, f_2, \cdots, f_d$ forms a d-sequence in A. In fact, let $1 \le i \le j \le d$ and let $\varphi \in (f_1, f_2, \cdots, f_{i-1}) : f_i f_j$. If $i = 1$, then since f_1 is A-regular, we get $f_j \varphi = 0$. Suppose $i > 1$ and look at the exact sequence

$$0 \to {}_p[D/(a_1, a_2, \cdots, a_{i-1})D] \xrightarrow{\iota} A/(f_1, f_2, \cdots, f_{i-1}) \to R/(a_1, a_2, \cdots, a_{i-1})R \to 0$$

(recall that $a_1, a_2, \cdots, a_{i-1}$ forms a regular sequence in R). We then have for some $x \in D$ $\overline{\varphi} = \overline{(0, x)}$ in $A/(f_1, f_2, \cdots, f_{i-1})$, where $\overline{*}$ denotes the image in $A/(f_1, f_2, \cdots, f_{i-1})$. Therefore $f_j \overline{\varphi} = \overline{(0, a_j x)} = 0$, because $a_j D = (0)$ in D. Hence $f_j \varphi \in (f_1, f_2, \cdots, f_{i-1})$, which shows that $f_1, f_2, \cdots, f_d$ is a d–sequence in A.

(4) This is because A is not a generalized Cohen-Macaulay ring. $\square$

We close this section with the following.

Proposition 1.16. *Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Let $G = \mathrm{gr}_Q(A) = \bigoplus_{n \ge 0} Q^n/Q^{n+1}$, $\mathcal{R} = \bigoplus_{n \ge 0} Q^n$, and $M = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$. Then the following assertions hold true.*

(1) *G_M is a Vasconcelos ring if and only if so is A.*

(2) *Suppose that A is a homomorphic image of a Cohen-Macaulay ring. Then $\mathcal{R}_M$ is a Vasconcelos ring, if so is A.*

Proof. (1) Recall that $\mathrm{e}^1_{(a_1^*, a_2^*, \cdots, a_d^*)G}(G) = \mathrm{e}^1_Q(A)$, where $a_i^* = a_i \bmod Q^2$ denotes the initial form of a_i.

(2) Let $U = \mathrm{U}_A(0)$ be the unmixed component of (0) in A. We may assume $U \ne (0)$. Then $B = A/U$ is a Cohen-Macaulay ring (Theorem 1.13 (3)). We look at the exact sequence

$$0 \to U^* \to \mathcal{R} \to \mathcal{S} \to 0,$$

where $\mathcal{S} = \mathcal{R}([Q+U]/U)$ denotes the Rees algebra of the ideal $[Q+U]/U$ in B. Then $\mathcal{R}([Q+U]/U)$ is a Cohen-Macaulay ring, since B is Cohen-Macaulay, while we have

$$\dim_{\mathcal{R}} U^* \le \dim_A U + 1 \le d - 1.$$

Thus $\mathcal{R}_M$ is a Vasconcelos ring by Theorem 1.13 (4). $\square$

2. Homological degrees

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We put

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

for each $0 \le i \le d$. With this notation we are interested in the following.

Question 2.1. (1) When is the set $\Lambda_1(A)$ finite?
(2) When $\sharp\Lambda_1(A) = 1$?

Notice that our characterization Theorem 1.13 of Vasconcelos rings shows that

$$0 \in \Lambda_1(A) \Rightarrow \Lambda_1(A) = \{0\}.$$

First of all, let me remind the estimation of $\mathrm{e}_Q^1(A)$ in terms of homological degrees ([38]). For simplicity, in the rest of this section let me assume that A is $\mathfrak{m}$–adically complete and the residue class field $A/\mathfrak{m}$ of A is infinite. Let M be a finitely generated A-module. For each $j \in \mathbb{Z}$ we put

$$M_j = \mathrm{Hom}_A(\mathrm{H}_{\mathfrak{m}}^j(M), E),$$

where $E = \mathrm{E}_A(A/\mathfrak{m})$ denotes the injective envelope of $A/\mathfrak{m}$. Then M_j is a finitely generated A-module and we have the following.

Fact 2.2. $\dim_A M_j \le j$ for all $j \in \mathbb{Z}$, where $\dim_A(0) = -\infty$.

Proof. Since A is complete, A is a homomorphic image of a Gorenstein complete local ring R with $\dim R = \dim A$. Passing to R, without loss of generality we may assume that A is a Gorenstein ring. Let $\mathfrak{p} \in \mathrm{Supp}_A M_j$. Then since

$$M_j \cong \mathrm{Ext}_A^{d-j}(M, A)$$

by the local duality theorem, we get

$$\mathrm{Ext}_{A_\mathfrak{p}}^{d-j}(M_\mathfrak{p}, A_\mathfrak{p}) \ne (0),$$

whence

$$d - j \le \mathrm{injdim}\, A_\mathfrak{p} = \dim A_\mathfrak{p}.$$

Hence $\dim A/\mathfrak{p} = d - \dim A_\mathfrak{p} \le j$, so that we have $\dim_A M_j \le j$. $\square$

Let I be a fixed $\mathfrak{m}$-primary ideal in A. The homological degree $\mathrm{hdeg}_I(M)$ of M with respect to I is defined, inductively, according to the dimension $s = \dim_A M$ of M.

Definition 2.3 ([38])**.** For each finitely generated A-module M with $s = \dim_A M$, we put

$$\mathrm{hdeg}_I(M) = \begin{cases} \ell_A(M) & (s = \dim_A M \leq 0), \\ \\ \mathrm{e}_I^0(M) + \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M_j) & (s > 0), \end{cases}$$

where $\mathrm{e}_I^0(M)$ denotes the multiplicity of M with respect to I.

Let me summarize some basic properties of $\mathrm{hdeg}_I(M)$.

Fact 2.4. (1) $0 \leq \mathrm{hdeg}_I(M) \in \mathbb{Z}$. $\mathrm{hdeg}_I(M) = 0$ if and only if $M = (0)$.

(2) (B. Ulrich) $\mathrm{hdeg}_I(M)$ depends only on $\overline{I}$. Namely, suppose that I, J are $\mathfrak{m}$-primary ideals in A. Then $\mathrm{hdeg}_I(*) = \mathrm{hdeg}_J(*)$ if and only if $\overline{I} = \overline{J}$, where $\overline{I}$ and $\overline{J}$ denote respectively the integral closures of I and J.

(3) If $M \cong M'$, then $\mathrm{hdeg}_I(M) = \mathrm{hdeg}_I(M')$.

(4) $\mathrm{hdeg}_I(M) = \mathrm{hdeg}_I(M/\mathrm{H}_\mathfrak{m}^0(M)) + \ell_A(\mathrm{H}_\mathfrak{m}^0(M))$.

(5) If M is a generalized Cohen-Macaulay A-module, then

$$\mathrm{hdeg}_I(M) = \mathrm{e}_I^0(M) + \mathbb{I}(M),$$

where $\mathbb{I}(M) = \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{h}^j(M)$ denotes the Stückrad-Vogel invariant of M.

Proof. (2) Let me check the *only if* part. We have $\mathrm{e}_I^0(A/\mathfrak{p}) = \mathrm{e}_J^0(A/\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec} A$ with $\dim A/\mathfrak{p} = 1$. Let $V = \overline{A/\mathfrak{p}}$ be the normalization of $A/\mathfrak{p}$. Then $IV = JV$, since V is a DVR with $\mathrm{e}_I^0(V) = \mathrm{e}_J^0(V)$. Therefore, as $(I+J)V = IV$, we get

$$\mathrm{e}_{[(I+J)+\mathfrak{p}]/\mathfrak{p}}^0(A/\mathfrak{p}) = \mathrm{e}_{[I+\mathfrak{p}]/\mathfrak{p}}^0(A/\mathfrak{p}),$$

whence the ideal $[(I+J)+\mathfrak{p}]/\mathfrak{p}$ is integral over $[I+\mathfrak{p}]/\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$ possessing $\dim A/\mathfrak{p} = 1$. As Ulrich showed in his lecture, this condition implies $I + J \subseteq \overline{I}$. Hence $J \subseteq \overline{I}$, so that $\overline{I} = \overline{J}$ by symmetry.

(3) We may assume $\dim_A M = s > 0$. Let $W = \mathrm{H}_\mathfrak{m}^0(M)$ and $M' = M/W$. Then

$$[M']_j \cong M_j \quad \text{for all} \quad j > 0 \quad \text{and} \quad [M']_0 = (0).$$

Hence

$$\begin{aligned}
\mathrm{hdeg}_I(M) &= \mathrm{e}_I^0(M) + \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M_j) \\
&= \mathrm{e}_I^0(M') + \sum_{j=1}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M') + \ell_A(\mathrm{Hom}_A(W, E)) \\
&= \mathrm{hdeg}_I(M') + \ell_A(W).
\end{aligned}$$

(4) Notice that $\mathrm{hdeg}\, M_j = \ell_A(M_j) = \ell_A(\mathrm{H}_{\mathfrak{m}}^j(M)) = \mathrm{h}^j(M)$ for all $j \neq s$. □

The following results play key roles in the analysis of homological degree.

Lemma 2.5 ([38, Proposition 3.18]). *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of finitely generated A-modules. Then the following assertions hold true.*

(1) *If $\ell_A(Z) < \infty$, then* $\mathrm{hdeg}_I(Y) \leq \mathrm{hdeg}_I(X) + \mathrm{hdeg}_I(Z)$.

(2) *If $\ell_A(X) < \infty$, then* $\mathrm{hdeg}_I(Y) = \mathrm{hdeg}_I(X) + \mathrm{hdeg}_I(Z)$.

Remark 2.6. In Lemma 2.5 (1) the equality

$$\mathrm{hdeg}_I(Y) = \mathrm{hdeg}_I(X) + \mathrm{hdeg}_I(Z)$$

does not hold true in general, even though $\ell_A(Z) < \infty$. For example, suppose that A is a Cohen-Macaulay local ring with $\dim A = 1$. We look at the exact sequence

$$0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0.$$

Then, since $\mathfrak{m}$ is a Cohen-Macaulay A–module, we get

$$\mathrm{hdeg}_I(A) = \mathrm{e}_I^0(A) = \mathrm{e}_I^0(\mathfrak{m}) = \mathrm{hdeg}_I(\mathfrak{m}).$$

Therefore, since $\mathrm{hdeg}_I(A/\mathfrak{m}) = 1$, we have

$$\mathrm{hdeg}_I(A) < \mathrm{hdeg}_I(A) + 1 = \mathrm{hdeg}_I(\mathfrak{m}) + \mathrm{hdeg}_I(A/\mathfrak{m}).$$

Let $\mathcal{R} = A[It] \subseteq A[t]$ be the Rees algebra of I, where t is an indeterminate. Let

$$f : I \to \mathcal{R}, \quad a \mapsto at$$

be the identification of I with $\mathcal{R}_1 = It$. We put

$$\mathrm{Proj}\, \mathcal{R} = \{\mathfrak{p} \mid \mathfrak{p} \text{ is a graded prime ideal of } \mathcal{R} \text{ such that } \mathfrak{p} \not\supseteq \mathcal{R}_+\}.$$

We then have the following.

Lemma 2.7. *Let M be a finitely generated A-module. Then there exists a finite subset $\mathcal{F} \subseteq \operatorname{Proj} \mathcal{R}$ such that*

(1) *every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m} I]$ is superficial for M with respect to I and*
(2) *for each $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m} I]$ we have $\operatorname{hdeg}_I(M/aM) \leq \operatorname{hdeg}_I(M)$.*

Proof. Induction on $s = \dim_A M$. If $s \leq 0$, choose $\mathcal{F} = \emptyset$. Suppose $s = 1$ and let $\mathcal{F} = \{\mathfrak{p} \in \operatorname{Ass}_{\mathcal{R}} \operatorname{gr}_I(M) \mid \mathfrak{p} \not\supseteq \mathcal{R}_+\}$. Then every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m} I]$ is superficial for M with respect to I. We have $\operatorname{hdeg}_I(M) = \mathrm{e}_I^0(M) + \mathrm{h}^0(M)$ and $\operatorname{hdeg}_I \overline{M} = \ell_A(\overline{M})$, where $\overline{M} = M/aM$. Let $W = \mathrm{H}^0_{\mathfrak{m}}(M)$ and look at the exact sequence

$$0 \to W \to M \to M' \to 0,$$

where $M' = M/W$. Then since M' is a Cohen-Macaulay A–module, the element a is M'–regular and we get

$$0 \to W/aW \to \overline{M} \to M'/aM' \to 0.$$

Hence

$$\begin{aligned} \ell_A(\overline{M}) &= \ell_A(W/aW) + \ell_A(M'/aM') \\ &\leq \ell_A(W) + \mathrm{e}^0_{(a)}(M') \\ &= \ell_A(W) + \mathrm{e}^0_I(M') \\ &= \mathrm{h}^0(M) + \mathrm{e}^0_I(M) \\ &= \operatorname{hdeg}_I(M). \end{aligned}$$

Suppose that $s > 1$ and our assertion holds true for $s-1$. Let $\mathcal{F}$ be a finite subset of $\operatorname{Proj} \mathcal{R}$ such that for every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m} I]$, a is superficial for M and M_j $(0 \leq j \leq s-2)$ and $\operatorname{hdeg}_I(M_j/aM_j) \leq \operatorname{hdeg}_I(M_j)$ for all $1 \leq j \leq s-1$. Then, since $\ell_A((0) :_M a) < \infty$, we get a long exact sequence

$$0 \to (0) :_M a \to \mathrm{H}^0_{\mathfrak{m}}(M) \xrightarrow{a} \mathrm{H}^0_{\mathfrak{m}}(M) \to \mathrm{H}^0_{\mathfrak{m}}(\overline{M}) \to \mathrm{H}^1_{\mathfrak{m}}(M) \xrightarrow{a} \mathrm{H}^1_{\mathfrak{m}}(M) \to \mathrm{H}^1_{\mathfrak{m}}(\overline{M}) \to \cdots$$

of local cohomology modules, where $\overline{M} = M/aM$. Hence

$$(\#) \qquad \begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M_{i+1}/aM_{i+1} & \longrightarrow & \overline{M}_i & \longrightarrow & (0):_{M_i} a & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & M_i & & \end{array}$$

for each $0 \le i \le s-2$. Because $\ell_A((0):_{M_i} a) < \infty$, thanks to Lemma 2.5, we get

$$\begin{aligned} \mathrm{hdeg}_I(\overline{M}_i) &\le \mathrm{hdeg}_I((0):_{M_i} a) + \mathrm{hdeg}_I(M_{i+1}/aM_{i+1}) \\ &\le \mathrm{hdeg}_I(M_i) + \mathrm{hdeg}_I(M_{i+1}), \end{aligned}$$

so that

$$\begin{aligned} \mathrm{hdeg}_I(\overline{M}) &= \mathrm{e}_I^0(\overline{M}) + \sum_{j=0}^{s-2} \binom{s-2}{j} \mathrm{hdeg}_I(\overline{M}_j) \\ &\le \mathrm{e}_I^0(M) + \sum_{j=0}^{s-2} \binom{s-2}{j} [\mathrm{hdeg}_I(M_j) + \mathrm{hdeg}_I(M_{j+1})] \\ &= \mathrm{e}_I^0(M) + \sum_{j=0}^{s-1} \binom{s-1}{j} \mathrm{hdeg}_I(M_j) \\ &= \mathrm{hdeg}_I(M) \end{aligned}$$

as claimed. □

Definition 2.8. Let M be a finitely generated A–module with $\dim_A M = s \ge 2$. We put

$$\mathrm{T}_I(M) = \sum_{j=1}^{s-1} \binom{s-2}{j-1} \mathrm{hdeg}_I(M_j).$$

Fact 2.9. Let M be a finitely generated A–module with $\dim_A M = s \ge 3$. Then the proof of Lemma 2.7 shows that there exists a finite subset $\mathcal{F} \subseteq \mathrm{Proj}\,\mathcal{R}$ such that for every $a \in I \setminus \bigcup_{\mathfrak{p} \in \mathcal{F}} [f^{-1}(\mathfrak{p}) + \mathfrak{m}I]$, a is superficial for M with respect to I and we have the inequality

$$\mathrm{T}_I(M/aM) \le \mathrm{T}_I(M).$$

We now come to the main result of this section.

Theorem 2.10. *Suppose that $d \geq 2$ and let Q be a parameter ideal in A. Then*

$$0 \geq \mathrm{e}_Q^1(M) \geq -\mathrm{T}_Q(M)$$

for every finitely generated A-module M with $\dim_A M = d$.

Proof. The inequality $0 \geq \mathrm{e}_Q^1(M)$ follows from Corollary 1.5 (if necessary, use the principle of idealization to reduce the problem to the ring case; the technique in the ring case, in fact, works also for modules). Let $M' = M/\mathrm{H}_\mathfrak{m}^0(M)$. Then, since $\mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(M')$ and $\mathrm{T}_Q(M) = \mathrm{T}_Q(M')$, to see that $\mathrm{e}_Q^1(M) \geq -\mathrm{T}_Q(M)$, passing to M', we may assume that $\mathrm{H}_\mathfrak{m}^0(M) = (0)$. Suppose $d = 2$ and choose $a \in Q \backslash \mathfrak{m}Q$ so that a is superficial for M with respect to Q and $\mathrm{hdeg}_Q(M_1/aM_1) \leq \mathrm{hdeg}_Q M_1$. Let $\overline{M} = M/aM$. Then since a is M–regular, we have $M_1/aM_1 = \overline{M}_0$. Hence

$$\mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(\overline{M}) = -\mathrm{h}^0(\overline{M}) = -\,\mathrm{hdeg}_Q(M_1/aM_1),$$

so that by the choice of a we get

$$\mathrm{e}_Q^1(M) \geq -\,\mathrm{hdeg}_Q(M_1) = -\mathrm{T}_Q(M)$$

as claimed. Suppose $d > 2$ and choose $a \in Q \setminus \mathfrak{m}Q$ so that a is superficial for M and $\mathrm{T}_Q(M/aM) \leq \mathrm{T}_Q(M)$. Then by induction on d we see

$$\mathrm{e}_Q^1(M) = \mathrm{e}_Q^1(M/aM) \geq -\mathrm{T}_Q(M/aM) \geq -\mathrm{T}_Q(M),$$

proving Theorem 2.10. □

Corollary 2.11. *If $d \geq 2$, then*

$$0 \geq \mathrm{e}_Q^1(A) \geq -\mathrm{T}_Q(A)$$

for every parameter ideal Q in A.

