MANIFOLDS, COHOMOLOGY, AND SHEAVES (VERSION 6)

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Contents

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To understand these lectures, it is essential to know some point-set topology, as in [3, Appendix A], and to have a passing acquaintance with the exterior calculus of differential forms on a Euclidean space, as in [3, Sections 1–4]. To be consistent with Eduardo Cattani's lectures at this summer school, the vector space of C^∞ differential forms on a manifold M will be denoted by $\mathcal{A}^*(M)$, instead of $\Omega^*(M)$.

1. Differential Forms on a Manifold

This section introduces smooth differential forms on a manifold and derives some of their basic properties. More details may be found in the reference [3].

1.1. Manifolds and Smooth Maps.

We will be following the convention of classical differential geometry in which vector fields X_1, X_2, $X_3, \ldots$ take on subscripts, differential forms ω^1, ω^2, $\omega^3, \ldots$ take on superscripts, and coefficient functions can have either superscripts or subscripts depending on whether they are coefficient functions of vector fields or of differential forms. See [3, §4.7, p. 42] for an explanation of this convention.

A manifold is a higher-dimensional analogue of a smooth curve or surface. Its prototype is the Euclidean space $\mathbb{R}^n$, with coordinates $r^1, \ldots, r^n$. Let U be an open subset of $\mathbb{R}^n$. A real-valued function $f\colon U \to \mathbb{R}$ is *smooth* on U if the partial derivatives $\partial^k f/\partial r^{j_1} \cdots \partial r^{j_k}$ exist on U for all integers $k \geq 1$ and all $j_1, \ldots, j_k$. A vector-valued function $f = (f^1, \ldots, f^m)\colon U \to \mathbb{R}^m$ is *smooth* if each component f^i is smooth on U. In these lectures we use the words "smooth" and "C^∞" interchangeably.

A topological space M is *locally Euclidean* of dimension n if, for every point p in M, there is a homeomorphism ϕ of a neighborhood U of p with an open subset of $\mathbb{R}^n$. Such a pair $(U, \phi\colon U \to \mathbb{R}^n)$ is called a *coordinate chart* or simply a *chart*. If $p \in U$, then we say that (U, ϕ) is a *chart about* p. A collection of charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ is C^∞ *compatible* if for every α and β, the transition function

$$\phi_\alpha \circ \phi_\beta^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

is C^∞. A collection of C^∞ compatible charts $\{(U_\alpha, \phi_\alpha\colon U_\alpha \to \mathbb{R}^n)\}$ that cover M is called a C^∞ *atlas*. A C^∞ atlas is said to be *maximal* if it contains every chart that is C^∞ compatible with all the charts in the atlas.

Definition 1.1. A *topological manifold* is a Hausdorff, second countable, locally Euclidean topological space. By "second countable," we mean that the space has a countable basis of open sets. A *smooth* or C^∞ *manifold* is a pair consisting of a topological manifold M and a maximal C^∞ atlas $\{(U_\alpha, \phi_\alpha)\}$ on M. In these lectures all manifolds will be smooth manifolds.

In the definition of a manifold, the Hausdorff condition excludes certain pathological examples, while the second countability condition guarantees the existence of a partition of unity, a useful technical tool that we will define shortly.

In practice, to show that a Hausdorff, second countable topological space is a smooth manifold it suffices to exhibit a C^∞ atlas, for by Zorn's lemma every C^∞ atlas is contained in a unique maximal atlas.

Example 1.2. The unit circle. Let S^1 be the circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$, with open sets (see Figure 1.1)

$$\begin{aligned} U_x^+ &= \{(x, y) \in S^1 \mid x > 0\}, \\ U_x^- &= \{(x, y) \in S^1 \mid x < 0\}, \\ U_y^+ &= \{(x, y) \in S^1 \mid y > 0\}, \\ U_y^- &= \{(x, y) \in S^1 \mid y < 0\}. \end{aligned}$$

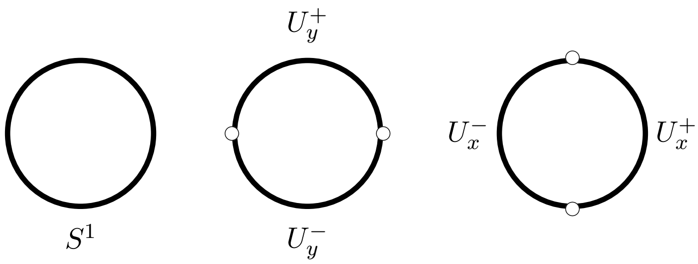

FIGURE 1.1. A C^∞ atlas on S^1.

Then $\{(U_x^+, y), (U_x^-, y), (U_y^+, x), (U_y^-, x)\}$ is a C^∞ atlas on S^1. For example, the transition function from

$$\text{the open interval } \;]0,1[\; = x(U_x^+ \cap U_y^-) \to y(U_x^+ \cap U_y^-) = \;]-1,0[$$

is $y = -\sqrt{1 - x^2}$, which is C^∞ on its domain.

A function $f\colon M \to \mathbb{R}^n$ on a manifold M is said to be *smooth* or C^∞ at $p \in M$ if there is a chart (U, ϕ) about p in the maximal atlas of M such that

$$f \circ \phi^{-1} \colon \mathbb{R}^m \supset \phi(U) \to \mathbb{R}^n$$

is C^∞. The function $f\colon M \to \mathbb{R}^n$ is said to be *smooth* or C^∞ on M if it is C^∞ at every point of M. Recall that an *algebra* over $\mathbb{R}$ is a vector space together with a bilinear map $\mu\colon A \times A \to A$, called *multiplication*, such that under addition and multiplication, A becomes a ring. Under pointwise addition, multiplication, and scalar multiplication, the set of all C^∞ functions $f\colon M \to \mathbb{R}$ is an algebra over $\mathbb{R}$, denoted $C^\infty(M)$.

A map $F\colon N \to M$ between two manifolds is *smooth* or C^∞ at $p \in N$ if there is a chart (U, ϕ) about $p \in N$ and a chart (V, ψ) about $F(p) \in M$ with $V \supset F(U)$ such that the composite map $\psi \circ F \circ \phi^{-1} \colon \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^m$ is C^∞ at $\phi(p)$. A smooth map $F\colon N \to M$ is called a *diffeomorphism* if it has a smooth inverse, i.e., a smooth map $G\colon M \to N$ such that $F \circ G = 1_M$ and $G \circ F = 1_N$.

A typical matrix in linear algebra is usually an $m \times n$ matrix, with m rows and n columns. Such a matrix represents a linear transformation $F\colon \mathbb{R}^n \to \mathbb{R}^m$. For this reason, we usually write a C^∞ map as $F\colon N \to M$, rather than $F\colon M \to N$.

1.2. **Tangent Vectors.** The derivatives of a function f at a point p in $\mathbb{R}^n$ depend only on the values of f in an arbitrarily small neighborhood of p. To make precise what is meant by an "arbitrarily small" neighborhood, we introduce the concept of the germ of a function. Decree two C^∞ functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$ defined on neighborhoods U and V of p to be *equivalent* if there is a neighborhood W of p contained in both U and V such that $f = g$ on W. The equivalence class of $f\colon U \to \mathbb{R}$ is called the *germ* of f at p.

It is fairly straightforward to verify that addition, multiplication, and scalar multiplication of functions induce well-defined operations on $C_p^\infty(M)$, the set of germs of C^∞ real-valued functions at p in M. These three operations make $C_p^\infty(M)$ into an algebra over $\mathbb{R}$.

Definition 1.3. A *derivation* at a point p of a manifold M is a linear map $D\colon C_p^\infty(M) \to C_p^\infty(M)$ such that for any $f, g \in C_p^\infty(M)$,

$$D(fg) = (Df)g(p) + f(p)Dg.$$

A derivation at p is also called a *tangent vector* at p. The set of all tangent vectors at p is a vector space T_pM, called the *tangent space* of M at p.

Example. If $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$ and $p \in \mathbb{R}^n$, then the usual partial derivatives

$$\left.\frac{\partial}{\partial r^1}\right|_p, \dots, \left.\frac{\partial}{\partial r^n}\right|_p$$

are tangent vectors at p that form a basis for the tangent space $T_p(\mathbb{R}^n)$.

