INTRODUCTION TO SHIMURA VARIETIES

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Introduction

These are notes for a short seminar at the Centre de Recerca Matemàtica in Bellaterra, Barcelona, delivered from September 20th to 23rd 2005 as part of the activities organized during the *Special year on Shimura varieties and Arakelov Geometry.* The goal of these notes is offering a rough introduction to hermitian symmetric domains and Shimura varieties attached to a reductive algebraic group over $\mathbb{Q}$. Comments and warnings of mistakes and misprints are welcome.

Chapter 1

Hermitian symmetric domains

1.1 Lie groups and Lie algebras

Definition 1.1.1. A real Lie group is a $\mathcal{C}^\infty$-real manifold G together with a group structure

$$G \times G \stackrel{\cdot}{\longrightarrow} G\,, \quad G \stackrel{-1}{\longrightarrow} G$$

defined by $\mathcal{C}^\infty$-morphisms.

By a Theorem of Lie, a real Lie group always admits a structure of real analytic manifold for which the group law is described by real analytic maps.

Similarly, we define a complex Lie group to be a complex analytical manifold endowed with a group structure described by holomorphic maps.

A Lie subgroup $H \subset G$ of a real Lie group is a real analytic submanifold of G for which the group law of G inherits on H the structure of a Lie group.

Definition 1.1.2. Let k be a field of characteristic $\neq 2$. A Lie algebra g over k is a k-vector space equipped with a bracket operation

$$[\,,\,] : \mathsf{g} \times \mathsf{g} \longrightarrow \mathsf{g}$$

such that $[X, Y] = -[Y, X]$ and $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$ for all $X, Y, Z \in \mathsf{g}$.

Throughout these notes, we always mean that g has finite dimension over k. A Lie subalgebra h of g is a vector subspace such that $[\mathsf{h}, \mathsf{h}] \subseteq \mathsf{h}$.

If G is a real (or complex) Lie group, the tangent space $\mathsf{g} = \mathrm{Lie}(G) = T_e(G)$ at the identity element e of G is naturally a real (complex) Lie algebra:

$$\begin{array}{rccc} [\,,\,]: & \mathsf{g}\times\mathsf{g} & \longrightarrow & \mathsf{g} \\ & (X,Y) & \mapsto & (\mathcal{X}\mathcal{Y}-\mathcal{Y}\mathcal{X})_e, \end{array}$$

where for any two tangent vectors $X, Y \in \mathsf{g}$, we let $\mathcal{X}, \mathcal{Y}$ denote the unique left G-invariant vector fields on G such that $\mathcal{X}_e = X$ and $\mathcal{Y}_e = Y$.

For a Lie group G, let G_0 denote the connected component of $e \in G$. Note that $\mathsf{g} = \mathrm{Lie}(\mathsf{G}) = \mathrm{Lie}(\mathsf{G}_0)$.

Adjoint representations. Inner conjugation

$$G \longrightarrow \mathrm{Aut}(G), \qquad g \mapsto c_g(h) = ghg^{-1}$$

induces the adjoint representation of G:

$$\begin{array}{rccc} \mathrm{Ad}: & G & \longrightarrow & \mathrm{GL}(\mathsf{g}) \\ & g & \mapsto & d_e(c_g) \end{array}$$

and the adjoint representation of g:

$$\mathrm{ad} = d_e(Ad): \quad \mathsf{g} \longrightarrow \mathrm{End}(\mathsf{g}).$$

Let $Z(G) = \{g \in G : gh = hg \text{ for all } h \in G\}$ and $z(\mathsf{g}) = \{\mathsf{X} \in \mathsf{g} : [\mathsf{X},\mathsf{Y}] = \mathsf{0} \text{ for all } \mathsf{Y} \in \mathsf{g}\}$ denote the centers of G and g, respectively. Each are a Lie subgroup of G and a Lie subalgebra of g.

The group $Ad(G) = G/Z(G) \subseteq \mathrm{GL}(\mathsf{g})$ is called the adjoint group of G and $ad(\mathsf{g}) = \mathsf{g}/\mathsf{z}(\mathsf{g})$ the adjoint algebra of g.

Proposition 1.1.3. *Let G be a real Lie group and $\mathsf{g} = \mathrm{Lie}(\mathsf{G})$ be its Lie algebra. There is a one-to-one correspondence*

$$\{H \subseteq G \text{ connected Lie subgroup }\} \leftrightarrow \{\mathsf{h} \subseteq \mathsf{g} \text{ Lie subalgebra }\}.$$

Let g be a real Lie algebra. The adjoint algebra $ad(\mathsf{g})$ is naturally a subalgebra of $\mathrm{End}(\mathsf{g}) = \mathrm{Lie}(\mathrm{GL}(\mathsf{g}))$ and thus there exists a connected Lie subgroup $\mathrm{Int}(\mathsf{g}) \subseteq \mathrm{GL}(\mathsf{g})$ such that $\mathrm{Lie}(\mathrm{Int}(\mathsf{g})) = ad(\mathsf{g})$.

Definition 1.1.4. A real Lie algebra g is compact if $\mathrm{Int}(\mathsf{g})$ is.

A Lie subalgebra h of g is compactly embedded in g if the Lie subgroup $H = \mathrm{Int}_{\mathsf{g}}(\mathsf{h}) \subset \mathrm{GL}(\mathsf{g})$ such that $\mathrm{Lie}(H) = ad_{\mathsf{g}}(\mathsf{h})$ is compact.

We have $\mathrm{Int}(\mathfrak{g}) = Ad(G)_0 = (G/Z(G))_0$. Hence, if G is compact then $\mathfrak{g}$ is a compact Lie algebra. But there may exist non compact Lie groups G such that $G/Z(G)$ is compact, and thus so $\mathfrak{g}$ is.

Simply connected Lie groups and the equivalence of categories. Let $\mathfrak{g}$ be a real Lie algebra. By the above construction we know that if $z(\mathfrak{g}) = \{0\}$ then $\mathfrak{g} = \mathsf{ad}(\mathfrak{g})$ and we can realize $\mathfrak{g} = \mathrm{Lie}(\mathrm{Int}(\mathfrak{g}))$ as the Lie algebra of a Lie group. What about if $z(\mathfrak{g}) \neq \{0\}$?

Besides, there might be plenty of connected Lie groups G such that $\mathrm{Lie}(G) = \mathfrak{g}$. Indeed, if $\tilde{G} \longrightarrow G$ is a topological covering of Lie groups, then G and $\tilde{G}$ are locally isomorphic and thus they share the same Lie algebra. Uniqueness is obtained when we consider the universal covering of G.

Theorem 1.1.5. *There is an equivalence of categories between the category of*

Simply connected real Lie groups

and the category of

Real Lie algebras.

Semisimple Lie algebras. Let k be a field of characteristic $\neq 2$ and let $\mathfrak{g}$ be a Lie algebra over k.

An *ideal* of $\mathfrak{g}$ is a vector subspace $\mathfrak{a} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. The ideal is abelian if $[\mathfrak{a}, \mathfrak{a}] = \{0\}$. Note that an ideal of $\mathfrak{g}$ is also a subalgebra.

Definition 1.1.6. A Lie algebra $\mathfrak{g}$ is simple if it is not abelian and it contains no ideals $\mathfrak{a} \neq \{0\}, \mathfrak{g}$. It is semisimple if it contains no abelian ideals $\mathfrak{a} \neq \{0\}$.

Define the Killing form on $\mathfrak{g}$ to be

$$\begin{array}{cccc} B_{\mathfrak{g}} : & \mathfrak{g} \times \mathfrak{g} & \longrightarrow & k \\ & (X, Y) & \mapsto & \mathrm{Tr}(ad(X) \cdot ad(Y)). \end{array}$$

Theorem 1.1.7 (Cartan). *Let k be a subfield of $\mathbb{C}$.*

(i) $\mathfrak{g}$ *is semisimple if and only if* $B_{\mathfrak{g}}$ *is nondegenerate.*

(ii) $\mathfrak{g}$ *is semisimple if and only if* $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r$ *is the direct sum of simple Lie algebras* $\mathfrak{g}_i$.

In fact, if a is an ideal of a semisimple Lie algebra g, then $\mathsf{g} = \mathsf{a} \oplus \mathsf{a}^{\perp}$, where $\mathsf{a}^{\perp} = \{\mathsf{X} \in \mathsf{g} : \mathsf{B}_{\mathsf{g}}(\mathsf{X}, \mathsf{a}) = 0\}$.

Definition 1.1.8. The Lie algebra g is reductive if for any ideal $\mathsf{a} \subseteq \mathsf{g}$ there exists an ideal $\mathsf{b} \subseteq \mathsf{g}$ such that $\mathsf{g} \simeq \mathsf{a} \oplus \mathsf{b}$.

Theorem 1.1.9. *The Lie algebra g is reductive if and only if $\mathsf{g} \simeq \mathsf{s} \oplus \mathsf{z}$, where s is a semisimple Lie algebra and z is an abelian Lie algebra.*

If g is reductive, in fact $s = [\mathsf{g}, \mathsf{g}]$ is semisimple and $\mathsf{g} \simeq s \oplus z(\mathsf{g})$.

Theorem 1.1.10. *Let k be a field of characteristic 0. Then g is reductive if and only if all representations $\varrho : \mathsf{g} \longrightarrow \mathrm{End}(V)$ on a finite dimensional vector space V over k are semisimple: if $W \subset V$ is $\varrho(\mathsf{g})$-invariant, there exists a $\varrho(\mathsf{g})$-invariant vector subspace $\tilde{W} \subset V$ such that $V = W \oplus \tilde{W}$.*

We say that a connected real Lie group G is :

- a **torus** if it can be embedded as a subgroup of $D_n = \{\mathrm{diag}(a_1, ..., a_n)\} \subset \mathrm{GL}_n(\mathbb{C})$, $D_n \simeq \mathbb{C}^* \times \overset{n)}{\cdots} \times \mathbb{C}^*$ for some n.
- **simple** if it is not abelian and contains no normal connected Lie subgroups $\neq \{1\}, G$.
- **semisimple** if it contains no normal connected abelian Lie subgroups $\neq \{1\}$.
- **reductive** if the only normal connected abelian Lie subgroups are tori.

Warning. The definitions of simple and semisimple Lie group are those naturally equivalent for the same definitions on Lie algebras. Note however that the definition of reductive Lie group is more restrictive: the only commutative algebraic subgroups that we allow in a reductive Lie group are tori. For instance, $G = \mathbb{R}$ with the addition law is not a torus and thus also not reductive, although $\mathrm{Lie}(\mathbb{R}) = \mathrm{Lie}(\mathbb{R}^*) = \mathbb{R}$.

Examples:

1. $\mathrm{SL}_n(\mathbb{R})$, $\mathsf{g} = \mathrm{sl}_n(\mathbb{R}) = \{X \in \mathrm{M}_n(\mathbb{R}) : \mathrm{Tr}(X) = 0\}$ is simple.
2. $\mathrm{GL}_n(\mathbb{R})$, $\mathsf{g} = \mathrm{M}_n(\mathbb{R}) = \langle 1_n \rangle \oplus \mathrm{sl}_n(\mathbb{R})$ is reductive but not semisimple.

3. $\mathrm{SO}(n) = \{A \in \mathrm{SL}_n(\mathbb{R}) : A^t A = 1\}$,
   $\mathfrak{g} = \{X \in \mathrm{M}_n(\mathbb{R}) : X^t + X = 0\}$ is simple.

   More generally, for any decomposition $n = p + q$, $p, q \geq 0$:
   $\mathrm{SO}(p,q) = \{A \in \mathrm{SL}_n(\mathbb{R}) : A^t \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} A = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\}$ is simple.

   When $p, q > 0$, $\mathrm{SO}(p,q)$ has two connected components.

4. $SU(p,q) = \{A \in \mathrm{SL}_n(\mathbb{C}) : \bar{A}^t \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix} A = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}\}$ is simple.

   It contains the Lie subgroup $S(U_p \times U_q) = \{A = \begin{pmatrix} A_p & 0 \\ 0 & A_q \end{pmatrix} : \bar{A}_p^t A_p = 1_p, \bar{A}_q^t A_q = 1_q, \det(A_p A_q) = 1\}$.

5. $\mathrm{SO}^*(2n) = \{A \in \mathrm{SU}(n,n) : A^t \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}\}$ is simple.

6. $\mathrm{Sp}_n(\mathbb{R}) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^t \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}\}$ is simple.

7. $\mathrm{Sp}(n) = \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{U}(2n)$ is simple.

1.2 Hermitian symmetric manifolds

A Riemannian real manifold is a $\mathcal{C}^\infty$-manifold (M, ϱ) equipped with a metric

$$\varrho : T(M) \otimes T(M) \longrightarrow \mathbb{R},$$

that is, a positive definite $\mathcal{C}^\infty$-tensor field.

A hermitian complex manifold is a complex analytic manifold (M, ϱ) equipped with a hermitian metric

$$\varrho : T(M) \otimes T(M) \longrightarrow \mathbb{C},$$

that is, an analytic positive definite hermitian tensor field: $\mathbb{C}$-linear on the first variable, $g_p(u,v) = \overline{g_p(v,u)}$ for all $u, v \in T_p(M)$ and $g(u,u) > 0$ for all $u \in T_p(M) \setminus \{0\}$.

Or equivalently, an $\mathbb{R}$-linear, symmetric, positive definite tensor field

$$\varrho_0 : T(M) \otimes T(M) \longrightarrow \mathbb{R}$$

such that $\varrho_{0,p}(iu, iv) = \varrho_{0,p}(u, v)$ for all $u, v \in T_p(M)$. It induces

$$\varrho(u, v) := \varrho_0(u, v) + i\varrho_0(u, iv).$$

Definition 1.2.1. A hermitian symmetric manifold is a connected hermitian manifold (M, ϱ) such that:

(i) M is homogenous: the group $\mathrm{Aut}(M, \varrho)$ of holomorphic isometries acts transitively on M.

(ii) M is symmetric: for all $p \in M$ there exists an involution $s_p \in \mathrm{Aut}(M, \varrho)$, $s_p^2 = 1$, such that p is an isolated fixed point of s_p.

Similarly one defines Riemannian symmetric manifolds. An important remark is that for each point $p \in M$, the symmetry s_p is always unique. Locally, s_p is the geodesic involution $s_p(\gamma(t)) = \gamma(-t)$, where γ denotes any geodesic on M with $\gamma(0) = p$.

Examples.

(a) $M = \mathcal{H}_1 = \{z \in \mathbb{C} : \text{ Im } (z) > 0\}$ with the hermitian metric $\varrho_0 = \frac{dxdy}{y^2}$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathcal{H}_1$ by Moebius transformations:

$$z \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

The group $\mathrm{PSL}_2(\mathbb{R})$ acts transitively by holomorphic isometries on $\mathcal{H}_1$ and a symmetry at $i \in \mathcal{H}_1$ is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(b) $M = \mathbb{P}^1(\mathbb{C}) \subset \mathbb{R}^3$ with the hermitian metric induced by $\varrho_0 = dxdydz$. The group $\mathrm{SO}_3(\mathbb{R})$ acts by rotations transitively on $\mathbb{P}^1(\mathbb{C})$. These are holomorphic isometries and for each $p \in \mathbb{P}^1(\mathbb{C})$, there is an obvious rotation which leaves p and $-p$ fixed.

(c) $M = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subset \mathbb{C}$ is a lattice, together with the metric $\varrho_0 = dxdy$. The group $\mathbb{C}/\Lambda$ itself acts transitively by translations on M. For any point $p \in M$, the involution $s_p : q \mapsto 2p - q$ is a symmetry which has exactly four fixed points, p being one of them.

For a point p in a $\mathcal{C}^\infty$-manifold M and vector subspace $E \subseteq T_p(M)$, $\dim(E) = 2$, recall Gauss' sectional curvature $K_p(E)$ of p along E. It may be defined as

- $K_p(E) = -g_p(R_p(X,Y)X,Y)$ where R is the $(3,1)$-curvature tensor field attached to the Riemannian connection ∇ on M and $\{X,Y\}$ is an orthonormal basis of E.
- Let $E \cap M$ be the local submanifold of M around p obtained by exponentiation of geodesics γ such that $\gamma(0) = p$ and $\dot{\gamma}(0) \in E$. Let C and c be the maximum and minimum of the curvatures of the signed curves on $E \cap M$ obtained by cutting the surface $E \cap M$ with planes through a normal line at p and define $K_p(E) = C \cdot c$.