Corollary 2.12 ([39])**.** *Suppose that $d \geq 2$ and let Q be a parameter ideal in A. Then the set*

$$\Lambda(Q) = \{\mathrm{e}_\mathfrak{q}^1(A) \mid \mathfrak{q} \textit{ is a parameter ideal of } A \textit{ such that } \overline{\mathfrak{q}} = \overline{Q}\}$$

is finite, where $\overline{\mathfrak{q}}$ and $\overline{Q}$ denote respectively the integral closures of $\mathfrak{q}$ and Q.

Proof. Since $\overline{\mathfrak{q}} = \overline{Q}$, we have $\mathrm{T}_{\mathfrak{q}}(A) = \mathrm{T}_Q(A)$. Hence $0 \geq \mathrm{e}^1_{\mathfrak{q}}(A) \geq -\mathrm{T}_{\mathfrak{q}}(A) = -\mathrm{T}_Q(A)$, so that the set $\Lambda(Q)$ is finite. $\square$

Corollary 2.13 ([15]). *Suppose that $d \geq 2$ and that A is a generalized Cohen-Macaulay ring. Then*

$$0 \geq \mathrm{e}^1_Q(A) \geq -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A)$$

for every parameter ideal Q in A, whence the set

$$\Lambda_1(A) = \{\mathrm{e}^1_Q(A) \mid Q \textit{ is a parameter ideal in } A\}$$

is finite.

3. When is the set $\Lambda_1(A)$ finite?

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We put

$$\Lambda_1(A) = \{\mathrm{e}^1_Q(A) \mid A \text{ is a parameter ideal in } A\}.$$

In this section we shall prove the following.

Theorem 3.1. *Suppose that A is unmixed and $d \geq 2$. If $\Lambda_1(A)$ is a finite set, then*

$$\mathfrak{m}^\ell \mathrm{H}^j_{\mathfrak{m}}(A) = (0) \textit{ for all } j \neq d,$$

where $\ell = -\min \Lambda_1(A)$, so that A is a generalized Cohen-Macaulay ring.

Before going ahead, let me remind what is known in the case where A is a generalized Cohen-Macaulay ring.

Proposition 3.2. *Suppose that A is a generalized Cohen-Macaulay ring and $d \geq 2$. Let Q be a parameter ideal of A. Then the following assertions hold true.*

(1) ([15, 28]) $0 \geq \mathrm{e}^1_Q(A) \geq -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A)$.

(2) ([33, Korollar 3.2]) *If Q is standard, then* $\mathrm{e}^1_Q(A) = -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A)$.

Hence the set $\Lambda_1(A)$ is finite.

Let me explain the notion of standard parameter ideal. Suppose that A is a generalized Cohen-Macaulay ring. Hence $\sup_Q [\ell_A(A/Q) - \mathrm{e}^1_Q(A)] < \infty$, which is equal to $\mathbb{I}(A) = \sum_{j=0}^{d-1} \binom{d-1}{j} \mathrm{h}^j(A)$. We say that a parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ is standard, if

$$\ell_A(A/Q) - \mathrm{e}_Q^0(A) = \mathbb{I}(A).$$

This condition is equivalent to saying that $a_1, a_2, \cdots, a_d$ form a strong d-sequence in any order, that is $a_1^{n_1}, a_2^{n_2}, \cdots, a_d^{n_d}$ is a d-sequence in A in any order for all integers $n_1, n_2, \cdots, n_d > 0$. For each generalized Cohen-Macaulay ring A, one can find an integer $\ell \gg 0$ such that every parameter ideal Q contained in $\mathfrak{m}^\ell$ is standard. Therefore a Noetherian local ring A is Buchsbaum if and only if A is a generalized Cohen-Macaulay ring and every parameter ideal of A is standard.

As for Schenzel's formula 3.2 (2) let me give a few comments. P. Schenzel [33] actually gave the following.

Theorem 3.3 ([33, Korollar 3.2]). *Suppose that A a generalized Cohen-Macaulay ring and let Q be a standard parameter ideal in A. Then we have*

$$(-1)^i \mathrm{e}_Q^i(A) = \begin{cases} \mathrm{h}^0(A) & (i = d), \\ \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \mathrm{h}^j(A) & (0 < i < d) \end{cases}$$

for $1 \le i \le d$.

Therefore the values $\{\mathrm{e}_Q^i(A)\}_{1 \le i \le d}$ are independent of the choice of standard parameter ideals Q, provided A is a generalized Cohen-Macaulay ring.

Theorem 3.3 follows by induction on d and the proof is not very complicated. We however do not know at this moment, except $i = 1, 2$, about the variation of values $\mathrm{e}_Q^i(A)$ of arbitrary parameter ideals Q, even in the case where A is a generalized Cohen-Macaulay ring.

Let me state a conjecture.

Conjecture 3.4. Let $\mathrm{T}_Q^i(A) = \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \operatorname{hdeg}_Q(A_j)$. Then $|\mathrm{e}_Q^i(A)| \le \mathrm{T}_Q^i(A)$ for $0 < i < d$.

To prove Theorem 3.1 I need the following observation. For a while, suppose that $d \ge 2$ and that A is a homomorphic image of a Gorenstein ring. Then by a theorem of

N. T. Cuong [5] (see [24] also) we have a system of parameters of A, say $x_1, x_2, \cdots, x_d$, which forms a strong d-sequence in A, that is the equality

$$(x_1^{n_1}, x_2^{n_2}, \cdots, x_{i-1}^{n_{i-1}} : x_i^{n_i} x_j^{n_j} = (x_1^{n_1}, x_2^{n_2}, \cdots, x_{i-1}^{n_{i-1}} : x_j^{n_j})$$

holds true for all integers $1 \le i \le j \le d$ and $n_1, n_2, \cdots, n_d > 0$. For each integer $q > 0$ let $\Gamma_q(A)$ denote the set of $\mathrm{e}^1_{(a_1,a_2,\cdots,a_d)}(A)$ where $a_1, a_2, \cdots, a_d$ runs through systems of parameters in A such that $(a_1, a_2, \cdots, a_d) \subseteq \mathfrak{m}^q$ and $a_1, a_2, \cdots, a_d$ forms a d-sequence in A. We notice that

$$\Lambda_1(A) \supseteq \Gamma_q(A) \supseteq \Gamma_{q+1}(A) \neq \emptyset$$

for all $q > 0$. With this notation we furthermore have the following.

Theorem 3.5. *Suppose that* $\operatorname{Ass} A = \operatorname{Assh} A$ *and that* $\Gamma_q(A)$ *is a finite set for some* $q > 0$*. Then* $\mathfrak{m}^\ell \mathrm{H}^j_\mathfrak{m}(A) = (0)$ *for all* $j \neq d$*, where* $\ell = -\min \Gamma_q(A)$*.*

Theorem 3.1 readily follows from Theorem 3.5, passing to the completion; notice that $-\min \Gamma_q(A) \le -\min \Lambda_1(A)$, since $\Gamma_q(A) \subseteq \Lambda_1(A)$.

Proof of Theorem 3.5. Suppose that $d = 2$. Then A is a generalized Cohen-Macaulay ring, since A is unmixed. Choose a standard parameter ideal $Q \subseteq \mathfrak{m}^q$. We then have

$$\mathrm{e}^1_Q(A) = -\mathrm{h}^1(A) = \min \Lambda_1(A)$$

by Proposition 3.3 (2). Hence $\ell = \mathrm{h}^1(A)$ and $\mathfrak{m}^\ell \mathrm{H}^1_\mathfrak{m}(A) = (0)$.

Suppose that $d > 2$ and that our assertion holds true for $d-1$. Recall that the set

$$\mathcal{F}_1 = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \neq \mathfrak{m},\ \operatorname{ht}_A \mathfrak{p} > 1 = \operatorname{depth} A_\mathfrak{p}\}$$

is finite (Lemma 1.7). We choose $x \in \mathfrak{m}$ so that

$$x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass} A} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}_1} \mathfrak{p}.$$

Let $n \ge q$ be any integer and put $y = x^n$. Let $\overline{A} = A/(y)$ and $B = \overline{A}/\mathrm{H}^0_\mathfrak{m}(\overline{A})$. We then have

$$\operatorname{Ass}_A \overline{A} \setminus \{\mathfrak{m}\} = \operatorname{Assh}_A \overline{A},$$

so that $\mathrm{H}^0_\mathfrak{m}(\overline{A}) = \mathrm{U}_{\overline{A}}(0)$ and $\operatorname{Ass}_A B = \operatorname{Assh}_A B$ by Lemma 1.7; hence B is unmixed.

Let $y_2, y_3, \cdots, y_d \in \mathfrak{m}^q$ be a system of parameters of $\overline{A}$ and assume that $y_2, y_3, \cdots, y_d$ form a d-sequence in $\overline{A}$. Then, since $y = y_1$ is A-regular, $y_1, y_2, \cdots, y_d$ forms a d-sequence in A, so that y_1 is superficial with respect to the ideal $(y_1, y_2, \cdots, y_d)$. Hence

$$\mathrm{e}^1_{(y_2, y_3, \cdots, y_d)}(\overline{A}) = \mathrm{e}^1_{(y_1, y_2, \cdots, y_d)}(A) \in \Gamma_q(A).$$

Thus $\Gamma_q(\overline{A}) \subseteq \Gamma_q(A)$ and $\Gamma_q(\overline{A})$ is a finite set.

Choose an integer $q' \geq q$ so that

$$\mathrm{H}^0_{\mathfrak{m}}(\overline{A}) \cap \mathfrak{n}^{q'} = (0),$$

where $\mathfrak{n} = \mathfrak{m}/(y)$ denotes the maximal ideal of $\overline{A}$. Let $y_2, y_3, \cdots, y_d \in \mathfrak{m}^{q'}$ be a system of parameters for B which form a d-sequence in B. Then, thanks to the condition $\mathrm{H}^0_{\mathfrak{m}}(\overline{A}) \cap \mathfrak{n}^{q'} = (0)$, the sequence $y_2, y_3, \cdots, y_d$ form a d-sequence also in $\overline{A}$ and we have

$$\mathrm{e}^1_{(y_2, y_3, \cdots, y_d)}(B) = \mathrm{e}^1_{(y_2, y_3, \cdots, y_d)}(\overline{A}) \in \Gamma_{q'}(\overline{A}).$$

Thus $\Gamma_{q'}(B)$ is a finite set (recall that $\Gamma_{q'}(B) \subseteq \Gamma_{q'}(\overline{A}) \subseteq \Gamma_q(\overline{A}) \subseteq \Gamma_q(A)$). Consequently, thanks to the hypothesis of induction, we get

$$\mathfrak{m}^{\ell'} \mathrm{H}^j_{\mathfrak{m}}(B) = (0)$$

for all $j \neq d-1$, where $\ell' = -\min \Gamma_{q'}(B) \leq \ell = -\min \Gamma_q(A)$. Hence

$$\mathfrak{m}^{\ell} \mathrm{H}^j_{\mathfrak{m}}(\overline{A}) = \mathfrak{m}^{\ell} \mathrm{H}^j_{\mathfrak{m}}(B) = (0)$$

for all $1 \leq j \leq d-2$.

We now look at the exact sequence

$$\cdots \to \mathrm{H}^j_{\mathfrak{m}}(\overline{A}) \to \mathrm{H}^{j+1}_{\mathfrak{m}}(A) \xrightarrow{x^n} \mathrm{H}^{j+1}_{\mathfrak{m}}(A) \to \cdots$$

of local cohomology modules, induced from the exact sequence

$$0 \to A \xrightarrow{x^n} A \to \overline{A} \to 0.$$

We then have

$$\mathfrak{m}^{\ell} \left[(0) :_{\mathrm{H}^{j+1}_{\mathfrak{m}}(A)} x^n \right] = (0)$$

for all $1 \leq j \leq d-2$ and $n \geq q$, where $\ell = -\min \Lambda_q(A)$. Because n and ℓ are independent of each other, this implies

$$\mathfrak{m}^{\ell} \mathrm{H}^{j+1}_{\mathfrak{m}}(A) = (0)$$

for all $1 \leq j \leq d-2$, that is $\mathfrak{m}^\ell \mathrm{H}^j_\mathfrak{m}(A) = (0)$ for $2 \leq j \leq d-1$. On the other hand, thanks to the exact sequence

$$\cdots \to \mathrm{H}^1_\mathfrak{m}(A) \xrightarrow{x^n} \mathrm{H}^1_\mathfrak{m}(A) \to \mathrm{H}^1_\mathfrak{m}(\overline{A}) \to \cdots$$

together with the fact that $\mathrm{H}^1_\mathfrak{m}(A)$ is a finitely generated A-module (Lemma 1.7 (1)), choosing the integer $n \geq q$ so that $x^n \mathrm{H}^1_\mathfrak{m}(A) = (0)$, we get

$$\mathrm{H}^1_\mathfrak{m}(A) \hookrightarrow \mathrm{H}^1_\mathfrak{m}(\overline{A}),$$

whence $\mathfrak{m}^\ell \mathrm{H}^1_\mathfrak{m}(A) = (0)$. Thus $\mathfrak{m}^\ell \mathrm{H}^i_\mathfrak{m}(A) = (0)$ for all $i \neq d$, which proves Theorem 3.5. □

Theorem 3.6. *Suppose that $d \geq 2$. Then the following conditions are equivalent.*

(1) $\Lambda_1(A)$ *is a finite set.*

(2) $\widehat{A}/U$ *is a generalized Cohen-Macaulay ring and* $\dim_{\widehat{A}} U \leq d-2$*, where* $U = \mathrm{U}_{\widehat{A}}(0)$.

When this is the case, we have $\Lambda_1(A) = \Lambda_1(\widehat{A}/U)$.

Proof. We may assume A is complete and $U \neq (0)$.

(1) ⇒ (2) Let $t = \dim_A U$ and $B = A/U$. Then by Observation 1.11, for every parameter ideal Q in A we have

$$(-1)^{d-i}\mathrm{e}^{d-i}_Q(A) = \begin{cases} (-1)^{d-i}\mathrm{e}^{d-i}_Q(B) + \mathrm{s}^{t-i}_Q(U) & (0 \leq i \leq t), \\ (-1)^{d-i}\mathrm{e}^{d-i}_Q(B) & (t < i \leq d). \end{cases}$$

Therefore, if $t = d-1$, we get

$$-\mathrm{e}^1_Q(A) = -\mathrm{e}^1_Q(B) + \mathrm{e}^0_Q(U).$$

Hence, choosing a system $a_1, a_2, \cdots, a_d$ of parameters of A so that $a_d U = (0)$ and taking $Q = (a_1^n, a_2^n, \cdots, a_d^n)$ $(n > 0)$, we get

$$-\mathrm{e}^1_{(a_1^n, a_2^n, \cdots, a_d^n)}(A) = -\mathrm{e}^1_{(a_1^n, a_2^n, \cdots, a_d^n)}(B) + n^{d-1}\mathrm{e}^0_{(a_1, a_2, \cdots, a_{d-1})}(U) \geq n^{d-1}$$

for all integers $n > 0$, which is impossible. Hence $t \leq d-2$, so that

$$-\mathrm{e}^1_Q(A) = -\mathrm{e}^1_Q(B)$$

for every parameter ideal Q in A. Consequently $\Lambda_1(A) = \Lambda_1(B)$ and B is a generalized Cohen-Macaulay ring by Theorem 3.1 (recall that $\Lambda_1(B)$ is a finite set and B is unmixed).

(2) $\Rightarrow$ (1) By Observation 1.11 we have $\Lambda_1(A) = \Lambda_1(B)$, since $\dim_A U \leq d-2$. Therefore $\Lambda_1(A)$ is finite, as so is $\Lambda_1(B)$. $\square$

Corollary 3.7. *Suppose that $\Lambda_i(A)$ is a finite set for all $1 \leq i \leq d$. Then A is a generalized Cohen-Macaulay ring.*

Proof. We may assume that $d > 1$, A is complete, and $U = \mathrm{U}_{\widehat{A}}(0) \neq (0)$. Then by Theorem 3.6 $B = A/U$ is a generalized Cohen-Macaulay ring and $\dim_A U \leq d-2$. We want to show $\ell_A(U) < \infty$, that is $t = 0$. Assume the contrary and choose a system $a_1, a_2, \cdots, a_d$ of parameters in A so that $a_1, a_2, \cdots, a_d$ is a standard system of parameters for B and

$$(a_{t+1}, a_{t+2}, \cdots, a_d)U = (0).$$

We look at the parameter ideal $Q = (a_1^n, a_2^n, \cdots, a_d^n)$ with $n > 0$. Then

$$(-1)^{d-t} e_Q^{d-t}(A) = (-1)^{d-t} \mathrm{e}_Q^{d-t}(B) + \mathrm{e}_Q^0(U)$$

by Observation 1.11. This is, however, impossible, because $(-1)^{d-t}\mathrm{e}_Q^{d-t}(B)$ is constant by Proposition 3.2 (2), $\mathrm{e}_Q^0(U) = \mathrm{e}^0_{(a_1^n, a_2^n, \cdots, a_t^n)}(U) \geq n^t \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \geq n^t$, and $\Lambda_{d-t}(A)$ is finite by our assumption. Hence $t = 0$ and A is a generalized Cohen-Macaulay ring. $\square$

There are left two natural questions.