At a point p in a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$, where $x^i = r^i \circ \phi$, we define the *coordinate vectors* $\partial/\partial x^i|_p \in T_pM$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} f \circ \phi^{-1} \qquad \text{for any } f \in C_p^\infty(M).$$

If $F\colon N \to M$ is a C^∞ map, then at each point $p \in N$ its *differential*

$$F_{*,p}\colon T_pN \to T_{F(p)}M, \tag{1.1}$$

is the linear map defined by

$$(F_{*,p}X_p)(h) = X_p(h \circ F)$$

for $X_p \in T_pN$ and $h \in C_{F(p)}^\infty(M)$. Usually the point p is clear from the context and we write F_* instead of $F_{*,p}$. It is easy to verify that if $F\colon N \to M$ and $G\colon M \to P$ are C^∞ maps, then for any $p \in N$,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p},$$

or, suppressing the points,

$$(G \circ F)_* = G_* \circ F_*.$$

A *vector field* X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$. At every p in a chart $(U, x^1, \dots, x^n)$, since the coordinate vectors $\partial/\partial x^i|_p$ form a basis of the tangent space T_pM, the vector X_p can be written as a linear combination

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \quad \text{with } a^i(p) \in \mathbb{R}.$$

As p varies over U, the coefficients $a^i(p)$ become functions on U. The vector field X is said to be *smooth* or C^∞ if M has a C^∞ atlas on each chart $(U, x^1, \ldots, x^n)$ of which the coefficient functions a^i in $X = \sum a^i \partial/\partial x^i$ are C^∞. We denote the set of all C^∞ vector fields on M by $\mathfrak{X}(M)$. It is a vector space under the addition of vector fields and scalar multiplication by real numbers. As a matter of notation, we write tangent vectors at p as $X_p, Y_p, Z_p \in T_pM$, or if the point p is understood from the context, as $v_1, v_2, \ldots, v_k \in T_pM$.

A *frame of vector fields* on an open set $U \subset M$ is a collection of vector fields $X_1, \ldots, X_n$ on U such that at each point $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ form a basis for the tangent space T_pM. For example, in a coordinate chart $(U, x^1, \ldots, x^n)$, the coordinate vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^n$ form a frame of vector fields on U.

If $f\colon N \to M$ is a C^∞ map, its differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ pushes forward a tangent vector at a point in N to a tangent vector in M. It should be noted, however, that in general there is no push-forward map $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ for vector fields. For example, when f is not one-to-one, say $f(p) = f(q)$ for $p \neq q$ in N, it may happen that for some $X \in \mathfrak{X}(N)$, $f_{*,p}X_p \neq f_{*,q}X_q$; in this case, there is no way to define f_*X so that $(f_*X)_{f(p)} = f_{*,p}X_p$ for all $p \in N$. Similarly, if $f\colon N \to M$ is not onto, then there is no natural way to define f_*X at a point of M not in the image of f. Of course, if $f\colon N \to M$ is a diffeomorphism, then $f_*\colon \mathfrak{X}(N) \to \mathfrak{X}(M)$ is well defined.

1.3. **Differential Forms.** For $k \geq 1$, a *differential k-form* or a *differential form of degree k* on M is the assignment to each p in M of an alternating k-linear function

$$\omega_p\colon \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ copies}} \to \mathbb{R}.$$

Here "alternating" means that for every permutation σ of $\{1, 2, \ldots, k\}$ and $v_1, \ldots, v_k \in T_pM$,

$$\omega_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn}\sigma)\omega_p(v_1, \ldots, v_k), \tag{1.2}$$

where $\operatorname{sgn}\sigma$, the sign of the permutation σ, is ± 1 depending on whether σ is even or odd. We often drop the adjective "differential" and call ω a *k-form* or simply a *form.* We define a 0-form to be the assignment of a real number to each $p \in M$; in other words, a 0-form on M is simply a real-valued function on M. When $k = 1$, the condition of being alternating is vacuous. Thus, a 1-form on M is the assignment of a linear function $\omega_p\colon T_pM \to \mathbb{R}$ to each p in M. For $k < 0$, a k-form is 0 by definition.

An alternating k-linear function on a vector space V is also called a *k-covector* on V. As above, a 0-covector is a constant and a 1-covector on V is a linear function $f\colon V \to \mathbb{R}$. Let $A_k(V)$ be the vector space of all k-covectors on V. Then $A_0(V) = \mathbb{R}$ and $A_1(V) = V^\vee := \operatorname{Hom}(V, \mathbb{R})$, the dual vector space of V. In this language, a k-form on M is the assignment of a k-covector

$\omega_p \in A_k(T_pM)$ to each point p in M. The addition and scalar multiplication of k-forms on a manifold are defined pointwise.

Let S_k be the group of all permutations of $\{1, 2, \ldots, k\}$. A (k, ℓ)-*shuffle* is a permutation $\sigma \in S_{k+\ell}$ such that

$$\sigma(1) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+\ell).$$

The *wedge product* of a k-covector α and an ℓ-covector β on a vector space V is by definition the $(k+\ell)$-linear function

$$(\alpha \wedge \beta)(v_1, \ldots, v_{k+\ell}) = \sum (\operatorname{sgn}\sigma)\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}), \tag{1.3}$$

where the sum is over all (k, ℓ)-shuffles. For example, if α and β are 1-covectors, then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

The wedge of a 0-covector, i.e., a constant c, with another covector ω is simply scalar multiplication. In this case, in keeping with the traditional notation for scalar multiplication, we often replace the wedge by a dot or even by nothing: $c \wedge \omega = c \cdot \omega = c\omega$.

The wedge product $\alpha \wedge \beta$ is a $(k+\ell)$-covector; moreover, the wedge operation $\wedge$ is bilinear, associative, and anticommutative in its two arguments. *Anticommutativity* means that

$$\alpha \wedge \beta = (-1)^{\deg\alpha \deg\beta} \beta \wedge \alpha.$$

Proposition 1.4. *If $\alpha^1, \ldots, \alpha^n$ is a basis for the 1-covectors on a vector space V, then a basis for the k-covectors on V is the set*

$$\{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}.$$

A k-tuple of integers $I = (i_1, \ldots, i_k)$ is called a *multi-index.* If $i_1 \leq \cdots \leq i_k$, we call I an *ascending multi-index*, and if $i_1 < \cdots < i_k$, we call I a *strictly ascending multi-index.* To simplify the notation, we will write $\alpha^I = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$.

As noted earlier, at a point p in a coordinate chart $(U, x^1, \ldots, x^n)$, a basis for the tangent space T_pM is

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

Let $(dx^1)_p, \ldots, (dx^n)_p$ be the dual basis for the cotangent space $A_1(T_pM) = T_p^*M$, i.e.,

$$(dx^i)_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \delta^i_j.$$

By Proposition 1.4, if ω is a k-form on M, then at each $p \in U$, ω_p is a linear combination:

$$\omega_p = \sum a_I(p)(dx^I)_p = \sum a_I(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p.$$

We say that the k-form ω is *smooth* if M has an atlas $\{(U, x^1, \ldots, x^n)\}$ such that on each U, the coefficients $a_I \colon U \to \mathbb{R}$ of ω are C^∞.

A *frame of k-forms* on an open set $U \subset M$ is a collection of k-forms $\omega_1, \ldots, \omega_r$ on U such that at each point $p \in U$, the k-covectors $(\omega_1)_p, \ldots, (\omega_r)_p$ form a basis for the vector space $A_k(T_pM)$ of k-covectors on the tangent space at p. For example, on a coordinate chart $(U, x^1, \dots, x^n)$, the k-forms $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, $1 \le i_1 < \cdots < i_k \le n$, constitute a frame of C^∞ k-forms on U, called the *coordinate frame* of k-forms on U.

Let R be a commutative ring. A subset B of a left R-module V is called a *basis* if every element of V can be written uniquely as a finite linear combination $\sum r_i b^i$, where $r_i \in R$ and $b^i \in B$. An R-module is said to be *free* if it has a basis, and if the basis is finite with n elements, then the free R-module is said to be of *rank* n. It can be shown that if a free R-module has a finite basis, then any two bases have the same number of elements, so that the rank is well defined. We denote the rank of V by $\operatorname{rk} V$.