  The curvature of a curve is $1/R$, where R is the radius of the circle that best approximates the curve. We give the sign $+$ or $-$ to the curvature of the curve depending whether the curve bends towards the normal line or not.

Definition 1.2.2. Let (M, ϱ) be a hermitian symmetric manifold. If for all points $p \in M$ and all vector subspaces $E \subseteq T_p(M)$, $\dim(E) = 2$, we have:

(a) $K_p(E) < 0$, we say that (M, ϱ) is of noncompact type.

(b) $K_p(E) > 0$, we say that (M, ϱ) is of compact type.

(c) $K_p(E) = 0$, we say that (M, ϱ) is of euclidean type.

Theorem 1.2.3. *Let (M, ϱ) be a hermitian symmetric manifold. Then*

$$(M, \varrho) \simeq (M^-, \varrho^-) \times (M^+, \varrho^+) \times (M^0, \varrho^0)$$

is isometric to the product of a hermitian symmetric space of noncompact type, compact type and euclidean type.

The group of holomorphic isometries $G = \mathrm{Aut}(M, \varrho)$ of a hermitian symmetric manifold is equipped with the compact-open topology for which a basis of open subsets is given by

$$W(C, U) = \{g \in G : g(C) \subset U\},$$

where $C \subset M$ is a compact and $U \subset M$ is open.

With this topology G admits a unique structure of real analytic Lie group and for each point $p \in M$, the isotropy group

$$K_p = \{g \in G : g(p) = p\}$$

is compact.

Moreover, if (M, ϱ) is of noncompact type or compact type, there exists an algebraic group $\mathcal{G} \subset \mathrm{GL}(\mathsf{g})$ over $\mathbb{R}$ such that

$$\mathcal{G}(\mathbb{R})_0 = G_0 = \mathrm{Hol}(M)_0 = \mathrm{Aut}(M^{\mathbb{R}}, g^{\mathbb{R}})_0.$$

Example 1.2.4. For $M = \mathcal{H}_1$, $g = \frac{dxdy}{y^2}$, we have $\mathrm{Aut}(M, \varrho) = \mathrm{Hol}(M) = \mathrm{PSL}_2(\mathbb{R})$, which is the set of real points of the algebraic group PSL_2. However, $\mathrm{Aut}(M^{\mathbb{R}}, g^{\mathbb{R}}) = \mathrm{PSL}_2(\mathbb{R}) \cup \mathrm{PSL}_2(\mathbb{R}) \cdot (z \mapsto \bar{z}^{-1})$.

We focus now on hermitian symmetric manifolds of noncompact type.

If $D \subset \mathbb{C}^n$ is a bounded open connected subset, there is canonical hermitian metric g_{Bgm} on D, the so-called Bergman's metric, which has negative sectional curvatures. If $\mathrm{Hol}(D)$ acts transitively on D and each point $p \in D$ admits a symmetry (in fact, if one then all), then it turns (D, g_{Bgm}) into a hermitian symmetric manifold of noncompact type. Conversely, any such space is isometric to (D, g_{Bgm}) for some bounded domain of $\mathbb{C}^n$.

Accordingly, hermitian symmetric manifolds of noncompact type are often called *hermitian symmetric domains*.

Let (M, ϱ) be a hermitian symmetric domain, $G = \mathrm{Aut}(M, \varrho)$ be the Lie group of holomorphic isometries of M, G_0 be the connected component of $e \in G$ and $\mathsf{g} = \mathrm{Lie}(G)$.

Fix a point $p \in M$ and let $K = K_p$ be the (compact) isotropy group of M at p.

The automorphism

$$\begin{array}{rccc} \sigma : & G & \longrightarrow & G \\ & g & \mapsto & s_p g s_p \end{array}$$

is an involution on G and we let $\mathsf{g} = \mathsf{g}^+ \oplus \mathsf{g}^-$ be the decomposition of g into ± 1-eigenspaces with respect to $d_e(\sigma)$.

With the above notations we have the following fundamental result.

Theorem 1.2.5. *Let (M, ϱ) be a hermitian symmetric domain. Then* *(i) M is simply connected and G is semisimple and noncompact.*

(ii) $Z(G) \subset K$ *is a finite group, and there is a diffeomorphism*

$$\begin{array}{rccc} \pi: & G_0/K & \xrightarrow{\simeq} & M \\ & g & \mapsto & g(p) \end{array}$$

(iii) $(K_\sigma)_0 \subseteq K \subseteq K_\sigma := \{g \in G : \sigma(g) = g\}$ *and* $\mathrm{Lie}(K) = \mathsf{g}^+$.

(iv) $d_e\pi : \mathsf{g}^- \xrightarrow{\simeq} T_p(M)$ *and* $Id \cdot exp : K \times \mathsf{g}^- \xrightarrow{\simeq} G_0$.

(v) The complex structure $J_p \in \mathrm{End}(T_p(M))$ *belongs to* $z(\mathrm{Lie}\, K)$*. In particular, K has non discrete center* $Z(K)$.

In item (v) above, we regard $\mathrm{Lie}(K)$ as a subalgebra of $\mathrm{End}(T_p(M))$ through the representation

$$\begin{array}{ccc} K & \longrightarrow & \mathrm{GL}(T_p(M)) \\ k & \mapsto & d_p(gK \mapsto kgK). \end{array}$$

We say that hermitian symmetric domain (M, ϱ) is *irreducible* if it is not isometric to the product of non zero hermitian symmetric domains. If (M, ϱ) is a hermitian symmetric domain, then

$$(M, \varrho) \simeq (M_1, \varrho_1) \times ...(M_r, \varrho_r),$$

where (M_i, ϱ_i) are irreducible.

If (M, ϱ) is irreducible, then $\mathsf{g} = \mathrm{Lie}(G)$ and $\mathsf{g} \otimes \mathbb{C}$ are simple Lie algebras and $\mathrm{Lie}(K)$ is a maximal proper subalgebra of g.

Remark 1.2.6. Although a real Lie algebra g is semisimple if and only $\mathsf{g} \otimes \mathbb{C}$ is, there exist simple real Lie algebras g such that $\mathsf{g} \otimes \mathbb{C}$ are not. Indeed, let $\mathsf{c}/\mathbb{C}$ be a simple complex Lie algebra and let $\mathsf{g} = \mathsf{c}^{\mathbb{R}}$ be c regarded as a Lie algebra. Then g is simple but $\mathsf{g} \otimes \mathbb{C}$ is not.

1.3 Cartan's classification

Cartan involutions.

Definition 1.3.1. Let g be a semisimple real Lie algebra. A Cartan involution on g is an automorphism $s \in \mathrm{End}(\mathsf{g})$ such that $s \neq \mathrm{Id}$, $s^2 = \mathrm{Id}$ and the $+1$-eigenspace $\mathsf{g}^+ \subseteq \mathsf{g}$ is a compactly embedded subalgebra.

Definition 1.3.2. We call (g, s) a symmetric Lie algebra, and we say that it is of compact or noncompact type depending whether g is compact or not. We say that (g, s) is an irreducible hermitian noncompact symmetric Lie algebra if

- g is non compact, g and $\mathsf{g} \otimes \mathbb{C}$ are simple
- g^+ is a maximal proper subalgebra of g and $z(\mathsf{g}^+) \neq \{0\}$.

Next proposition can be found in [2, p. 292, 303, 385].

Proposition 1.3.3. *Let g be a semisimple real Lie algebra. The Killing form B_{g} on g is $Ad(G)$-invariant:*

$$B_{\mathsf{g}}(Ad(g)X, Ad(g)Y) = B_{\mathsf{g}}(X, Y), \qquad \text{for all } g \in G, X, Y \in \mathsf{g}.$$

Moreover, it is negative definite on g^+ and positive definite on g^-.

Symmetric domains versus symmetric Lie algebras.

If (M, ϱ) is an irreducible hermitian symmetric domain, then Theorem 1.2.5 implies that $(\mathsf{g}, d_e\sigma)$ is an irreducible noncompact hermitian symmetric Lie algebra, where $\mathsf{g} = \mathrm{Lie}(G)$, $G = \mathrm{Aut}(M, \varrho)$ and $\sigma : G \longrightarrow G$, $g \mapsto s_p g s_p$.

Conversely, let (g, s) be an irreducible noncompact hermitian symmetric Lie algebra. Let $\tilde{G}$ be the simply connected Lie group such that $\mathrm{Lie}(\tilde{G}) = \mathsf{g}$ (which can be obtained as the universal covering of $\mathrm{Int}(\mathsf{g})$) and let $\sigma \in \mathrm{Aut}(\tilde{G})$ be an involution of $\tilde{G}$ such that $d_e\sigma = s$. The existence of σ is guaranteed by Theorem 1.1.5.

Let $\tilde{K}$ be the connected component of e in $K_\sigma = \{g \in \tilde{G} : \sigma(g) = g\}$. Since $\mathrm{Lie}(\tilde{K}) = \mathsf{g}^+$, it follows from the prescribed properties of (g, s) that $\tilde{K}$ is a maximal connected proper Lie subgroup of $\tilde{G}$. Moreover, $Ad_{\tilde{G}}(\tilde{K}) \subset \mathrm{GL}(\mathsf{g})$ is compact.

The coset space $M = \tilde{G}/\tilde{K}$ is naturally a real analytic manifold. Let $\pi : \tilde{G} \longrightarrow M$ be the projection map and let $p_0 = \pi(e) \in M$ be the base point. The Killing form B_{g} is positive definite on $\mathsf{g}^- \simeq T_{p_0}(M)$ and it is $\tilde{G}$-invariant:

$$B_{\mathsf{g}}(X, Y) = B_{\mathsf{g}}(d_e Ad(g)X, d_e Ad(g)Y) \quad \text{for any } g \in \tilde{G}.$$

By means of the transitive action of $\tilde{G}$ on M, it can be extended to a metric ϱ on M which turns (M, ϱ) into a Riemannian manifold on which $\tilde{G}$ acts by isometries.

The action of $\tilde{G}$ on M is obviously transitive and for each $p \in M$ there is a symmetry $s_p \in \tilde{G}$ which leaves p fixed. Indeed, the symmetry at p_0 is $s_{p_0} : g\tilde{K} \mapsto \sigma(g)\tilde{K}$ (and this is again an isometry because B_{g} is invariant under automorphisms of $\tilde{G}$).

Recall now that $Z(\tilde{K})$ is a non discrete abelian (compact) Lie subgroup of $\tilde{K}$. Hence $Z(\tilde{K})_0 = S^1 \times ... \times S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. In fact, we have the following lemma (see [3, Thm 1.9]).

Lemma 1.3.4. *There exists a homomorphism*

$$u : S^1 \longrightarrow Z(\tilde{K})$$

such that the isometry

$$\begin{array}{rccl} c_{u(-1)} : & \tilde{G}/\tilde{K} & \longrightarrow & \tilde{G}/\tilde{K} \\ & g\tilde{K} & \mapsto & u(-1)g\tilde{K} \quad (= u(-1)gu(-1)^{-1}\tilde{K}) \end{array}$$

induced by conjugation by $u(-1)$ fixes p_0 and satisfies

$$d_{p_0} c_{u(-1)} = -\ Id\ \in \text{End}\ T_{p_0}(M).$$

Since the symmetry s_{p_0} satisfies the same properties, by uniqueness we have $c_{u(-1)} = s_{p_0}$ and thus also $\text{ad}(u(-1)) = s$.

Also, $J_0 = d_e c_{u(i)} \in \text{End}\ T_{p_0}(M)$ is an endomorphism such that $J_0^2 = -\ \text{Id}$ and $d_e(c_k)J_0 = J_0 d_e(c_k)$ for all $k \in \tilde{K}$.

It follows that there can be constructed a unique $\tilde{G}$-invariant almost complex structure J on M such that $J_{p_0} = J_0$ for which ϱ is hermitian. In fact, J is integrable and (M, ϱ) is therefore an irreducible hermitian symmetric domain.

Remark 1.3.5. And now the lemma can be reformulated to ensure the stronger statement that $d_{p_0}c_{u(z)} = z \cdot \mathrm{Id} \in \mathrm{End}\, T_{p_0}(M)$ for all $z \in S^1$. It implies that in the representation $\mathrm{Ad}(u) \otimes \mathbb{C} : S^1 \longrightarrow \mathrm{GL}(\mathsf{g} \otimes \mathbb{C})$, the only characters $\xi \in \mathrm{Hom}(S^1, \mathbb{C}^*) \simeq \mathbb{Z}$ that occur are 1, z and z^{-1}.

We have thus established a one-to-one correspondence between

Irreducible hermitian symmetric domains

and

Irreducible noncompact hermitian symmetric Lie algebras.

If we start with (M, ϱ), Theorem 1.2.5 produces $G = \mathrm{Aut}(M, \varrho)_0$, $K = K_p$ and $\sigma \in \mathrm{Aut}(G)$ so that $(\mathsf{g} = \mathrm{Lie}(G), s = d_e(\sigma))$ is an irreducible noncompact hermitian symmetric Lie algebra. Starting now with the resulting pair (g, s), we have attached to it a simply connected Lie group $\tilde{G}$ and a compact subgroup $\tilde{K}$. The automorphism group G of M does not need to be simply connected, but we do have that $\tilde{G}/\tilde{K}$ is the universal covering of G/K (see [1, p. 178]). Since M is simply connected by Theorem 1.2.5, we deduce that $\tilde{G}/\tilde{K} \simeq G/K = M$.

Classification of simple Lie algebras.

For a real symmetric Lie algebra (g_0, s), let $\mathsf{g}_0 = \mathsf{g}_0^+ \oplus \mathsf{g}_0^-$ be the decomposition into ± 1-eigenspaces with respect to s. Define the dual symmetric Lie algebra (g_0^*, s^*) to be the subalgebra $\mathsf{g}_0^* = \mathsf{g}_0^+ \oplus i\mathsf{g}_0^-$ of the complex Lie algebra $\mathsf{g} := \mathsf{g}_0 \otimes \mathbb{C}$, and $s^* : X^+ + iX^- \mapsto X^+ - iX^-$.

Theorem 1.3.6. *(i) If $\mathsf{g}/\mathbb{C}$ is a semisimple Lie algebra, there exists a compact real Lie algebra g_0 such that $\mathsf{g} = \mathsf{g}_0 \otimes \mathbb{C}$. Any two compact real forms of g are isomorphic and there is a one-to-one correspondence between*

$$\{ \textit{ Semisimple Lie algebras } \mathsf{g}/\mathbb{C}\}/\simeq$$

and

$$\{ \textit{ Compact Lie algebras } \mathsf{g}_0/\mathbb{R}\}/\simeq$$

which preserves simplicity.

(ii) If (g_0, s) is a symmetric simple Lie algebra of compact type, then its dual symmetric Lie algebra (g_0^*, s^*) is of noncompact type, and conversely.

Remark that in general it is not true that $\mathfrak{g}_0$ is simple over $\mathbb{R}$ if and only if $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ is simple over $\mathbb{C}$. This holds however true for compact real Lie algebras (see [1, p. 308]).

Combining the two statements of the above theorem we obtain that in order to classify symmetric Lie algebras of noncompact type and irreducible hermitian symmetric domains, we can proceed as follows:

1. Classify the isomorphism classes of simple Lie algebras $\mathfrak{g}$ over $\mathbb{C}$.
2. Find a compact real form $\mathfrak{g}_0$ for them and classify all possible Cartan involutions s on them: $(\mathfrak{g}_0^*, s^*)$ will recover all symmetric Lie algebras of noncompact type up to isomorphism.
3. Compute $z(\mathfrak{g}_0^{*+})$.

Theorem 1.3.7 (Cartan). *Let (M, g) be an irreducible hermitian symmetric domain. Then M is isometric to either*

Label	*Compact form*	*Domain*	*dim*
A_{III}	$SU(n)$	$SU(p,q)/S(U_p \times U_q)$	$2pq$
$BD_I(q=2)$	$SO(n+2)$	$SO_0(n,2)/SO(n) \times SO(2)$	$2n$
C_I	$Sp(n)$	$Sp_n(\mathbb{R})/U(n)$	$n(n+1)$
D_{III}	$SO(2n)$	$SO^*(2n)/U(n)$	$n(n-1)$

or to the exceptional cases $EIII$ of dimension 32 *or $EVII$ of dimension* 54.