Question 3.8. (1) How about the converse of Corollary 3.7?
(2) What happen in the case where $\sharp\Lambda_1(A) = 1$?

Later I will discuss question (1). As for the second question, if A is a Buchsbaum ring with $d = \dim A \geq 2$, then by Proposition 3.2 (2) we get

$$\mathrm{e}_Q^1(A) = -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A)$$

for every parameter ideal Q in A; hence

$$\Lambda_1(A) = \left\{ -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A) \right\}.$$

The converse is also true, as we show in the following.

Theorem 3.9. *Suppose that A is unmixed. Then A is a Buchsbaum ring, if $\sharp\Lambda_1(A) = 1$.*

The general answer is the following.

Theorem 3.10. *Suppose that $d \geq 2$. Then the following conditions are equivalent.*

(1) $\sharp\Lambda_1(A) = 1$.

(2) $\widehat{A}/U$ *is a Buchsbaum ring and* $\dim_{\widehat{A}} U \leq d-2$, *where* $U = \mathrm{U}_{\widehat{A}}(0)$.

When this is the case, we have

$$\Lambda_1(A) = \left\{ -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(\widehat{A}/U) \right\}.$$

Corollary 3.11. *Suppose that $\sharp\Lambda_i(A) = 1$ for all $1 \leq i \leq d$. Then $A/\mathrm{H}^0_{\mathfrak{m}}(A)$ is a Buchsbaum ring.*

Let me talk a little bit about the proof of Theorem 3.9.

Sketch of Proof of Theorem 3.9. We may assume A is complete and $d \geq 2$. Then A is a generalized Cohen-Macaulay ring by Theorem 3.5, because $\Lambda_1(A)$ is a finite set. Since $\mathrm{e}^1_Q(A) = -\sum_{j=1}^{d-1}\binom{d-2}{j-1}\mathrm{h}^j(A)$ for every standard parameter ideal Q in A (Proposition 3.2 (2)), we get $\Lambda_1(A) = \left\{-\sum_{j=1}^{d-1}\binom{d-2}{j-1}\mathrm{h}^j(A)\right\}$, so that

$$\mathrm{e}^1_Q(A) = -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A)$$

for every parameter ideal Q in A. Then apply the following result of K. Ozeki.

Theorem 3.12 (K. Ozeki [17])**.** *Suppose that A is a generalized Cohen–Macaulay ring, $d \geq 2$, and* $\operatorname{depth} A > 0$. *Let Q be a parameter ideal in A. Then Q is standard if and only if*

$$\mathrm{e}^1_Q(A) = -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{h}^j(A).$$

Thanks to Theorem 3.12, every parameter ideal Q in A is standard. Hence A is a Buchsbaum ring. $\square$

Question 3.13. Let Q be a parameter ideal in A. Find a criterion for the equality

$$\mathrm{e}_Q^1(A) = -\sum_{j=1}^{d-1} \binom{d-2}{j-1} \mathrm{hdeg}_Q(A_j),$$

assuming that A is complete, $d \geq 2$, and the residue class field $A/\mathfrak{m}$ of A is infinite.

4. How about $\mathrm{e}_Q^2(A)$? – uniform bounds for the sets $\Lambda_i(A)$ $(1 \leq i \leq d)$

We have just proved that A is a generalized Cohen-Macaulay ring, if

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

is a finite set for all $1 \leq i \leq d$. The converse is also true and we have the following.

Theorem 4.1. *Let A be a Noetherian local ring with $d = \dim A > 0$. Then the following conditions are equivalent.*

(1) *A is a generalized Cohen-Macaulay ring.*

(2) *$\Lambda_i(A)$ is a finite set for all $1 \leq i \leq d$.*

To prove the implication (1) $\Rightarrow$ (2) we need the notion of regularity. Let Q be a parameter ideal of A and let

$$G = \mathrm{gr}_Q(A) = \bigoplus_{n \geq 0} Q^n/Q^{n+1}$$

be the associated graded ring of Q. Let $M = \mathfrak{m}G + G_+$ be the graded maximal ideal of G. For each $i \in \mathbb{Z}$ let

$$\mathrm{a}_i(G) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}_M^i(G)]_n \neq (0)\},$$

where $[\mathrm{H}_M^i(G)]_n$ $(n \in \mathbb{Z})$ denotes the homogeneous component of the graded local cohomology module $\mathrm{H}_M^i(G)$ with degree n.

Definition 4.2. We put

$$\mathrm{reg}(G) = \sup\{i + \mathrm{a}_i(G) \mid i \in \mathbb{Z}\}$$

and call it the regularity of G. Notice that $0 \le \operatorname{reg} G \in \mathbb{Z}$.

The notion of regularity plays an important role in the analysis of graded rings and modules. In our case we have the following.

Theorem 4.3 ([18])**.** *Suppose that A is a generalized Cohen-Macaulay ring and let Q be a parameter ideal in A. Then the following assertions hold true.*

(1) $|\mathrm{e}_Q^1(A)| \le \mathbb{I}(A)$.
(2) $|\mathrm{e}_Q^i(A)| \le (r+1)^{i-1}{\cdot}\mathbb{I}(A){\cdot}3{\cdot}2^{i-2}$ *for all* $2 \le i \le d$, *where* $r = \operatorname{reg}(\mathrm{gr}_Q(A))$.

The right hand side of the inequality in Theorem 4.3 (2)is a huge number but once we agree with this, we can apply the following result to our case in order to see the finiteness of the sets $\Lambda_i(A)$.

Theorem 4.4 ([25])**.** *Suppose that A is a generalized Cohen-Macaulay ring and let Q be a parameter ideal in A. Then*

$$\operatorname{reg}(\mathrm{gr}_Q(A)) \le \begin{cases} \max\{\mathbb{I}(A)-1,\ 0\} & (d=1), \\ \max\{4{\cdot}\mathbb{I}(A)^{(d-1)!} - \mathbb{I}(A) - 1,\ 0\} & (d>1). \end{cases}$$

The second number appearing in the right hand side of the estimation of Theorem 4.4 is still very huge, but anyway, combining these two theorems, we see $\mathrm{e}_Q^i(A)$ ($1 \le i \le d$) has a uniform bound independent of the choice of parameter ideals Q, if A is a generalized Cohen-Macaulay ring.

Question 4.5. What are the sharp bounds for $\mathrm{e}_Q^i(A)$?

This is a problem different from the question of the finiteness of the sets $\Lambda_i(A)$. Our guess is the following.

Guess 4.6. We have

$$|\mathrm{e}_Q^i(A)| \le \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \mathrm{h}^j(A)$$

for all $0 < i < d$, if A is a generalized Cohen-Macaulay ring. More generally, for an arbitrary Noetherian local ring A, we have

$$|\mathrm{e}_Q^i(A)| \leq \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} \mathrm{hdeg}_Q(A_j)$$

for $0 < i < d$, provided A is complete and the residue class field $A/\mathfrak{m}$ of A is infinite.

Let me state study $\mathrm{e}_Q^2(A)$.

Theorem 4.7. *Suppose that A is complete with infinite residue class field and $d \geq 3$. Let Q be a parameter ideal in A. Then for every finitely generated unmixed A-module M with $\dim_A M = d$, we have th following eestimation*

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} \mathrm{hdeg}_Q(M_j) \leq \mathrm{e}_Q^2(M) \leq \mathrm{T}_Q^2(M).$$

In Theorem 4.7, for the latter inequality we do not need the unmixedness assumption on the modules M. However, unless M is unmixed, the former inequality in Theorem 4.7 does not hold true in general. Later we will explore an example (Example 4.9).

As a direct consequence of Theorem 4.7 we have

Corollary 4.8. *Suppose that A is complete with infinite residue class field and $d \geq 3$. Assume that A is unmixed. Then*

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} \mathrm{hdeg}_Q(A_j) \leq \mathrm{e}_Q^2(A) \leq \mathrm{T}_Q^2(A)$$

for every parameter ideal Q in A. Therefore, for a fixed parameter ideal Q in A, the set

$$\{\mathrm{e}_{\mathfrak{q}}^2(A) \mid \mathfrak{q} \textit{ is a parameter ideal in } A \textit{ such that } \overline{\mathfrak{q}} = \overline{Q}\}$$

is finite.

Example 4.9. Let R be a complete regular local ring with maximal ideal $\mathfrak{n}$, infinite residue class field, and $\dim R = 3$. Let $\mathfrak{n} = (X, Y, Z)$ and put $S = R/(Z^n)$ $(n > 0)$. Then

$$\mathrm{e}_{\mathfrak{n}}^0(S) = n, \ \mathrm{e}_{\mathfrak{n}}^1(S) = \frac{n(n-1)}{2}, \text{ and } \mathrm{e}_{\mathfrak{n}}^2(S) = \frac{n(n-1)(n-2)}{6}.$$

We look at the idealization $A = R \ltimes S$ of S over R and put $Q = \mathfrak{n}A$. Then A is mixed with $\dim A = 3$, $\operatorname{depth} A = 2$,

$$\operatorname{hdeg}_Q(A_2) = n, \quad \text{and} \quad \mathrm{e}_Q^2(A) = -\mathrm{e}_{\mathfrak{n}}^1(S) = -\frac{n(n-1)}{2},$$

whence

$$-\operatorname{hdeg}_Q(A_2) > \mathrm{e}_Q^2(A), \quad \text{if} \quad n \geq 4.$$

Proof. Since

$$\mathrm{H}(\mathrm{gr}_{\mathfrak{n}}(S), \lambda) = \frac{1 + \lambda + \cdots + \lambda^{n-1}}{(1-\lambda)^2},$$

we get $\mathrm{e}_{\mathfrak{n}}^0(S) = n$, $\mathrm{e}_{\mathfrak{n}}^1(S) = \frac{n(n-1)}{2}$, and $\mathrm{e}_{\mathfrak{n}}^2(S) = \frac{n(n-1)(n-2)}{6}$. On the other hand, since S is a Gorenstein ring and since

$$\mathrm{H}_{\mathfrak{m}}^2(A) \cong {}_p[\mathrm{H}_{\mathfrak{n}}^2(S)]$$

(here $p : A \to R, p(a,x) = a$ denotes the projection), we have

$$\operatorname{hdeg}_Q(A_2) = \operatorname{hdeg}_Q(S_2) = \operatorname{hdeg}_{\mathfrak{n}}(S_2) = \mathrm{e}_{\mathfrak{n}}^0(S) = n.$$

Recall now that

$$\begin{aligned} \ell_A(A/Q^{\ell+1}) &= \ell_R(R/\mathfrak{n}^{\ell+1}) + \ell_S(S/\mathfrak{n}^{\ell+1}S) \\ &= \binom{\ell+3}{3} + \left[\mathrm{e}_{\mathfrak{n}}^0(S)\binom{\ell+2}{2} - \mathrm{e}_{\mathfrak{n}}^1(S)\binom{\ell+1}{1} + \mathrm{e}_{\mathfrak{n}}^2(S)\right] \end{aligned}$$

for all $\ell \gg 0$ and we readily have

$$(-1)^i \mathrm{e}_Q^i(A) = \begin{cases} 1 & (i=0), \\ \mathrm{e}_{\mathfrak{n}}^0(S) = n & (i=1), \\ -\mathrm{e}_{\mathfrak{n}}^1(S) = -\frac{n(n-1)}{2} & (i=2), \\ \mathrm{e}_{\mathfrak{n}}^2(S) = \frac{n(n-1)(n-2)}{6} & (i=3). \end{cases}$$

□

Let me note a little bit about Proof of Corollary 4.8 in order to explain why I cannot extend this result, say for $\mathrm{e}_Q^3(A)$.

In the case of $\mathrm{e}_Q^1(A)$ the key of our argument is the following fact [15, Lemma 2.4 (1)]

$$\mathrm{e}_Q^1(A) = -\mathrm{h}^0(A), \text{ if } d = 1.$$

For the estimation of $\mathrm{e}_Q^2(A)$ the key is the following.

Proposition 4.10. *Suppose that A is unmixed and $d = 2$. Then*

$$-\mathrm{h}^1(A) \le \mathrm{e}_Q^2(A) \le 0$$

for every parameter ideal Q in A.

Proof. We may assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Take an integer $\ell \gg 0$ and put $I = Q^\ell$, $a = x^\ell$, and $b = y^\ell$. Let $G = \mathrm{gr}_I(A)$. Then, thanks to a theorem of L. T. Hoa [19], we see that

(1) $[\mathrm{H}_M^i(G)]_n = (0)$ for all $i \in \mathbb{Z}$ and $n > 0$, where $M = \mathfrak{m}G + G_+$ and
(2) $I^2 = \mathfrak{q}I$, where $\mathfrak{q} = (a, b)$.

The element a is still superficial with respect to I and we furthermore have the following.

Claim 1.

$$\mathrm{e}_Q^2(A) = \mathrm{e}_I^2(A) = -\ell_A([((a) : b) \cap I]/(a)) \le 0.$$

Proof of Claim 1. We have $\mathrm{e}_Q^2(A) = \mathrm{e}_I^2(A)$ (in fact, $\mathrm{e}_{Q^\ell}^2(A) = \mathrm{e}_Q^2(A)$ for all integers $\ell > 0$), while $\mathrm{a}_2(G) < 0$ by condition (2). Therefore $\mathrm{a}_0(G) < 0$, since $\mathrm{a}_1(G) \le 0$ and $\operatorname{depth} A > 0$. Hence $\mathrm{H}_M^0(G) = (0)$, so that we have

$$\mathrm{e}_I^2(A) = -\ell_A([\mathrm{H}_M^1(G)]_0),$$

thanks to a classical theorem of Serre. Let $\overline{G} = \mathrm{gr}_{I/(a)}(A/(a))$. Then since the initial form $a^* = a \bmod Q^2$ of a is regular on G, we get $\overline{G} \cong G/a^*G$, $[\mathrm{H}_M^1(G)]_0 \cong [\mathrm{H}_M^0(\overline{G})]_1$, and $[\mathrm{H}_M^0(\overline{G})]_n = (0)$ for all $n \ge 2$. It is now standard to show that

$$[\mathrm{H}_M^0(\overline{G})]_1 \cong [((a) : b) \cap I]/(a) \subseteq [(a) : b]/(a) \subseteq \mathrm{H}_\mathfrak{m}^0(A/(a)) \cong (0) :_{\mathrm{H}_\mathfrak{m}^1(A)} a,$$

whence

$$\mathrm{e}_Q^2(A) = \mathrm{e}_I^2(A) = -\ell_A([((a):b)\cap I]/(a)) \le 0$$

which proves Claim 1. $\square$

Proposition 4.10 now readily follows from Claim 1, since

$$\ell_A([((a):b)\cap I]/(a)) \le \mathrm{h}^1(A).$$

$\square$

We are in a position to prove Corollary 4.8.

Proof of Corollary 4.8. Let $C = \mathrm{Hom}_A(K_A, K_A)$ and look at the exact sequence

$$0 \to A \xrightarrow{\varphi} C \to X \to 0,$$

where $\varphi(a) = a1_{K_A}$ for all $a \in A$. Let us choose an element $a \in Q \setminus \mathfrak{m}Q$ so that

(1) a is superficial for all of A, C, and X with respect to Q and
(2) a is superficial for A_j with respect to Q and $\mathrm{hdeg}_Q(A_j/aA_j) \le \mathrm{hdeg}_Q(A_j)$ for all $j \ge 0$.