Let $\mathcal{A}^k(M)$ denote the vector space of C^∞ k-forms on M and let

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M).$$

If $(U, x^1, \dots, x^n)$ is a coordinate chart on M, then $\mathcal{A}^k(U)$ is a free module over $C^\infty(U)$ of rank $\binom{n}{k}$, with coordinate frame $\{dx^I\}$ as above.

An algebra A is said to be *graded* if it can be written as a direct sum $A = \bigoplus_{k=0}^\infty A^k$ of vector spaces such that under multiplication, $A^k \cdot A^\ell \subset A^{k+\ell}$. A graded algebra $A = \bigoplus_{k=0}^\infty A^k$ is said to be *graded commutative* or *anticommutative* if for all $x \in A^k$ and $y \in A^\ell$,

$$x \cdot y = (-1)^{k\ell} y \cdot x.$$

The wedge product $\wedge$ makes $\mathcal{A}^*(M)$ into an anticommutative graded algebra over $\mathbb{R}$.

1.4. **Exterior Differentiation.** On any manifold M there is a linear operator $d\colon \mathcal{A}^*(M) \to \mathcal{A}^*(M)$, called *exterior differentiation*, uniquely characterized by three properties:

(1) d is an antiderivation of degree 1, i.e., d increases the degree by 1 and for $\omega \in \mathcal{A}^k(M)$ and $\tau \in \mathcal{A}^\ell(M)$,

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau;$$

(2) $d^2 = d \circ d = 0$;
(3) on a 0-form $f \in C^\infty(M)$,

$$(df)_p(X_p) = X_p f \text{ for } p \in M,\ X_p \in T_pM.$$

By induction the antiderivation property (1) extends to more than two factors; for example,

$$d(\omega \wedge \tau \wedge \eta) = d\omega \wedge \tau \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\tau \wedge \eta + (-1)^{\deg \omega \wedge \tau} \omega \wedge \tau \wedge d\eta.$$

The existence and uniqueness of exterior differentiation on a general manifold is established in [3, Section 19, p. 189]. To develop some facility with this operator, we will examine the case when M is covered by a single coordinate chart $(U, x^1, \dots, x^n)$. To prove its existence on U, we define d by the two formulas:

(i) if $f \in \mathcal{A}^0(U)$, then $df = \sum (\partial f / \partial x^i)\, dx^i$;
(iii) if $\omega = \sum a_I\, dx^I \in \mathcal{A}^k(U)$ for $k \ge 1$, then $d\omega = \sum da_I \wedge dx^I$.

Next we check that so defined, d satisfies the three properties of exterior differentiation.

(1) For $\omega \in \mathcal{A}^k(U)$ and $\tau \in \mathcal{A}^\ell(U)$,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau. \tag{1.4}$$

Proof. Suppose $\omega = \sum a_I \, dx^I$ and $\tau = \sum b_J \, dx^J$. On functions, $d(fg) = (df)g + f(dg)$ is simply another manifestation of the ordinary product rule, since

$$\begin{aligned} d(fg) &= \sum \frac{\partial}{\partial x^i}(fg)\, dx^i \\ &= \sum \left(\frac{\partial f}{\partial x^i} g + f \frac{\partial g}{\partial x^i} \right) dx^i \\ &= \left(\sum \frac{\partial f}{\partial x^i}\, dx^i \right) g + f \sum \frac{\partial g}{\partial x^i}\, dx^i \\ &= (df)\, g + f\, dg. \end{aligned}$$

Next suppose $k \geq 1$. Since d is linear and $\wedge$ is bilinear over $\mathbb{R}$, we may assume that $\omega = a_I \, dx^I$ and $\tau = b_J \, dx^J$ each consist of a single term. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d(a_I b_J \, dx^I \wedge dx^J) \\ &= d(a_I b_J) \wedge dx^I \wedge dx^J \quad \text{(definition of } d) \\ &= (da_I) b_J \wedge dx^I \wedge dx^J + a_I \, db_J \wedge dx^I \wedge dx^J \\ &\qquad \text{(by the degree 0 case)} \\ &= da_I \wedge dx^I \wedge b_J \, dx^J + (-1)^k a_I \, dx^I \wedge db_J \wedge dx^J \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau. \end{aligned}$$

□

(2) $d^2 = 0$ on $\mathcal{A}^k(U)$.

Proof. This is a consequence of the fact that the mixed partials of a function are equal. For $f \in \mathcal{A}^0(U)$,

$$d^2 f = d\left(\sum_{i=1}^n \frac{\partial f}{\partial x^i}\, dx^i \right) = \sum_{j=1}^n \sum_{i=1}^n \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i.$$

In this double sum, the factors $\partial^2 f / \partial x^j \partial x^i$ are symmetric in i, j, while $dx^j \wedge dx^i$ are skew-symmetric in i, j. Hence, for each pair $i < j$ there are two terms

$$\frac{\partial^2 f}{\partial x^i \partial x^j}\, dx^i \wedge dx^j, \qquad \frac{\partial^2 f}{\partial x^j \partial x^i}\, dx^j \wedge dx^i$$

that add up to zero. It follows that $d^2 f = 0$.

For $\omega = \sum a_I \, dx^I \in \mathcal{A}^k(U)$, where $k \geq 1$,

$$\begin{aligned} d^2\omega &= d\left(\sum da_I \wedge dx^I \right) \quad \text{(by the definition of } d\omega) \\ &= \sum (d^2 a_I) \wedge dx^I + da_I \wedge d(dx^I) \\ &= 0. \end{aligned}$$

In this computation, $d^2 a_I = 0$ by the degree 0 case, and $d(dx^I) = 0$ follows by the antiderivation property (1) and the degree 0 case.

(3) Suppose $X = \sum a^j \, \partial/\partial x^j$. Then

$$(df)(X) = \left(\sum \frac{\partial f}{\partial x^i} \, dx^i\right) \left(\sum a^j \frac{\partial}{\partial x^j}\right) = \sum a^i \frac{\partial f}{\partial x^i} = X(f). \qquad \square$$

The exterior derivative d generalizes the gradient, curl, and divergence of vector calculus.

1.5. **Pullback of Differential Forms.** Unlike vector fields, which in general cannot be pushed forward under a C^∞ map, differential forms can always be pulled back. Let $F\colon N \to M$ be a C^∞ map. The *pullback* of a C^∞ function f on M is the C^∞ function $F^* f := f \circ F$ on N. This defines the pullback on C^∞ 0-forms. For $k > 0$, the *pullback* of a k-form ω on M is the k-form $F^*\omega$ on N defined by

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(F_{*,p} v_1, \dots, F_{*,p} v_k)$$

for $p \in N$ and $v_1, \dots, v_k \in T_p M$. From this definition, it is not obvious that the pullback $F^*\omega$ of a C^∞ form ω is C^∞. To show this, we first derive a few basic properties of the pullback.

Proposition 1.5. *Let $F\colon N \to M$ be a C^∞ map of manifolds. If ω and τ are k-forms and σ is an ℓ-form on M, then*

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$;
(ii) *for any real number a,* $F^*(a\omega) = aF^*\omega$;
(iii) $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$;
(iv) *for any C^∞ function h,* $dF^*h = F^*dh$.

Proof. The first three properties (i), (ii), (iii) follow directly from the definitions. To prove (iv), let $p \in N$ and $X_p \in T_p N$. Then

$$\begin{aligned}(dF^*h)_p(X_p) &= X_p(F^*h) && \text{(property (3) of } d\text{)}\\ &= X_p(h \circ F) && \text{(definition of } F^*h\text{)}\end{aligned}$$

and

$$\begin{aligned}(F^*dh)_p(X_p) &= (dh)_{F(p)}(F_{*,p}X_p) && \text{(definition of } F^*\text{)}\\ &= (F_{*,p}X_p)h && \text{(property (3) of } d\text{)}\\ &= X_p(h \circ F). && \text{(definition of } F_{*,p}\text{)}\end{aligned}$$

Hence,

$$dF^*h = F^*dh. \qquad \square$$

We now prove that the pullback of a C^∞ form is C^∞. On a coordinate chart $(U, x^1, \dots, x^n)$ in M, a C^∞ k-form ω can be written as a linear combination

$$\omega = \sum a_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients a_I are C^∞ functions on U. By the preceding proposition,

$$\begin{aligned} F^*\omega &= \sum (F^*a_I)\, d(F^*x^{i_1}) \wedge \cdots \wedge d(F^*x^{i_k}) \\ &= \sum (a_I \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F), \end{aligned}$$

which shows that $F^*\omega$ is C^∞, because it is a sum of products of C^∞ functions and C^∞ 1-forms.