- The compact real simple algebra of SO(n) admits two Cartan involutions when n is even. These give raise to two different noncompact forms. A similar phenomenon happens for the compact group $SU(n)$: for each decomposition $n = p + q$ with p, $q > 0$ there is Cartan involution on $SU(n)$ which gives raise to each of the non-compact forms of the third column.
- There are several coincidences among the above Lie groups. More precisely: $A_{III}(p = q = 1) \simeq C_I(n = 1)$, $BD_I(p = 3, q = 2) \simeq C_I(n = 2)$, $A_{III}(p = q = 2) \simeq BD_I(p = 4, q = 2)$, $A_{III}(p = 3, q = 1) \simeq D_{III}(n = 3)$ and $BD_I(p = 6, q = 2) \simeq D_{III}(n = 4)$.

- The Siegel space is $\mathcal{H}_n = \{Z \in \mathrm{M}_n(\mathbb{C}) : Z = Z^t, \mathrm{Im}(Z) > 0\} = \mathrm{Sp}_n(\mathbb{R})/\mathrm{U}(n)$.

Chapter 2

Locally symmetric varieties

The aim of this chapter is considering quotients $D(\Gamma) := \Gamma\backslash D$ of hermitian symmetric domains D by discrete subgroups $\Gamma \subset \mathrm{Aut}(D)_0$ and studying under what conditions the coset space $D(\Gamma)$ is the set of complex points of an algebraic variety defined over a number field.

The starting point is the following proposition.

Proposition 2.0.8. *Let D be a hermitian symmetric domain and let $G = \mathrm{Aut}(D)$ be the group of holomorphic isometries of D. Let $\Gamma \subset G_0$ be a discrete* torsion-free *subgroup. There is a unique complex analytic structure on $D(\Gamma)$ for which $\pi : D \twoheadrightarrow D(\Gamma)$ is a local isomorphism.*

A map of complex analytic varieties $D(\Gamma) \longrightarrow V$ is analytic if and only if the composition $D \twoheadrightarrow D(\Gamma) \longrightarrow V$ is.

Proof. With the quotient topology, $D(\Gamma)$ is a separated space[1].

For any $p \in D$, $\Gamma_p = \{\gamma \in \Gamma : \gamma \cdot p = p\} \subset K_p$ is a discrete subgroup of a compact group, hence finite. If Γ is torsion-free, $\Gamma_p = \{1\}$ for all $p \in D$. There exists $p \in U_p$ such that $\gamma U_p \cap U_p = \emptyset$ for all $\gamma \in \Gamma \setminus \{1\}$ so that $\pi_{|U_p} : U_p \longrightarrow \pi(U_p)$ is a homeomorphism, producing a complex analytic atlas on $D(\Gamma)$. □

Assume Γ is torsion-free. Since D is simply connected, it is the universal covering of the complex manifold $D(\Gamma)$. For any $p \in D$ we have

$$\begin{array}{ccc} \Gamma & \xrightarrow{\simeq} & \pi_1(D(\Gamma), \pi(p)) \\ g & \mapsto & [\pi(c)] \end{array}$$

[1] For any $p, q \in D$ not in the same orbit under Γ, we can find open subsets $p \in U_p$, $q \in U_q \subset D$ such that $\gamma U_p \cap U_q = \emptyset$ for all $\gamma \in \Gamma$.

where c is any path on D joining p and $g \cdot p$.

2.1 Algebraic groups

Let k denote a field of characteristic 0 and $\bar{k}$ a fixed algebraic closure of k.

Definition 2.1.1. An algebraic group over a field k is an algebraic variety G over k together with a group structure

$$G \times G \xrightarrow{\cdot} G\,, \quad G \xrightarrow{-1} G$$

defined by algebraic morphisms defined over k.

Examples of algebraic groups over a field k are

- $\mathbb{G}_m = \mathrm{Spec} k[X,Y]/(XY-1)$ is the multiplicative group such that for any k-algebra A, $\mathbb{G}_m(A) = A^*$.
- $\mathbb{G}_a = \mathrm{Spec} k[X]$ is the additive group such that for any k-algebra A, $\mathbb{G}_m(A) = A$.
- M_n, GL_n, SL_n, SO_n, Sp_n, ...
- Elliptic curves and abelian varieties over k.

The Lie algebra of G is $\mathsf{g} = \mathrm{Lie}(G) = T_e(G)$, and it is a Lie algebra over k.

All examples of algebraic groups we will consider are affine. We will not consider abelian varieties, for instance.

The notion of torus, semisimple group and reductive group are similarly defined as in the previous chapter for Lie groups. That is:

Definition 2.1.2. A connected algebraic group G over k is

- a **torus** if $G \otimes \bar{k} \simeq \mathbb{G}_m \times \overset{n)}{\cdots} \times \mathbb{G}_m$ for some n. The minimal field extension K/k for which $G \otimes K \simeq \mathbb{G}_m \times ... \times \mathbb{G}_m$ is called the splitting field of G.
- **semisimple** if it contains no smooth connected normal commutative algebraic subgroups $\neq \{1\}$.

- **reductive** if it contains no smooth connected normal commutative algebraic subgroups other than tori.

Note for instance that abelian varieties or $\mathbb{G}_a$ are not reductive algebraic groups.

A typical example of a non split torus is constructed as follows. Let K/k be a finite Galois extension and $T = \mathrm{Res}_{K/k}(\mathbb{G}_m)$. This is an algebraic group over k of dimension $n = [K : k]$ characterized by $T(A) = (A \otimes_k K)^*$ for any k-algebra A. In particular $T(k) = K^*$ and $T(K) = (K^*)^n$. It is split over K.

For a reductive algebraic group G over k, let T stand for the largest commutative quotient of G. Since T is connected, it is a torus. We define the derived group G^{der} of G to be the kernel of ν. The Lie algebra of G^{der} is $\mathrm{Lie}(G^{der}) = [\mathsf{g}, \mathsf{g}]$ and according to Theorem 1.1.9 it is a semisimple group.

Let Z denote the centre of G. Since $\mathsf{g} = [\mathsf{g}, \mathsf{g}] \oplus z(\mathsf{g})$, we have that $\mathrm{Lie}(Z) = \mathrm{Lie}(T)$. In fact $Z/Z' \simeq T$, where $Z' := Z \cap G^{der}$ is a finite group. We thus have exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G^{der} & \longrightarrow & G & \xrightarrow{\nu} & T & \longrightarrow & 1 \\ 1 & \longrightarrow & Z & \longrightarrow & G & \xrightarrow{ad} & G^{ad} & \longrightarrow & 1 \\ 1 & \longrightarrow & Z' & \longrightarrow & Z & \longrightarrow & T & \longrightarrow & 1. \end{array}$$

When $G = \mathrm{GL}_n$, these are

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{SL}_n & \longrightarrow & \mathrm{GL}_n & \xrightarrow{\det} & \mathbb{G}_m & \longrightarrow & 1 \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GL}_n & \xrightarrow{ad} & \mathrm{PGL}_n & \longrightarrow & 1 \\ 1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{G}_m & \xrightarrow{x \mapsto x^n} & \mathbb{G}_m & \longrightarrow & 1. \end{array}$$

2.2 Arithmetic and congruence groups

Two subgroups $S_1, S_2 \subset S$ of a group S are *commensurable* if $S_1 \cap S_2$ has finite index both in S_1 and S_2.

Definition 2.2.1. Let G be an algebraic group over $\mathbb{Q}$. A subgroup $\Gamma \subset G(\mathbb{Q})$ is *arithmetic* if it is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for some embedding $G \hookrightarrow \mathrm{GL}_n$.

A *congruence subgroup* of $G(\mathbb{Q})$ is a subgroup $\Gamma \subset G(\mathbb{Q})$ which for some embedding $G \hookrightarrow \mathrm{GL}_n$ contains

$$\Gamma(N) = G(\mathbb{Q}) \cap \{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv Id_n \bmod N\}$$

as a subgroup of finite index.

For a reductive group there always exist an embedding $G \hookrightarrow \mathrm{GL}_n$ and it can be shown that the above definitions do not depend on the chosen embedding $G \hookrightarrow \mathrm{GL}_n$.[2]

Let

$$\mathbb{A}_f = \hat{\prod} \mathbb{Q}_\ell = \{(a_\ell) : a_\ell \in \mathbb{Q}_\ell, a_\ell \in \mathbb{Z}_\ell \text{ for almost all } \ell\}$$

be the ring of finite adeles of $\mathbb{Q}$. It is a topological ring when we regard it as a subring of $\prod \mathbb{Q}_\ell$ with the product topology, in which a basis of (compact) open subsets of 0 are $\{K(N) = \prod K_\ell(N)\}_{N \geq 1}$, where $K_\ell(N) = \mathbb{Z}_\ell$ if $\ell \nmid N$; $K_\ell(N) = \ell^{r_\ell}\mathbb{Z}_\ell$ if $r_\ell = \mathrm{ord}_\ell(N) \geq 1$.

For an algebraic group G over $\mathbb{Q}$, let $G(\mathbb{A}_f) = \hat{\prod} G(\mathbb{Q}_\ell)$.[3] If $G \hookrightarrow \mathrm{GL}_n$ is an embedding, a basis of (compact) open neighbourhoods of 1 is given by $K(N) = \prod K_\ell$ where

$$K_\ell = \begin{cases} G(\mathbb{Z}_\ell) & \text{if } \ell \nmid N \\ \{g \in G(\mathbb{Z}_\ell) : g \equiv Id_n \bmod \ell^{r_\ell}\} & \text{if } r_\ell = \mathrm{ord}_\ell(N) \geq 1. \end{cases}$$

The topology does not depend of the choice of the embedding.[4] For instance $\mathbb{G}_a \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \mathrm{GL}_2$ and $\mathbb{G}_m = \mathrm{GL}_1$.

Proposition 2.2.2. *Let G be a reductive group over $\mathbb{Q}$. For any compact open subgroup $K \subset G(\mathbb{A}_f)$, $K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$.*[5]

Proof. Let $G \hookrightarrow \mathrm{GL}_n$ be an embedding. Then $K(N)$ is a compact open subgroup of $G(\mathbb{A}_f)$ and

$$K(N) \cap G(\mathbb{Q}) = \Gamma(N).$$

[2]Congruence groups are arithmetic and the *classical congruence problem* asks whether any arithmetic group $\Gamma \subset G(\mathbb{Q})$ is congruence. By definition we know that Γ is commensurable with $\Gamma(1)$ but we do not know whether $\Gamma(N) \subset \Gamma$ for some N. The answer is *yes* if G is simply connected (that is, if any isogeny $G' \to G$ from a connected algebraic group G' is the identity) and $G \not\simeq \mathrm{SL}_2$. Otherwise, SL_2 and non simply connected reductive groups have many non-congruence arithmetic subgroups. See [4].

[3]In order to talk about $G(\mathbb{Z}_\ell)$, this definition implies the choice of a model of G over $\mathbb{Z}$. However, any two such models will become isomorphic over $\mathbb{Z}[\frac{1}{d}]$ for some $d \geq 1$. Since there are finitely many primes $\ell \mid d$, there is no ambiguity in our definition.

[4]It does always exist for reductive groups.

[5]And every congruence subgroup arises in this way.

If K is a compact open subgroup of $G(\mathbb{A}_f)$, there exists $N \geq 1$ such that $K \supseteq K(N)$ and $\Gamma := K \cap G(\mathbb{Q}) \supseteq \Gamma(N)$. Thus $\Gamma/\Gamma(N) \subset K/K(N)$ is a discrete subgroup of a compact group: the index of $\Gamma(N)$ in Γ is finite and Γ is congruence. □

Definition 2.2.3. If G is an arbitrary connected real Lie group, we still define *arithmetic subgroups* of G as follows. A subgroup $\Gamma \subset G$ is *arithmetic* if there exists

- An algebraic group $\mathcal{G}$ over $\mathbb{Q}$.
- An arithmetic subgroup $\tilde{\Gamma} \subset \mathcal{G}(\mathbb{Q})$
- A surjective morphism $\pi : \mathcal{G}(\mathbb{R})_0 \longrightarrow G$ with compact kernel

such that $\pi(\tilde{\Gamma}) = \Gamma$.

2.3 The theorem of Baily-Borel

Let D be a hermitian symmetric domain and $G = \mathrm{Aut}(D)$ be the (semisimple) real Lie group of holomorphic isometries of D. Let $\Gamma \subset G_0$ be an arithmetic torsion-free subgroup. This means that there exists an algebraic group $\mathcal{G}$ over $\mathbb{Q}$, an arithmetic subgroup $\tilde{\Gamma} \subset \mathcal{G}(\mathbb{Q})$ and a surjective morphism $\pi : \mathcal{G}(\mathbb{R})_0 \longrightarrow G$ with compact kernel such that $\pi(\tilde{\Gamma}) = \Gamma$.

Theorem 2.3.1. *(i) [Baily-Borel] Then $D(\Gamma)$ has a canonical realization as a smooth Zariski-open subset of a projective algebraic variety $D(\Gamma)^*$. If $\mathcal{G}(\mathbb{Q})$ contains no unipotent elements*[6]*, then $D(\Gamma)$ is compact.*

(ii) [Borel] Let V be a nonsingular quasi-projective variety over $\mathbb{C}$. Then every holomorphic map of complex analytic manifolds

$$f : V(\mathbb{C}) \longrightarrow D(\Gamma)(\mathbb{C})$$

is regular algebraic.

- For $D = \mathcal{H}_1$ the proof of (i) works as follows.

[6] An unipotent element is an element $\gamma \in \mathcal{G}(\mathbb{Q})$ such that $\varrho(\gamma) - 1$ is nilpotent for all representations $\varrho : \mathcal{G} \hookrightarrow \mathrm{GL}(V)$ of $\mathcal{G}$.

1. Let $D^* = \mathcal{H}_1 \cup \mathbb{P}^1(\mathbb{Q})$ with a suitable topology.
2. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ be an arithmetic subgroup (commensurable with $\mathrm{SL}_2(\mathbb{Z})$). It acts on $\mathcal{H}_1^*$ by Moebius transformations on $\mathcal{H}_1$ and by linear projective transformations on $\mathbb{P}^1(\mathbb{Q})$ so that $D(\Gamma)^* = \Gamma\backslash\mathcal{H}_1^*$ is a compact complex surface.
3. Modular forms (i.e. regular differentials) of large weight enough produce an embedding of $D(\Gamma)^*$ into a projective space.
4. Chow's theorem asserts that $D(\Gamma)^*$ is naturally a projective variety and $D(\Gamma) = D(\Gamma)^* \setminus (\Gamma\backslash\mathbb{P}^1(\mathbb{Q}))$, the complementary of a finite set of points.

- For arbitrary D, the proof of (i) follows the same pattern.

  1. $D^* = D \bigcup \cup B_i$, where B_i are so-called rational boundary components, endowed with the Satake topology.
  2. Automorphic forms of large weight embed $D(\Gamma)^* = \Gamma\backslash D^*$ into a projective space, so that $D(\Gamma) \subset D(\Gamma)^*$ is a Zariski-open subvariety.

Remark 2.3.2. 1. If PGL_2 is not a quotient of the algebraic group $\mathcal{G}$ over $\mathbb{Q}$, then the components of $D(\Gamma)^* \setminus D(\Gamma)$ have codimension ≥ 2.[7]

2. If Γ is an arithmetic group with torsion, Proposition 2.0.8 does not apply and $D(\Gamma)$ is even not a complex manifold. However, there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index in Γ which is torsion free. Thus $D(\Gamma_0)$ is by Baily-Borel a smooth algebraic variety and we can construct $D(\Gamma) = D(\Gamma_0)/(\Gamma/\Gamma_0)$. By Hilbert's theorem on invariant algebras under finite groups, $D(\Gamma)$ still has the structure of an algebraic variety, with *quotient* singularities at those points $p \in D(\Gamma)$ for which the stabilizer Γ_p is not trivial. Hence the

[7] The example $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}_1$ is explained as follows: the boundary is a nonempty set of finite points, thus of codimension 1. We already expected this since $\mathrm{SL}_2(\mathbb{Z})$ contains unipotent elements. But PGL_2 is not a quotient of SL_2 over $\mathbb{Q}$! That's true, but $\mathrm{PGL}_2(\mathbb{R})_0 = \mathrm{SL}_2(\mathbb{R})$ and we can also choose $\mathcal{G} = \mathrm{PGL}_2$ instead of SL_2. With this choice we also have $\mathcal{H}_1 = \mathrm{PGL}_2(\mathbb{R})_0/K$ and now the remark makes sense. Since the natural inclusion $\mathcal{G} = \mathrm{PSL}_2 \longrightarrow \mathrm{PGL}_2$ is an isomorphism on the connected components of 1 of the real points, $\mathrm{SL}_2(\mathbb{Z})$ is an arithmetic subgroup of $\mathrm{PGL}_2(\mathbb{Q})$.

singularities of $D(\Gamma)^*$ are found at these points in $D(\Gamma)$ and at the boundary $D(\Gamma)^* \setminus D(\Gamma)$.