We put $\overline{A} = A/aA$, $\overline{C} = C/aC$, and $\overline{X} = X/aX$. Then since a is C-regular, we have the exact sequence

$$0 \to (0):_X a \to \overline{A} \xrightarrow{\overline{\varphi}} \overline{C} \to \overline{X} \to 0.$$

Let $L = \mathrm{Im}\overline{\varphi}$. Then since $\ell_A((0):_X a) < \infty$, we have $\dim_A L = d-1$ and L is unmixed (recall that $\overline{C}$ is unmixed, since $\mathrm{depth}_{A_\mathfrak{p}} C_\mathfrak{p} \ge \inf\{2, \dim A_\mathfrak{p}\}$ for all $\mathfrak{p} \in \mathrm{Spec}\, A$), whence $(0):_X a = \mathrm{H}^0_\mathfrak{m}(\overline{A})$. Therefore, if $d = 3$, then L is a generalized Cohen-Macaulay A-module with $\dim_A L = 2$ and $\mathrm{depth}_A L > 0$, whence

$$\mathrm{e}_Q^2(A) = \mathrm{e}_Q^2(\overline{A}) = \mathrm{e}_Q^2(L) + \ell_A((0):_X a).$$

Consequently, thanks to Proposition 4.10, we have

$$\ell_A((0):_X a) - \mathrm{h}^1(\overline{A}) \le \mathrm{e}_Q^2(A) = \mathrm{e}_Q^2(L) + \ell_A((0):_X a) \le \ell_A((0):_X a),$$

because $\mathrm{h}^1(L) = \mathrm{h}^1(\overline{A})$. Since $A_1/aA_1 \cong \overline{A}_0$, we also have

$$\begin{aligned} \ell_A((0) :_X a) = \mathrm{h}^0(\overline{A}) &= \ell_A(\overline{A}_0) \\ &= \mathrm{hdeg}_Q(\overline{A}_0) \\ &= \mathrm{hdeg}_Q(A_1/aA_1) \\ &\leq \mathrm{hdeg}_Q A_1. \end{aligned}$$

Look now at the exact sequence

$$0 \to (0) :_{A_1} a \to A_1 \xrightarrow{a} A_1 \to \overline{A}_0 \to 0.$$

We then have $\ell_A(\overline{A}_0) = \ell_A((0) :_{A_1} a)$, whence $\ell_A((0) :_X a) = \ell_A((0) :_{A_1} a)$. Therefore we get

$$\begin{aligned} \ell_A((0) :_{A_1} a) - \mathrm{h}^1(\overline{A}) &= \ell_A((0) :_{A_1} a) - [\mathrm{hdeg}_Q(A_2/aA_2) + \mathrm{hdeg}_Q((0) :_{A_1} a)] \\ &= -\mathrm{hdeg}_Q(A_2/aA_2) \geq -\mathrm{hdeg}_Q(A_2), \end{aligned}$$

because $\mathrm{h}^1(\overline{A}) = \mathrm{hdeg}_Q(A_2/aA_2) + \mathrm{hdeg}_Q((0) :_{A_1} a)$ by the exact sequence

$$0 \to A_2/aA_2 \to \overline{A}_1 \to (0) :_{A_1} a \to 0.$$

Hence $-\mathrm{hdeg}_Q(A_2) \leq \mathrm{e}_Q^2(A) \leq \mathrm{hdeg}_Q(A_1)$.

Suppose that $d > 3$ and that our assertion holds true for $d-1$. Then since a is A–regular, we have the long exact sequence

$$\begin{aligned} 0 \quad \to \quad & \mathrm{H}^0_{\mathfrak{m}}(\overline{A}) \to \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(\overline{A}) \to \mathrm{H}^2_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^2_{\mathfrak{m}}(A) \to \cdots \\ \to \quad & \mathrm{H}^j_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^j_{\mathfrak{m}}(A) \to \mathrm{H}^j_{\mathfrak{m}}(\overline{A}) \to \mathrm{H}^{j+1}_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^{j+1}_{\mathfrak{m}}(A) \to \cdots . \end{aligned}$$

Taking the Matlis dual $\mathrm{Hom}_A(*, \mathrm{E}_A(A/\mathfrak{m}))$ of it, we get short exact sequences

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & A_{j+1}/aA_{j+1} & \longrightarrow & \overline{A}_j & \longrightarrow & (0) :_{A_j} a & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & A_j & & \end{array}$$

for each $1 \leq j \leq d-2$. Hence

$$\mathrm{hdeg}_Q(\overline{A}_j) \leq \mathrm{hdeg}_Q((0) :_{A_j} a) + \mathrm{hdeg}_Q A_{j+1} \leq \mathrm{hdeg}_Q(A_j) + \mathrm{hdeg}_Q(A_{j+1})$$

by Lemma 2.5, because $\ell_A((0) :_{A_j} a) < \infty$. We then have by the hypothesis of induction that

$$\begin{aligned}
-\sum_{j=2}^{d-2} \binom{d-4}{j-2} \mathrm{hdeg}_Q(\overline{A}_j) \leq \mathrm{e}_Q^2(\overline{A}) &\leq \mathrm{T}_Q^2(\overline{A}) \\
&= \sum_{j=1}^{d-3} \binom{d-4}{j-1} \mathrm{hdeg}_Q(\overline{A}_j) \\
&\leq \sum_{j=1}^{d-3} \binom{d-4}{j-1} [\mathrm{hdeg}_Q(A_j) + \mathrm{hdeg}_Q(A_{j+1})] \\
&= \sum_{j=1}^{d-2} \binom{d-3}{j-1} \mathrm{hdeg}_Q(A_j) \\
&= \mathrm{T}_Q^2(A),
\end{aligned}$$

while we similarly get

$$\begin{aligned}
-\sum_{j=2}^{d-1} \binom{d-3}{j-2} \mathrm{hdeg}_Q(A_j) &= -\sum_{j=2}^{d-2} \binom{d-4}{j-2} [\mathrm{hdeg}_Q(A_j) + \mathrm{hdeg}_Q(A_{j+1})] \\
&\leq -\sum_{j=2}^{d-2} \binom{d-4}{j-2} \mathrm{hdeg}_Q(\overline{A}_j).
\end{aligned}$$

Hence

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} \mathrm{hdeg}_Q(A_j) \leq \mathrm{e}_Q^2(A) \leq \mathrm{T}_Q^2(A),$$

because $\mathrm{e}_Q^2(A) = \mathrm{e}_Q^2(\overline{A})$. $\square$

Question 4.11. When does the equality $\mathrm{e}_Q^2(A) = \mathrm{T}_Q^2(A)$ hold true?

Here is an answer in the case where A is a generalized Cohen-Macaulay ring.

Theorem 4.12 ([18]). *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$, $\operatorname{depth} A > 0$, and infinite residue class field. Let Q be a parameter ideal in A. Then the following conditions are equivalent.*

(1) $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} \mathrm{h}^j(A)$.

(2) *There exist elements* $a_1, a_2, \cdots, a_d \in A$ *such that* (a) $Q = (a_1, a_2, \cdots, a_d)$, (b) $a_1, a_2, \cdots, a_d$ *is a d-sequence in* A, *and* (c) $Q\mathrm{H}^j_{\mathfrak{m}}(A/(a_1, a_2, \cdots, a_k)) = (0)$, *whenever* $j + k \le d - 2$, $0 < j$, *and* $0 \le k$.

Remark 4.13. The parameter ideal Q is not necessarily standard, even if

$$\mathrm{e}^2_Q(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} \mathrm{h}^j(A).$$

For example, suppose that A is a generalized Cohen-Macaulay ring with $d = 3$ and $\operatorname{depth} A = 2$. Assume that $\mathfrak{m}\mathrm{H}^2_{\mathfrak{m}}(A) \neq (0)$ and choose $a \in \mathfrak{m}$ so that a is regular but $a\mathrm{H}^2_{\mathfrak{m}}(A) \neq (0)$. Let $b, c \in \mathfrak{m}$ be a standard system of parameters for $A/(a)$. Then a, b, c forms a d-sequence in A, so that

$$\mathrm{e}^2_{(a,b,c)}(A) = 0 = \mathrm{h}^1(A).$$

The ideal Q is, however, not standard, because $Q{\cdot}\mathrm{H}^2_{\mathfrak{m}}(A) \neq (0)$.

5. A method to compute $\mathrm{e}^1_Q(A)$

In this section let A be a Noetherian local ring with $\dim A = 2$ and assume that A is a homomorphic image of a Gorenstein local ring, say $A = R/\mathfrak{a}$ with R a Gorenstein local ring and $\mathfrak{a}$ an ideal in it. We assume that A is unmixed. Hence $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module (Lemma 1.7 (1)). Let $Q = (a, b)$ be a parameter ideal in A. Then, thanks to a lemma of Davis [23, Theorem 124], we get a regular sequence x, y in R so that $a = x \bmod \mathfrak{a}$ and $b = y \bmod \mathfrak{a}$. We put $\mathfrak{q} = (x, y)R$; hence $Q = \mathfrak{q}A$. Let $B = \operatorname{Hom}_A(\mathrm{K}_A, \mathrm{K}_A)$ be the endomorphism ring of the canonical module K_A and look at the exact sequence

$$(E) \qquad 0 \to A \xrightarrow{\varphi} B \to C \to 0$$

of A-modules, where $\varphi(a)$ is defined, for each $a \in A$, to be the homothety $a{\cdot}1_{K_A}$ of a. Then, since $\operatorname{depth}_A \mathrm{K}_A = 2$, B is a Cohen-Macaulay A-module with $\dim_A B = 2$ and we get $C \cong \mathrm{H}^1_{\mathfrak{m}}(A)$ as A-modules (cf. [1, Theorem 3.2, Proof of Theorem 4.2], [2, Theorem

1.6]). Let $n \geq 0$ be an integer and let $\mathbb{M}$ denote the $n+1$ by $n+2$ matrix defined by

$$\mathbb{M} = \begin{pmatrix} x & y & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & y & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & y & 0 & \cdots & 0 \\ & & & \cdots & & & \\ 0 & 0 & \cdots & 0 & 0 & x & y \end{pmatrix}.$$

Then the ideal $\mathfrak{q}^{n+1}$ is generated by the maximal minors of the matrix $\mathbb{M}$ and, thanks to the theorem of Hilbert–Burch ([23, Exercises 8, p. 148]), the R-module $R/\mathfrak{q}^{n+1}$ has the resolution of the form

$$0 \longrightarrow F_2 = R^{n+1} \xrightarrow{{}^t\mathbb{M}} F_1 = R^{n+2} \xrightarrow{\partial} F_0 = R \longrightarrow R/\mathfrak{q}^{n+1} \longrightarrow 0,$$

in which the homomorphism ∂ is defined by

$$\partial(\mathbf{e}_j) = (-1)^j{\cdot}\det \mathbb{M}_j$$

for all $1 \leq j \leq n+2$ (here $\mathbb{M}_j$ denotes the matrix obtained by deleting from $\mathbb{M}$ the j-th column and $\{\mathbf{e}_j\}_{1\leq j\leq n+2}$ denotes the standard basis of R^{n+2}). Consequently, for each R-module X, $\operatorname{Tor}_j^R(R/\mathfrak{q}^{n+1}, X)$ is computed as the j-th homology module of the complex

$$0 \to X^{n+1} = F_2 \otimes_R X \xrightarrow{{}^t\mathbb{M}\otimes_R 1_X} X^{n+2} = F_1 \otimes_R X \xrightarrow{\partial\otimes_R 1_X} X = F_0 \otimes_R X \longrightarrow 0.$$

Setting $X = C$, we therefore have, since $\operatorname{Tor}_1^R(R/\mathfrak{q}^{n+1}, B) = (0)$ (see [3, Theorem 9.1.6]; notice that the ideal $\mathfrak{q} = (x, y)R$ is generated by a B-regular sequence of length 2), the exact sequence

$$0 \to \operatorname{Tor}_1^R(R/\mathfrak{q}^{n+1}, C) \to A/Q^{n+1} \to B/Q^{n+1}B \to C/Q^{n+1}C \to 0.$$

Therefore

$$(2) \qquad \ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(\operatorname{Tor}_1^R(R/\mathfrak{q}^{n+1}, C)) - \ell_A(C/Q^{n+1}C)$$

for all $n \geq 0$. On the other hand, since the alternating sum of the length of homology modules of the complex

$$0 \to C^{n+1} = F_2 \otimes_R C \xrightarrow{{}^t\mathbb{M}\otimes_R 1_C} C^{n+2} = F_1 \otimes_R C \xrightarrow{\partial\otimes_R 1_C} C = F_0 \otimes_R C \longrightarrow 0$$

is 0, we get

$$\ell_R(\operatorname{Tor}_1^R(R/\mathfrak{q}^{n+1}, C)) = \ell_R(\operatorname{Tor}_2^R(R/\mathfrak{q}^{n+1}, C)) + \ell_A(C/Q^{n+1}C).$$

Hence by equation (2) we have for all $n \geq 0$ that

$$(3) \quad \ell_A(A/Q^{n+1}) = \mathrm{e}_Q^0(A)\binom{n+2}{2} + \ell_R(\mathrm{Tor}_2^R(R/\mathfrak{q}^{n+1}, C)),$$

because $\mathrm{e}_Q^0(A) = \mathrm{e}_Q^0(B) = \ell_A(B/QB)$ (see exact sequence (E); recall that B is a Cohen-Macaulay A-module with $\dim_A B = 2$ and $\ell_A(C) < \infty$) and $\ell_A(B/Q^{n+1}B) = \ell_A(B/QB)\binom{n+2}{2}$ for all $n \geq 0$. We remember the isomorphism

$$\mathrm{Tor}_2^R(R/\mathfrak{q}^{n+1}, C) \cong \mathrm{Ker}(C^{n+1} \xrightarrow{{}^t\mathbb{M}} C^{n+2}),$$

that is

$$\mathrm{Tor}_2^R(R/\mathfrak{q}^{n+1}, C) \cong \left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in C^{n+1} \;\middle|\; a\alpha_i + b\alpha_{i-1} = 0 \text{ for all } 0 \leq i \leq n+1 \right\},$$

where $\alpha_{-1} = \alpha_{n+1} = 0$ for convention.

Summarizing these observations, we get the following, which we will use very frequently in this paper. The same method of computation of $\mathrm{e}_Q^i(A)$ is given in [10, Example 3.8] and [28, Section 3].

Proposition 5.1. *Let $n \geq 0$ be an integer and let*

$$T_n = \left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in C^{n+1} \;\middle|\; a\alpha_i + b\alpha_{i-1} = 0 \text{ for all } 0 \leq i \leq n+1 \right\}.$$

Then the following assertions hold true.

(1) $\ell_A(A/Q^{n+1}) = \mathrm{e}_Q^0(A)\binom{n+2}{2} + \ell_A(T_n)$ *for all* $n \geq 0$.

(2) $-\ell_A(C) \leq \mathrm{e}_Q^1(A) \leq -\ell_A((0):_C Q)$.

(3) *Suppose* $aC = (0)$. *Then* $\mathrm{e}_Q^1(A) = -\ell_A((0):_C b) = \ell_A(C/bC)$ *and* $\mathrm{e}_Q^2(A) = 0$.

(4) ([10, Example 3.8], [28, Section 3]) *Suppose* $QC = (0)$. *Then* $\mathrm{e}_Q^1(A) = -\ell_A(C)$ *and* $\mathrm{e}_Q^2(A) = 0$.

Proof. See equation (3) for assertion (1). We see $\ell_A((0):_C Q)(n+1) \leq \ell_A(T_n) \leq \ell_A(C)(n+1)$, since $[(0):_C Q]^{n+1} \subseteq T_n \subseteq C^{n+1}$. Hence we have assertion (2). If $aC = (0)$, then $T_n = [(0):_C b]^{n+1}$, so that $\ell_A(A/Q^{n+1}) = \mathrm{e}_Q^0(A)\binom{n+2}{2} + \ell_A((0):_C b)\binom{n+1}{1}$ by assertion (1). Hence assertion (3) follows, because $\ell_A((0) :_C b) = \ell_A(C/bC)$. Assertion (4) is now obvious. $\square$

Example 5.2. Let $R = k[[X, Y, Z, W]]$ be the formal power series ring over a field k and we look at the local ring

$$A = R/[(X, Y)^\ell \cap (Z, W)],$$

where $\ell \geq 1$ is an integer. Then A is a 2-dimensional generalized Cohen-Macaulay local ring with $\operatorname{depth} A = 1$. In this local ring A, the following assertions hold true.

(1) Let a, b be a system of parameters in A. Then a, b or b, a forms a d-sequence in A. Hence every parameter ideal of A is generated by a d-sequence.
(2) $\Lambda_1(A) = \{-\frac{(2\ell-n+1)n}{2} \mid 0 < n \in \mathbb{Z}\}$ and $\Lambda_2(A) = \{0\}$.

Proof. Let $\mathfrak{m}$ be the maximal ideal in A and let x, y, z, and w be the images of X, Y, Z, and W in A. Then $\mathfrak{m} = (x, y, z, w)$. Thanks to the exact sequence

$$0 \to A \to A/(x, y)^\ell \oplus A/(z, w) \to A/[(x, y)^\ell + (z, w)] \to 0,$$

we have $\dim A = 2$, $\operatorname{depth} A = 1$, and $\mathrm{H}^1_\mathfrak{m}(A) \cong A/[(x, y)^\ell + (z, w)]$. Hence A is a generalized Cohen-Macaulay local ring. Let $C = A/[(x, y)^\ell + (z, w)]$.

Now choose a system a, b of parameters in A and put $Q = (a, b)$. Suppose that $aC = (0)$. If $bC = (0)$, then Q is standard and so, $\mathrm{e}^1_Q(A) = -\ell_A(C) = \frac{(\ell+1)\ell}{2}$ and $\mathrm{e}^2_Q(A) = 0$ by Proposition 5.1 (4). Suppose $bC \neq (0)$. Then $[(b) : (a^2)]/(b) \subseteq \mathrm{U}(b)/(b) \cong (0) :_C b$. Therefore, since $a[(0) :_C b] = (0)$, we get that b, a is a d-sequence. Let $n = v_{\mathfrak{m}_C}(\overline{b})$ denote the order of the image $\overline{b}$ of b in C with respect to the maximal ideal $\mathfrak{m}_C$ of C. Then $0 < n < \ell$ and $(0) :_C b = \mathfrak{m}_C^{\ell-n}$, whence $\mathrm{e}^1_Q(A) = -\ell_A((0) :_C b) = -\ell_A(\mathfrak{m}_C^{\ell-n}) = -\frac{(2\ell-n+1)n}{2}$ and $\mathrm{e}^2_Q(A) = 0$ by Proposition 5.1 (3)

Suppose that $aC \neq (0)$ and $bC \neq (0)$. We may assume that

$$n = v_{\mathfrak{m}_C}(\overline{a}) \leq m = v_{\mathfrak{m}_C}(\overline{b}).$$

Then $b[(0) :_C a] \subseteq \mathfrak{m}_C^m{\cdot}\mathfrak{m}_C^{\ell-n} \subseteq \mathfrak{m}_C^\ell = (0)$, so that $b\mathrm{U}(a) \subseteq (a)$, whence a, b is a d-sequence in A. We have

$$\begin{aligned} T &= \left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix} \in C^{q+1} \,\middle|\, a\alpha_i + b\alpha_{i-1} = 0 \text{ for all } 0 \le i \le q+1 \right\} \\ &= [(0) :_C a]^{q+1} \end{aligned}$$

for all $q \geq 0$, whence $\mathrm{e}_Q^1(A) = -\ell_A((0) :_C a) = -\frac{(2\ell-n+1)n}{2}$ and $\mathrm{e}_Q^2(A) = 0$.

