Algebraic cycles

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Contents:

Prerequisites

We assume some basic knowledge of algebraic geometry which can be found in Hartshorne's book on Algebraic Geometry and in Griffiths-Harris Principles of Algebraic Geometry for the complex algebraic geometry. Of course we only assume the basics from these books.

At the end of each lecture are the more specific references for that lecture (lecture I and II are combined).

Lecture I (outline) Algebraic cycles. Chow groups.

In lecture I and II k is an algebraically closed field (otherwise arbitrary).

We work with algebraic varieties X, Y, etc. defined over k (i.e. k-schemes which are reduced). We assume moreover (unless otherwise stated) that our varieties are smooth, quasi-projective and irreducible.

If X is such a variety, let $d = \dim X$; in the following we often write for such a variety shortly X_d.

(A) Algebraic cycles.

Let X_d be such a variety.

Let $0 \leq i \leq d$ and put $q = d - i$.

Let $Z_q(X) = Z^i(X)$ be the group of algebraic cycles of dim q on X, i.e. the free abelian group generated by the k-irreducible subvarieties W ~~of dimension~~ on X of dimension q.

Explicitly:

$$Z_q(X) = Z^i(X) = \left\{ Z = \sum_{\text{finite}} n_\alpha W_\alpha ;\ n_\alpha \in \mathbb{Z} \text{ and } W_\alpha \subset X \text{ irreducible, } \dim W_\alpha = q \right\}$$

To be precise: here we assume $W_\alpha \subset X$ to be k-irreducible but not necessarily smooth.

Examples

1. $Z^1(X) = \mathrm{Div}(X)$ the group of (Weil) divisors on X
2. $Z^d(X) = Z_0(X)$ the group of 0-dimensional cycles on X, i.e. $Z = \sum n_\alpha P_\alpha$ with P_α points on X. Put $\deg(Z) = \sum n_\alpha$.
3. $Z_1(X_3) = Z^2(X_3)$ the group of curves on the threefold X_3.

Operations on algebraic cycles

There are 3 basic operations:

1. Cartesian product of cycles
2. push forward
3. intersection product

1. Cartesian product

$$Z_{q_1}(X_1) \times Z_{q_2}(X_2) \longrightarrow Z_{q_1+q_2}(X_1 \times X_2)$$

via $W \times V \longrightarrow W \times V$ for irred. subvar., further by linearity

2 Push-forward of cycles ([F], p. 11)

Given $f: X \longrightarrow Y$

$$f_*: Z_q(X) \longrightarrow Z_q(Y)$$

By linearity sufficient to define $f_*(W)$ if $W \subset X$ is q-dimensional irreducible subvariety on X.

Then settheoretical image $f(W)$ is algebraic subvariety of Y and $\dim f(W) \leq \dim W$

Def
$$f_*(W) = \begin{cases} 0 & \text{if } \dim f(W) < \dim W \\ [k(W):k(f(W))]\cdot f(W) & \text{if } \dim f(W) = \dim W \end{cases}$$

Where $k(W)$, etc. is the function field of W (i.e. the field of rational functions on W)

3. Intersection product (⚠ Not always defined)

Let $V \subset X$ (resp $W \subset X$) of codim i (resp j).

Facts: settheoretically $V \cap W = \cup A_\ell$ with A_ℓ irreducible subvarieties of X of codimension (in X) $\leq i+j$.

Def. Intersection of V and W at A_ℓ (or in A_ℓ) is proper or good if $\operatorname{codim}_X A_\ell = i+j$.

Then define:

$$i(V.W; A_\ell) := \sum_{r=0}^{\dim X} (-1)^r \operatorname{length}_{\mathcal{O}} \{\operatorname{Tor}_r^{\mathcal{O}}(\mathcal{O}/J(V), \mathcal{O}/J(W))\},$$

where $\mathcal{O} = \mathcal{O}_{X, A_\ell}$ and $\mathcal{I}(V)$, resp. $\mathcal{I}(W)$, is the ideal
~~defined~~ defining V (resp. W) in $\mathcal{O}$
(see [5], p. 144 or [H], p. 427)
$i(V.W; A_\ell)$ is the intersection multiplicity of A_ℓ in $V \cap W$

Now if all intersections A_ℓ of $V \cap W$ are good define
$$V.W := \sum_\ell i(V.W; A_\ell)\, A_\ell \in Z^{i+j}(X)$$
Next define $Z_1 . Z_2$ for $Z_1 \in Z^i(X)$, $Z_2 \in Z^j(X)$
by linearity