- The proof of (ii) works as follows. Let $\mathcal{D}_1 = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{D}_1^* = \mathcal{D}_1 \setminus 0$.

  1. Big Picard Theorem: If a holomorphic function $f : \mathcal{D}_1^* \longrightarrow \mathbb{C}$ has an essential singularity at 0, it takes all values of $\mathbb{C}$ except possibly one. (thus, if there are two values not in the image, f has a pole at 0).
  2. Any holomorphic $f : \mathcal{D}_1^* \longrightarrow \mathbb{P}^1(\mathbb{C}) \setminus \{p_1, p_2, p_3\}$ extends to an holomorphic function $f : \mathcal{D}_1 \longrightarrow \mathbb{P}^1(\mathbb{C})$.
  3. Borel's extension of big Picard's Theorem: every holomorphic map $\mathcal{D}_1^{*r} \times \mathcal{D}_1^s \hookrightarrow D(\Gamma)$ extends to a holomorphic map $\mathcal{D}_1^{r+s} \hookrightarrow D(\Gamma)^*$.
  4. Hironaka: $V \subset V^*$ where V^* is (possible singular) projective and $V^* \setminus V$ is a divisor with normal crossings: locally for the complex topology, $V \hookrightarrow V^*$ is of the form $\mathcal{D}_1^{*r} \times \mathcal{D}_1^s \hookrightarrow \mathcal{D}_1^{r+s}$.
  5. By Borel's lemma, $f : V(\mathbb{C}) \hookrightarrow D(\Gamma)$ extends to a holomorphic map $V^*(\mathbb{C}) \hookrightarrow D(\Gamma)^*$.
  6. Chow: any holomorphic map between projective varieties is algebraic.

Corollary 2.3.3. *(i) The structure of an algebraic variety on $D(\Gamma)$ is unique.*

(ii) For any other compactification $D(\Gamma) \hookrightarrow D(\Gamma)^\dagger$ with $D(\Gamma)^\dagger$ a projective variety and $D(\Gamma)^\dagger \setminus D(\Gamma)$ a divisor with normal crossings, there is a unique regular map $D(\Gamma)^\dagger \longrightarrow D(\Gamma)^*$ commuting with $D(\Gamma) \hookrightarrow D(\Gamma)^*$.

Proof: As for (i), let V be a complex algebraic variety such that $D(\Gamma) \simeq V(\mathbb{C})$ as complex manifolds. By Borel's theorem, this is an algebraic isomorphism. Statement (ii) follows similarly from the proof of Borel's theorem, considering $V = D(\Gamma) \subset V^* = D(\Gamma)^\dagger$ and the map $f = Id : D(\Gamma) \longrightarrow D(\Gamma)$, which extends to $D(\Gamma)^\dagger \longrightarrow D(\Gamma)^*$. □

For this reason, $D(\Gamma) \hookrightarrow D(\Gamma)^*$ is often called the *minimal* compactification on $D(\Gamma)$. Other names are the Satake-Baily-Borel or Baily-Borel compactification.

Examples:

1. $D = \mathcal{H}_1$, $\mathrm{Aut}(\mathcal{H}_1) = \mathrm{SL}_2(\mathbb{R})$, $G = \mathrm{SL}_2$ is a semisimple group over $\mathbb{Q}$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $D(\Gamma) \overset{\simeq}{\to} \mathbb{A}^1_{\mathbb{C}}$ is an algebraic variety over $\mathbb{C}$. It is not projective: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ is unipotent.

2. $D = \mathcal{H}_n$, $\mathrm{Aut}(\mathcal{H}_n) = \mathrm{Sp}_n(\mathbb{R})$, $G = \mathrm{Sp}_n$ is a semisimple group over $\mathbb{Q}$, $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$. Hence $\mathrm{Sp}_n(\mathbb{Z})\backslash\mathcal{H}_n$ is the set of complex points of a (non-projective) algebraic variety. We will see later that it is a moduli space defined over $\mathbb{Q}$.

3. Let F be a totally real field, $n = [F : \mathbb{Q}]$. Let $B = F \oplus Fi \oplus Fj \oplus Fk$, $i^2 = a, j^2 = b \in F^*$, $ij = -ji = k$ be a quaternion algebra over F.

   Let $v_1, ..., v_n : F \hookrightarrow \mathbb{R}$ be the real archimedean places of B and write $B \otimes_{\mathbb{Q}} \mathbb{R} = B \otimes_F F \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_v (B \otimes_F \mathbb{R}_v) = \mathrm{M}_2(\mathbb{R}) \overset{r}{\ldots} \mathrm{M}_2(\mathbb{R}) \bigoplus \mathbb{H} \oplus \overset{s}{\ldots} \oplus \mathbb{H}$, $n = r + s$.

   Let $\mathrm{n} : B^* \longrightarrow F^*$ be the reduced norm, which coincides with *det* on any matrix representation. Let $B^1 = \{b \in B^* : \mathrm{n}(b) = 1\}$, which may be regarded as an algebraic group over F because it is given by a polynomial equation in $F[x_1, x_2, x_3, x_4]$. Let $\mathcal{G}_B = \mathrm{Res}_{F/\mathbb{Q}} B^1$, which as algebraic group over $\mathbb{Q}$ such that $\mathcal{G}(K) = (B \otimes_{\mathbb{Q}} K)^1$ for any extension field $K/\mathbb{Q}$. In particular, $\mathcal{G}_B(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \times \overset{r}{\ldots} \times \mathrm{SL}_2(\mathbb{R}) \times \mathbb{H}^1 \times \overset{s}{\ldots} \times \mathbb{H}^1$.

   Let $\mathcal{O}$ be a ring of integers of B: a subring of B of rank 4 over the ring of integers $\mathcal{O}_F$ of F. Let $\mathcal{O}^1 \subset B^1$ the subgroup of elements of reduced norm 1. This is a discrete subgroup of $\mathcal{G}_B(\mathbb{R})$ and according to Definition 2.2.3, it is an arithmetic subgroup of $G = \mathrm{SL}_2(\mathbb{R}) \times \overset{r}{\ldots} \times \mathrm{SL}_2(\mathbb{R})$ because the kernel of $\mathcal{G}_B(\mathbb{R}) \twoheadrightarrow G$ is compact. Note that $\mathrm{Aut}(D)$, where we let

$$D = \mathcal{H}_1 \times \overset{r}{\ldots} \times \mathcal{H}_1.$$

   By the theorem of Baily-Borel $D(\Gamma)$ this is a quasi-projective algebraic variety.

   - If $B \simeq \mathrm{M}_2(F)$, there are unipotent elements in Γ and $D(\Gamma)$ is not compact.

If $F = \mathbb{Q}$, $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ and $X = D(\Gamma)^* = D(\Gamma) \cup \{\infty_1, ..., \infty_h\}$ is a *modular curve*. Prominent examples are the congruence subgroups $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, giving raise to the modular curves $X(N) \twoheadrightarrow X_1(N) \twoheadrightarrow X_0(N) \twoheadrightarrow X(1) = \mathbb{P}^1_{\mathbb{C}}$.

If $F \neq \mathbb{Q}$, $D(\Gamma)^*$ is called a *Hilbert(-Blumenthal) modular variety*, a singular projective variety of dimension $n = [F : \mathbb{Q}]$. The difference with the case $F = \mathbb{Q}$ is that now PGL_2 is not a quotient of $\mathcal{G}_B$ over $\mathbb{Q}$ and we can apply Remark 2.3.2: the singular locus, which is contained in $D(\Gamma)^* \setminus D(\Gamma)$ has codimension ≥ 2.

- If $B \not\simeq \mathrm{M}_2(F)$, there are no unipotent elements in Γ and $D(\Gamma) = D(\Gamma)^*$ is already a *projective* variety of dimension r. It is smooth unless Γ has torsion. Prominent examples are Shimura curves X_D attached to a maximal order $\mathcal{O}$ in the quaternion algebra B over $\mathbb{Q}$ of discriminant $D = p_1 \cdots p_{2d}$, and their covers $X(D, N) \twoheadrightarrow X_1(D, N) \twoheadrightarrow X_0(D, N) \twoheadrightarrow X_D$, for any integer $N \geq 1$, $(D, N) = 1$.

Chapter 3

Shimura varieties

3.1 Connected Shimura varieties

Definition 3.1.1. A **connected** Shimura datum is a pair $(G, D = \{h\})$ where

- G is a semisimple algebraic group over $\mathbb{Q}$.
- $h : S^1 \longrightarrow G^{ad}_{\mathbb{R}}$ is a homomorphism such that

SV1: Only the characters 1, z and $z^{-1} = \bar{z}$ occur in the adjoint representation of S^1 on $V = \mathrm{Lie}(G^{ad})_{\mathbb{C}}$,[1]

SV2: $Ad(h(-1))$ is a Cartan involution on $\mathrm{Lie}\, G^{ad}_{\mathbb{R}}$,

SV3: There exists no factor H of G^{ad} over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact.

- $D = \{g \cdot h \cdot g^{-1}\}_{g \in G^{ad}(\mathbb{R})_0} \subset \mathrm{Hom}(S^1, G^{ad}_{\mathbb{R}})$.

Let (G, D) be a connected Shimura datum. Write

$$G^{ad}(\mathbb{R}) \sim G_1 \times ... \times G_r \times H_1 \times ... \times H_s$$

as a product[2] of simple *real* Lie groups, and label them so that G_i are non-compact and H_j are compact.

[1] That is, $V = V^0 \oplus V^+ \oplus V^-$ in such a way that for any $z \in S^1$ we have $Ad\, h(z) \cdot v_0 = v_0$, $Ad\, h(z) \cdot v_+ = z \cdot v_+$, $Ad\, h(z) \cdot v_- = z^{-1} \cdot v_-$ for any $v_0 \in V^0$, $v_+ \in V^+$, $v_- \in V^-$.

[2] This is an isomorphism up to a finite group, and the factors G_i, H_j correspond to the decomposition of $\mathfrak{g}$ into simple Lie algebras.

By SV3 we have that $r \geq 1$: otherwise $G(\mathbb{R})$ would be compact.

If we write $h = (h_1, ..., h_{r+s})$, we know by SV2 that $s_i = Ad(h_i(-1)) \neq Id$ for the factors G_i.[3]

For $i = 1, ..., r$, let $K_i = K_{s_i} \subset G_{i,0}$ and $\mathcal{D}_i = G_{i,0}/K_i$ be the irreducible[4] hermitian symmetric domain attached to (G_i, s_i).

The $\mathcal{D}_i$ are indeed hermitian thanks to the existence of h_i satisfying $SV1, 2$: $h_i(S^1)$ lies in the compact group K_i because the elements obviously commute with $h(-1)$.

Moreover, $h_i(S^1)$ is contained[5] in the centre of K_i and thus there is an integrable complex structure J on $\mathcal{D}_i$ (cf. the discussion in p. 12, 13).

The domain $\mathcal{D}_i$ is naturally identified with the $G_{i,0}$-conjugacy class $D_i = \{g h_i g^{-1} : g \in G_{i,0}\} \subset \mathrm{Hom}(S^1, G_i^{ad})$ of h_i. The one-to-one correspondence is

$$\begin{array}{ccc} \mathcal{D}_i = G_{i,0}/K_i & \xrightarrow{\simeq} & D_i \\ g & \mapsto & g h_i g^{-1} \end{array}$$

This allows us to regard $D = \prod D_i$ as a product of irreducible hermitian domains and there is a natural surjective map

$$G^{ad}(\mathbb{R})_0 \longrightarrow \mathrm{Aut}(D)_0$$

whose kernel is $H_{1,0} \times ... \times H_{s,0}$, which is compact.

Let $\Gamma \subset G^{ad}(\mathbb{Q})_0$ be[6] an arithmetic subgroup of G^{ad}. It follows from Definition 2.2.3 that the image $\bar{\Gamma}$ of Γ in $\mathrm{Aut}(D)$ is again an arithmetic group.

Moreover, since the kernel of $\Gamma \longrightarrow \bar{\Gamma}$ is finite (being discrete in a compact group), if Γ is torsion-free we then have that $\Gamma \simeq \bar{\Gamma}$ and $\Gamma \backslash D \simeq \bar{\Gamma} \backslash D$ is a smooth algebraic variety. If $\Gamma' \subset \Gamma$, we obtain regular maps $D(\Gamma') \longrightarrow D(\Gamma)$ of algebraic varieties (by Borel's Theorem 2.3.1).

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[3]This implies in particular that all three characters 1, z, z^{-1} do actually occur in the representation of S^1, because if only 1 appeared, G_i would be compact. By the way, the converse is also true (see Milne 1.17(a)): for H_j, $s_j = Id$.

[4]$\mathcal{D}_i$ are irreducibles. See my comment in p.11 previous to Remark 1.2.5 or Milne's Lemma 4.7.

[5]If $k \in K_i$, we need to show that $h_k := k h k^{-1} h^{-1} = 1 \in \mathrm{Hom}(S^1, G_i^{ad})$, where we already know that $h_k(-1) = 1$. But then h_k factors through $S^1 \xrightarrow{2} S^1$ and the only possible characters that may occur are $z^{2\mathbb{Z}}$. Since we only allow 1, z and z^{-1}, we obtain that $h_k = 1$.

[6]By $G^{ad}(\mathbb{Q})_0$ we mean $G^{ad}(\mathbb{Q}) \cap G^{ad}(\mathbb{R})_0$.

Also, any $g \in G^{ad}(\mathbb{Q})_0$ defines a holomorphic map $g : D \longrightarrow D$ and induces a regular morphism between algebraic varieties

$$g : D(\Gamma) \longrightarrow D(g\Gamma g^{-1}).$$

Let $\pi : G \longrightarrow G^{ad}$ denote the natural projection of algebraic groups.

Definition 3.1.2. Let (G, D) be a connected Shimura data. The *connected Shimura variety* $Sh^0(G, D)$ is the inverse system of locally symmetric varieties $D(\Gamma)$, where Γ runs over the congruence subgroups of $G(\mathbb{Q})_0$ such that[7] the image in $G^{ad}(\mathbb{Q})_0$ is torsion-free.

For a given such congruence subgroup Γ, we also denote $Sh^0_\Gamma(G, D) = D(\Gamma)$. For a compact open subgroup $K \subset G(\mathbb{A}_f)$, Proposition 2.2.2 shows that $\Gamma = K \cap G(\mathbb{Q})_0$ is congruence and we also set $Sh^0_K(G, D) = D(\Gamma)$.

Many examples are already obtained by those mentioned at the end of Chapter II.

Remark 3.1.3. Let $\tilde{\Gamma} \subset G^{ad}(\mathbb{Q})_0$ be an arithmetic subgroup. As it shown in Milne's easy Lemma 4.12, $\pi^{-1}(\tilde{\Gamma}) \subset G(\mathbb{Q})_0$ is congruence if and only if $\tilde{\Gamma}$ contains the image $\pi(\Gamma)$ of a congruence subgroup $\Gamma \subset G(\mathbb{Q})_0$. But usually the map $\pi : G(\mathbb{Q})_0 \rightarrow G^{ad}(\mathbb{Q})_0$ is not surjective, and the family

$$\{\pi(\Gamma) : \Gamma \subset G(\mathbb{Q})_0 \text{ congruence}\} \subset \{\tilde{\Gamma} \subset G^{ad}(\mathbb{Q})_0 : \tilde{\Gamma} \supseteq \pi(\Gamma), \Gamma \text{ congruence}\}$$

is smaller than the latter. A more general family of varieties is thus obtained when considering this second family of groups.