Let $0 < n < \ell$ be integers and look at the system $a = x^\ell - z, b = y^n - w$ of parameters in A. Then $aC = (0)$, $bC \neq (0)$, and $v_{\mathfrak{m}_C}(\overline{b}) = n$. Hence

$$\Lambda_1(A) = \{-\frac{(2\ell - n + 1)n}{2} \mid 0 < n \leq \ell\}$$

as claimed. $\square$

As for the following question, I do not know the answer in general. The answer is affirmative, if $\ell \leq 3$, or $d \leq 2$, or the parameter ideals are homogeneous.

Question 5.3. Let $\ell, d > 0$ be integers and let $R = k[[X_1, X_2, \cdots, X_d, Y_1, Y_2, \cdots, Y_d]]$ be the formal power series ring over a filed k. We look at the local ring

$$A = R/[(X_1, X_2, \cdots, X_d)^\ell \cap (Y_1, Y_2, \cdots, Y_d)].$$

Then, is every parameter ideal in A generated by a d-sequence of length d?

Thanks to Proposition 5.1, we similarly have the following.

Example 5.4. Let $\ell \geq 1$ be an integer and $R = k[[X, Y, Z, W]]$ be the formal power series ring over a field k and we look at the local ring

$$A = R/[(X^\ell, Y^\ell) \cap (Z, W)].$$

Then A is a 2-dimensional generalized Cohen-Macaulay local ring with $\operatorname{depth} A = 1$. Let $\mathfrak{q} = (X - Z, Y - W)$. Then $Q = \mathfrak{q}A$ is a parameter ideal in A and $\mathrm{e}_Q^0(A) = \ell^2 + 1$, $\mathrm{e}_Q^1(A) = -\ell$, and $\mathrm{e}_Q^2(A) = -\frac{\ell(\ell-1)}{2}$. Hence $\mathrm{e}_Q^2(A) < 0$ if $\ell \geq 2$, so that Q cannot be generated by a d-sequence of length 2 (Proposition **??** (2)).

Proof. Let us discuss only the case where $\ell \geq 2$. Let $C = k[X,Y,Z,W]/(X^\ell, Y^\ell, Z, W)$ $(\cong \mathrm{H}^1_{\mathfrak{m}}(A))$ and let $n \geq \ell + 1$ be an integer. We look at the graded C-module

$$T = \left\{ \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in C^{n+1} \,\middle|\, x\alpha_i + y\alpha_{i-1} = 0 \text{ for all } 0 \leq i \leq n+1 \right\},$$

where x, y be the images of $X - Z, Y - W$ in C. Let T_q $(q \in \mathbb{Z})$ denote the homogeneous component of the graded module T. Then $T_q = (0)$ if $q \leq \ell - 2$, because $(0) :_C x = x^{\ell-1}C$. Suppose $\ell - 1 \leq q \leq 2\ell - 2$. Let $\{c_i\}_{0 \leq i \leq n+1}$ be a family of elements in k such that $c_i = 0$ for all $n + 1 \leq i \leq n - 2\ell + q + 3$. We put

$$(*) \quad \alpha_i = \begin{cases} \sum_{j=1}^{i+1} (-1)^{j-1} c_{i-j+1} x^{\ell-j} y^{q-\ell+j} & \text{if} \quad 0 \leq i \leq \ell - 1, \\ \sum_{j=1}^{\ell} (-1)^{j-1} c_{i+1-j} x^{\ell-j} y^{q-\ell+j} & \text{if} \quad \ell \leq i \leq n. \end{cases}$$

Then $\begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in T_q$ and it is routine to check that T_q consists of all those elements which are defined by the above equation $(*)$. Hence $\dim_k T_q = n - 2\ell + q + 3$, if $\ell - 1 \leq q \leq 2\ell - 2$. Consequently, we have

$$\begin{aligned} \dim_k T &= \sum_{q=0}^{2\ell-2} \dim_k T_q \\ &= \sum_{q=\ell-1}^{2\ell-2} (n - 2\ell + q + 3) \\ &= (n+1)\ell - \frac{(\ell-1)\ell}{2}. \end{aligned}$$

Hence $\mathrm{e}^1_Q(A) = -\ell$ and $\mathrm{e}^2_Q(A) = -\frac{\ell(\ell-1)}{2}$ by Proposition 5.1. As $\mathrm{e}^0_Q(A) = \mathrm{e}^0_{\mathfrak{q}}(R/(X^\ell, Y^\ell)) + \mathrm{e}^0_{\mathfrak{q}}(R/(Z,W)) = \ell^2 + 1$, this completes the computation. $\square$

6. Constancy of $\mathrm{e}^1_Q(A)$ with the common $\overline{Q}$

Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. In this section we study the question, raised by Wolmer V. Vasconcelos, of whether $\mathrm{e}^1_Q(A)$ is independent of the choice of minimal reductions Q of I, where I is an $\mathfrak{m}$-primary ideal in A.

We begin with the following general result.

Proposition 6.1. *Let M be a finitely generated A-module with $\dim_A M = s$ and let Q and Q' be parameter ideals for M with $\overline{Q} = \overline{Q'}$ in A. Suppose that there exists an exact sequence*

$$0 \to L \to M \to M/L \to 0$$

of A-modules such that $L \neq (0)$, $\dim_A L = t < s$, and M/L is a Cohen-Macaulay A-module. Then

$$\mathrm{e}^1_Q(M) = \mathrm{e}^1_{Q'}(M),$$

where $\mathrm{e}^1_Q(M)$ (resp. $\mathrm{e}^1_{Q'}(M)$) denote the first Hilbert coefficients of M with respect to Q (resp. Q').

Proof. Passing to the ring $A/[(0) : M]$, we may assume that $(0) : M = (0)$, whence $s = d$ and both Q and Q' are parameter ideals of A. Let $C = M/L$. Then C is a maximal Cohen-Macaulay A-module. Hence we get the exact sequence

$$0 \to L/Q^{n+1}L \to M/Q^{n+1}M \to C/Q^{n+1}C \to 0$$

of A-modules, so that

$$(4) \qquad \begin{aligned} \ell_A(M/Q^{n+1}M) &= \ell_A(C/Q^{n+1}C) + \ell_A(L/Q^{n+1}L) \\ &= \ell_A(C/QC)\binom{n+s}{s} + \ell_A(L/Q^{n+1}L) \end{aligned}$$

for $n \geq 0$. We write

$$\ell_A(L/Q^{n+1}L) = \mathrm{e}^0_Q(L)\binom{n+t}{t} - \mathrm{e}^1_Q(L)\binom{n+t-1}{t-1} + \cdots + (-1)^t\mathrm{e}^t_Q(L)$$

for $n \gg 0$, where $\{\mathrm{e}^i_Q(L)\}_{0 \leq i \leq t}$ are integers with $\mathrm{e}^0_Q(L) \geq 1$. We then have $\mathrm{e}^1_Q(M) = -\mathrm{e}^0_Q(L)$, if $t = s-1$ and $\mathrm{e}^1_Q(M) = 0$, if $t < s-1$. Thus, from equation (4) the equality $\mathrm{e}^1_Q(M) = \mathrm{e}^1_{Q'}(M)$ follows, because $e^0_Q(L) = e^0_{Q'}(L)$ once $\overline{Q} = \overline{Q'}$. $\square$

Let M $(\neq (0))$ be a finitely generated A-module. We say that M is a sequentially Cohen-Macaulay A-module, if M possesses a Cohen-Macaulay filtration, that is a filtration

$$L_0 = (0) \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_\ell = M$$

of A-submodules $\{L_i\}_{0 \le i \le \ell}$ such that $\dim_A L_i > \dim_A L_{i-1}$ and L_i/L_{i-1} is a Cohen-Macaulay A-module for all $1 \le i \le \ell$ ([34], [6], [9]). Therefore, applying Proposition 6.1, we readily get the following.

Corollary 6.2. *Suppose that M is a sequentially Cohen-Macaulay A-module with $\dim_A M > 0$ and let Q and Q' be parameter ideals for M. Then $\mathrm{e}^1_Q(M) = \mathrm{e}^1_{Q'}(M)$, if $\overline{Q} = \overline{Q'}$ in A.*

Let us note a typical example.

Example 6.3 ([?]). Let R be a regular local ring of dimension 3 and let X, Y, Z be a regular system of parameters of R. We look at the two-dimensional local ring $A = R/(X) \cap (Y, Z)$. Then A is not Cohen-Macaulay but sequentially Cohen-Macaulay. Let x, y, z be the images of X, Y, Z in A, respectively, and put $C = A/(y, z)$ and $B = A/(x)$. Then C is a DVR and B is a two-dimensional regular local ring. Let $Q = (a, b)$ be a parameter ideal in A. Then a, b forms a B-regular sequence and, thanks to the exact sequence $0 \to C \to A \to B \to 0$, we get

$$\ell_A(A/Q^{n+1}) = \mathrm{e}^0_{QB}(B)\binom{n+2}{2} + \mathrm{e}^0_{QC}(C)\binom{n+1}{1}$$

for all $n \gg 0$, so that $\mathrm{e}^0_Q(A) = \ell_B(B/QB)$, $\mathrm{e}^1_Q(A) = -\mathrm{e}^0_{QC}(C)$, and $\mathrm{e}^2_Q(A) = 0$. Therefore, if Q' is a parameter ideal in A with $\overline{Q'} = \overline{Q}$, we always have $\mathrm{e}^i_Q(A) = \mathrm{e}^i_{Q'}(A)$ for all $0 \le i \le 2$, because $QC = Q'C$.

We now assume that $\dim A = 2$, $\operatorname{depth} A = 1$, and $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module. For simplicity, we assume that the residue class field $k = A/\mathfrak{m}$ of A is infinite. We put

$$C = \mathrm{H}^1_{\mathfrak{m}}(A) \quad \text{and} \quad \mathfrak{c} = (0) : C.$$

Proposition 6.4. *Let I be an $\mathfrak{m}$-primary ideal in A and assume that the scheme $\operatorname{Proj} \mathcal{R}(Q)$ is Cohen-Macaulay for every minimal reduction Q of I. Then $\mathrm{e}^1_Q(A)$ is independent of the choice of minimal reductions Q of I and is an invariant of I.*

Proof. Let $Q = (a, b)$ and $Q' = (a', b')$ be reductions of I. Then, since the ideal I contains an element x such that (b, x) and (x, a') are reductions of I, without loss of generality we may assume that $a = a'$. Then the element a is superficial for both Q and Q', because the schemes $\operatorname{Proj}\mathcal{R}(Q)$ and $\operatorname{Proj}\mathcal{R}(Q')$ are Cohen-Macaulay. In fact, let $G = \mathrm{G}(Q)$. Then $\operatorname{Proj} G$ is a Cohen-Macaulay scheme, since so is $\operatorname{Proj}\mathcal{R}(Q)$. Hence the local ring G_P is Cohen-Macaulay for every prime ideal $P \in \operatorname{Spec} G \setminus \{\mathfrak{M}\}$, where $\mathfrak{M} = \mathfrak{m}G + G_+$. Therefore, every system f, g of parameters of the local ring $R = G_{\mathfrak{M}}$ forms a filter regular sequence, that is equivalent to saying that the R-modules $(0) :_R f$ and $[(f) :_R g]/(f)$ have finite length. Applying this observation to the homogeneous system $f = \overline{at}, g = \overline{bt}$ of parameters for the graded ring G (here $\overline{at}$ and $\overline{bt}$ denote the image of at and bt in G, respectively), by definition of superficial elements we see that a and b are always superficial for the ideal Q, once $Q = (a, b)$. Consequently, since a is A-regular, we get

$$\mathrm{e}^1_Q(A) = \mathrm{e}^1_{Q/(a)}(A/(a)) = -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A/(a))) = -\ell_A((0) :_C a),$$

which depends on the element a only, so that we have $\mathrm{e}^1_Q(A) = \mathrm{e}^1_{Q'}(A)$. $\square$

We now come to the main result of this section.

Theorem 6.5. *Suppose that the ideal $\mathfrak{c}$ is not integrally closed. Then for each reduction $Q = (a, b)$ of $\mathfrak{c}$, there exists a reduction $Q' = (a', b')$ of $\overline{Q}$ such that*

$$0 > \mathrm{e}^1_{Q'}(A) > \mathrm{e}^1_Q(A) = -\ell_A(C).$$

Proof. We put $I = \overline{Q}$ and let $\ell = \mu_A(I)$. We write $I = (x_1, x_2, \cdots, x_\ell)$ so that every two elements x_i, x_j $(1 \le i, j \le \ell, i \ne j)$ generate a reduction of I. Then, since $\mathfrak{c} \subsetneq I = \overline{\mathfrak{c}}$, we have $x_i \notin \mathfrak{c}$ for some $1 \le i \le \ell$. Choose an integer $1 \le j \le \ell$ so that $j \ne i$ and put $Q' = (x_i, x_j)$. Then Q' is a reduction of $I = \overline{Q} = \overline{\mathfrak{c}}$ but $Q' \not\subseteq \mathfrak{c}$. Therefore, choosing elements a', b' of Q' so that $Q' = (a', b')$ and both a', b' are superficial for the ideal Q', we may assume that $a' \notin \mathfrak{c} = (0) : C$. We then have

$$\mathrm{e}^1_{Q'}(A) = \mathrm{e}^1_{Q'/(a')}(A/(a')) = -\ell_A((0) :_C a') > -\ell_A(C),$$

while by Proposition 5.1 (4)

$$\mathrm{e}_Q^1(A) = -\ell_A(C),$$

because $QC = (0)$. Thus

$$0 > \mathrm{e}_{Q'}^1(A) > \mathrm{e}_Q^1(A) = -\ell_A(C).$$

$\square$

Let us note concrete examples.

Example 6.6. Let R be a regular local ring with maximal ideal $\mathfrak{n}$ and $\dim R = 4$. Let X, Y, Z, W be a regular system of parameters for R and let

$$\mathfrak{a} = (X^n, Y^n) \cap (Z, W),$$

where $n \geq 2$ is an integer. We look at the local ring $A = R/\mathfrak{a}$. Then $\dim A = 2$, $\operatorname{depth} A = 1$, and

$$\mathrm{H}_{\mathfrak{m}}^1(A) \cong A/(x^n, y^n, z, w),$$

where $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$ is the maximal ideal of A and x, y, z, and w denote the images of X, Y, Z, and W in A, respectively. Let $Q = (x^n - z, y^n - w)$ and $Q' = (xy^{n-1} - z, x^n + y^n - w)$. Then we have the following, where $\mathfrak{c} = (0) : \mathrm{H}_{\mathfrak{m}}^1(A) = (x^n, y^n, z, w)$.

(1) $\overline{Q} = \overline{Q'} = \overline{\mathfrak{c}} = \mathfrak{m}^n + (z, w)$.

(2) $\mathrm{e}_Q^0(A) = \mathrm{e}_{Q'}^0(A) = 2n^2$.

(3) $0 > \mathrm{e}_{Q'}^1(A) = -(n^2 - n + 1) > \mathrm{e}_Q^1(A) = -n^2$.

(4) $\ell_A(A/Q^{\ell+1}) = 2n^2\binom{\ell+2}{2} + n^2\binom{\ell+1}{1}$ and $\ell_A(A/{Q'}^{\ell+1}) = 2n^2\binom{\ell+2}{2} + (n^2 - n + 1)\binom{\ell+1}{1}$ for all integers $\ell \geq 0$.

(5) The element $x^n + y^n - w$ is not superficial for Q', whence the scheme $\operatorname{Proj} \mathcal{R}(Q')$ is not Cohen-Macaulay.

(6) Let $S = \mathrm{S}_Q(I)$ (resp. $S' = \mathrm{S}_{Q'}(I)$) denote the Sally module of $I = \overline{Q} = \overline{Q'}$ with respect to Q (resp. Q') and let $T = \mathcal{R}(Q)$ (resp. $T' = \mathcal{R}(Q')$) be the Rees algebra of Q (resp. Q'). We put $\mathfrak{p} = \mathfrak{m}T$ and $\mathfrak{p}' = \mathfrak{m}T'$. Then

$$\ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = \ell_{T'_{\mathfrak{p}'}}(S'_{\mathfrak{p}'}) + (n - 1).$$

To prove assertions in Example 6.6 we need the following.

Lemma 6.7. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and* $\dim A = 2$*. Suppose that* $\operatorname{depth} A = 1$ *and that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module. We put* $\mathfrak{c} = (0) : \mathrm{H}^1_{\mathfrak{m}}(A)$*. Let a, b be a system of parameters in A. Then the following assertions hold true.*

(1) *If $b \in \mathfrak{c}$, then a, b forms a d-sequence in the sense of C. Huneke* [20].

(2) $(a) : b \subseteq \overline{(a)}$.