Now using these basic operations we define further:

4. Pull-back of cycles (⚠ Not always defined)

Given $f: X \to Y$
$$f^*: Z^i(Y) \to Z^i(X)$$
defined as follows: for $Z \in Z^i(Y)$

Def $f^*(Z) := (pr_X)_* \{\Gamma_f . (X \times Z)\}$ Γ_f graph of f
provided $\Gamma_f . (X \times Z)$ ~~defined~~ is defined

Remark. This is OK if f flat ([F], p.18)

5. Operation of correspondences on algebraic cycles (⚠ only partially defined)

$T \in Z^n(X_d \times Y_e)$ is called correspondence from X to Y
(and its transpose is denoted by ${}^tT \in Z^n(Y \times X)$)

Define a homomorphism
$$T: Z'^i(X_d) \longrightarrow Z^{i+n-e}(Y_e)$$
by the formula $T(Z) := (pr_Y)_* \{T . (Z \times Y)\}$

on the subgroup $Z'^i(X) \subset Z^i(X)$ of those $Z \in Z^i(X)$
for which $T.(Z \times Y)$ is defined (on $X \times Y$).

(Note for $f: X \to Y$ we get for $T = \Gamma_f$ the f_*
and for $T = {}^t\Gamma_f$ the f^*).

(B) Equivalence relations

One wants to introduce on the groups of algebraic cycles equivalence relations such that - in particular - the above operations are always defined on the corresponding classes

Samuel introduced in 1958 the notion of "adequate equivalence relation"

We shall discuss this concept in the lecture.

Such an equivalence relation is defined via subgroups $Z^i_\sim(X) \subset Z^i(X)$ and for the corresponding quotient groups $C^i_\sim(X) = Z^i(X)/Z^i_\sim(X)$ the above operations are defined.

In particular

$$C_\sim(X) = \bigoplus_{i=0}^{\dim X} C^i_\sim(X)$$

is a ring w.r.t. the intersection product.

We shall discuss:

1. Rational equivalence (Samuel, Chow 1956)
2. Algebraic equivalence (Weil 1952)
3. Smash-nilpotent equivalence (Voevodsky 1995)
4. Homological equivalence
5. Numerical equivalence.

(C) Rational equivalence. Chow groups

Rational equivalence, introduced independently in 1956 by Samuel and Chow, is a generalization of the concept of linear equivalence for divisors.

C1: Linear equivalence for divisors

Let $X = X_d$ but not necessarily smooth for the moment (for technical reasons).
Let $\varphi \in k(X)^*$ be a rational function on X
"Recall":

$$\operatorname{div}(\varphi) := \sum_{\substack{Y \subset X \\ \text{codim } 1}} \operatorname{ord}_Y(\varphi) \cdot Y$$

where $\operatorname{ord}_Y(\varphi)$ is defined as:

a) if $\varphi \in \mathcal{O}_{X,Y}$ then $\operatorname{ord}_Y(\varphi) := \operatorname{length}_{\mathcal{O}_{X,Y}} \left(\mathcal{O}_{X,Y} / (\varphi) \right)$

b) otherwise write $\varphi = \varphi_1 / \varphi_2$ with $\varphi_1, \varphi_2 \in \mathcal{O}_{X,Y}$ and $\operatorname{ord}_Y(\varphi) = \operatorname{ord}_Y(\varphi_1) - \operatorname{ord}_Y(\varphi_2)$ (well-defined!)

Note: if X is smooth at Y then $\mathcal{O}_{X,Y}$ is a discrete valuation ring and $\operatorname{ord}_Y(\varphi) = \operatorname{val}_Y(\varphi)$

Now $\operatorname{div}(\varphi)$ is a Weil divisor and put $\operatorname{Div}_l(X) \subseteq \operatorname{Div}(X)$ for the subgroup

$$\operatorname{Div}_l(X) := \{ D = \operatorname{div}(\varphi) ; \varphi \in k(X)^* \}$$

and $CH^1(X) := \operatorname{Div}(X) / \operatorname{Div}_l(X)$

i.e. the group of divisor classes (w.r.t. linear equiv.)

(see lect. II for further discussion): $CH^1(X)$ is isomorphic to the Picard group).

C2. Rational equivalence (see [F], chap. 1)

Let $X = X_d$ be as before (smooth, q.-projective, irreducible and defined over $k = \bar{k}$).
let $0 \leq i \leq d$, put $q = d - i$.