A semisimple group G over a field k is said to be *simply connected* if any isogeny $G' \rightarrow G$ with G connected is an isomorphism. With this definition, SL_2 is simply connected, whereas PGL_2 is not simply connected because it admits the isogeny $\mathrm{GL}_2 \rightarrow \mathrm{PGL}_2$.

The *Strong Approximation Theorem* asserts that if G is a semisimple, simply connected algebraic group over $\mathbb{Q}$ of non compact type, then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$.[8]

[7] By $G(\mathbb{Q})_0$ we mean those elements of $G(\mathbb{Q})$ that map to $G^{ad}(\mathbb{Q})_0$.

[8] For instance, $G = \mathbb{G}_m$ and PGL_2 are not simply connected, and it can be checked that $\mathbb{Q}^* \subset \mathbb{A}_f^*$ is not dense, nor it is $\mathrm{PGL}_2(\mathbb{Q})$ in $\mathrm{PGL}_2(\mathbb{A}_f)$.

Proposition 3.1.4. *Let (G, D) be a connected Shimura datum with G simply connected. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ and let $\Gamma = K \cap G(\mathbb{Q})$. There is a homeomorphism*

$$\begin{array}{ccc} \Gamma\backslash D & \xrightarrow{\simeq} & G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \\ x & \mapsto & (x, 1) \end{array}$$

In the proposition, $G(\mathbb{Q})$ acts on $D \times G(\mathbb{A}_f)$ on the left, and K acts on $G(\mathbb{A}_f)$ on the right (and trivially on D): for $(x, \{g_\ell\}) \in D \times G(\mathbb{A}_f)$ the action described by $g \in G(\mathbb{Q})$, $k \in K$ is

$$g \cdot (x, \{g_\ell\}) \cdot k = (gx, g\{g_\ell\}k).$$

Proof. Since $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$ and K is open, $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K$.[9] Thus every element in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$ is represented by $(x, 1)$ for some $x \in D$ and this shows that our map is surjective. For $x, x' \in D$, we have $[x, 1] = [x', 1]$ if and only if there exists $g \in G(\mathbb{Q}) \cap K = \Gamma$ such that $gx = x'$: our map is a bijection of sets.

Since $G(\mathbb{A}_f)/K$ is discrete because K is open, the map $D \longrightarrow D \times (G(\mathbb{A}_f)/K)$, $x \mapsto (x, [1])$ is a homeomorphism between D and its image, which is open in $D \times G(\mathbb{A}_f)/K$. From this a routine exercise shows that the quotient map

$$\Gamma\backslash D \longrightarrow G(\mathbb{Q})\backslash(D \times (G(\mathbb{A}_f)/K))$$

is bi-continuous. □

Remark 3.1.5. As K runs among compact open subgroups of $G(\mathbb{A}_f)$, the inverse limit of $Sh^0_K(G, D)$ is $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$, something which contains D and may be regarded as a kind of completion of it. We will not prove this.

3.2 Shimura varieties

Alternative definition of connected Shimura datum.

The exact sequence of real Lie groups

$$1 \to \mathbb{R}^* \xrightarrow{t \mapsto t^{-1}} \mathbb{C}^* \xrightarrow{z \mapsto z/\bar{z}} S^1 \to 1$$

[9]If $\{g_\ell\} \in G(\mathbb{A}_f)$, for any open set $\{g_\ell\} \in U$ there exists $g \in G(\mathbb{Q})$ such that $g \in U$. For $U = \{g_\ell\} \cdot K$ this implies that $\{g_\ell\} = g \cdot k$ for some $k \in K$.

arises from the exact sequence of algebraic groups

$$1 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \longrightarrow U_1 \to 1,$$

where $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ and U_1 denotes simply the quotient of tori $\mathbb{S}/\mathbb{G}_m$.

A connected Shimura datum can be alternatively defined to be a pair $(G, D = \{h\})$, where G and D are as in Definition 3.1.1 except that

$$h : \mathbb{S}_\mathbb{R} \longrightarrow G^{ad}_\mathbb{R}$$

is a homomorphism of real algebraic groups satisfying the conditions

SV1: Only the characters 1, $z/\bar{z}$ and $\bar{z}/z$ occur in the representation of $\mathbb{S}_\mathbb{R}$ on $\mathrm{Lie}(G^{ad})_\mathbb{C}$,

SV2: $Ad(h(i))$ is a Cartan involution on $\mathrm{Lie}\, G^{ad}_\mathbb{R}$,

SV3: There exists no factor H of G^{ad} over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact.

Following this approach, we define (non-connected) Shimura varieties.

Definition 3.2.1. A Shimura datum is a pair $(G, D = \{h\})$ where

- G is a **reductive** algebraic group over $\mathbb{Q}$.
- $h : \mathbb{S}_\mathbb{R} \longrightarrow G_\mathbb{R}$ is a homomorphism of algebraic groups over $\mathbb{R}$ such that

  SV1: Only the characters 1, $z/\bar{z}$ and $\bar{z}/z$ occur in the representation of $\mathbb{S}_\mathbb{R}$ on $\mathrm{Lie}(G^{ad})_\mathbb{C}$,

  SV2: $Ad(h(i))$ is a Cartan involution on $\mathrm{Lie}\, G^{ad}_\mathbb{R}$,

  SV3: There exists no factor H of G^{ad} over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact.
- $D = \{g \cdot h \cdot g^{-1}\}_{g \in G_\mathbb{R}} \subset \mathrm{Hom}(S^1, G_\mathbb{R})$.

Note that we extend the definition to arbitrary reductive algebraic groups over $\mathbb{Q}$. Another remarkable difference in our definition is that D is the conjugation class of h under the *possibly non-connected*[10] real Lie group $G_\mathbb{R}$.

[10] A theorem of Cartan asserts that if G is reductive, $G(\mathbb{R})$ has finitely many connected components for the real topology. In fact, a more powerful theorem of Whitney proves that the set of real points of an algebraic variety over $\mathbb{R}$ has finitely many connected components.

Proposition 3.2.2. *Let (G, D) be a Shimura datum. Let D_0 be a connected component of D and $K \subset G(\mathbb{A}_f)$ be a compact open subgroup. Then*

- *The set $G(\mathbb{Q})_0 \backslash G(\mathbb{A}_f)/K$ is finite. Let $\mathcal{C}$ be a set of representatives of the double coset.*
- *There is an homeomorphism*

$$\begin{array}{ccc} \bigcup_{c\in\mathcal{C}} \Gamma_c \backslash D_0 & \xrightarrow{\simeq} & G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K \\ x & \mapsto & (x, c_x), \end{array}$$

where $\Gamma_c = c \cdot K \cdot c^{-1} \cap G(\mathbb{Q})_0$ and c_x denotes the connected component to which x belongs.

Proof. In order to prove the first item, it suffices to show that $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite, because $G(\mathbb{Q})_0\backslash G(\mathbb{Q}) \hookrightarrow G^{ad}(\mathbb{R})_0\backslash G^{ad}(\mathbb{R})$ is already finite. If G is semisimple and simply connected, we already showed that $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite: we actually saw that the cardinality is 1. We will not explain here why the finiteness statement is also true in the general case (cf. [3, p. 48, 50]).

The second item follows similarly as in Proposition 3.1.4. □

Example 3.2.3.
- $G = \mathrm{GL}_2$,
 $h : \mathbb{S}_{\mathbb{R}} \to \mathrm{GL}_{2,\mathbb{R}},\ a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$,
 $D = \{h_g := ghg^{-1}\}_{g\in \mathrm{GL}_2(\mathbb{R})} \xrightarrow{\cong} \mathbb{C} \setminus \mathbb{R} = \mathcal{H}_1^+ \cup \mathcal{H}_1^-$,
 $g \mapsto$ Fixed point p of $h_g(\mathbb{C}^*)$ on $\mathbb{C}$ such that $h(z)$ acts on the tangent space of p as $z/\bar{z}$ (and not as $\bar{z}/z$).[11]
- B quaternion algebra over a totally real number field F, $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$,

$$h(a+bi) = (\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \overset{r}{\ldots}, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, ..., 1), \quad D = \mathcal{H}_1^{\pm} \times \overset{r}{\ldots} \times \mathcal{H}_1^{\pm}.$$

[11] The pair $(\mathrm{GL}_2, \mathcal{H}_1^{\pm})$ satisfies the conditions SV1, SV2, SV3. As for SV1, $h_{ad} : \mathbb{C}^* \longrightarrow \mathrm{PGL}_2(\mathbb{R})$ factors through the circle unit U_1 by means of the map $u : U_1 \to \mathrm{PGL}_2(\mathbb{R})_0 = \mathrm{PSL}_2(\mathbb{R}),\ a + bi \mapsto \pm\sqrt{a + bi} = \pm(x + yi) \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$.
The action of $h_{ad}(\mathbb{C}^*) \subset \mathrm{PSL}_2(\mathbb{R})$ on the Lie algebra $\mathrm{sl}_2(\mathbb{R}) = \mathrm{so}_2(\mathbb{R}) \oplus T_i(\mathcal{H}_1)$ is trivial on the first factor, and acts on the second through the character $z/\bar{z} = \frac{a+bi}{a-bi}$, because $d(\tau \mapsto \frac{a\tau+b}{-b\tau+a})_{|i} = \frac{a(-b\tau+a)+b(a\tau+b)}{(-b\tau a)^2}_{|i} = \frac{a^2+b^2}{(a-bi)^2} = z/\bar{z}$. Hence the only characters that occur in $\mathrm{Lie}(G(\mathbb{C})^{ad})$ are 1, $z/\bar{z}$ and $\bar{z}/z$.

- $G = T$ a torus over $\mathbb{Q}$, $h : \mathbb{S} \longrightarrow T$ any morphism.
  Since $T^{ad} = \{1\}$, conditions SV1, 2, 3 hold trivially true.
  $D = \{x\}$, a single point.
  For any compact open subgroup $K \subset G(\mathbb{A}_f)$,

$$Sh_K(T, x) = T(\mathbb{Q})\backslash\{x\} \times T(\mathbb{A}_f)/K \simeq T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K,$$

a finite set of points by Proposition 3.2.2 -note that $T(\mathbb{Q})_0 = T(\mathbb{Q})$.

Definition 3.2.4. Let (G, D) be a Shimura data. The *Shimura variety* $Sh(G, D)$ is the inverse system $Sh_K(G, D) = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$, where K runs among compact open subgroups of $G(\mathbb{A}_f)$ such that the image of all congruence subgroups $\Gamma_c = cKc^{-1} \cap G(\mathbb{Q})_0 \subset G(\mathbb{Q})$ in $G^{ad}(\mathbb{Q})$ are torsion free.

The Shimura variety $Sh(G, D)$ comes equipped with a right action of $G(\mathbb{A}_f)$. Indeed, if $g = \{g_\ell\} \in G(\mathbb{A}_f)$, there is a natural morphism[12]

$$\begin{array}{cccc} T_g: & Sh_K(D, G) & \longrightarrow & Sh_{g^{-1}Kg}(D, G) \\ & (x, a) & \mapsto & (x, ag) \end{array}$$

Remark 3.2.5. Write $\mathcal{C} = \{c_1, ..., c_h\}$. Any element of $Sh_K(D, G)$ can be written as (x, c_i) for some $x \in D$, $1 \leq i \leq h$. An element $g \in G(\mathbb{A}_f)$ induces a permutation $i \mapsto g(i)$ on $\{1, ..., h\}$ as follows: since $G(\mathbb{A}_f) = \bigcup G(\mathbb{Q})_0 c_i K$, $c_i g = \gamma c_{g(i)} k$ for some $\gamma \in G(\mathbb{Q})_0$ and $k \in K$. We then have that T_g maps the connected component $\Gamma_{c_i}\backslash D$ to $\Gamma_{c_{g(i)}}\backslash D$. More precisely, $T_g(x, c_i) = (x, c_i g) = (x, \gamma c_{g(i)} k) = \gamma(\gamma^{-1}x, c_{g(i)})k \equiv (\gamma^{-1}x, c_{g(i)})$.

The weight homomorphism and additional axioms.

Recall that exact sequence of tori

$$1 \rightarrow \mathbb{G}_m \xrightarrow{w} \mathbb{S} \longrightarrow U_1 \rightarrow 1.$$

Let (G, D) be a Shimura datum. An element of D is a homomorphism $h : \mathbb{S}_{\mathbb{R}} \longrightarrow G_{\mathbb{R}}$ that factors through U_1. When we restrict h to $\mathbb{G}_{m,\mathbb{R}} = \mathbb{R}^*$, $h(r)$ must act trivially on $\mathrm{Lie}(G_{\mathbb{R}}) \otimes \mathbb{C}$ for any $r \in \mathbb{R}^*$, because it lies in the kernel of the above exact sequence. Since the only elements of $G(\mathbb{R})$ which

[12] It is an easy exercise that these maps are well defined.

act trivially by the adjoint representation are those lying in the center, we obtain that $h(r) \in Z(\mathbb{R})$.

Hence $h_{|\mathbb{G}_m}$ does not depend of the choice of $h \in D$ and we can define the weight homomorphism of (G, D) to be

$$\begin{array}{rccc} \omega : & \mathbb{G}_{m,\mathbb{R}} & \longrightarrow & G_{\mathbb{R}} \\ & r & \mapsto & 1/h(r) \end{array}$$

Note that both $\mathbb{G}_{m,\mathbb{R}}$ and $G_{\mathbb{R}}$ are the sets of real points of algebraic groups over $\mathbb{Q}$. Since ω is a morphism of algebraic groups, it must actually be defined over some finite extension of $\mathbb{Q}$. Together with the axioms SV1, SV2 and SV3, sometimes the Shimura datum satisfies some or all of the following further axioms:

SV4: The weight homomorphism ω is *rational*, that is, it is defined over $\mathbb{Q}$,

SV5: The group $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$,

SV6: The torus Z_0 is split over a CM-field, that is, over a totally imaginary quadratic extension of a totally real number field.

3.3 The Siegel modular variety

Let (V, Ψ) be a symplectic space over $\mathbb{Q}$ of dimension $2n$, for some $n \geq 1$. That is, a vector space over $\mathbb{Q}$ equipped with an alternating non-degenerate bilinear form $\Psi : V \times V \xrightarrow{\Psi} \mathbb{Q}$.

Define $G = GSp(V, \Psi) \subset \mathrm{GL}(V)$ to be the algebraic group of transformations of V preserving Ψ up to scalar, so that

$$G(\mathbb{Q}) \;=\; \{g \in \mathrm{GL}(V) : \Psi(gu, gv) = \nu(g)\Psi(u, v)\}$$

for all $u, v \in V$ and some $\nu(g) \in \mathbb{Q}^*$.[13] Define $S = Sp(V, \Psi)$ by the exact sequence

$$1 \to S \to G \xrightarrow{\nu} \mathbb{G}_m.$$

The center of G is $\mathbb{G}_m$, $G^{ad} = S/\{\pm 1\}$ and $G^{der} = S$.

[13] One checks that automatically ν defines a homomorphism $\nu : G \to \mathbb{G}_m$.

A symplectic complex structure J on $(V(\mathbb{R}), \Psi)$ is an endomorphism $J \in S(\mathbb{R})$ such that $J \neq Id$, $J^2 = -Id$. For a given J, the bilinear form

$$\begin{array}{cccc} \Psi_J : & V(\mathbb{R}) \times V(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & (u, v) & \mapsto & \Psi(u, Jv) \end{array}$$

is symmetric. We say that J is positive (negative) if Ψ_J is positive (negative) definite.

Let $D = D^+ \cup D^-$ be the set of positive or negative symplectic complex structures on $V(\mathbb{R})$. The group $G(\mathbb{R})$ acts on D by $(g, J) \mapsto gJg^{-1}$. The stabilizer of D^+ in $G(\mathbb{R})$ is its identity's connected component $G(\mathbb{R})_0 = \{g \in G(\mathbb{R}) : \nu(g) > 0\}$. One easily shows also that $G(\mathbb{R})$ acts transitively on D, and $S(\mathbb{R})$ acts transitively on D^+.