Proof. (1) We have $[(a) : b^2]/(a) \subseteq \mathrm{H}^0_{\mathfrak{m}}(A/(a)) \cong (0) :_{\mathrm{H}^1_{\mathfrak{m}}(A)} a$, so that $\mathfrak{c}\cdot [((a) : b^2)/(a)] = (0)$. Hence $(a) : b^2 \subseteq (a) : \mathfrak{c} \subseteq (a) : b$. Thus a, b forms a d-sequence, because a is A-regular.

(2) Let $B = \widetilde{A}$ be the Cohen-Macaulayfication of A ([2]). We then have $[(a) : b]B = aB$, since a, b is a regular sequence in B. Therefore, $\frac{x}{a} \in B$ for all $x \in (a) : b$, whence $x \in \overline{(a)}$, because B is a module-finite extension of A. Thus $(a) : b \subseteq \overline{(a)}$. $\square$

Let us check the assertions in Example 6.6.

Proof of the assertions in Example 6.6. We have $Q \subseteq \mathfrak{c} = Q + (z, w)$ and $\mathfrak{c}^2 = Q\mathfrak{c}$. Hence $\overline{Q} = \overline{\mathfrak{c}}$. Since $A/(z, w)$ is a regular local ring of dimension 2, we have $\overline{\mathfrak{m}^n + (z, w)} = \mathfrak{m}^n + (z, w)$. Therefore, because

$$Q \subseteq \mathfrak{m}^n + (z, w) = (x, y)^n + (z, w) \subseteq \overline{(x^n, y^n) + (z, w)} = \overline{\mathfrak{c}},$$

we get $\overline{Q} = \mathfrak{m}^n + (z, w) = \overline{\mathfrak{c}}$, whence $\mathfrak{c} \neq \overline{\mathfrak{c}}$, because $xy^{n-1} \not\in \mathfrak{c}$ (recall that $n \geq 2$). Let $\mathfrak{p}_1 = (x, y)$ and $\mathfrak{p}_2 = (z, w)$. Then $\operatorname{Ass} A = \operatorname{Assh} A = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and the associative formula of multiplicity says that the equality

$$\mathrm{e}^0_{\mathfrak{q}}(A) = \sum_{\mathfrak{p} \in \operatorname{Assh} A} \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \mathrm{e}^0_{\mathfrak{q}\cdot(A/\mathfrak{p})}(A/\mathfrak{p})$$

holds true for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$ in A. Applying it to our ideals Q and Q', we readily get that

$$\mathrm{e}^0_{Q}(A) = \mathrm{e}^0_{Q'}(A) = 2n^2.$$

Hence Q' is also a reduction of $\overline{\mathfrak{c}}$ by a theorem of D. Rees [32], because $Q' \subseteq \overline{\mathfrak{c}}$ and $\mathrm{e}^0_{\overline{\mathfrak{c}}}(A) = \mathrm{e}^0_{Q}(A) = \mathrm{e}^0_{Q'}(A)$. Thus $\overline{Q} = \overline{Q'}$ but $Q' \not\subseteq \mathfrak{c}$. We put $C = A/(x^n, y^n, z, w)$. Then

$C \cong \mathrm{H}^1_{\mathfrak{m}}(A)$ and $(x^n + y^n - w)C = (0)$. Hence

$$\mathrm{e}^1_{Q'}(A) = -(n^2 - n + 1)$$

by Proposition 5.1 (3), because

$$\ell_A((0) :_C xy^{n-1} - z) = \ell_A(C/(xy^{n-1} - z)C) = \ell_A(A/(x^n, y^n, xy^{n-1}, z, w)) = n^2 - n + 1,$$

whence by Proposition 5.1 (1) we have for all integers $\ell \geq 0$

$$\begin{aligned} \ell_A(A/Q'^{\ell+1}) &= 2n^2\binom{\ell+2}{2} + \ell_A((0) :_C xy^{n-1} - z)\binom{\ell+1}{1} \\ &= 2n^2\binom{\ell+2}{2} + (n^2 - n + 1)\binom{\ell+1}{1}. \end{aligned}$$

We similarly have

$$\mathrm{e}^1_Q(A) = -\ell_A(C) = -n^2,$$

because $QC = (0)$, whence

$$\ell_A(A/Q^{n+1}) = 2n^2\binom{\ell+2}{2} + n^2\binom{\ell+1}{1}$$

for all $\ell \geq 0$.

If $x^n + y^n - w$ is superficial for the ideal Q', we must have

$$\mathrm{e}^1_{Q'}(A) = \mathrm{e}^1_{Q'/(x^n+y^n-w)}(A/(x^n + y^n - w)) = \ell_A((0) :_C x^n + y^n - w) = -\ell_A(C) = -n^2,$$

which is impossible, because $n \geq 2$. Hence $x^n + y^n - w$ is not superficial for Q'. Therefore the scheme $\mathrm{Proj}\,\mathcal{R}(Q')$ is not Cohen-Macaulay (see Proof of Proposition 6.4).

To see assertion (6), notice that by [16, Proposition 2.5] we get the equalities

$$\begin{aligned} \mathrm{e}^1_I(A) &= \mathrm{e}^0_I(A) + \mathrm{e}^1_Q(A) - \ell_A(A/I) + \ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) \\ &= \mathrm{e}^0_I(A) + \mathrm{e}^1_{Q'}(A) - \ell_A(A/I) + \ell_{T'_{\mathfrak{p}}}(S'_{\mathfrak{p}'}) \end{aligned}$$

for the ideal $I = \overline{\mathfrak{c}}$, because by Lemma 4.7 all conditions (C_0) and (C_2) in [16] are satisfied for the ideals Q, Q', and $I = \overline{\mathfrak{c}}$. This completes the proof of all the assertions. $\square$

Remark 6.8. In Example 6.6 assume that the residue class field $R/\mathfrak{n}$ of R is infinite. Then $\mathrm{e}^1_{\mathfrak{q}}(A) = -n$ for every minimal reduction $\mathfrak{q} = (a, b)$ of the maximal ideal $\mathfrak{m}$ of A.

Proof. Let $\overline{A} = A/(z,w)$ and let $\overline{c}$ denote, for each $c \in A$, the image of c in $\overline{A}$. Then $\overline{A}$ is a two-dimensional regular local ring with $\overline{x}, \overline{y}$ a regular system of parameters. Let $\mathfrak{q} = (a,b)$ be a minimal reduction of $\mathfrak{m}$. Then $\mathfrak{m} = \mathfrak{q} + (z,w)$, since the local ring $\overline{A}$ is regular. We may assume that a is superficial for $\mathfrak{q}$. Hence

$$\mathrm{e}^1_{\mathfrak{q}}(A) = \mathrm{e}^1_{\mathfrak{q}/(a)}(A/(a)) = -\ell_A((0) :_C a) = -\ell_A(C/aC) = -\ell_A(A/(x^n, y^n, z, w, a)).$$

Let us check that $\ell_A(A/(x^n, y^n, z, w, a)) = n$. We write $\overline{a} = \alpha\overline{x} + \beta\overline{y}$ with $\alpha, \beta \in \overline{A}$. We may assume that α is a unit of $\overline{A}$, because $a \notin \mathfrak{m}^2 + (z,w)$. Therefore

$$(\overline{x}^n, \overline{y}^n, \overline{a}) = (\overline{x}^n, \overline{y}^n, \overline{x} + \beta'\overline{y}) = (\overline{y}^n, \overline{x} + \beta'\overline{y})$$

with $\beta' = \alpha^{-1}\beta$, whence

$$\ell_A(A/(x^n, y^n, z, w, a)) = \ell_A(\overline{A}/(\overline{x} + \beta'\overline{y}, \overline{y}^n)) = n.$$

Thus $\mathrm{e}^1_{\mathfrak{q}}(A) = -n$ as is claimed. $\square$

Before closing this section, let us note the following example, which shows that the rank of Sally modules depends on the choice of minimal reductions.

Example 6.9. Choose $n = 2$ in Example 6.6 and put $I = \mathfrak{m}^2 + (z,w)$. We denote by $S = \mathrm{S}_Q(I)$ (resp. $S' = \mathrm{S}_{Q'}(I)$) the Sally module of I with respect to Q (resp. Q'). Let $T = \mathcal{R}(Q) = A[Qt]$ (resp. $T' = \mathcal{R}(Q') = A[Q't]$), where t is an indeterminate over A. We put $B = T/\mathfrak{m}T$ and $B' = T'/\mathfrak{m}T'$. Then

(1) $S \cong B_+$ as graded T-modules,
(2) $S' \cong B'/(x^2 + y^2 - w)t{\cdot}B'$ as graded T'-modules, and
(3) $\ell_A(A/I^{n+1}) = 8\binom{n+2}{2} - 2\binom{n+1}{1} - 4$ for all $n \geq 1$.

Hence $\mathrm{rank}_B\, S = 1$ but $\mathrm{rank}_{B'}\, S' = 0$.

Proof. (1) We put $a = x^2 - z$ and $b = y^2 - w$. It is routine to check that $I^2 = QI + (xyz, xyw), xyz \notin Q, I^3 = QI^2$, and $\mathfrak{m}I^2 \subseteq QI$. Hence $S \neq (0)$ and $\mathfrak{m}S = (0)$, because $S = TS_1$ and $S_1 \cong I^2/QI$ (see [16, Lemma 2.1]), where S_1 stands for the homogeneous component of S with degree 1. Therefore we have an epimorphism

$$\varphi : B(-1) \to S$$

of graded B-modules defined by $\varphi(\mathbf{e}_1) = \widetilde{xyzt}$ and $\varphi(\mathbf{e}_2) = \widetilde{xywt}$, where $\widetilde{xyzt}$ and $\widetilde{xywt}$ denote the images of $xyzt$ and $xywt$ in S, respectively, and $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $B(-1)^2$. Let $\overline{f}$ denote, for each $f \in T$, the image of f in B. Then, since

$$b(xyz) = a(xyw) = -xyzw,$$

we see $\overline{bt}\mathbf{e}_1 - \overline{at}\mathbf{e}_2 \in \operatorname{Ker} \varphi$. Therefore, we get an epimorphism

$$\overline{\varphi} : B_+ \to S$$

induced from φ (notice that $B = k[\overline{at}, \overline{bt}]$ and $B_+ \cong B(-1)^2/B{\cdot}[\overline{bt}\mathbf{e}_1 - \overline{at}\mathbf{e}_2]$, since $\overline{at}, \overline{bt}$ are algebraically independent over the residue class field $k = A/\mathfrak{m}$ of A), which must be an isomorphism, because $S \neq (0)$ and by [16, Lemma 2.3] S is a torsionfree B-module (notice that conditions (C_0) and (C_2) in [16] are satisfied by Lemma 4.7). Thus $S \cong B_+$ as graded B-modules. We have condition (C_1) in [16] also satisfied, since $QC = (0)$ (see [36, Theorem 2.5]). Therefore by [16, Theorem 1.3 (iii)] we get

$$\begin{aligned} \ell_A(A/I^{n+1}) &= \mathrm{e}_I^0(A)\binom{n+2}{2} - \left\{\mathrm{e}_I^0(A) + \mathrm{e}_Q^1(A) - \ell_A(A/I) + 1\right\}\binom{n+1}{1} \\ &+ \left\{\mathrm{e}_Q^1(A) + \mathrm{e}_Q^2(A)\right\} \\ &= 8\binom{n+2}{2} - 2\binom{n+1}{1} - 4 \end{aligned}$$

for all $n \geq 1$.

(2) This time we have $I^2 = Q'I + (xyz), I^3 = Q'I^2$, and $\mathfrak{m}I^2 \subseteq Q'I$. Notice that $S' \neq (0)$, since $xyz \notin Q'$. Let $a' = z - xy$ and $b' = w - (x^2 + y^2)$. We then have

$$b'(xyz) = a'(xyw) = xyzw$$

and $xyw = b'z - a'w \in {Q'}^2I$. Hence we get an epimorphism

$$\varphi' : (B'/b't{\cdot}B')(-1) \to S'$$

such that $\varphi'(1) = \widetilde{xyzt}$.

We now want to show that φ' is an isomorphism. Suppose that $\operatorname{Ker} \varphi' \neq (0)$. Then the homogeneous component $[\operatorname{Ker}\varphi']_n$ of $\operatorname{Ker} \varphi'$ is non-zero for some integer n. Choose such an integer n as small as possible. Then $n \geq 2$ and $\overline{a't}^{n-1} \in \operatorname{Ker} \varphi'$, since

$B' = k[\overline{a't}, \overline{b't}]$. Therefore

$$a'^{n-1}(xyz) \in Q'^n I = a'Q'^{n-1}I + b'^n I.$$

Let $a'^{n-1}(xyz) = a'i + b'^n j$ with $i \in Q'^{n-1}I$ and $j \in I$. We then have

$$j \in (a') : b'^n = (a') : b',$$

since a', b' is a d-sequence by Lemma 4.7 (1). Let $b'j = a'h$ with $h \in A$. Then $h \in (b') : a' \subseteq I$ by Lemma 4.7 (2) and $a'^{n-1}(xyz) = a'i + a'(b'^{n-1}h)$, whence

$$a'^{n-2}(xyz) = i + b'^{n-1}h \in Q'^{n-1}I,$$

because a' is A-regular. Therefore

$$\overline{a't}^{n-2} \in [\mathrm{Ker}\ \varphi']_{n-1},$$

which contradicts the minimality of n. Hence φ' is a monomorphism and $S' \cong B'/\overline{b't}{\cdot}B'$ as graded T'-modules. $\square$

7. The case where $\overline{Q} = \mathfrak{m}$

The value $\mathrm{e}_Q^1(A)$ depends on the choice of minimal reductions Q, even in the case where $\overline{Q} = \mathfrak{m}$. To see this, we need some technique of reduction.

Let B be a Noetherian local ring with maximal ideal $\mathfrak{n}$ and assume that B contains a field k such that the composite map $k \xrightarrow{\iota} B \xrightarrow{\varepsilon} B/\mathfrak{n}$ is bijective, where $\iota : k \to B$ denotes the embedding and $\varepsilon : B \to B/\mathfrak{n}$ denotes the canonical epimorphism. Let J be an $\mathfrak{n}$-primary ideal in B and put $A = k + J$. Then A is a local k-subalgebra of B with maximal ideal $\mathfrak{m} = J$ and B is a module-finite extension of A, because $\ell_A(B/A) = \ell_B(B/J) - 1$. Hence A is a Noetherian local ring with $\dim A = \dim B$, thanks to Eakin-Nagata's theorem.

Suppose now that $d = \dim B > 0$. Let $\mathfrak{q} = (a_1, a_2, \cdots, a_d)B$ be a parameter ideal in B and assume that $\mathfrak{q}$ is a reduction of J. We put

$$Q = (a_1, a_2, \cdots, a_d)A.$$

Then Q is a reduction of $\mathfrak{m}$. Hence Q is a parameter ideal in A. We have the canonical isomorphism between the Sally module $\mathrm{S}_Q(\mathfrak{m}) = \bigoplus_{n\geq 1} \mathfrak{m}^{n+1}/Q^n\mathfrak{m}$ of $\mathfrak{m}$ with respect to

Q and the Sally module $\mathrm{S}_{\mathfrak{q}}(J) = \bigoplus_{n \geq 1} J^{n+1}/\mathfrak{q}^n J$ of J with respect to $\mathfrak{q}$, because

$$\mathfrak{m}^{n+1}/Q^n\mathfrak{m} = J^{n+1}/\mathfrak{q}^n J$$

for all $n \geq 1$:

Fact 7.1. $\mathrm{S}_Q(\mathfrak{m}) \cong \mathrm{S}_{\mathfrak{q}}(J)$ as graded $\mathcal{R}(\mathfrak{q})$-modules.

We put $T = \mathcal{R}(Q)$ and $\mathfrak{p} = \mathfrak{m}T$. Then, thanks to [16, Remark 2.6], the sum

$$\mathrm{e}_Q^1(A) + \ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = \mathrm{e}_{\mathfrak{m}}^1(A) - \mathrm{e}_{\mathfrak{m}}^0(A) + 1$$

is an invariant of $\mathfrak{m}$, whence we have the following.

Proposition 7.2. *Let $\mathfrak{q} = (a_1, a_2, \cdots, a_d)B$ and $\mathfrak{q}' = (a_1', a_2', \cdots, a_d')B$ be parameter ideals of B and assume that $\mathfrak{q}$ and $\mathfrak{q}'$ are reductions of J. Let $Q = (a_1, a_2, \cdots, a_d)A$ and $Q' = (a_1', a_2', \cdots, a_d')A$. Then one has the equality*

$$\mathrm{e}_Q^1(A) + \ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = \mathrm{e}_{Q'}^1(A) + \ell_{T'_{\mathfrak{p}'}}(S'_{\mathfrak{p}'}),$$

where $T = \mathcal{R}(Q)$, $T' = \mathcal{R}(Q')$, $\mathfrak{p} = \mathfrak{m}T$, and $\mathfrak{p}' = \mathfrak{m}T'$.

The following example shows that $\mathrm{e}_Q^1(A)$ depends on the choice of minimal reductions Q, even in the case where $\overline{Q} = \mathfrak{m}$. This eventually shows that the rank, or the multiplicity of Sally modules depend on the choice of minimal reductions, as well.