$Z^i_{rat}(X) := Z^{rat}_q(X) \subset Z_q(X)$ is defined as the subgroup generated by those $Z \in Z_q(X)$ which are of the following type: $Z = \mathrm{div}(\varphi)$ with $\varphi \in k(Y)^*$ with $Y \subset X$ an irreducible subvariety of X of dimension $(q+1)$ and defined over k, but Y is not necessarily smooth.

Equivalently:

$Z^{rat}_q(X) = \{ Z \in Z_q(X)$; $\exists$ finite collection $(Y_\alpha, \varphi_\alpha)$ with $Y_\alpha \subset X$ irred. (not nec. smooth) of dimension $(q+1)$ and $\varphi_\alpha \in k(Y_\alpha)^*$ and $Z = \sum_\alpha \mathrm{div}(\varphi_\alpha) \}$.

Clearly $Z^1_{rat}(X) = Z^1_{lin}(X) = \mathrm{Div}_\ell(X)$, i.e. rational equivalence in codimension 1 is linear equivalence.

There is an alternative definition for rational equivalence

Lemma ([F], p. 15)

$Z \in Z_q(X)$. Equivalent conditions:

i) $Z \sim 0$ rational equivalence

ii) $\exists$ $T \in$ ~~$Z_{q+1}(X \times \mathbb{P}^1)$~~ $Z_{q+1}(\mathbb{P}^1 \times X)$ and two points $a, b \in \mathbb{P}^1$ such that $T(a) = Z$ and $T(b) = 0$

Recall $T(t) = (pr_X)_*(T \cdot (t \times X))$ for $t \in \mathbb{P}^1$.

C3. Properties of rational equivalence

Proposition
Rational equivalence is an adequate equivalence relation.
In particular there is the important
Moving lemma:

$X = X_d$ smooth, quasi-projective
Given $Z \in \mathcal{Z}^i(X)$ and a finite number of subvarieties $W_\alpha \subset X$, then there exists $Z' \in \mathcal{Z}^i(X)$ such that $Z' \underset{rat}{\sim} Z$ and such that all $Z' \cap W_\alpha$ are proper.

C4. Chow groups

Put $CH^i(X) := \mathcal{Z}^i(X) / \mathcal{Z}^i_{rat}(X)$ i-th Chow group of X

and $CH(X) = \bigoplus_{i=0}^{\dim X} CH^i(X)$

Theorem (Chow, Samuel 1956)
Let X_d, Y_e be smooth, projective (irreducible) varieties
Then
i) $CH(X)$ is a ring with respect to the intersection product.
ii) For $f: X \to Y$ proper we have additive homomorphisms
$f_*: CH_q(X) \to CH_q(Y)$
iii) For $f: X \to Y$ arbitrary we have additive homomorphisms
$f^*: CH^i(Y) \to CH^i(X)$ which in fact (put together) give a ring homomorphism $f^*: CH(Y) \to CH(X)$.

Further properties of Chow groups

Theorem (localization sequence) ([F], p. 21)

Let $Y \overset{i}{\hookrightarrow} X$ be a closed (arbitrary) subvariety; put $j: U \hookrightarrow X$ for $U = X - Y$.
Then the following sequence is exact:

$$CH_q(Y) \xrightarrow[i_*]{} CH_q(X) \xrightarrow[j^*]{} CH_q(U) \longrightarrow 0$$

Theorem (homotopy property)

X smooth, projective and $\mathbb{A}^n$ affine n-space.
Let $p: X \times \mathbb{A}^n \longrightarrow X$ be the projection.
Then $p^*: CH^i(X) \longrightarrow CH^i(X \times \mathbb{A}^n)$ is an isomorphism.

Remark ([F], p. 22)
This holds in fact for X arbitrary

$$p^*: CH_q(X) \xrightarrow{\sim} CH_{q+n}(X \times \mathbb{A}^n)$$

Note that p is flat.

Lecture II (Outline)

Part 1

Equivalence Relations continued

Always we assume X smooth, q-projective, irreducible $/k$

(D) Algebraic equivalence (Weil 1952)

Def. $Z \in \mathcal{Z}^i(X)$

Z algebraically equivalent to zero if there exists a smooth curve C (irreducible!), a cycle $T \in \mathcal{Z}^i(X \times C)$ and two points $a, b \in C$ s.t. ~~${}^tZ(a) = Z$~~ ${}^tT(a) = Z$ and ${}^tT(b) = 0$

Let $\mathcal{Z}^i_a(X) \subset \mathcal{Z}^i(X)$ be the subgroup generated by the $Z \sim_a$ algebr. equivalent to zero

Clearly $\mathcal{Z}^i_{rat}(X) \subsetneqq \mathcal{Z}^i_{alg}(X)$

(Indeed in general $\neq$, ex. $X = E$ elliptic curve $\mathcal{Z}^1_{lin}(E) \neq \mathcal{Z}^1_{alg}(E)$)

Put $CH^i_{alg}(X) := \mathcal{Z}^i_{alg}(X) / \mathcal{Z}^i_{rat}(X) \subset CH^i(X)$

Remark

Due to the theory of the Hilbert schemes (or of the Chow varieties) we know that $\mathcal{Z}^i(X) / \mathcal{Z}^i_{alg}(X)$ is a discrete group.