Proposition 1.6. *Suppose $F\colon N \to M$ is a smooth map. On C^∞ k-forms, $dF^* = F^*d$.*

Proof. Let $\omega \in \mathcal{A}^k(M)$ and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p in M. On U,

$$\omega = \sum a_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

As computed above,

$$F^*\omega = \sum (a_I \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F).$$

Hence,

$$\begin{aligned} dF^*\omega &= \sum d(a_I \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \\ &= \sum d(F^*a_I) \wedge d(F^*x^{i_1}) \wedge \cdots \wedge d(F^*x^{i_k}) \\ &= \sum F^*da_I \wedge F^*dx^{i_1} \wedge \cdots \wedge F^*dx^{i_k} \\ &\quad (dF^* = F^*d \text{ on functions by Proposition 1.5(iv)}) \\ &= \sum F^*(da_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &\quad (F^* \text{ preserves the wedge product by Proposition 1.5(iii)}) \\ &= F^*d\omega. \end{aligned}$$

□

In summary, for any C^∞ map $F\colon N \to M$, the pullback map $F^*\colon \mathcal{A}^*(M) \to \mathcal{A}^*(N)$ is an algebra homomorphism that commutes with the exterior derivative d.

Example 1.7. Pullback under the inclusion of an immersed submanifold. Let N and M be manifolds. A C^∞ map $f\colon N \to M$ is called an *immersion* if for all $p \in N$, the differential $f_{*,p}\colon T_pN \to T_{f(p)}M$ is injective. A subset S of M with a manifold structure such that the inclusion map $i\colon S \to M$ is an immersion is called an *immersed submanifold* of M. An example is the image of a line with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$. An immersed submanifold need not have the subspace topology.

If $\omega \in \mathcal{A}^k(M)$, $p \in S$, and $v_1, \ldots, v_k \in T_pS$, then by the definition of the pullback,

$$(i^*\omega)_p(v_1, \ldots, v_k) = \omega_{i(p)}(i_*v_1, \ldots, i_*v_k) = \omega_p(v_1, \ldots, v_k).$$

Thus, the pullback of ω under the inclusion map i is simply the restriction of ω to the submanifold S. We also adopt the more suggestive notation $\omega|_S$ for $i^*\omega$.

1.6. **Real Projective Space.** To conclude, we give another example of a manifold, the *real projective space* $\mathbb{R}P^n$. It is defined as the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation:

$$x \sim y \iff y = tx \text{ for some nonzero real number } t,$$

where $x, y \in \mathbb{R}^{n+1} - \{0\}$. We denote the equivalence class of a point $(a^0, \dots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ by $[a^0, \dots, a^n]$ and let $\pi\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a^0, \dots, a^n]$ the *homogeneous coordinates* on $\mathbb{R}P^n$.

Geometrically, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Each line through the origin in

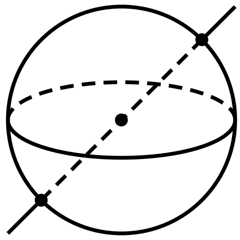

FIGURE 1.2. A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

$\mathbb{R}^{n+1}$ meets the unit sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin (Figure 1.2). This suggests that we define an equivalence relation $\sim$ on S^n by identifying antipodal points

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\!\sim$. As a quotient space of a sphere, the real projective space $\mathbb{R}P^n$ is the image of a compact space under a continuous map and is therefore compact.

Next we construct a C^∞ atlas on $\mathbb{R}P^n$. Let $[a^0, \dots, a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is independent of the choice of a representative for $[a^0, \dots, a^n]$. Hence, the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^0 \neq 0\}.$$

Similarly, for each $i = 1, \dots, n$, let

$$U_i := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}.$$

Define

$$\phi_0\colon U_0 \to \mathbb{R}^n$$

by

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^1}{a^0}, \dots, \frac{a^n}{a^0}\right).$$

This map has a continuous inverse

$$(b^1, \dots, b^n) \mapsto [1, b^1, \dots, b^n]$$

and is therefore a homeomorphism. Similarly, for $i = 1, \dots, n$ there are homeomorphisms

$$\phi_i \colon U_i \to \mathbb{R}^n,$$

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^0}{a^i}, \dots, \widehat{\frac{a^i}{a^i}}, \dots, \frac{a^n}{a^i} \right),$$

where the caret sign $\widehat{\ }$ over a^i/a^i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

On the intersection $U_0 \cap U_1$, $a^0 \neq 0$ and $a^1 \neq 0$, and there are two coordinate systems

$$[a^0, a^1, a^2, \dots, a^n]$$

$$\phi_0 \swarrow \qquad \searrow \phi_1$$

$$\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \dots, \frac{a^n}{a^0} \right) \qquad \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \dots, \frac{a^n}{a^1} \right).$$

We will refer to the coordinate functions on U_0 as $x^1, \dots, x^n$, and the coordinate functions on U_1 as $y^1, \dots, y^n$. On U_0,

$$x^i = \frac{a^i}{a^0}, \quad i = 1, \dots, n,$$

and on U_1,

$$y^1 = \frac{a^0}{a^1}, \ y^2 = \frac{a^2}{a^1}, \ \dots, \ y^n = \frac{a^n}{a^1}.$$

Then on $U_0 \cap U_1$,

$$y^1 = \frac{1}{x^1}, y^2 = \frac{x^2}{x^1}, y^3 = \frac{x^3}{x^1}, \dots, y^n = \frac{x^n}{x^1},$$

so

$$(\phi_1 \circ \phi_0^{-1})(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \dots, \frac{x^n}{x^1} \right).$$

This is a C^∞ function because $x^1 \neq 0$ on $\phi_0(U_0 \cap U_1)$. On any other $U_i \cap U_j$ an analogous formula holds. Therefore, the collection $\{(U_i, \phi_i)\}_{i=0,\dots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the *standard atlas*. For a proof that $\mathbb{R}P^n$ is Hausdorff and second countable, see [3, Cor. 7.15 and Prop. 7.16, p. 71]. It follows that $\mathbb{R}P^n$ is a C^∞ manifold.

Problems

1.1. Connected Components

(a) The *connected component* of a point p in a topological space S is the largest connected subset of S containing p. Show that the connected components of a manifold are open.

(b) Let $\mathbb{Q}$ be the set of rational numbers considered as a subspace of the real line $\mathbb{R}$. Show that the connected component of $p \in \mathbb{Q}$ is the singleton set $\{p\}$, which is not open in $\mathbb{Q}$. Which condition in the definition of a manifold does $\mathbb{Q}$ violate?

1.2. Connected Components Versus Path Components

The *path component* of a point p in a topological space S is the set of all points $q \in S$ that can be connected to p via a continuous path. Show that for a manifold, the path components are the same as the connected components.

1.3. Unit n-Sphere

The unit n-sphere S^n in $\mathbb{R}^{n+1}$ is the solution set of the equation

$$(x^0)^2 + \cdots + (x^n)^2 = 1.$$

Generalizing Example 1.2, find a C^∞ atlas on S^n.

2. The de Rham Complex

A basic goal in algebraic topology is to associate to a manifold M an algebraic object $F(M)$ so that the algebraic properties of $F(M)$ reflect the topological properties of M. Such an association is formalized in the notion of a functor. In this section we define the de Rham complex and the de Rham cohomology of a manifold. It will turn out to be one of the most important functors from manifolds to algebras.

2.1. Categories and Functors.

A *category* $\mathcal{K}$ consists of a collection of *objects* and for any two objects A and B in $\mathcal{K}$ a set $\mathrm{Mor}(A, B)$ of *morphisms* from A to B, satisfying the following properties:

(i) If $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, C)$, then there is a law of composition so that the *composite morphism* $g \circ f \in \mathrm{Mor}(A, C)$ is defined.
(ii) The composition of morphisms is associative: $(h \circ g) \circ f = h \circ (g \circ f)$.
(iii) For every object A there is a morphism $1_A \in \mathrm{Mor}(A, A)$ that serves as the identity under composition: for every morphism $f \in \mathrm{Mor}(A, B)$, $f = f \circ 1_A = 1_B \circ f$.