Attached to any $J \in D$ there is the morphism[14]

$$\begin{array}{cccc} h_J : & \mathbb{C}^* & \longrightarrow & G(\mathbb{R}) \\ & z = a + bi & \mapsto & a + bJ. \end{array}$$

Since $h_{gJg-1} = gh_Jg^{-1}$ for any $g \in G(\mathbb{R})$, there is a natural identification

$$D \leftrightarrow \{G(\mathbb{R}) - \text{Conjugation class of } h_J : \mathbb{C}^* \to G(\mathbb{R})\}.$$

Lemma 3.3.1. *The Shimura datum (G, D) satisfies the axioms SV1, SV2 and SV3.*

Proof. **SV1:** Only the characters 1, $z/\bar{z}$ and $\bar{z}/z$ occur in the representation of $\mathbb{S}_{\mathbb{R}}$ on $\mathrm{Lie}(G^{ad})_{\mathbb{C}}$:
Write $V(\mathbb{C}) = V^+ \oplus V^-$ as a direct sum of $\pm i$-eigenspaces under the action of J. Then $h_J(z)$ acts on V^+ as multiplication by z, and it acts on V^- as multiplication by $\bar{z}$. Thus[15]

$$\begin{array}{cccccc} \mathrm{End}(V(\mathbb{C})) & \simeq & \mathrm{End}(V^+)\oplus & \mathrm{Hom}(V^+, V^-)\oplus & \mathrm{Hom}(V^-, V^+)\oplus & \mathrm{End}(V^-) \\ ad(h_J(z)) & \mapsto & (1, & z/\bar{z}, & \bar{z}/z, & 1). \end{array}$$

[14]Indeed, $a + bJ \in G(\mathbb{R})$, with $\nu(a + bJ) = a^2 + b^2 = |z|^2$.

[15]In order to check this, the exercise is a generalization from PSL_2 (as in example 3.2.3) to the algebraic group of symplectic similitudes of a vector space of arbitrary even dimension. Indeed, the matrix expression of $h_J(z)$ in $n \times n$-blocks is $\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, which is a complex form of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

SV2: $Ad(h(i))$ is a Cartan involution on Lie $G^{ad}_{\mathbb{R}}$: It is an involution because $ad(h(i)) : G^{ad}(\mathbb{R}) \longrightarrow G^{ad}(\mathbb{R}),\ g \mapsto JgJ^{-1}$ and $J^2 = -Id$. It is Cartan because the subgroup of fixed elements in $G(\mathbb{R})$ is $\{g \in G(\mathbb{R}) : Jg = gJ\} = \{g \in G(\mathbb{R}) : \Psi_J(gu, gv) = \Psi(u, v)\}$ for all $u, v \in V(\mathbb{R})$. Since Ψ_J is positive definite, the subgroup is compact.

SV3: There exists no factor H of G^{ad} over $\mathbb{Q}$ such that $H(\mathbb{R})$ is compact: Indeed, G^{ad} is simple over $\mathbb{Q}$ and $G^{ad}(\mathbb{R})$ itself is not compact (a factor of it is Siegel's upper half space $\mathcal{H}_n$). □

We define the *Siegel modular variety* attached to (V, Ψ) to be the Shimura variety $Sh(G, D)$ associated to the Shimura datum (G, D).

3.3.1 Modular interpretation

Let (V, Ψ) be a symplectic space over $\mathbb{Q}$ of dimension $2n$, for some $n \geq 1$ as before. Let (G, D) be the Shimura datum attached to (V, Ψ) and $K \subset G(\mathbb{A}_f)$ a compact open subgroup. Let $Sh_K(G, D)$ be the Shimura variety attached to (G, D) and K.

Let $A/\mathbb{C}$ be a complex abelian variety of dimension n. There exists a lattice $\Lambda \subset T_0(A)$ of rank $2n$ over $\mathbb{Z}$ such that $A(\mathbb{C}) \simeq T_0(A)/\Lambda$. Under this isomorphism, there is a natural identification $H_1(A, \mathbb{Z}) = \Lambda$.

The isomorphism $V(\mathbb{R}) \simeq \Lambda \otimes \mathbb{R} \simeq T_0(A)$ induces a complex structure J on the real vector space $V(\mathbb{R})$.

Definition 3.3.2. A polarization on A is a non-degenerate alternating form $s : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ such that $s(Ju, Jv) = \Psi(u, v)$ for all $u, v \in V(\mathbb{R})$ and $s_J(u, v) := s(u, Jv)$ is positive definite.

Let $V(\mathbb{A}_f) = \Lambda \otimes_{\mathbb{Z}} \mathbb{A}_f \simeq \mathbb{A}_f^{2n}$. Let $V_f(A) = H_1(A, \mathbb{A}_f) \simeq \Lambda \otimes \mathbb{A}_f$ the Tate module of A.

Let AV^0 be the category of abelian varieties up to isogeny and let $M_K(G, D) = \{A, s, \eta \cdot K\}$ be the set of triples where

- A is a complex abelian variety of dimension n, $A(\mathbb{C}) = V(\mathbb{R})/\Lambda$,
- s is an alternating form on $H_1(A, \mathbb{Z})$ such that s or $-s$ is a polarization on A,
- $\eta : V(\mathbb{A}_f) \xrightarrow{\simeq} V_f(A)$ such that $\eta_*(\Psi) = a \cdot s$ for some $a \in \mathbb{A}_f^*$.

Two triples $(A, s, \eta \cdot K)$, $(A', s', \eta' \cdot K)$ are isomorphic if there is an isogeny $f : A \to A'$ such that $f^*(s') = q \cdot s$, $q \in \mathbb{Q}^*$ and $f^*(\eta' K) = \eta K$.

Theorem 3.3.3. *The Shimura variety $Sh_K(G, D)$ is the coarse moduli space over $\mathbb{C}$ that classifies triples in $M_K(G, D)$ up to isomorphism. In particular, there is a canonical bijection of sets*

$$M_K(G, D)/\simeq \quad \leftrightarrow \quad G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K.$$

3.4 Shimura varieties of Hodge type

3.4.1 Hodge structures

A *Hodge estructure* is a real vector space V together with a decomposition of $V(\mathbb{C})$ into complex vector subspaces

$$V(\mathbb{C}) = \oplus V^{p,q}, \quad (p, q) \in \mathbb{Z} \times \mathbb{Z} : \quad \bar{V}^{p,q} = V^{q,p}.$$

The *type* of the Hodge structure is the set of pairs (p, q) for which $V^{p,q} \neq \{0\}$. The *weight decomposition* of a Hodge structure is the decomposition

$$V = \oplus_{n \in \mathbb{Z}} V_n$$

where V_n is the real vector subspace of V such that $V_n(\mathbb{C}) = \oplus_{p+q=n} V^{p,q}$. If $V = V_n$, the Hodge structure is said to be of *weight* n.

A *rational* Hodge structure is a vector space V over $\mathbb{Q}$ together with a Hodge structure for $V(\mathbb{R})$ such that V_n is defined over $\mathbb{Q}$ for any $n \in \mathbb{Z}$.

Example 3.4.1.

- Let V be a real vector space. To give a complex structure J on V is equivalent to give a Hodge structure $V(\mathbb{C}) = V^{-1,0} \oplus V^{0,-1}$ of type $(-1, 0)$, $(0, -1)$ on V.

- Let $\mathbb{Q}(r)$ denote the rational Hodge structure $V = \mathbb{Q}$, $V(\mathbb{C}) = \mathbb{C}^{-r,-r}$, the unique possible rational Hodge structure on $\mathbb{Q}$ of weight $-2r$.

- Let X be a non-singular projective variety. Let $V = H^n(X, \mathbb{Q})$, a vector space over $\mathbb{Q}$. Then V admits the following Hodge structure of weight n: $V(\mathbb{C}) = \oplus_{p+q=n} H^q(X, \Omega^p)$.

A *morphism of Hodge structures* $t : V = \oplus_{(p,q)} V^{p,q} \longrightarrow W = \oplus_{(p,q)} W^{p,q}$ is a linear map $t : V \to W$ such that $t(V^{p,q}) \subseteq W^{p,q}$.

3.4.2 Shimura varieties of Hodge type and its modular interpretation

Definition 3.4.2. A Shimura datum (G, D) is *of Hodge type* if there exists a symplectic vector space (V, Ψ) over $\mathbb{Q}$ and a monomorphism $G \hookrightarrow GSp(V, \Psi)$ such that D maps to $D_{(V,\Psi)}$. The Shimura variety $Sh(G, D)$ is then called *of Hodge type.*

Recall the character $\nu : GSp(V, \Psi) \longrightarrow \mathbb{G}_m$. We shall still denote $\nu : G \hookrightarrow GSp(V, \Psi) \longrightarrow \mathbb{G}_m$. For any $r \in \mathbb{Z}$, denote (also) by $\mathbb{Q}(r)$ the vector space $\mathbb{Q}$ with the action $(g, q) \mapsto \nu(g)^r \cdot q$.

Proposition 3.4.3. *Let (G, D) be a Shimura datum of Hodge type, $(G, D) \hookrightarrow GSp(V, \Psi)$ where V is a vector space over $\mathbb{Q}$ of dimension $2n$. Then there exist non-zero multilinear maps*

$$t_i : V \times \overset{2r_i)}{\cdots} \times V \longrightarrow \mathbb{Q}(r_i), \quad i = 1, ..., k$$

such that for any field extension $k/\mathbb{Q}$,

$$G(k) = \{g \in \mathrm{GL}_k(V) : t_i(gv_1, ..., gv_{2r_i}) = \nu(g)^{r_i} \cdot t_i(v_1, ..., v_{2r_i})\}$$

for all $v_j \in V(k)$, $i = 1, ..., k$.

That is, we demand to g to be equivariant with respect to the actions of G on V and $\mathbb{Q}(r_i)$, respectively.

If $J \in D$ is a complex structure on $V(\mathbb{R})$, it induces a Hodge structure of weight -1 on $V(\mathbb{R})$. For any $r \geq 1$, there is a natural[16] Hodge structure of weight $-r$ on $V(\mathbb{R})^{\otimes r}$. The map $t_i : V^{\otimes 2r_i} \to \mathbb{Q}(r_i)$ is a morphism of Hodge structures of weight $-2r_i$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subset. Let $M_K(G, D)$ be the set of triples $(A, \{s_i\}_{i=0,...,k}, \eta \cdot K)$ where

- A is a complex abelian variety of dimension n,
- s_0 is an alternating form on $H_1(A, \mathbb{Z})$ such that s_0 or $-s_0$ is a polarization on A,

[16] $V(\mathbb{R})^{\otimes r} = V(\mathbb{R})^{-r,0} \oplus V(\mathbb{R})^{-r+1,-1} \oplus ... \oplus V(\mathbb{R})^{0,-r}$.

- $s_i \in H^{2r_i}(A,\mathbb{Q}) \simeq \mathrm{Hom}(\wedge^{2r_i}\Lambda,\mathbb{Q})$ such that $V^{\otimes 2r_i} \to \wedge^{2r_i} V \to \mathbb{Q}(r_i)$ is a morphism of Hodge structures,[17] for $i = 1, ..., k$,
- $\eta : V(\mathbb{A}_f) \xrightarrow{\simeq} V_f(A)$ such that $\eta_*(\Psi) = a \cdot s$ for some $a \in \mathbb{A}_f^*$, and $\eta_*(t_i) = s_i$ for $i = 1, ..., k$,

satisfying the following condition:

There exists an isomorphism $\alpha : H_1(A,\mathbb{Q}) \simeq V$ of vector spaces over $\mathbb{Q}$ such that $\alpha^*(\Psi) = q \cdot s$ for some $q \in \mathbb{Q}^*$, $\alpha^*(t_i) = s_i$ for $i = 1, ..., k$, and $\alpha_*(J) \in D_{(V,\Psi)}$.[18]

Theorem 3.4.4. *The Shimura variety $Sh_K(G,D)$ is the coarse moduli space over $\mathbb{C}$ that classifies triples in $M_K(G,D)$ up to isomorphism. In particular, there is a canonical bijection of sets*

$$M_K(G,D)/\simeq \quad \leftrightarrow \quad G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K.$$

3.4.3 Shimura varieties of PEL type

Let $(B,*)$ be a simple algebra over $\mathbb{Q}$ together with a positive involution $* : B \to B$: an anti-involution such that $\mathrm{Tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(b^* \cdot b) > 0$ for all $b \in B\backslash\{0\}$. Let F denote the center of B and $F_0 = \{b \in F : b^* = b\}$.

Assume that for every embedding $\varphi : F_0 \hookrightarrow \bar{\mathbb{Q}}$, $(B \otimes_{F_0} \bar{\mathbb{Q}}, *)$ is isomorphic to a product of algebras with involution either of the form

$$(A) \qquad \mathrm{M}_n(\bar{\mathbb{Q}}) \times \mathrm{M}_n(\bar{\mathbb{Q}}), \quad (b_1,b_2)^* = (b_2^t, b_1^t),$$

or of the form

$$(C) \qquad \mathrm{M}_n(\bar{\mathbb{Q}}), \quad b^* = b^t,$$

but not a mixture of them.

[17]The map $s : V^{\otimes 2r} \to \mathbb{Q}(r)$ is a morphism of Hodge structures if and only if $s((V^{\otimes 2r})^{p,q}) = 0$ for all pairs $(p,q) \neq (-r,-r)$, because the type of $\mathbb{Q}(r)$ is $\{(-r,-r)\}$. In other words: if we write $H = H^{2r}(A,\mathbb{Q}) \simeq \wedge^{2r} V$, H is endowed with a Hodge structure of weight $2r$ (see Example 3.4.1) and we require that $s \in H^{2r}(A,\mathbb{Q}) \cap H(\mathbb{C})^{r,r} \subset H^{2r}(A,\mathbb{C})$. The *Hodge conjecture* for abelian varieties asserts that all elements in this intersection are the cohomology classes of algebraic cycles on A with coefficients in $\mathbb{Q}$. This is known for $r = 1$.

[18]J is the complex structure on $H_1(A,\mathbb{R})$ induced by the isomorphism with $T_0(A)$. We require that $\alpha_*(J)$ is a symplectic complex structure on (V,Ψ) such that $\Psi_{\alpha_*(J)}$ is positive or negative definite.

Let (V, Ψ) be a $(B, *)$-module: a vector space over $\mathbb{Q}$ together with an action $B \subset \mathrm{End}(V)$ and a non-degenerate alternating bilinear form

$$\Psi : V \times V \to \mathbb{Q}, \qquad \Psi(bu, v) = \Psi(u, b^* v), \quad u, v \in V, b \in B.$$

Let $G \subset \mathrm{GL}(V)$ be the algebraic group over $\mathbb{Q}$ such that for any field extension $k/\mathbb{Q}$:

$$G(k) = \{g \in \mathrm{Aut}_B(V \otimes k) : \Psi(gu, gv) = \mu(g)\Psi(u, v)\}$$

for all $u, v \in V \otimes k$ and some $\mu(g) \in k^*$.

Let $G' = \{g \in G : \mu(g) = 1, \det(g) = 1\} \subset G$.

Proposition 3.4.5. *The algebraic group G is reductive and G' is semisimple and simply connected.*[19]

In fact, we have

$$G'_{\bar{\mathbb{Q}}} \simeq \mathrm{SL}_m^d \qquad \text{if } B \text{ is of type A;}$$

$$G'_{\bar{\mathbb{Q}}} \simeq \mathrm{Sp}_m^d \qquad \text{if } B \text{ is of type C;}$$

where $m = \dim_F(V)/\sqrt{[B : F]}$ and $d = [F_0 : \mathbb{Q}]$.

Proposition 3.4.6. *There exists a (unique) $G(\mathbb{R})$-conjugacy class D of homomorphisms $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ such that each h induces a symplectic complex structure $J = h(i)$ on $V(\mathbb{R})$ such that Ψ_J is positive or negative definite.*

The Shimura datum (G, D) satisfies the axioms SV1, SV2, SV3.

The corresponding Shimura varieties $Sh(G, D)$ are called of PEL-type (A or C).

Let $b_1, ..., b_k$ be a set of generators of B as a $\mathbb{Q}$-algebra and let

$$\begin{array}{rccc} t_{b_i} : & V \times V & \longrightarrow & \mathbb{Q} \\ & (u, v) & \mapsto & \Psi(u, bv). \end{array}$$

Then (G, D) is the Shimura datum of Hodge type associated to $(V, \Psi, \{t_{b_i}\})$.