(E) Smash-nilpotent equivalence

Voevodsky introduced (± 1995) the notion of Smash-nilpotence also denoted by $\otimes$-nilpotence (see [A])

$Z \in Z^i(X)$ is is called smash-nilpotent to zero on X if there exists an integer $N > 0$ such that $Z \times Z \times \cdots \times Z$ (N-times) is rational equivalent to zero on $X \times X \times \cdots \times X$. Let $Z^i_{\otimes}(X) \subset Z^i(X)$ be the subgroup generated by such $Z \sim_{\otimes} 0$. This turns out to be an adequate equivalence relation. There is the following important theorem (which goes however too far for the lectures):

Theorem (Voisin, Voevodsky)

$$Z^i_{alg}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq Z^i_{\otimes}(X) \otimes \mathbb{Q}$$

Remark

Very recently B. Kahn and Sebastian showed that we have $\neq$ in the above theorem.

(F) Homological equivalence

Let $H(X)$ be a "good" (= so-called Weil-) cohomology theory. For instance if $k = \mathbb{C}$ we could take $H(X) = H_B(X_{an}, \mathbb{Q})$ or $H_B(X_{an}, \mathbb{C})$, i.e. the "usual" Betti-cohomology theory on the underlying analytic manifold X_{an}; for arbitrary k one can take the étale cohomology $H(X) = H_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)$ ($\ell \neq \mathrm{char}(k)$) (Remark: we always assume $k = \bar{k}$, so in our case the base change from k to $\bar{k}$ is not necessary).

For such a cohomology theory there is a cycle map (see lecture 3)

$$\gamma_X : Z^i(X) \longrightarrow H^{2i}(X)$$

having "nice" properties.

<u>Def</u> Let $Z \in Z^i(X)$, then Z is <u>homologically equivalent to zero</u> if $\gamma_X(Z) = 0$. Let $Z^i_{hom}(X) \subset Z^i(X)$ be the subgroup generated by cycles homologically equivalent to zero.

This gives indeed again an adequate equivalence relation.

<u>Remarks</u>

1. This $Z^i_{hom}(X)$ depends – at least a-priori — on the choice of the cohomology theory $H(X)$.
2. Clearly we have $Z^i_{alg}(X) \subseteq Z^i_{hom}(X)$. For divisors we have $Z^1_{alg}(X) = Z^1_{hom}(X)$ (<u>theorem of Matsusaka</u>) however for $1 < i < d$ we have in general $Z^i_{alg}(X) \subsetneqq Z^i_{hom}(X)$ by a famous <u>theorem of Griffiths</u> (see lecture 4)
3. Using the Künneth theorem for cohomology one can see $Z^i_{\otimes}(X) \otimes \mathbb{Q} \subseteq Z^i_{hom}(X) \otimes \mathbb{Q}$. Voevodsky <u>conjectured</u> that in fact we have $=$ (which would imply – in particular – that $Z^i_{hom} \otimes \mathbb{Q}$ is independent of the choice of $H(X)$).

(G) <u>Numerical equivalence</u>

Let X be smooth, irreducible and <u>projective</u> (or proper). Let $Z \in Z^i(X)$, then for $W \in Z^{d-i}(X)$ we have that the intersection product $V = Z.W$ (provided defined, but this is always the case if we take their classes in the Chow group $CH(X)$) is a zero cycle Z in $Z_0(X)$, i.e. $V = \sum n_\alpha P_\alpha$ with P_α points on X and we have a <u>degree</u> $\deg(V) = \sum n_\alpha$.

Def.: $Z \in \mathcal{Z}^i(X)$ is numerically equivalent to zero if $\deg(Z.W) = 0$ for all $W \in \mathcal{Z}^{d-i}(X)$ (for which $Z.W$ is defined) and $\mathcal{Z}^i_{num}(X) \subset \mathcal{Z}^i(X)$ is the subgroup generated by such cycles.