If $f \in \mathrm{Mor}(A, B)$, we also write $f\colon A \to B$.

Example. The collection of groups together with group homomorphisms is a category.

Example. The collection of smooth manifolds together with C^∞ maps between manifolds is a category.

A *covariant functor* from a category $\mathcal{K}$ to a category $\mathcal{L}$ associates to every object A in $\mathcal{K}$ an object $F(A)$ in $\mathcal{L}$ and to every morphism $f\colon A \to B$ in $\mathcal{K}$ a morphism $F(f)\colon F(A) \to F(B)$ in $\mathcal{L}$ such that F preserves composition and identity:

$$F(g \circ f) = F(g) \circ F(f),$$
$$F(1_A) = 1_{F(A)}.$$

If F reverses the arrows, i.e., $F(f)\colon F(B) \to F(A)$ such that $F(g \circ f) = F(f) \circ F(g)$ and $F(1_B) = 1_{F(B)}$, then it is said to be a *contravariant functor*.

Example. A *pointed manifold* is a pair (M, p) where M is a manifold and p is a point in M. For any two pointed manifolds (M, p) and (N, q), define a morphism $f\colon (N, q) \to (M, p)$ to be a C^∞ map $f\colon N \to M$ such that $f(q) = p$. To every pointed manifold (M, p), we associate its tangent space $F(M, p) = T_pM$, and to every morphism of pointed manifolds $f\colon (N, q) \to (M, p)$ we associate the differential $F(f) = f_{*,q}\colon T_qN \to T_pM$. Then F is a covariant functor from the category of pointed manifolds and morphisms of pointed manifolds to the category of finite-dimensional vector spaces and linear maps.

A morphism $f\colon A \to B$ in a category is called an *isomorphism* if it has a two-sided inverse, that is, a morphism $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Two objects A and B in a category are said to be *isomorphic* if there is an isomorphism $f\colon A \to B$ between them.

Proposition 2.1. *A functor F from a category $\mathcal{K}$ to a category $\mathcal{L}$ takes an isomorphism in $\mathcal{K}$ to an isomorphism in $\mathcal{L}$.*

PROOF. We prove the proposition for a covariant functor F. The proof is equally valid, *mutatis mutandis*, for a contravariant functor. Let $f\colon A \to B$ be an isomorphism in $\mathcal{K}$ with two-sided inverse $g\colon B \to A$. By functoriality, i.e., since F is a covariant functor,

$$F(g \circ f) = F(g) \circ F(f).$$

On the other hand,

$$F(g \circ f) = F(1_A) = 1_{F(A)}.$$

Hence, $F(g) \circ F(f) = 1_{F(A)}$. Similarly, reversing the roles of f and g gives $F(f) \circ F(g) = 1_{F(B)}$. This shows that $F(f)$ has a two-sided inverse $F(g)$ and is therefore an isomorphism. □

Remark. It follows from this proposition that under a functor F from a category $\mathcal{K}$ to a category $\mathcal{L}$, if two objects $F(A)$ and $F(B)$ are not isomorphic in $\mathcal{L}$, then the two objects A and B are not isomorphic in $\mathcal{K}$. In this way a functor distinguishes nonisomorphic objects in the category $\mathcal{K}$.

2.2. **De Rham Cohomology.** A functor F from the category of smooth manifolds and smooth maps to another category $\mathcal{L}$ associates to each manifold M a well-defined object $F(M)$ in $\mathcal{L}$. For the functor to be useful, it should be complex enough to distinguish many nondiffeomorphic manifolds and yet simple enough to be computable.

As a first candidate, one might consider the vector space $\mathfrak{X}(M)$ of all C^∞ vector fields on M. One problem with vector fields is that in general they cannot be pushed forward or pulled back under smooth maps. Thus, $\mathfrak{X}(M)$ is not a functor on the category of smooth manifolds.

A great advantage of differential forms is that they pull back under smooth maps. Assigning to each manifold M the algebra $\mathcal{A}^*(M)$ of C^∞ forms on M and to each smooth map $f\colon N \to M$ the pullback map $f^*\colon \mathcal{A}^*(M) \to \mathcal{A}^*(N)$ gives a contravariant functor from the category of smooth manifolds and smooth maps to the category of anticommutative graded algebras and their homomorphisms. However, the algebra $\mathcal{A}^*(M)$ is too large to be a computable invariant. In fact, unless M is a finite set of points, $\mathcal{A}^0(M)$ is already an infinite-dimensional vector space.

The *de Rham complex* of the manifold M is the sequence of vector spaces and linear maps

$$0 \longrightarrow \mathcal{A}^0(M) \xrightarrow{d_0} \mathcal{A}^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{k-1}} \mathcal{A}^k(M) \xrightarrow{d_k} \cdots,$$

where $d_k = d|_{\mathcal{A}^k}$ is exterior differentiation. Since $d_k \circ d_{k-1} = 0$, the image $\operatorname{im} d_{k-1}$ is a subspace of the kernel $\ker d_k$, and so it is possible to take the quotient of $\ker d_k$ by $\operatorname{im} d_{k-1}$ with the hope of obtaining a finite-dimensional quotient space. A differential k-form ω is said to be *closed* if $d\omega = 0$; ω is said to be *exact* if there is a $(k-1)$-form τ such that $\omega = d\tau$. Let $Z^k(M) = \ker d_k$ denote the vector space of closed k-forms on M, and $B^k(M) = \operatorname{im} d_{k-1}$ the vector space of exact k-forms on M. The *kth de Rham cohomology* of M is by definition the quotient vector space

$$H^k(M) := \frac{Z^k(M)}{B^k(M)} = \frac{\{\text{closed } k\text{-forms on } M\}}{\{\text{exact } k\text{-forms on } M\}}.$$

If ω is a closed k-form on M, its equivalence class in $H^k(M)$, denoted $[\omega]$, is called the *cohomology class* of ω. It can be shown that the de Rham cohomology $H^k(M)$ of a compact manifold M is a finite-dimensional vector space for all k [1, Prop. 5.3.1, p. 43].

The letter Z for closed forms comes from the German word *Zyklus* for a cycle and the letter B for exact forms comes from the English word *boundary,* as such elements are called in the general theory of homology.

Let $H^*(M) = \bigoplus_{k=0}^{\infty} H^k(M)$. A priori, $H^*(M)$ is a vector space. By the antiderivation property

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau,$$

if ω is closed, then

$$\omega \wedge d\tau = \pm d(\omega \wedge \tau),$$

i.e., the wedge product of a closed form with an exact form is exact. It follows that the wedge product induces a well-defined product in cohomology

$$\begin{aligned} \wedge \colon H^k(M) \times H^\ell(M) &\to H^{k+\ell}(M), \\ [\omega] \wedge [\tau] &= [\omega \wedge \tau]. \end{aligned} \tag{2.1}$$

This makes the de Rham cohomology $H^*(M)$ into an anticommutative graded algebra.

2.3. **Cochain Complexes and Cochain Maps.** A *cochain complex* $\mathcal{C}$ in the category of vector spaces is a sequence of vector spaces and linear maps

$$\cdots \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots, \qquad k \in \mathbb{Z}$$

such that $d_k \circ d_{k-1} = 0$. In principle, this sequence extends to infinity in both directions; in practice, we are interested only in cochain complexes for which $C^k = 0$ for all $k < 0$, called *nonnegative cochain complexes.* Effectively, nonnegative cochain complexes will start with

$$0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots.$$

To simplify the notation, we often omit the subscript and write d instead d_k. An element $c \in C^k$ is a *cocycle of degree* k or a k*-cocycle* if $dc = 0$. It is a k*-coboundary* if there exists an element $b \in C^{k-1}$ such that $c = db$. Let $Z^k(\mathcal{C})$ be the space of k-cocycles and $B^k(\mathcal{C})$ the space of k-coboundaries in $\mathcal{C}$. The *kth cohomology* of $\mathcal{C}$ is defined to be the quotient vector space

$$H^k(\mathcal{C}) = \frac{Z^k(\mathcal{C})}{B^k(\mathcal{C})} = \frac{\{k\text{-cocycles}\}}{\{k\text{-coboundaries}\}}.$$

An element of $H^k(\mathcal{C})$ determined by a k-cocycle $c \in Z^k(\mathcal{C})$ is denoted $[c]$.