Let $K \subset G(\mathbb{A}_f)$ be a compact open subset. Let $M_K(G, D)$ be the set of quadruples $(A, \iota, s, \eta \cdot K)$ where

[19] There is a remaining possible type for $(B, *)$, which is usually called BD. In this case, G is not connected, but its connected component is reductive. The group G' is a special orthogonal group.

- A is a complex abelian variety,
- $\iota : B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$,
- s is an alternating form on $H_1(A, \mathbb{Z})$ such that s or $-s$ is a polarization on A,
- $\eta : V(\mathbb{A}_f) \xrightarrow{\simeq} V_f(A)$ such that $\eta_*(\Psi) = a \cdot s$ for some $a \in \mathbb{A}_f^*$.

satisfying the following condition:

There exists a B-linear isomorphism $\alpha : H_1(A, \mathbb{Q}) \simeq V$ of vector spaces over $\mathbb{Q}$ such that $\alpha^*(\Psi) = q \cdot s$ for some $q \in \mathbb{Q}^*$.

Theorem 3.4.7. *The Shimura variety $Sh_K(G, D)$ is the coarse moduli space over $\mathbb{C}$ that classifies quadruples in $M_K(G, D)$ up to isomorphism.*

Chapter 4

Canonical models of Shimura varieties

4.1 The reciprocity map for abelian extensions of number fields

Let E be a number field and let E^{ab} be the maximal abelian extension of E inside a fixed algebraic closure of E. Global class field theory provides a continuous surjective homomorphism

$$rec_E : \mathbb{A}_E^* \twoheadrightarrow \mathrm{Gal}\,(E^{ab}/E)$$

which is called the *Artin reciprocity map* and it is such that for any finite abelian extension E'/E of E, we have a commutative diagram:

$$\begin{array}{ccc} E^*\backslash\mathbb{A}_E^* & \overset{rec_E}{\twoheadrightarrow} & \mathrm{Gal}\,(E^{ab}/E) \\ \downarrow & \circlearrowleft & \downarrow \\ E^*\backslash\mathbb{A}_E^*/N(\mathbb{A}_{E'}^*) & \overset{rec_{E'/E}}{\underset{\simeq}{\longrightarrow}} & \mathrm{Gal}\,(E'/E) \end{array}$$

The identity component of $\mathbb{A}_E^*$ (and thus in particular the identity component of $\prod_{v|\infty} E_v^*$) lies in the kernel of the reciprocity map rec_E [1].

Hence, if E is totally imaginary, rec_E factors through $\mathbb{A}_{E,f}^*$.

[1]because $\mathrm{Gal}\,(E^{ab}/E)$ is totally disconnected and $1 \in \mathrm{Gal}\,(E^{ab}/E)$ is its own connected component.

When $E=\mathbb{Q}$, $rec_{\mathbb{Q}}$ factors through $\{\pm\}\times\mathbb{A}_{\mathbb{Q},f}$ and $\mathbb{Q}^{ab}=\bigcup_{N\geq 1}\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N^{th}-root of the unity.

There is a commutative diagram

$$\begin{array}{ccc} \mathbb{Q}^*\backslash(\{\pm 1\}\times\mathbb{A}^*_{\mathbb{Q},f}) & \overset{rec_{\mathbb{Q}}}{\twoheadrightarrow} & \mathrm{Gal}\,(\mathbb{Q}^{ab}/\mathbb{Q}) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathbb{Q}^*\backslash\mathbb{A}^*_{\mathbb{Q}}/N(\mathbb{A}^*_{\mathbb{Q}(\zeta_N)})=(\mathbb{Z}/N\mathbb{Z})^* & \overset{a\mapsto\{\zeta_N\mapsto\zeta_N^{a^{-1}}\}}{\underset{\simeq}{\longrightarrow}} & \mathrm{Gal}\,(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \end{array}$$

4.2 Abelian varieties with complex multiplication

A CM-field is a totally imaginary quadratic extension E of a totally real number field F. Let $[E:\mathbb{Q}]=2g$ for some $g\geq 1$. Each of the g embeddings $F\hookrightarrow\mathbb{R}$ extends to two conjugate embeddings $\varphi,\bar{\varphi}:E\hookrightarrow\mathbb{C}$. A CM-type Φ for E is a subset $\Phi=\{\varphi_1,...,\varphi_g\}\subset\mathrm{Hom}(E,\mathbb{C})$ such that $\mathrm{Hom}(E,\mathbb{C})=\Phi\sqcup\bar{\Phi}$, that is, a full set of representatives of embeddings $E\hookrightarrow\mathbb{C}$ up to complex conjugation.

An abelian variety of CM-type (E,Φ) is an abelian variety $A/\mathbb{C}$ of dimension g such that there exists a monomorphism

$$i:E\hookrightarrow\mathrm{End}^0(A)$$

such that for any $a\in E$, we have $Tr(a_{|T_0(A)})=\sum_{\varphi\in\Phi}\varphi(a)$.[2]

Definition 4.2.1. Let (E,Φ) be a CM-type. The reflex field of (E,Φ) is the number field $\tilde{E}$ characterized for any of the following equivalent conditions:

- $\tilde{E}$ is the fixed field of $\{\sigma\in\mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q}):\Phi^\sigma=\Phi\}$.
- $\tilde{E}=\mathbb{Q}(\sum_{\varphi\in\Phi}\varphi(a)\ :\ a\in E)$.

[2]For any $i:E\hookrightarrow\mathrm{End}^0(A)$, there always exists a CM-type Φ for E such that A is of CM-type (E,Φ). Indeed, write $A=\mathbb{C}/\Lambda$. Then $\Lambda\otimes\mathbb{Q}$ is an E-vector space of dimension 1. We have that $\Lambda\otimes\mathbb{C}$ is a 1-dimensional module over $E\otimes\mathbb{C}\simeq\oplus_{\varphi\in\mathrm{Hom}(E,\mathbb{C})}\mathbb{C}_\varphi$, where E acts on $\mathbb{C}_\varphi$ through φ. Since $\Lambda\otimes\mathbb{R}=T_0(A)$ and it is well known that $\Lambda\otimes\mathbb{C}=T_0(A)\oplus\overline{T_0(A)}$, we obtain that any $a\in E$ acts on $T_0(A)$ as $\mathrm{diag}(\varphi_1(a),...,\varphi_g(a))$ for some CM-type $\Phi=\{\varphi_i,...,\varphi_g\}$.

- It is the smallest subfield of $\bar{\mathbb{Q}}$ for which there exists an $\tilde{E}$-vector space V together with a monomorphism $E \hookrightarrow \mathrm{End}_{\tilde{E}}(V)$ for which $Tr(a) = \sum_{\varphi\in\Phi}\varphi(a)$ for all $a \in E$.

Observe that when $E/\mathbb{Q}$ is a Galois extension, then $\tilde{E} \subseteq E$ by the second definition.

As it follows from the third definition of $\tilde{E}$, if $(A,i)/k$ is an abelian variety of CM-type (E,Φ) defined over a number field k, then $V = T_0(A)$ is a k-vector space for which $Tr(a) = \sum_{\varphi\in\Phi}\varphi(a)$ for all $a \in E$. Hence $\tilde{E} \subset k$ for any possible field of definition of A.

Note that in the definition above, V can also be regarded as an E-vector space on which $\tilde{E}$ acts. Under this point of view, for any $a \in \tilde{E}$ we shall denote $\mathrm{Tr}_E(a)$ and $\det_E(a)$ for the trace and determinant of the E-linear endomorphism $a : V \to V$, $v \mapsto a\cdot v$.

Theorem 4.2.2 (Shimura-Taniyama). *Let (A,i) be an abelian variety of CM-type (E,Φ) and let $\sigma \in \mathrm{Aut}(\mathbb{C}/\tilde{E})$. For any idele $s \in \mathbb{A}^*_{\tilde{E},f}$ such that $rec_{\tilde{E}}(s) = \sigma_{|\tilde{E}^{ab}}$, there exists a unique E-linear isogeny*

$$\alpha : A \longrightarrow A^\sigma$$

such that

$$x^\sigma = \alpha\left(\det_{\mathbb{A}_{E,f}}(s^{-1})\cdot x\right), \qquad \textit{for all } x \in V_f(A).$$

In particular, Theorem 4.2.2 asserts that the isogeny class of (A,i) is defined over $\tilde{E}$.

4.3 The reflex field of a Shimura datum

Let (G,D) be a Shimura datum. For any subfield $k \subset \mathbb{C}$, define

$$\mathcal{C}(k) = G(k)\backslash \mathrm{Hom}_k(\mathbb{G}_m, G_k),$$

that is, the set of co-characters of G defined over k up to inner conjugation by elements in $G(k)$.

The group $\mathrm{Aut}(\mathbb{C}/k)$ naturally acts on $\mathcal{C}(k)$: $c^\sigma(a) := c(a)^\sigma$ for any $c \in \mathcal{C}(k), a \in \mathbb{G}_m(\mathbb{C}^*)$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$.

As an example, let $x \in D$ and $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ be the associated morphism. It induces

$$\begin{array}{rcccc} \mu_x : & \mathbb{C}^* & \to & \mathbb{G}_{m,\mathbb{C}} = \mathbb{C}^*\times\mathbb{C}^* & \longrightarrow & G_{\mathbb{C}} \\ & z & \mapsto & (z,1) & \mapsto & h_{\mathbb{C}}(z,1) \end{array}$$

Note for any other point in D, the associated morphism is conjugated to h and therefore $\mu_x \in \mathcal{C}(\mathbb{C})$ is independent of the choice of x and we shall denote the co-character as $c_D \in \mathcal{C}(\mathbb{C})$. It can be seen that actually c_D is defined over a number field[3].

Definition 4.3.1. The reflex field $E(G, D)$ of the Shimura datum (G, D) is the field of definition of c_D as an element of $\mathcal{C}(\bar{\mathbb{Q}})$, that is, the field fixed by $\{\sigma \in \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q}) : c_D^\sigma \equiv c_D \in \mathcal{C}(\bar{\mathbb{Q}})\}$.

Example 4.3.2.

- (T, h) where T is a torus and $h : \mathbb{S} \longrightarrow T_{\mathbb{R}}$ is any morphism. Then $E(T, h)$ is simply the field of definition of μ_h, because it is the single element in its conjugacy class under $T(\mathbb{C})$.

- Let (E, Φ), $\Phi = \{\varphi_1, ..., \varphi_g\}$, be a CM-type.
  $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m,\ T(\mathbb{Q}) = E^*,\ T(\mathbb{R}) = (E \otimes_{\mathbb{Q}} \mathbb{R})^* = \mathbb{C}^*_{\varphi_1} \times ...\mathbb{C}^*_{\varphi_g}$.[4]
  Define $h_\Phi : \mathbb{S}(\mathbb{R}) = \mathbb{C}^* \to T(\mathbb{R}),\ z \mapsto (z, \overset{g}{\ldots}, z)$. Then

$$h_\Phi \otimes \mathbb{C} : \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* \to T(\mathbb{C}), z \mapsto (z, \overset{g}{\ldots}, z, \bar{z}, \overset{g}{\ldots}, \bar{z})$$

  and

$$\begin{array}{cccccc} \mu_h : & \mathbb{C}^* & \to & \mathbb{S}(\mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^* & \to & T(\mathbb{C}) = \mathbb{C}^*_{\varphi_1} \times ...\mathbb{C}^*_{\varphi_g} \times \mathbb{C}^*_{\bar{\varphi}_1} \times ...\mathbb{C}^*_{\bar{\varphi}_g} \\ & z & \mapsto & (z, 1) & \mapsto & (z, \overset{g}{\ldots}, z, 1, \overset{g}{\ldots}, 1). \end{array}$$

  Hence $E(T, h_\Phi)$ is the field fixed by $\{\sigma \in \mathrm{Aut}(\bar{\mathbb{Q}}/\mathbb{Q}) : \Phi^\sigma = \Phi\}$, that is, the reflex field $\tilde{E}$ of (E, Φ) that we already defined.
  At the level of $\tilde{E}$-rational points, we have $\mu_h : \tilde{E}^* \to T(\tilde{E}) = (E \otimes_{\mathbb{Q}} \tilde{E})^*$.

- *The Siegel modular variety.* Let (V, Ψ) be a symplectic vector space over $\mathbb{Q}$ of dimension $2n$, $G = GSp(V, \Psi)$, $D = D^+ \cup D^-$ be the set of positive or negative symplectic complex structures J on $V(\mathbb{R})$.

  Let $V = W \oplus \tilde{W}$ be a decomposition into totally isotropic vector spaces over $\mathbb{Q}$: $\Psi(W, W) = \{0\}$, $\Psi(\tilde{W}, \tilde{W}) = \{0\}$. Let $W =< e_1, ..., e_n >$, $\tilde{W} =< \tilde{e}_1, ..., \tilde{e}_n >$ be symplectic basis and define $J \in \mathrm{End}(V)$, $J(e_i) = \tilde{e}_i$, $J(\tilde{e}_i) = -e_i$.

[3] See [3, Lemma 12.1].
[4] For any choice of g-inequivalent embeddings $\{\varphi_1, ..., \varphi_g\}$ we have an isomorphism $E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{C}^g,\ a \otimes r \mapsto (\varphi_1(a)r, ..., \varphi_g(a)r)$.

We had $h_J(a+bi) = a+bJ$. Hence $(h_J \otimes \mathbb{C})(z_1, z_2) = (a_1+b_1J, a_2-b_2J)$ and

$$\begin{array}{rccc} c_{h_J}: & \mathbb{C}^* & \longrightarrow & G_{\mathbb{C}} \subset \mathrm{End}(V(\mathbb{C})) = \mathrm{End}(V^+ \oplus V^-) \\ & z & \mapsto & (h_J \otimes \mathbb{C})(z,1) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \end{array}$$

where V^+, V^- are the $\pm i$-eigenspaces under J.

Since the decomposition $V = W \oplus \tilde{W}$ is defined over $\mathbb{Q}$, the conjugation class of c_{h_J} remains invariant under the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. Thus $E(G, D) = \mathbb{Q}$.

- Let (G, D) be the Shimura datum attached to a quaternion algebra B over a totally real number field F.
  $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$, $h(a+bi) = (\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \overset{r}{\ldots}, \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, 1, \overset{s}{\ldots}, 1)$.
  Up to inner conjugation: $h_{\mathbb{C}}(z_1, z_2) = (\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, \overset{r}{\ldots}, \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}, 1, \overset{s}{\ldots}, 1)$.
  $$\begin{array}{rccc} \mu_h: & \mathbb{C}^* & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \times \overset{g=r+s}{\ldots} \times \mathrm{GL}_2(\mathbb{C}) \\ & z & \mapsto & (z,1) \mapsto (\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \overset{r}{\ldots}, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, 1, \overset{s}{\ldots}, 1). \end{array}$$
  $E(G, D) = \bar{\mathbb{Q}}^{\{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) : \{\infty_1, \ldots, \infty_r\}^\sigma = \{\infty_1, \ldots, \infty_r\}\}} \subseteq F$
  - If $B = \mathrm{M}_2(F)$, $r = g, s = 0$, $\dim_{\mathbb{C}}(D) = g$, $E(G, D) = \mathbb{Q}$.
  - If $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R}) \times \overset{g}{\ldots} \times \mathrm{M}_2(\mathbb{R})$, $\dim_{\mathbb{C}}(D) = g$, $E(G, D) = \mathbb{Q}$.
  - If $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathrm{M}_2(\mathbb{R}) \times \mathbb{H} \overset{g-1}{\ldots} \times \mathbb{H}$, $\dim_{\mathbb{C}}(D) = 1$, $E(G, D) = F$.
- Let $i : (G, D) \hookrightarrow (G', D')$ be an inclusion of Shimura data (i.e. $G \hookrightarrow G'$ is a monomorphism and i maps D to D'). Then there is a natural morphism of Shimura varieties $Sh(G, D) \hookrightarrow Sh(G', D')$ which is a closed immersion (Deligne) and it is an easy exercise to show that $E(G, D) \supseteq E(G', D')$.

4.4 Canonical models of Shimura varieties

Let (G, D) be a Shimura datum.