Remarks

1. $\deg(Z.W)$ is called the intersection number of Z and W and is sometimes denoted by $\#(Z.W)$

2. Because of the compatibility between the intersection product of cycles and their cup product of their cycleclasses

$\gamma_X(\alpha.\beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$ for $\alpha \in \mathcal{Z}^i(X)$, $\beta \in \mathcal{Z}^j(X)$

we have $\mathcal{Z}^i_{hom}(X) \subseteq \mathcal{Z}^i_{num}(X)$

For divisors we have $\mathrm{Div}_{hom}(X) = \mathrm{Div}_{num}(X)$ (theorem of Matsusaka). It is a fundamental conjecture that this should hold for all i:

B(X) Conjecture: $\mathcal{Z}^i_{hom}(X) \overset{?}{=} \mathcal{Z}^i_{num}(X)$

If $k = \mathbb{C}$ this would follow from the famous Hodge conjecture (see lecture 3)

Remark There are more adequate equivalence relations for algebraic cycles (see for instance [J1]) but in the present lecture we restrict ourselves to the above

Résumé of relations and notations

X_d smooth, projective, irreducible defined over $k = \bar{k}$.

We have subgroups:

$$Z^i_{rat}(X) \subsetneq Z^i_{alg}(X) \underset{\substack{\uparrow \\ \text{Griffiths}}}{\subsetneq} Z^i_{hom}(X) \underset{\substack{\uparrow \\ ?? \\ \text{conject } B(X)}}{\subseteq} Z^i_{num}(X) \subset Z^i(X)$$

~~Def~~ Dividing out by $Z^i_{rat}(X)$ we get in the Chow group

$$CH^i_{alg}(X) \subsetneq CH^i_{hom}(X) \subseteq CH^i_{num}(X) \subset CH^i(X)$$

For any $\sim$ adequate:

$$C^i_{\sim}(X) := Z^i(X) / Z^i_{\sim}(X) \cong CH^i(X) / CH^i_{\sim}(X)$$

and $C_{\sim}(X) = \bigoplus_{i=0}^{d} C^i_{\sim}(X)$ has a ring structure w.r.t. the intersection product

As to $\otimes$-equivalence with $\mathbb{Q}$-coefficients:

$$Z^i_{alg}(X) \otimes \mathbb{Q} \underset{\substack{\uparrow \\ \text{Kahn + Sebastian}}}{\subsetneq} Z^i_{\otimes}(X) \otimes \mathbb{Q} \underset{\substack{\uparrow \\ ?? \\ \text{Conj. Voevodsky}}}{\subseteq} Z^i_{hom}(X) \otimes \mathbb{Q}$$

In fact Voevodsky conjectures

$$Z^i_{\otimes}(X) \otimes \mathbb{Q} = Z^i_{num}(X) \otimes \mathbb{Q}$$

Lecture II. Part 2

Short survey for divisors

In this part we give a short survey of the main points and main results for divisors (without proofs). We shall see then (in lectures 4 and 5) that for cycles in codimension $i > 1$ the situation is very different!

We intend to discuss here:

1. Cartier divisors (see [H], p 140-145)
2. Cartier divisors versus Weil divisors ([H], p 141)
3. Invertible sheaves (line bundles) and the Picard group
4. Structure of the $CH^1(X)$:

Theorem $\quad CH^1_{alg}(X) = \mathrm{Pic}^0(X)_{red} \quad$ Picard variety

$$CH^1_{alg}(X) = CH^1_{hom}(X) = CH^1_{num}(X) \quad \text{(Matsusaka)}$$

$$NS(X) := CH^1(X)/CH^1_{alg}(X) \quad \text{Néron-Severi group, finitely generated}$$

In case $k = \mathbb{C}$ most of the above results follow from

(a) the famous "GAGA" theorem of Serre (comparison between X and the complex manifold X_{an})

+ (b) the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X_{an}} \xrightarrow{\exp} \mathcal{O}^*_{X_{an}} \to 1 \quad \text{exact}$$

References for lecture I and II.

In the lectures we assume the "usual" basic knowledge of algebraic geometry which can be found amply (for instance) in Hartshorne's book [H], chap 1 and parts of chap 2 and chap. 3.

For the algebraic cycles and Chow groups a good introduction is Appendix A of [H]. The basic is the book of Fulton [F], but we need here mostly only chap 1. One can also look in the book of Claire Voisin [V], part VII (French), part III vol 2 (English)

For the definition of the intersection multiplicities see [S], chap 5C.

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(very advanced!)