If $\mathcal{A}$ and $\mathcal{B}$ are two cochain complexes, then a *cochain map* $h \colon \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $h_k \colon A^k \to B^k$ such that $h_{k+1} \circ d = d \circ h_k$ for all k.

This is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} A^{k+1} & \xrightarrow{h_{k+1}} & B^{k+1} \\ d \uparrow & & \uparrow d \\ A^k & \xrightarrow[h_k]{} & B^k \end{array}$$

for all k.

A cochain map $h\colon \mathcal{A} \to \mathcal{B}$ takes cocycles in $\mathcal{A}$ to cocycles in $\mathcal{B}$, because if $a \in Z^k(\mathcal{A})$, then

$$d(h_k a) = h_{k+1}(da) = h_{k+1}(0) = 0.$$

Similarly, it takes coboundaries in $\mathcal{A}$ to coboundaries in $\mathcal{B}$, since $h_k(da) = d(h_{k-1}a)$. Therefore, a cochain map $h\colon \mathcal{A} \to \mathcal{B}$ induces a linear map in cohomology,

$$h^{\#}\colon H^*(\mathcal{A}) \to H^*(\mathcal{B}),$$
$$h^{\#}[a] = [h(a)].$$

Returning to the de Rham complex, a C^∞ map $f\colon N \to M$ of manifolds induces a pullback map $f^*\colon \mathcal{A}^k(M) \to \mathcal{A}^k(N)$ of differential forms, which preserves degree. The commutativity of f^* with the exterior derivative d says precisely that $f^*\colon \mathcal{A}^*(M) \to \mathcal{A}^*(N)$ is a cochain map of degree 0. Therefore, it induces a linear map in cohomology:

$$f^{\#}\colon H^k(M) \to H^k(N),$$
$$f^{\#}[\omega] = [f^*\omega].$$

Since the pullback f^* of differential forms is an algebra homomorphism,

$$\begin{aligned} f^{\#}[\omega \wedge \tau] &= [f^*(\omega \wedge \tau)] \\ &= [f^*\omega \wedge f^*\tau] = [f^*\omega] \wedge [f^*\tau] \quad \text{(by (2.1))} \\ &= f^{\#}[\omega] \wedge f^{\#}[\tau]. \end{aligned}$$

Similar computations show that $f^{\#}$ preserves addition and scalar multiplication in cohomology. Hence, the pullback $f^{\#}$ in cohomology is also an algebra homomorphism. Thus, the de Rham cohomology $H^*(M)$ gives a contravariant functor from the category of smooth manifolds and smooth maps to the category of anticommutative graded algebras and algebra homomorphisms. In practice, we write f^* also for the pullback in cohomology, instead of $f^{\#}$. By Proposition 2.1, the de Rham cohomology algebras of diffeomorphic manifolds are isomorphic as algebras. In this sense, de Rham cohomology is a diffeomorphism invariant of C^∞ manifolds.

2.4. **Cohomology in Degree Zero.** A function $f\colon S \to T$ from a topological space S to a to topological space T is said to be *locally constant* if every point $p \in S$ has a neighborhood U on which f is constant. Since a constant function is continuous, a locally constant function on S is continuous at every point $p \in S$ and therefore continuous on S.

Lemma 2.2. *On a connected topological space S, a locally constant function is constant.*

PROOF. Problem 2.2. □

Proposition 2.3. *If a manifold M has m connected components, then $H^0(M) = \mathbb{R}^m$.*

PROOF. Since there are no forms of degree -1 other than 0, the only exact 0-form is 0. A closed C^∞ 0-form is a C^∞ function $f \in \mathcal{A}^0(M)$ such that $df = 0$. If f is closed, on any coordinate chart $(U, x^1, \dots, x^n)$,

$$df = \sum \frac{\partial f}{\partial x^i}\, dx^i = 0.$$

Because $dx^1, \dots, dx^n$ are linearly independent at every point of U, $\partial f/\partial x^i = 0$ on U for all i. By the mean-value theorem from calculus, f is locally constant on U (see Problem 2.1). Hence, f is locally constant on M.

By Lemma 2.2, f is constant on each connected component of M. If $M = \bigcup_{i=1}^m M_i$ is the decomposition of M into its connected components, then

$$\begin{aligned} H^0(M) \simeq Z^0(M) &= \{\text{locally constant functions } f \text{ on } M\} \\ &= \{(r_1, r_2, \dots, r_m) \mid r_i \in \mathbb{R},\ f = r_i \text{ on } M_i\} \\ &= \mathbb{R}^m. \end{aligned}$$

Because a manifold M is by definition second countable, every open cover of M has a countable subcover [3, Problem A.8, p. 297]. Since every connected component of a manifold is open (Problem 1.1), a manifold must have countably many components. If a manifold M has infinitely many components, say $M = \bigcup_{i=1}^\infty M_i$, then

$$\begin{aligned} H^0(M) \simeq Z^0(M) &= \{\text{locally constant functions } f \text{ on } M\} \\ &= \{(r_1, r_2, \dots) \mid r_i \in \mathbb{R},\ f = r_i \text{ on } M_i\} \\ &= \prod_{i=1}^\infty \mathbb{R}. \end{aligned}$$

□

2.5. **Cohomology of $\mathbb{R}^n$.** Since $\mathbb{R}^1$ is connected, by Theorem 2.3, $H^0(\mathbb{R}^1) = \mathbb{R}$ with generator the constant function 1. Since $\mathbb{R}^1$ is 1-dimensional, there are no nonzero k-forms on $\mathbb{R}^1$ for $k \geq 2$. Hence, $H^k(\mathbb{R}^1) = 0$ for $k \geq 2$. It remains to compute $H^1(\mathbb{R}^1)$.

The space of closed 1-forms on $\mathbb{R}^1$ is

$$Z^1(\mathbb{R}^1) = \mathcal{A}^1(\mathbb{R}^1) = \{f(x)\, dx \mid f(x) \in C^\infty(\mathbb{R}^1)\}.$$

The space of exact 1-forms on $\mathbb{R}^1$ is

$$B^1(\mathbb{R}^1) = \{dg \mid g \in C^\infty(\mathbb{R}^1)\} = \{g'(x)\, dx \mid g(x) \in C^\infty(\mathbb{R}^1)\}.$$

The question then becomes the following: for every C^∞ function f on $\mathbb{R}^1$, is there a C^∞ function g on $\mathbb{R}^1$ such that $f(x) = g'(x)$?

Define

$$g(x) = \int_0^x f(t)\, dt.$$

By the fundamental theorem of calculus, $g'(x) = f(x)$. Hence, every closed 1-form on $\mathbb{R}^1$ is exact. Therefore,

$$H^1(\mathbb{R}^1) = \frac{Z^1(\mathbb{R}^1)}{B^1(\mathbb{R}^1)} = \frac{B^1(\mathbb{R}^1)}{B^1(\mathbb{R}^1)} = 0.$$

When $n > 1$, the computation of the de Rham cohomology of $\mathbb{R}^n$ is not as straightforward. Henri Poincaré first computed $H^k(\mathbb{R}^n)$ for $k = 1, 2, 3$ in 1887. The general result on the cohomology of $\mathbb{R}^n$ now bears his name (see Corollary 4.11).

Problems

2.1. Vanishing of All Partial Derivatives
Let f be a differentiable function on a coordinate neighborhood $(U, x^1, \ldots, x^n)$ in a manifold M. Prove that if $\partial f/\partial x^i \equiv 0$ on U for all i, then f is locally constant on U. (*Hint*: First consider the case $U \subset \mathbb{R}^n$. For any $p \in U$, choose a convex neighborhood V of p contained in U. If $x \in V$, define $h(t) = f(p + t(x - p))$ for $t \in [0, 1]$. Apply the mean-value theorem to $h(t)$.)

2.2. Locally Constant Functions
Prove that on a connected topological space, a locally constant function is constant.

2.3. Cohomology of a Disjoint Union
A manifold is the disjoint union of its connected components. Prove that the cohomology of a disjoint union is the Cartesian product of the cohomology groups of the components:

$$H^*\left(\coprod_\alpha M_\alpha\right) = \prod_\alpha H^*(M_\alpha).$$

3. Mayer–Vietoris Sequences

When a manifold M is covered by two open subsets U and V, the Mayer–Vietoris sequence provides a tool for calculating the cohomology vector space of M from those of U, V, and $U \cap V$. It is based on a basic result of homological algebra: a short exact sequence of cochain complexes induces a long exact sequence in cohomology. To illustrate the technique, we will compute the cohomology of a circle.