Example 7.3. Let $R = k[[X, Y, Z, W]]$ be the formal power series ring over a field k and let $B = R/(X^2, Y^2) \cap (Z, W)$. Let $J = (x, y)^2 + (z, w)$, where x, y, z and w denote the images of X, Y, Z, and W in B, respectively. We put $A = k + J$. Then A is a Noetherian local ring with maximal ideal $\mathfrak{m} = J$ and B is a module-finite extension of A. Let $Q = (x^2 - z, y^2 - w)A$ and $Q' = (xy - z, x^2 + y^2 - w)A$. Then Q and Q' are minimal reductions of $\mathfrak{m}$ such that

$$\mathrm{e}_{Q'}^1(A) = \mathrm{e}_Q^1(A) + 1 = -5, \quad \ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = 1, \quad \text{and} \quad \ell_{T'_{\mathfrak{p}'}}(S'_{\mathfrak{p}'}) = 0,$$

where $S = \mathrm{S}_Q(\mathfrak{m})$, $S' = \mathrm{S}_{Q'}(\mathfrak{m})$, $T = \mathcal{R}(Q)$, $T' = \mathcal{R}(Q')$, $\mathfrak{p} = \mathfrak{m}T$, and $\mathfrak{p}' = \mathfrak{m}T'$.

Proof. Since $\ell_A(A/\mathfrak{m}^{n+1}) = \ell_A(B/J^{n+1}) - \ell_A(B/A)$ and $\ell_A(B/A) = \ell_B(B/J) - 1$, by Example 6.9 (3) we have

$$\ell_A(A/\mathfrak{m}^{n+1}) = 8\binom{n+2}{2} - 2\binom{n+1}{1} - 6$$

for all $n \geq 1$, whence

$$\mathrm{e}^0_{\mathfrak{m}}(A) = 8, \quad \mathrm{e}^1_{\mathfrak{m}}(A) = 2, \quad \text{and} \quad \mathrm{e}^2_{\mathfrak{m}}(A) = -6.$$

Therefore

$$\mathrm{e}^1_Q(A) + \ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = \mathrm{e}^1_{Q'}(A) + \ell_{T'_{\mathfrak{p}'}}(S'_{\mathfrak{p}'}) = \mathrm{e}^1_{\mathfrak{m}}(A) - \mathrm{e}^0_{\mathfrak{m}}(A) + 1 = -5$$

by Proposition 7.2. On the other hand, thanks to Fact 2 and Example 6.9 (1), (2), we see that $\ell_{T_{\mathfrak{p}}}(S_{\mathfrak{p}}) = 1$ and $\ell_{T'_{\mathfrak{p}'}}(S'_{\mathfrak{p}'}) = 0$, whence $\mathrm{e}^1_{Q'}(A) = \mathrm{e}^1_Q(A) + 1 = -5$. □

8. A structure theorem for local rings possessing $\mathrm{e}^1_Q(A) = -1$

The condition $\mathrm{e}^1_Q(A) = -1$ for some parameter ideal Q in A is a rather strong condition. In this section we shall explore this phenomenon. Similarly as in Section 6 let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\dim A = 2$. We assume that $\operatorname{depth} A = 1$ and that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is a finitely generated A-module. We put $C = \mathrm{H}^1_{\mathfrak{m}}(A)$ and $\mathfrak{c} = (0) : \mathrm{H}^1_{\mathfrak{m}}(A)$. Suppose that the residue class field $k = A/\mathfrak{m}$ of A is infinite. We then have the following.

Theorem 8.1. *We consider the following two conditions.*

(1) $\mu_A(\mathfrak{m}) = 4$, *the Cohen-Macaulayfication* $\widetilde{A}$ *of* A *is not a local ring, and* A *contains a parameter ideal* Q *such that* $\mathrm{e}^1_Q(A) = -1$.

(2) $A \cong R/(F, Y) \cap (Z, W)$ *as rings, where* R *is a regular local ring of dimension* 4, X, Y, Z, W *is a regular system of parameters in* R, *and* $F \in R$ *such that* $F = X^n + \xi$ *for some integer* $n \geq 1$ *and* $\xi \in (Z, W)$.

Then the implication (2) $\Rightarrow$ (1) *is always true and we have* $\mathrm{e}^1_{\mathfrak{q}}(A) = -1$ *for every minimal reduction* $\mathfrak{q}$ *of* $\mathfrak{m}$. *When* A *is* $\mathfrak{m}$*-adically complete, the implication* (1) $\Rightarrow$ (2) *also holds true, so that conditions* (1) *and* (2) *are equivalent to each other.*

We divide the proof of Theorem 8.1 into two parts.

Let us consider the implication (2) $\Rightarrow$ (1). Let R be a regular local ring of dimension 4 and let X, Y, Z, W be a regular system of parameters in R. Let $n \geq 1$ be an integer and $\xi \in (Z, W)$. We put $F = X^n + \xi$. Then $(F, Y, Z, W) = (X^n, Y, Z, W)$ and F, Y, Z, W forms a system of parameters in R. Let

$$A = R/(F, Y) \cap (Z, W)$$

and let $\mathfrak{m}$ be the maximal ideal of A. We denote by $f, x, y, z,$ and w the images of $F, X, Y, Z,$ and W in A, respectively. Then, thanks to the exact sequence

$$0 \to A \to A/(f, y) \oplus A/(z, w) \to A/(x^n, y, z, w) \to 0,$$

we have $\operatorname{depth} A = 1$ and $\mathrm{H}^1_{\mathfrak{m}}(A) \cong A/(x^n, y, z, w)$. Hence

$$\widetilde{A} = A/(f, y) \times A/(z, w)$$

by [2, Theorem 1.6]. We put $C = \mathrm{H}^1_{\mathfrak{m}}(A)$ and $\mathfrak{c} = (x^n, y, z, w)$. Let $a = f - z$ and $b = y - w$. We look at the parameter ideal $Q = (a, b)$ in A. Then $(a) : b = (a, z)$ and $(b) : a = (b, w)$. Hence

$$[(a) : b] + [(b) : a] = (a, b, z, w) = \mathfrak{c},$$

so that by [13, Theorem 1.1] we get the following.

Fact 8.2. The Rees algebra $\mathcal{R}(Q^2)$ of Q^2 is a Gorenstein ring.

We now assume that the residue class field of R is infinite and let $\mathfrak{q} = (a, b)$ be any minimal reduction of $\mathfrak{m}$, where we choose the system a, b of generators of the ideal $\mathfrak{q}$ so that both a, b are superficial for $\mathfrak{q}$. Let $\overline{A} = A/(z, w)$. Then, since $\mathfrak{q}\overline{A}$ is a reduction of the maximal ideal in the two-dimensional regular local ring $\overline{A}$, we get $\mathfrak{q}\overline{A} = \mathfrak{m}/(z, w)$, whence $\mathfrak{q} + (z, w) = \mathfrak{m}$. We want to show that $\mathrm{e}^1_{\mathfrak{q}}(A) = -1$. Here we may assume that $n > 1$. In fact, if $n = 1$, then $\mathrm{H}^1_{\mathfrak{m}}(A) \cong A/\mathfrak{m}$, so that A is a Buchsbaum local ring and $\mathrm{e}^1_{\mathfrak{q}}(A) = -1$ by Schenzel's formula 3.3. Suppose that $n > 1$. Then, since $\mathfrak{m} = \mathfrak{q} + \mathfrak{c}$, without loss of generality we may assume that $a \notin \mathfrak{c} + \mathfrak{m}^2 = (x^2, y, z, w)$, whence $\ell_A(C/aC) = 1$. Thus $\mathrm{e}^1_{\mathfrak{q}}(A) = -\ell_A((0) :_C a) = -1$.

Let us note one remark.

Remark 8.3. Suppose that $\xi \in (Z, W)$. Let $a = x^\ell - z$, $b = y - w$ with $1 \le \ell \le n$. We put $Q = (a, b)$. Then Q is a parameter ideal in A and, since $bC = (0)$, by Proposition 5.1 (3) we get

$$\mathrm{e}_Q^1(A) = -\ell_A(C/aC) = -\ell_A(A/(x^\ell, y, z, w)) = -\ell.$$

This shows that the value $\mathrm{e}_Q^1(A)$ varies between $-n$ and -1 with $-n$ the least (cf. Proposition 5.1 (2)).

Let us prove the implication (1) $\Rightarrow$ (2) in Theorem 8.1. With the notation in the preamble of this section we assume that A is $\mathfrak{m}$-adically complete. Let $B = \widetilde{A}$ be the Cohen-Macaulayfication of A, whence $B \cong \mathrm{End}_A(\mathrm{K}_A)$ as A-algebras ([2, Theorem 1.6]), where K_A denotes the canonical module of A. We then have the exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow C \longrightarrow 0$$

of A-modules, where $\varphi(x)$ is, for each $x \in A$, the homothety of x. Let $Q = (a, b)$ be a parameter ideal in A such that $\mathrm{e}_Q^1(A) = -1$. We may assume that a, b are both superficial for Q. Then, since $\ell_A(C/aC) = \ell_A((0) :_C a) = 1$, we get $\mu_A(C) = 1$. Therefore $C \cong A/\mathfrak{c}$ and $\mathfrak{c} + (a) = \mathfrak{m}$, whence $\mu_A(B) = 2$. Consequently, because B is not a local ring and A is complete, we have the canonical decomposition

$$B = A/\mathfrak{a}_1 \times A/\mathfrak{a}_2$$

of the A-algebra B, where $\mathfrak{a}_i$ is an ideal in A such that $A/\mathfrak{a}_i$ is a two-dimensional Cohen-Macaulay local ring for each $i = 1, 2$. Hence $\mathfrak{a}_1 \cap \mathfrak{a}_2 = (0)$ and $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{c}$, thanks to the exact sequence

$$0 \to A \to A/\mathfrak{a}_1 \oplus A/\mathfrak{a}_2 \to A/(\mathfrak{a}_1 + \mathfrak{a}_2) \to 0.$$

Let $V = [\mathfrak{c} + \mathfrak{m}^2]/\mathfrak{m}^2 \subseteq \mathfrak{m}/\mathfrak{m}^2$. Then $\dim_k V \ge 3$, because $\mu_A(\mathfrak{m}) = 4$ and $\mathfrak{c} + (a) = \mathfrak{m}$. Since $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{c}$, we may assume that $\dim_k [\mathfrak{a}_2 + \mathfrak{m}^2]/\mathfrak{m}^2 \ge 2$. Therefore the ideal $\mathfrak{a}_2$ contains a part z, w of a minimal system of generators of the maximal ideal $\mathfrak{m}$. We then have $\mu_{A/(z,w)}(\mathfrak{m}/(z, w)) = 2$, whence the epimorphism

$$A/(z, w) \to A/\mathfrak{a}_2 \to 0$$

is an isomorphism, because $\dim A/(z,w) \geq \dim A/\mathfrak{a}_2 = 2$. Thus $\mathfrak{a}_2 = (z,w)$. Therefore $\dim_k[\mathfrak{a}_1 + \mathfrak{m}^2]/\mathfrak{m}^2 \geq 1$, because $\dim_k V \geq 3$. Choose $y \in \mathfrak{a}_1$ so that y, z, w forms a part of a minimal system of generators of $\mathfrak{m}$ and write $\mathfrak{m} = (x, y, z, w)$. Then $A/(y,z,w)$ is a DVR, because $A/(z,w)$ is a two-dimensional regular local ring with the images of x, y in it a regular system of parameters. Consequently, since $\mathfrak{c} = \mathfrak{a}_1 + \mathfrak{a}_2 \supsetneq (y,z,w)$, we have

$$\mathfrak{c}/(y,z,w) = (\overline{x}^n)$$

for some $n \geq 1$, where $\overline{x}$ stands for the image of x in $A/(y,z,w)$. Hence $\mathfrak{c} = (x^n, y, z, w)$ and $n = \ell_A(A/\mathfrak{c})$. On the other hand, because

$$\mathfrak{c}/(y,z,w) = [\mathfrak{a}_1 + (z,w)]/(y,z,w) = (\overline{x}^n),$$

we find some element $\eta \in \mathfrak{a}_1$ so that $x^n - \eta \in (y,z,w)$. Let

$$x^n - \eta = \alpha y + \beta z + \gamma w$$

with $\alpha, \beta, \gamma \in A$. We then have $x^n - f \in (z,w)$ where $f = \eta + \alpha y$. Hence $\mathfrak{a}_1 = (f, y)$, because

$$\mathfrak{c} = \mathfrak{a}_1 + \mathfrak{a}_2 \supseteq (f,y) \oplus (z,w) \supseteq \mathfrak{c}.$$

Now we choose a regular local ring R with maximal ideal $\mathfrak{n}$ and $\dim R = 4$ together with a surjective homomorphism

$$R \xrightarrow{\phi} A \to 0$$

of rings. Let X, Y, Z, and W be elements of R such that $\phi(X) = x, \phi(Y) = y, \phi(Z) = z$, and $\phi(W) = w$. Then $\mathfrak{n} = (X, Y, Z, W)$, since $\mathrm{Ker}\phi \subseteq \mathfrak{n}^2$. Notice that

$$R/(Z,W) \cong A/(z,w),$$

because $A/(z,w)$ is a two-dimensional regular local ring. Hence $K := \mathrm{Ker}\ \phi \subseteq (Z,W)$. We look at the exact sequence

$$(*) \qquad 0 \to L \to R/(Z,W) \to A/(z,w) \to 0$$

of R-modules. Let $F \in R$ such that $\phi(F) = f$. We then have $X^n - F \in (Z,W)$, because $x^n - f \in (z,w)$ and $K \subseteq (Z,W)$. Therefore $(F, Y, Z, W) = (X^n, Y, Z, W)$, so that F, Y, Z, W is a system of parameters of R. Hence, because z, w is a regular sequence in the two-dimensional regular local ring $A/\mathfrak{a}_2 = A/(z,w)$, from exact sequence $(*)$ we get the exact sequence

$$0 \longrightarrow L/(Z,W)L \longrightarrow R/(F,Y,Z,W) \xrightarrow{\varepsilon} A/(f,y,z,w) \longrightarrow 0,$$

in which the homomorphism ε has to be an isomorphism, because

$$\ell_R(R/(F,Y,Z,W)) = \ell_R(R/(X^n,Y,Z,W)) = n$$

and

$$\ell_A(A/(f,y,z,w)) = \ell_A(A/(x^n,y,z,w)) = \ell_A(A/\mathfrak{c}) = n.$$

Thus $L = (0)$ by Nakayama's lemma, so that we have $R/(F,Y) \cong A/(f,y)$. This shows that $K := \operatorname{Ker} \phi \subseteq (F,Y)$, whence $K = (F,Y) \cap (Z,W)$, because $(F,Y) \cap (Z,W)$ is certainly included in K (recall that $(f,y) \cap (z,w) = \mathfrak{a}_1 \cap \mathfrak{a}_2 = (0)$). Thus

$$A \cong R/(F,Y) \cap (Z,W),$$

with $X^n - F \in (Z,W)$ and $n \geq 1$. This proves the implication (1) $\Rightarrow$ (2) in Theorem 8.1 under the assumption that A is complete.

9. Appendix: when $\overline{\mathrm{e}}_I^1(R) \geq 0$?

This is a joint work with J. Hong and M. Mandal [8].

Throughout this section let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Assume that R is analytically unramified, whence the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is reduced. We fix an $\mathfrak{m}$-primary ideal I in R and denote by $\overline{I^{n+1}}$ (resp. $\ell_R(R/\overline{I^{n+1}})$) the integral closure of I^{n+1} (resp. the length of $R/\overline{I^{n+1}}$) for each $n \geq 0$. Then the normalized Hilbert function

$$\ell_R(R/\overline{I^{n+1}})$$

of R with respect to I is of polynomial type with degree d and we have integers $\{\overline{\mathrm{e}}_I^i(R)\}_{0 \leq i \leq d}$ such that the equality

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_I^0(R)\binom{n+d}{d} - \overline{\mathrm{e}}_I^1(R)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{\mathrm{e}}_I^d(R)$$

holds true for all $n \gg 0$. We call these integers $\overline{\mathrm{e}}_I^i(R)$ the normalized Hilbert coefficients of R with respect to I.

In this section we are interested in the analysis of the first normalized Hilbert coefficient $\overline{\mathrm{e}}_I^1(R)$. The main purpose is to study the positivity conjecture on $\overline{\mathrm{e}}_I^1(R)$ posed by Wolmer V. Vasconcelos [39] and our result is stated as follows.

Theorem 9.1. *Let R be an analytically unramified local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. If R is unmixed, then*

$$\overline{\mathrm{e}}_I^1(R) \geq 0$$

for every $\mathfrak{m}$-primary ideal I in R.

Here we should note that the conjecture holds true in the case where R is a Cohen-Macaulay local ring ([41, Theorem 2.2]). In fact, generally we have

$$\overline{\mathrm{e}}_I^0(R) = \mathrm{e}_I^0(R),$$

where $\mathrm{e}_I^0(R)$ stands for the ordinary Hilbert-Samuel multiplicity of R with respect to I. Therefore $\overline{\mathrm{e}}_I^1(R) \geq \mathrm{e}_I^1(R)$ and so, if R is a Cohen-Macaulay local ring, we get

$$\overline{\mathrm{e}}_I^1(R) \geq \mathrm{e}_I^1(R) \geq 0,$$

because $\mathrm{e}_I^1(R) \geq 0$ ([30, Corollary 1]). Mainly based on this fact, the third author M. Mandal, B. Singh, and J. Verma [26] gave several interesting answers in certain special cases and our Theorem 9.1 now affirmatively settles the conjecture in full generality.

We shall prove Theorem 9.1 in Section 2. In Section 3 we will discuss a few results related to the positivity conjecture. We suspect if the integral closure $\overline{R}$ of R is a regular ring and $I\overline{R}$ is normal, that is, $I^n\overline{R}$ is integrally closed for all $n \geq 1$, once $\overline{\mathrm{e}}_I^1(R) = 0$ for some $\mathfrak{m}$-primary ideal I in R. We shall give an affirmative answer in the case where $\overline{R}$ is a Cohen-Macaulay ring.