Definition 4.4.1. A point $x \in D$ is a *special point* if there exists a torus $T \subset G$ such that $h_x(\mathbb{C}^*) \subset T(\mathbb{R})$.

The pair (T, x) is also called a *special pair* and $(T, x) \subset (G, D)$ is an inclusion of Shimura data. Note that since $h_x(\mathbb{C}^*) \subset T(\mathbb{R})$, we have $t h_x t^{-1} = h_x$ for any $t \in T(\mathbb{R})$ and thus $T(\mathbb{R})$ fixes the point $x \in D$. Almost conversely, if $T \subset G$ is a *maximal* torus and $x \in D$ is a point fixed by the elements of $T(\mathbb{R})$, then $h(\mathbb{C}^*) \subset \{g \in G(\mathbb{R}) : gt = tg \text{ for all } t \in T(\mathbb{R})\} = T(\mathbb{R})$, because T is its own centralizer in G; and hence (T, x) is a special pair.

Example 4.4.2.

- $G = \mathrm{GL}_2$, $D = \mathcal{H}_1^{\pm}$. The tori in GL_2 are $\mathbb{G}_m \overset{a \mapsto a \cdot Id_2}{\hookrightarrow} \mathrm{GL}_2$ and $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \overset{i}{\hookrightarrow} \mathrm{GL}_2$, for any embedding $E \overset{i}{\hookrightarrow} \mathrm{GL}_2(\mathbb{R})$ of a quadratic field $E/\mathbb{Q}$. Among these, only imaginary quadratic fields provide special pairs (because only in this case we have $T(\mathbb{R}) = (E \otimes \mathbb{R})^* = \mathbb{C}^*$), and for each $E \overset{i}{\hookrightarrow} \mathrm{GL}_2(\mathbb{R})$, $T(\mathbb{R})$ has exactly two fixed points $x, \bar{x}$ on $\mathcal{H}_1^{\pm}$.
- $G = \mathrm{Res}_{F/\mathbb{Q}}(B^*)$, B a quaternion algebra over F. Embeddings $E \hookrightarrow B$ of quadratic extensions E/F with at least one non-real archimedean place provide special pairs (T, x).

Let $(T, x) \subset (G, D)$ be a torus in (G, D). Let $E(x) = E(T, x)$ be the field of definition of μ_x, which is a finite extension of the reflex field $E = E(G, D)$.[5] Define the homomorphism

$$\begin{array}{rccccc} r_x : & \mathbb{A}^*_{E(x)} & \longrightarrow & T(\mathbb{A}_{\mathbb{Q}}) & \overset{\pi}{\twoheadrightarrow} & T(\mathbb{A}_{\mathbb{Q},f}) \\ & s & \mapsto & \prod_{\sigma : E(x) \hookrightarrow \bar{\mathbb{Q}}} \mu_x(s)^{\sigma} & \mapsto & r_x(s). \end{array}$$

Definition 4.4.3. Let (G, D) be a Shimura datum and let $K \subset G(\mathbb{A}_{\mathbb{Q},f})$ be a compact open subgroup. A *canonical model* of $Sh_K(G, D)$ is an algebraic variety $M_K = M_K(G, D)$ defined over $E(G, D)$ with

$$M_K(G, D)(\mathbb{C}) \simeq Sh_K(G, D)$$

such that for any special pair $(T, x) \subset (G, D)$ and any $a \in G(\mathbb{A}_f)$:

- $(x, a) \in M_K(E(x))^{ab}$,
- $(x, a)^{\mathrm{rec}(s)} = (x, r_x(s^{-1}) \cdot a)$ for any $s \in \mathbb{A}^*_{E(x)}$.

[5] Watch out that we still have not defined the notion of field of definition of a Shimura variety, though clearly E will become a field over which $Sh(G, D)$ admits a canonical model and $E(x)$ will turn out to be the field generated by the coordinates of the point $x \in Sh(G, D)$.

Example 4.4.4. • (T,x), $K \subset T(\mathbb{A}_f)$, $E = E(x)$,
$Sh_K(T,x) = T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$.
A model over E of a finite set of points $=$ Action of $\mathrm{Gal}\,(\bar{E}/E)$ on it.
Define it through

$$\mathrm{Gal}\,(\bar{E}/E) \twoheadrightarrow \mathrm{Gal}\,(E^{ab}/E) \twoheadrightarrow \mathrm{Gal}\,(H_K/E) \overset{\mathrm{rec}_{H_K/E}}{\overset{\simeq}{\leftarrow}} E^*\backslash\mathbb{A}_E^*/K$$

by $a^s := r_x(s^{-1})\cdot a, \quad a \in T(\mathbb{A}_f),\ s \in \mathbb{A}_E^*$.

- (E,Φ) CM-type, $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$, $E(h_\Phi) = \tilde{E}$,
$Sh_K(T,h_\Phi) = E^*\backslash\mathbb{A}_{E,f}^*/K$.
The action of $\tilde{E}^*\backslash\mathbb{A}_{\tilde{E},f}^*/K \overset{\mathrm{rec}}{\simeq} \mathrm{Gal}\,(H_K/\tilde{E})$ on $Sh_K(T,h_\Phi)$ is provided by
$$a^s := r_{h_\Phi}(s^{-1})\cdot a = \prod_{\sigma:\tilde{E}\hookrightarrow\bar{\mathbb{Q}}} \mu_\Phi(s^{-1})^\sigma \cdot a = \det{}_{\mathbb{A}_{E,f}}(s^{-1})\cdot a$$
for $a \in \mathbb{A}_{E,f}^*$, $s \in \mathbb{A}_{\tilde{E},f}^*$, as in the theorem of Shimura-Taniyama.[6]

Modular interpretation of the example: For $T = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ as above, and any compact open subgroup $K \subset T(\mathbb{A}_f)$, $Sh_K(T,h_\Phi)$ is a finite set of points which can be regarded as a Shimura variety of PEL-type associated to the commutative algebra E.

It is the moduli space of triples $(A,\iota,s,\eta K)$ as in Section 3.4.3, where we recall that (A,ι) is an abelian variety of CM-type (E,Φ) up to isogeny.

The automorphism group $\mathrm{Aut}(\mathbb{C}/\tilde{E})$ acts on the finite set $Sh_K(T,h_\Phi)(\mathbb{C}) = \{[A,\iota,s,\eta K]\}$:

$(A,\iota,s,\eta K)^\sigma = (A^\sigma, \iota^\sigma : E \hookrightarrow \mathrm{End}(A^\sigma), s^\sigma, \eta^\sigma : V(\mathbb{A}_f)\to V_f(A)\to V_f(A^\sigma))$.

Since σ fixes $\tilde{E}$ one checks that $(A^{\sigma,\iota^\sigma})$ is again an abelian variety of CM-type (E,Φ).

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[6] The first equality is the definition of r_{h_Φ}. The second equality follows by tracing the definitions of μ_Φ, $\tilde{E}$ and $\det_E$, but it is not immediate. Check the example $E \subset \mathbb{C}$ imaginary quadratic, $\Phi = \{\varphi\}$ given by the inclusion: Then $\tilde{E} = E$ and

$$\begin{array}{cccc} \mu_\Phi : & \tilde{E}^* & \longrightarrow & T(\tilde{E}) = (E\otimes_{\mathbb{Q}}\tilde{E})^* = \tilde{E}^*\times\tilde{E}^* \\ & s & \mapsto & (s,1). \end{array}$$

The Galois action of $\mathrm{Gal}\,(\tilde{E}/\mathbb{Q})$ on $T(\tilde{E})$ is given by $\overline{(z_1,z_2)} = (\bar{z}_2,\bar{z}_1)$, so that $T(\mathbb{Q}) = E^* \overset{\iota}{\hookrightarrow} T(\tilde{E})$ as $\{(s,\bar{s}) : s \in \tilde{E}\}$. Thus $\prod_{\sigma:\tilde{E}\hookrightarrow\bar{\mathbb{Q}}}\mu_\Phi(s)^\sigma = (s,1)\cdot(1,\bar{s}) = (s,\bar{s}) = \iota(s)$. Since we can choose $V = \tilde{E} = E$, we also have $\det_E(V \overset{\cdot s}{\to} V) = s$, as we wished to show. For general E, the details are those of the proof of Theorem 4.2.2 of Shimura-Taniyama.

A reformulation of the main Theorem of Complex Multiplication 4.2.2 shows the following proposition:

Proposition 4.4.5. *By means of the identification of sets*

$$Sh_K(T, h_\Phi) = T(\mathbb{Q})\backslash\{x\} \times T(\mathbb{A}_f)/K \xrightarrow{\sim} \{[A, \iota, s, \eta K]\},$$

the Galois action induced on $Sh_K(T, h_\Phi)$ *is:*

$$(x, a)^\sigma = (x, r_{h_\Phi}(s^{-1}) \cdot a),$$

where $\sigma_{|\tilde{E}^{ab}} = \mathrm{rec}(s)$.

We are now able to state the main theorem of this chapter.

Theorem 4.4.6. *Let* (G, D) *be a Shimura datum. For any compact open subgroup* $K \subseteq G(\mathbb{A}_f)$*, there exists a unique canonical model* $M_K(G, D)$ *of* $Sh_K(G, D)$ *over* $E(G, D)$*, up to (unique) isomorphism over* $E(G, D)$.

Outline of the proof for Shimura varieties of Hodge type.

We first discuss the case of **Siegel modular varieties** $\mathcal{S}_K(G, D)$ attached to a symplectic vector space (V, Ψ) of dimension $2n$:

We already saw that the reflex field of (G, D) is $E(G, D) = \mathbb{Q}$. Thus we wish to prove that $\mathcal{S}_K(G, D)$ admits a canonical model $\mathcal{M}_K/\mathbb{Q}$ over $\mathbb{Q}$.

Thanks to the moduli interpretation, we are able to describe an action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on the set of complex points $\mathcal{S}_K(\mathbb{C})$.

Indeed, there is a one-to-one correspondence

$$\mathcal{S}_K(\mathbb{C}) \quad \leftrightarrow \quad \{[A, s, \eta K]\}$$

and $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ acts on it:

$$[A, s, \eta K] \mapsto [A^\sigma, s^\sigma, \eta^\sigma K], \text{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}).$$

Recall that we already saw that PEL-Shimura varieties are of Hodge type and hence Shimura subvarieties of a Siegel modular variety.

In our case, we have that for any CM-type (E, ϕ) with $[E : \mathbb{Q}] = 2n$, the Shimura variety of PEL-type attached to $(\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m, h_\Phi)$ is a subvariety of $\mathcal{S}_K$. These are finite sets of points and they are special. Proposition 4.4.5

above shows that the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on these special points on $\mathcal{S}_K$ behaves according to the rule required in the definition of a canonical model for $\mathcal{S}_K$.

Almost conversely, any special point $[A, s, \eta K] \in \mathcal{S}_K(\mathbb{C})$ corresponds to an abelian variety $A/\mathbb{C}$ such that $\mathrm{End}(A) = E_1 \times ... \times E_m$, where E_j are CM-fields and $\sum [E_j : \mathbb{Q}] = 2\dim(A) = 2n$. Proposition 4.4.5 readily extends to this more general case and hence we obtain that the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on all special points of $\mathcal{S}_K(\mathbb{C})$ is as it should according to Definition 4.4.3

Descent criteria. It remains left proving that the action of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ on $\mathcal{S}_K(G, D)$ is a true Galois action on a certain algebraic variety $\mathcal{M}_K/\mathbb{Q}$ such that $\mathcal{M}_K \otimes \mathbb{C} \simeq \mathcal{S}_K(G, D)$.

Let k be a field of characteristic 0 and let $\mathcal{K}/k$ be an algebraically closed field containing k. Let $X/\mathcal{K}$ be an algebraic variety.

For any $\sigma \in \mathrm{Aut}(\mathcal{K}/k)$ there is a well-defined algebraic variety X^σ and a map of sets

$$\begin{array}{ccc} X(\mathcal{K}) & \longrightarrow & X^\sigma(\mathcal{K}) \\ x & \mapsto & x^\sigma. \end{array}$$

Locally at affine open subsets, X^σ is obtained from X by conjugating by σ the defining polynomials of X. If $x \in X(\mathcal{K})$, the coordinates of the point $x^\sigma \in X^\sigma$ are the conjugate coordinates of x by σ.

Assume that X is equipped together with an action of $\mathrm{Aut}(\mathcal{K}/k)$ on the set $X(\mathcal{K})$. Let us denote the action by $(\sigma, x) \mapsto \sigma(x) \in X(\mathcal{K})$.

For instance, if X_0 is a variety over k, then $\sigma(x) := x^\sigma$ defines an action of $\mathrm{Aut}(\mathcal{K}/k)$ on $X = X_0 \otimes \mathcal{K}$.

Theorem 4.4.7. *Let X be a quasi-projective variety over $\mathcal{K}$ together with an action of* $\mathrm{Aut}(\mathcal{K}/k)$ *on $X(\mathcal{K})$ such that*

- *(Regularity) The morphism of sets*

$$\begin{array}{cccc} f_\sigma : & X & \longrightarrow & X^\sigma \\ & x & \mapsto & \sigma^{-1}(x^\sigma) \end{array}$$

  is a regular algebraic isomorphism.

- *(Continuity) There exist points x_1, ..., $x_n \in X(\mathcal{K})$ and a finitely generated extension L/k in $\mathcal{K}$ such that*

  - $\sigma(x_i) = x_i$ *for all* $\sigma \in \mathrm{Aut}(\mathcal{K}/L)$,

– *The only automorphism* $\alpha \in \mathrm{Aut}(X)$ *fixing all* x_i *simultaneously is* $\alpha = Id$.

Then there exists a model X_0 *over* k *of* X.

Theorem 4.4.7 applies to the Siegel modular variety $\mathcal{S}_K = \mathcal{S}_K(G, D)/\mathbb{C}$ in order to show that it admits a model over $\mathbb{Q}$:

- $\mathcal{S}_K$ is a quasi-projective variety over $\mathbb{C}$ by Baily-Borel's Theorem 2.3.1.

- **Regularity:** Checking this condition exploits the moduli interpretation of $\mathcal{S}_K$ and uses the theory of local systems on topological manifolds and families of abelian varieties. We refer the interested reader to [3, p. 100-103].

- **Continuity:** Let $x \in D$ be a special point. The real approximation Theorem (cf. [4, Theorem 7.7]) asserts that for any connected algebraic group over $\mathbb{Q}$, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. Hence, since D is a quotient of $G(\mathbb{R})$, the set of points $\{[x, a] : a \in G(\mathbb{A}_f)\} \subset G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$ is dense in $\mathcal{S}_K$ for the analytic and thus also the Zariski topology.

  Hence only the trivial automorphism $Id \in \mathrm{Aut}(\mathcal{S}_K)$ fixes all points $\{[x, a] : a \in G(\mathbb{A}_f)\}$ simultaneously. It can be shown that $\mathrm{Aut}(\mathcal{S}_K)$ is a finite group -recall we only consider $K \subset G(\mathbb{A}_f)$ such that the resulting congruence groups are torsion free!. Therefore there exists a finite set $\{[x, a_1], ..., [x, a_n]\}$ such that only $\alpha = Id$ fixes all them.

  Finally, the main Theorem of Complex Multiplication implies that $[x, a_1], ..., [x, a_n]$ are fixed by $\mathrm{Aut}(\mathbb{C}/E(x)')$, where $E(x)'$ denotes a finite abelian extension of $E(x)$.

Hence $\mathcal{S}_K$ admits a model $\mathcal{M}_K$ over $\mathbb{Q}$, which is canonical in the sense of Definition 4.4.3.

The proof of the existence of a canonical model for Shimura varieties of Hodge type follows the same pattern, since they are subvarieties of Siegel modular varieties and they also have a moduli interpretation in terms of abelian varieties.

The main technical difficulty is the definition of the Galois action on the Hodge tensors s_i on the abelian varieties A arising in the moduli interpretation. If the Hodge conjecture were true, any Hodge tensor s on A would be the cohomology class $c(Z)$ of an algebraic cycle Z on A and one could define

$s^\sigma = c(Z^\sigma)$. But the conjecture is not known to hold and Deligne succeeded to give a definition of s^σ not in terms of algebraic cycles - check [3, Theorem 14.13].

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