3.1. **Exact Sequences.** A sequence of vector spaces and linear maps

$$\cdots \to V^{k-1} \xrightarrow{f_{k-1}} V^k \xrightarrow{f_k} V^{k+1} \to \cdots$$

is said to be *exact at* V^k if the kernel of f_k is equal to the image of its predecessor f_{k-1}. The sequence is *exact* if it is exact at V^k for all k. Note that a cochain complex $\mathcal{C}$ is exact if and only if its cohomology $H^k(\mathcal{C}) = 0$ for all k. Thus, the cohomology of a cochain complex may be viewed as a measure of the deviation of the complex from exactness.

An exact sequence of vector spaces of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is called a *short exact sequence*. In such a sequence, $\ker i = \operatorname{im} 0 = 0$, so that i is injective, and $\operatorname{im} j = \ker 0 = C$, so that j is surjective. Moreover, by exactness and the first isomorphism theorem of linear algebra,

$$\frac{B}{i(A)} = \frac{B}{\ker j} \simeq \operatorname{im} j = C.$$

These three properties, the injectivity of i, the surjectivity of j, and the isomorphism $C \simeq B/i(A)$, characterize a short exact sequence (Problem 3.1).

Now suppose $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are cochain complexes and $i\colon \mathcal{A} \to \mathcal{B}$ and $j\colon \mathcal{B} \to \mathcal{C}$ are cochain maps. The sequence

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0 \tag{3.1}$$

is a *short exact sequence of complexes* if in every degree k,

$$0 \to A^k \xrightarrow{i} B^k \xrightarrow{j} C^k \to 0$$

is a short exact sequence of vector spaces.

In the short exact sequence of complexes (3.1), since $i\colon \mathcal{A} \to \mathcal{B}$ and $j\colon \mathcal{B} \to \mathcal{C}$ are cochain maps, they induce linear maps $i^*\colon H^k(\mathcal{A}) \to H^k(\mathcal{B})$ and $j^*\colon H^k(\mathcal{B}) \to H^k(\mathcal{C})$ in cohomology by the formulas

$$i^*[a] = [i(a)], \quad j^*[b] = [j(b)].$$

There is in addition a linear map

$$d^*\colon H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A}),$$

called the *connecting homomorphism* and defined as follows.

The short exact sequence of complexes (3.1) is in fact an infinite diagram of commutative squares

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{i} & B^{k+1} & \xrightarrow{j} & C^{k+1} & \longrightarrow & 0 \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & A^{k} & \xrightarrow[i]{} & B^{k} & \xrightarrow[j]{} & C^{k} & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

To reduce visual clutter, we will often omit the parentheses around the argument of a map and write, for example, ia and db instead of $i(a)$ and $d(b)$. Let $[c] \in H^k(\mathcal{C})$ with c a cocycle in C^k. By the surjectivity of $j\colon B^k \to C^k$, there is an element $b \in B^k$ such that $j(b) = c$. Because $j(db) = dj(b) = dc = 0$ and because the rows are exact, $db = i(a)$ for some $a \in A^{k+1}$. By the injectivity of i, the element a is unique. This a is a cocycle since

$$i(da) = d(ia) = d(db) = 0,$$

from which it follows by the injectivity of i again that $da = 0$. Therefore, a determines a cohomology class $[a] \in H^{k+1}(A)$. We define $d^*[c] = [a]$.

Remark 3.1. In making this definition, we have made two choices: the choice of a cocycle $c \in C^k$ to represent the class $[c] \in H^k(\mathcal{C})$ and the choice of an element $b \in B^k$ such that $j(b) = c$. It is not difficult to show that $[a]$ is independent of these choices (see [3, Exercise 24.6, p. 317]), so that $d^*\colon H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$ is a well-defined map. As easily verified, it is in fact a linear map.

The construction of the connecting homomorphism d^* can be summarized by the diagrams

$$\begin{array}{ccc}
A^{k+1} & \rightarrowtail & B^{k+1} & & \\
 & & d\uparrow & & \\
 & & B^k & \twoheadrightarrow & C^k,
\end{array}
\qquad\qquad
\begin{array}{ccc}
a & \longmapsto & db & & \\
 & & \uparrow & & \\
 & & b & \longmapsto & c.
\end{array}$$

Proposition 3.2 (Zig-zag lemma). *A short exact sequence of cochain complexes*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

gives rise to a long exact sequence in cohomology:

$$\cdots \to H^{k-1}(\mathcal{C}) \xrightarrow{d^*} H^k(\mathcal{A}) \xrightarrow{i^*} H^k(\mathcal{B}) \xrightarrow{j^*} H^k(\mathcal{C}) \xrightarrow{d^*} H^{k+1}(\mathcal{A}) \to \cdots .$$

The proof consists of unravelling the definitions and is an exercise in what is commonly called *diagram-chasing.* See [3, p. 247] for more details. The long exact sequence extends to infinity in both directions. For cochain complexes for which the terms in negative degrees are zero, the long exact sequence will start with

$$0 \to H^0(\mathcal{A}) \xrightarrow{i^*} H^0(\mathcal{B}) \xrightarrow{j^*} H^0(\mathcal{C}) \xrightarrow{d^*} H^1(\mathcal{A}) \to \cdots .$$

In such a sequence the map $i^*\colon H^0(\mathcal{A}) \to H^0(\mathcal{B})$ is injective.

3.2. **Partitions of Unity.** In order to prove the exactness of the Mayer–Vietoris sequence, we will need a C^∞ partition of unity. The *support* of a real-valued function f on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$:

$$\operatorname{supp} f = \operatorname{cl}_M(f^{-1}(\mathbb{R}^\times)) = \text{closure of } \{q \in M \mid f(q) \neq 0\} \text{ in } M.$$

If $\{U_i\}_{i \in I}$ is a finite open cover of M, a C^∞ *partition of unity subordinate to* $\{U_i\}$ is a collection of nonnegative C^∞ functions $\{\rho_i \colon M \to \mathbb{R}\}_{i \in I}$ such that $\operatorname{supp} \rho_i \subset U_i$ and

$$\sum \rho_i = 1. \tag{3.2}$$

When I is an infinite set, for the sum in (3.2) to make sense, we will impose a *local finiteness* condition. A collection $\{A_\alpha\}$ of subsets of a topological space S is said to be *locally finite* if every point q in S has a neighborhood that meets only finitely many of the sets A_α. In particular, every q in S is contained in only finitely many of the A_α's.

Example. An open cover that is not locally finite. Let $U_{r,n}$ be the open interval $\left]r - \frac{1}{n}, r + \frac{1}{n}\right[$ in the real line $\mathbb{R}$. The open cover $\{U_{r,n} \mid r \in \mathbb{Q}, n \in \mathbb{Z}^+\}$ of $\mathbb{R}$ is not locally finite.

Definition 3.3. A C^∞ *partition of unity* on a manifold is a collection of nonnegative C^∞ functions $\{\rho_\alpha \colon M \to \mathbb{R}\}_{\alpha \in \mathrm{A}}$ such that

(i) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in \mathrm{A}}$, is locally finite,
(ii) $\sum \rho_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha \in \mathrm{A}}$ is *subordinate to the open cover* $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for every $\alpha \in \mathrm{A}$.

Since the collection of supports, $\{\operatorname{supp} \rho_\alpha\}$, is locally finite (Condition (i)), every point q lies in finitely many of the sets $\operatorname{supp} \rho_\alpha$. Hence $\rho_\alpha(q) \neq 0$ for only finitely many α. It follows that the sum in (ii) is a finite sum at every point.

Example. Let U and V be the open intervals $]-\infty, 2[$ and $]-1, \infty[$ in $\mathbb{R}$ respectively, and let ρ_V be a C^∞ function with graph as in Figure 3.1. Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp} \rho_V \subset V$ and $\operatorname{supp} \rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

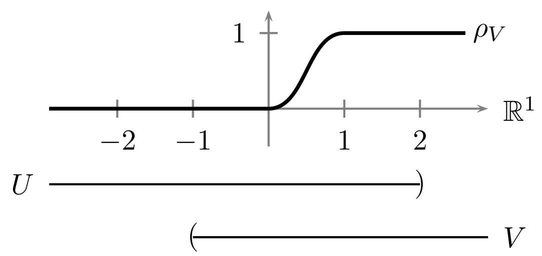

FIGURE 3.1. A partition of unity $\{\rho_U, \rho_V\}$ subordinate to an open cover $\{U, V\}$.