Proof of Theorem 9.1. We have $\overline{\mathrm{e}}_{I\widehat{R}}^1(\widehat{R}) = \overline{\mathrm{e}}_I^1(R)$, since $\overline{\mathfrak{a}\widehat{R}} = \overline{\mathfrak{a}}\widehat{R}$ for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in R. Therefore, passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, without loss of generality we may assume that R is complete. If $d = 1$, we then have

$$\overline{\mathrm{e}}_I^1(R) = \ell_R(\overline{R}/R) \geq 0.$$

Suppose that $d \geq 2$ and let $S = \overline{R}$. For each $\mathfrak{p} \in \operatorname{Ass} R$ we put $S(\mathfrak{p}) = \overline{R/\mathfrak{p}}$. Then $S(\mathfrak{p})$ is a module-finite extension of $R/\mathfrak{p}$ and we get

$$S = \prod_{\mathfrak{p} \in \operatorname{Ass} R} S(\mathfrak{p}) \quad \text{and} \quad \overline{I^{n+1}} = \overline{I^{n+1}S} \cap R$$

for all $n \geq 0$. Hence

$$\begin{aligned}
\ell_R(R/\overline{I^{n+1}}) \leq \ell_R(S/\overline{I^{n+1}S}) &= \sum_{\mathfrak{p} \in \operatorname{Ass} R} \ell_R(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}) \\
&= \sum_{\mathfrak{p} \in \operatorname{Ass} R} \ell_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \ell_{S(\mathfrak{p})}(S(\mathfrak{p})/\overline{I^{n+1}S(\mathfrak{p})}),
\end{aligned}$$

where $\mathfrak{m}_{S(\mathfrak{p})}$ denotes the maximal ideal of $S(\mathfrak{p})$. Notice that, since $\dim S(\mathfrak{p}) = d$ for each $\mathfrak{p} \in \operatorname{Ass} R$, we have

$$\begin{aligned}
\overline{\mathrm{e}}_I^0(R) = \mathrm{e}_I^0(R) = \mathrm{e}_I^0(S) &= \sum_{\mathfrak{p} \in \operatorname{Ass} R} \mathrm{e}_I^0(S(\mathfrak{p})) \\
&= \sum_{\mathfrak{p} \in \operatorname{Ass} R} {}_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \mathrm{e}_{IS(\mathfrak{p})}^0(S(\mathfrak{p})) \\
&= \sum_{\mathfrak{p} \in \operatorname{Ass} R} \ell_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \overline{\mathrm{e}}_{IS(\mathfrak{p})}^0(S(\mathfrak{p})),
\end{aligned}$$

whence

$$\begin{aligned}
0 &\leq \ell_R(S/\overline{I^{n+1}S}) - \ell_R(R/\overline{I^{n+1}}) \\
&= \left[\overline{\mathrm{e}}_I^1(R) - \sum_{\mathfrak{p} \in \operatorname{Ass} R} \ell_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \overline{\mathrm{e}}_{IS(\mathfrak{p})}^1(S(\mathfrak{p}))\right] \binom{n+d-1}{d-1} \\
&+ \text{(terms of lower degree)},
\end{aligned}$$

so that

$$\overline{\mathrm{e}}_I^1(R) \geq \sum_{\mathfrak{p} \in \operatorname{Ass} R} \ell_R(S(\mathfrak{p})/\mathfrak{m}_{S(\mathfrak{p})}) \cdot \overline{\mathrm{e}}_{IS(\mathfrak{p})}^1(IS(\mathfrak{p})).$$

Thus, in order to see $\overline{\mathrm{e}}_I^1(R) \geq 0$, it suffices to show that $\overline{\mathrm{e}}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq 0$ for each $\mathfrak{p} \in \operatorname{Ass} R$. If $d = 2$, we get

$$\overline{\mathrm{e}}_{IS(\mathfrak{p})}^1(S(\mathfrak{p})) \geq \mathrm{e}_{IS(\mathfrak{p})}^1(IS(\mathfrak{p})) \geq 0,$$

because $S(\mathfrak{p})$ is a Cohen-Macaulay local ring. Hence $\overline{\mathrm{e}}_I^1(R) \geq 0$.

Suppose that $d \geq 3$ and that our assertion holds true for $d-1$. Then thanks to the above observation, passing to the ring $S(\mathfrak{p})$, we may assume that R is a normal complete local ring. Let $I = (a_1, a_2, \dots, a_\ell)$ with $a_i \in R$, where $\ell = \mu_R(I)$. Let

$$T = R[Z_1, Z_2, \dots, Z_\ell], \quad \mathfrak{q} = \mathfrak{m}T, \quad x = \sum_{i=1}^{\ell} a_i Z_i, \quad \text{and} \quad D = T/xT,$$

where $Z_1, Z_2, \dots, Z_\ell$ are indeterminates over R. Let

$$R' = T_\mathfrak{q}, \quad I' = IR', \quad \text{and} \quad D' = D_\mathfrak{q}.$$

We then have $\overline{I^{n+1}R'} = \overline{I^{n+1}}R'$ for all $n \geq 0$, so that $\ell_{R'}(R'/\overline{I^{n+1}R'}) = \ell_R(R/\overline{I^{n+1}})$, whence

$$\overline{\mathrm{e}}_I^1(R) = \overline{\mathrm{e}}_{I'}^1(R').$$

Here we notice that $\operatorname{Ass} D' = \operatorname{Assh} D'$, because R' is catenary and normal; hence D' is unmixed, as D' is a homomorphic image of a Cohen-Macaulay ring. The ring D' is analytically unramified. To see this, since D' is a Nagata local ring, it suffices to show that D is reduced, that is, $D_P = T_P/xT_P$ is an integral domain for every $P \in \operatorname{Ass}_T D$. Let $\mathfrak{p} = P \cap R$. Then since $\mathrm{ht}_T P = 1$, we have $\mathrm{ht}_R \mathfrak{p} \leq 1$, so that $I \not\subseteq \mathfrak{p}$, because $\mathrm{ht}_R \mathfrak{p} \leq 1 < d = \dim R$. Without loss of generality we may assume that $a_\ell \notin \mathfrak{p}$. Then, because $x = \sum_{i=1}^{\ell} a_i Z_i$ and a_ℓ is a unit of $R_\mathfrak{p}$, we get

$$T_\mathfrak{p} = R_\mathfrak{p}[Z_1, Z_2, \dots, Z_\ell] = R_\mathfrak{p}[Z_1, Z_2, \dots, Z_{\ell-1}, x],$$

whence the ring

$$T_\mathfrak{p}/xT_\mathfrak{p} = R_\mathfrak{p}[Z_1, Z_2, \dots, Z_\ell]/xR_\mathfrak{p}[Z_1, Z_2, \dots, Z_\ell] = R_\mathfrak{p}[Z_1, Z_2, \dots, Z_{\ell-1}]$$

is an integral domain, as it is the polynomial ring with $\ell - 1$ indeterminates over $R_\mathfrak{p}$. Therefore for all $P \in \operatorname{Ass}_T D$ the ring $D_P = T_P/xT_P$ is an integral domain, because it is a localization of $R_\mathfrak{p}[Z_1, Z_2, \dots, Z_{\ell-1}]$. Thus D is reduced, whence D' is analytically unramified and unmixed.

Let us denote by $\mathcal{A}$ the extended Rees ring of IT and by $\overline{\mathcal{A}}$ the integral closure of $\mathcal{A}$ in $T[t, t^{-1}]$, where t denotes an indeterminate. Similarly, let us denote by T the extended Rees ring of ID and by $\overline{\mathsf{T}}$ the integral closure of T in $D[t, t^{-1}]$. We put $N = (t^{-1}, It)$ in $\mathcal{A}$. We look at the homomorphism

$$\psi : T[t, t^{-1}] \to D[t, t^{-1}]$$

of graded T-algebras such that $\psi(t) = t$. Since $\psi(\mathcal{A}) = \mathsf{T}$ and $\overline{\mathsf{T}}$ is a module-finite extension of T, the homomorphism ψ gives rise to the finite homomorphism

$$\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow \overline{\mathsf{T}}$$

of graded T-algebras. Let $\overline{\mathcal{B}}$ (resp. $\overline{\mathsf{U}}$) denote the integral closure of $\mathcal{B} = \mathcal{A}_{\mathsf{q}}$ (resp. $\mathsf{U} = \mathsf{T}_{\mathsf{q}}$). Then we get the homomorphism

$$\varphi_{\mathsf{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \to \overline{\mathsf{U}}$$

of graded R'-algebras and, thanks to Proof of [22, Theorem 2.1], we furthermore have the following. Let us include a brief proof for the sake of completeness.

Claim 2. *The homomorphism*

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\mathsf{T}}]_P$$

is an isomorphism for all $P \in \operatorname{Spec} \mathcal{A} \setminus V(N)$. Hence the kernel and the cokernel of the homomorphism $\varphi_{\mathsf{q}} : \overline{\mathcal{B}}/xt\overline{\mathcal{B}} \longrightarrow \overline{\mathsf{U}}$ of graded $\mathcal{B}$-modules are finitely graded.

Proof. Because $\overline{\mathcal{A}}[t] = T[t, t^{-1}]$ and $xt\overline{\mathcal{A}}[t] = xT[t, t^{-1}]$, the homomorphism $\varphi_{t^{-1}}$ is an isomorphism, whence so is the homomorphism φ_P, if $t^{-1} \notin P$.

Suppose now that $It \not\subseteq P$. We may assume $a_\ell t \notin P$. Notice that

$$\begin{aligned}
[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_\ell t} &= \left[\overline{R[It, t^{-1}]}[Z_1, Z_2, \ldots, Z_\ell]/xt{\cdot}\overline{R[It, t^{-1}]}[Z_1, Z_2, \ldots, Z_\ell]\right]_{a_\ell t} \\
&= \left(\overline{R[It, t^{-1}][\frac{1}{a_\ell t}]}\right)[Z_1, Z_2, \ldots, Z_\ell]/\left(\sum_{i=1}^{\ell-1} \frac{a_i Z_i t}{a_\ell t} + Z_\ell\right) \\
&= \left(\overline{R[It, t^{-1}][\frac{1}{a_\ell t}]}\right)[Z_1, Z_2, \ldots, Z_{\ell-1}]
\end{aligned}$$

and that

$$\begin{aligned}
D[t,t^{-1}]_{a_\ell t} &= T[t,t^{-1},\frac{1}{a_\ell t}]/x{\cdot}T[t,t^{-1},\frac{1}{a_\ell t}] \\
&= T[t,t^{-1},\frac{1}{a_\ell}]/x{\cdot}T[t,t^{-1},\frac{1}{a_\ell}] \\
&= R[\frac{1}{a_\ell},Z_1,Z_2,\ldots,Z_\ell,t,t^{-1}]/x{\cdot}R[\frac{1}{a_\ell},Z_1,Z_2,\ldots,Z_\ell,t,t^{-1}] \\
&= \left(R[\frac{1}{a_\ell},t,t^{-1}]\right)[Z_1,Z_2,\ldots,Z_\ell]/\left(\sum_{i=1}^{\ell-1}\frac{a_iZ_i}{a_\ell}+Z_\ell\right) \\
&= \left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}].
\end{aligned}$$

Then we get the following commutative diagram

$$\begin{array}{ccccc}
[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_{a_\ell t} & \xrightarrow{\varphi_{a_\ell t}} & [\overline{\mathsf{T}}]_{a_\ell t} & \longrightarrow & D[t,t^{-1}]_{a_\ell t} \\
\downarrow\simeq & & & & \downarrow\simeq \\
\left([\overline{R[It,t^{-1}]}][\frac{1}{a_\ell t}]\right)[Z_1,\ldots,Z_{\ell-1}] & & \longrightarrow & & \left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,\ldots,Z_{\ell-1}],
\end{array}$$

where the vertical homomorphisms are isomorphisms, so that the horizontal homomorphism $\varphi_{a_\ell t}$ is injective. Because $\left([\overline{R[It,t^{-1}]}][\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}]$ is integrally closed in $\left([R[t,t^{-1}]]\,[\frac{1}{a_\ell t}]\right)[Z_1,Z_2,\ldots,Z_{\ell-1}]$ and $\varphi_{a_\ell t}$ is finite, $\varphi_{a_\ell t}$ is an isomorphism, whence

$$\varphi_P : [\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_P \longrightarrow [\overline{\mathsf{T}}]_P$$

is an isomorphism too. This proves Claim 2. $\square$

Because t^{-1}, xt form a regular sequence in the normal ring $\overline{\mathcal{B}}$ and because $\dim D' = \dim R' - 1 = d - 1 \geq 2$, thanks to Claim 2, we have

$$\overline{\mathrm{e}}_I^1(R) = \overline{\mathrm{e}}_{I'}^1(R') = \overline{\mathrm{e}}_{ID'}^1(D').$$

Thus the hypothesis of induction on d yields the assertion that $\overline{\mathrm{e}}_I^1(R) \geq 0$, which completes the proof of Theorem 9.1. $\square$

The condition in Theorem 9.1 that R is unmixed is not superfluous. Let us note the simplest example. See [26, Example 2.4] for more examples.

Example 9.2. We look at the local ring

$$R = k[[X,Y,Z]]/\mathfrak{a},$$

where $k[[X,Y,Z]]$ is the formal power series ring over a field k and $\mathfrak{a} = (X) \cap (Y,Z)$. Then $\dim R = 2$, R is mixed, and $\overline{\mathrm{e}}_{\mathfrak{m}}^1(R) = \overline{\mathrm{e}}_{\mathfrak{m}}^2(R) = -1$. Hence the famous bad example [29, p. 203, Example 2] of Nagata which is a non-regular local integral domain $(A, \mathfrak{n})$ of dimension 2 with $\mathrm{e}_{\mathfrak{n}}^0(A) = 1$ possess $\overline{\mathrm{e}}_{\mathfrak{n}}^1(A) = \overline{\mathrm{e}}_{\mathfrak{n}}^2(A) = -1$, because

$$\widehat{A} \cong k[[X,Y,Z]]/[(X) \cap (Y,Z)]$$

for some field k.

Proof. We put $T = k[[X,Y,Z]]$ and $\mathfrak{q} = (X,Y,Z)$ in T. Then $\overline{R} = T/(X) \oplus T/(Y,Z)$ and we have the exact sequence

$$(E) \quad 0 \to R \to T/(X) \oplus T/(Y,Z) \to T/\mathfrak{q} \to 0$$

of T-modules; hence $\mathfrak{m}\overline{R} \subseteq R$. Recall that $\mathfrak{m}$ is a normal ideal in R, that is, $\overline{\mathfrak{m}^n} = \mathfrak{m}^n$ for all $n \geq 1$, since the associated graded ring

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[X,Y,Z]/[(X) \cap (Y,Z)]$$

of $\mathfrak{m}$ is reduced. Therefore, as

$$\mathfrak{m}^{n+1} = \overline{\mathfrak{m}^{n+1}} = \overline{\mathfrak{m}^{n+1}\overline{R}} \cap R = \mathfrak{m}^{n+1}\overline{R} \cap R,$$

thanks to exact sequence (E) above, we get

$$0 \to R/\overline{\mathfrak{m}^{n+1}} \to T/[(X) + \mathfrak{q}^{n+1}] \oplus T/[(Y,Z) + \mathfrak{q}^{n+1}] \to T/\mathfrak{q} \to 0$$

for all $n \geq 0$. Hence

$$\ell_R(R/\overline{\mathfrak{m}^{n+1}}) = \binom{n+2}{2} + \binom{n+1}{1} - 1,$$

so that $\overline{\mathrm{e}}_{\mathfrak{m}}^1(R) = \overline{\mathrm{e}}_{\mathfrak{m}}^2(R) = -1$. □

Let us note a consequence of Theorem 9.1.

Corollary 9.3 ([27, Theorem 1]). *Let R be an analytically unramified unmixed local ring with maximal ideal $\mathfrak{m}$ and $d = \dim R > 0$. Let I be a parameter ideal in R. If $\overline{\mathrm{e}}_I^1(R) = \mathrm{e}_I^1(R)$, then R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$, whence I is normal*

Proof. We get $\mathrm{e}_I^1(R) \geq 0$ by Theorem 9.1, whence by Theorem 1.8 R is a Cohen-Macaulay local ring with $\mathrm{e}_I^1(R) = 0$. Because $\overline{\mathrm{e}}_I^1(R) \geq \mathrm{e}_{\overline{I}}^1(R)$ and

$$\mathrm{e}_{\overline{I}}^1(R) \geq 0$$

([30, Corollary 1]), we furthermore have $\mathrm{e}_{\overline{I}}^1(R) = 0$, whence $\overline{I}$ is a parameter ideal in R ([30, Corollary 2]). Because parameter ideals contain no proper reductions ([31]), we get $\overline{I} = I$, whence by [7, Theorem (3.1)] R is a regular local ring with $\mu_R(\mathfrak{m}/I) \leq 1$ and I is normal. $\square$

Remark 9.4. In Corollary 9.3, unless I is a parameter ideal, R is not necessarily a regular local ring, even though $\overline{\mathrm{e}}_I^1(R) = \mathrm{e}_I^1(R)$. Let us note an example. We look at the local ring

$$R = k[[X, Y, Z]]/(Z^2 - XY),$$

where $k[[X, Y, Z]]$ is the formal power series ring over a field k of characteristic 0. Then R is a rational singularity, so that $\overline{\mathrm{e}}_I^1(R) = \mathrm{e}_I^1(R)$ for every integrally closed $\mathfrak{m}$-primary ideal I in R.

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