Theorem 3.4 (Existence of a C^∞ partition of unity). *Let $\{U_\alpha\}_{\alpha\in \mathrm{A}}$ be an open cover of a manifold M.*

(i) *There is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with every φ_k having compact support such that for each k, $\operatorname{supp}\varphi_k \subset U_\alpha$ for some $\alpha \in \mathrm{A}$.*

(ii) *If we do not require compact support, then there is a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.*

3.3. **The Mayer–Vietoris Sequence for de Rham Cohomology.** Suppose a manifold M is the union of two open subsets U and V. There are four inclusion maps

$$\begin{array}{ccccc} & & U & & \\ & \overset{j_U}{\nearrow} & & \overset{i_U}{\searrow} & \\ U\cap V & & & & M. \\ & \underset{j_V}{\searrow} & & \underset{i_V}{\nearrow} & \\ & & V & & \end{array}$$

They induce four restriction maps on differential forms

$$\begin{array}{ccccc} & & \mathcal{A}^k(U) & & \\ & \overset{j_U^*}{\swarrow} & & \overset{i_U^*}{\nwarrow} & \\ \mathcal{A}^k(U\cap V) & & & & \mathcal{A}^k(M). \\ & \underset{j_V^*}{\nwarrow} & & \underset{i_V^*}{\swarrow} & \\ & & \mathcal{A}^k(V) & & \end{array}$$

Define $i\colon \mathcal{A}^k(M) \to \mathcal{A}^k(U) \oplus \mathcal{A}^k(V)$ to be the restriction

$$i(\sigma) = (i_U^*\sigma, i_V^*\sigma) = (\sigma|_U, \sigma|_V)$$

and $j\colon \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \to \mathcal{A}^k(U\cap V)$ to be the difference of restrictions

$$j(\omega_U, \omega_V) = j_V^*\omega_V - j_U^*\omega_U = \omega_V|_{U\cap V} - \omega_U|_{U\cap V}.$$

To simplify the notation, we will often suppress the restrictions and simply write $j(\omega_U, \omega_V) = \omega_V - \omega_U$.

Proposition 3.5 (Mayer–Vietoris sequence for forms). *If $\{U, V\}$ is an open cover of a manifold M, then*

$$0 \to \mathcal{A}^*(M) \xrightarrow{i} \mathcal{A}^*(U) \oplus \mathcal{A}^*(V) \xrightarrow{j} \mathcal{A}^*(U\cap V) \to 0 \tag{3.3}$$

is a short exact sequence of cochain complexes.

Proof. The exactness is clear except at $\mathcal{A}^*(U\cap V)$ (see Problem 3.4). We will prove exactness at $\mathcal{A}^*(U\cap V)$, i.e., the surjectivity of j. Consider first the case of C^∞ functions on $M = \mathbb{R}^1$. Let f be a C^∞ function on $U \cap V$ with graph as in Figure 3.2.
We need to write f as the difference of a C^∞ function g_V on V and a C^∞ function g_U on U.

Let $\{\rho_U, \rho_V\}$ be a C^∞ partition of unity on M subordinate to the open cover $\{U, V\}$. Thus, $\operatorname{supp}\rho_U \subset U$, $\operatorname{supp}\rho_V \subset V$, and $\rho_U + \rho_V = 1$. Note that $\rho_U f$, a priori a function on $U \cap V$, can be extended by zero to a C^∞ function on V, which we still denote by $\rho_U f$. Similarly, $\rho_V f$ can be extended by zero to a C^∞ function on U—to get a function on an open set in the cover, we multiply by the partition function of the other open set. On $U \cap V$, since

$$j(-\rho_V f, \rho_U f) = \rho_U f - (-\rho_V f) = f,$$

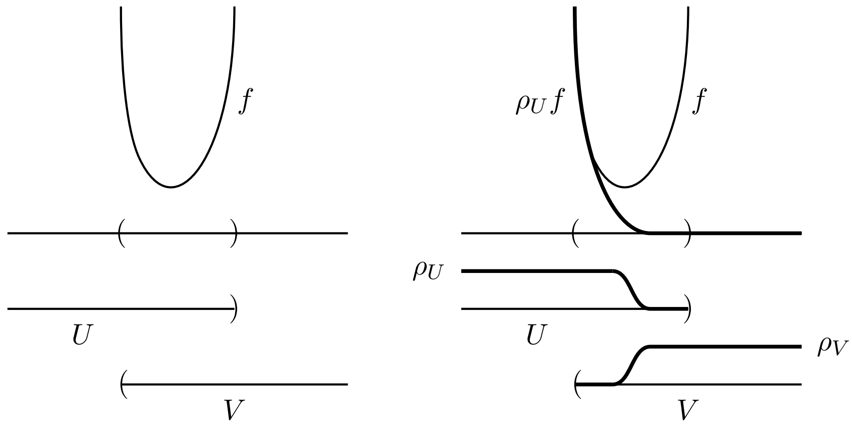

FIGURE 3.2. Writing f as the difference of a C^∞ function on V and a C^∞ function on U.

the map $j\colon \mathcal{A}^0(U) \oplus \mathcal{A}^0(V) \to \mathcal{A}^0(U \cap V)$ is surjective.

For a general manifold M, again let $\{\rho_U, \rho_V\}$ be a C^∞ partition of unity on M subordinate to the open cover $\{U, V\}$. If $\omega \in \mathcal{A}^k(U \cap V)$, we define $\xi_V \in \mathcal{A}^k(V)$ to be the extension by zero of $\rho_U \omega$ from $U \cap V$ to V:

$$\xi_V = \begin{cases} \rho_U \omega & \text{on } U \cap V, \\ 0 & \text{on } V - (U \cap V). \end{cases} \tag{3.4}$$

Similarly, define $\xi_U \in \mathcal{A}^k(U)$ to be the extension by zero of $-\rho_V \omega$ from $U \cap V$ to U:

$$\xi_U = \begin{cases} -\rho_V \omega & \text{on } U \cap V, \\ 0 & \text{on } U - (U \cap V). \end{cases}$$

On $U \cap V$,

$$j(\xi_U, \xi_V) = \xi_V - \xi_U = \rho_U\, \omega - (-\rho_V\, \omega) = \omega.$$

This proves the surjectivity of $j\colon \mathcal{A}^k(U) \oplus \mathcal{A}^k(V) \to \mathcal{A}^k(U \cap V)$. □

By Theorem 3.2, the short exact Mayer–Vietoris sequence (3.3) induces a long exact sequence in cohomology, also called a *Mayer–Vietoris sequence*,

$$\begin{array}{l} \cdots \xrightarrow{j^*} H^{k-1}(U \cap V) \xrightarrow{d^*} \\ \to H^k(M) \xrightarrow{i^*} H^k(U) \oplus H^k(V) \xrightarrow{j^*} H^k(U \cap V) \xrightarrow{d^*} \\ \to H^{k+1}(M) \xrightarrow{i^*} \cdots . \end{array} \tag{3.5}$$

Since the de Rham complex $\mathcal{A}^*(M)$ is a nonnegative cochain complex, in the long exact sequence $H^k = 0$ for all $k < 0$. Hence, the Mayer–Vietoris sequence in cohomology starts with

$$0 \to H^0(M) \overset{i^*}{\to} H^0(U) \oplus H^0(V) \overset{j^*}{\to} \cdots .$$

Example 3.6. Cohomology of a circle. Cover the circle S^1 with two open sets U and V as in Figure 3.3. The intersection $U \cap V$ has two connected components that we call A and B. By Theorem 2.3 and Problem 2.3,

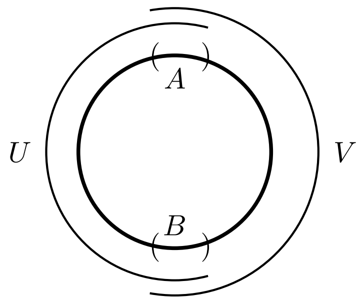

FIGURE 3.3. An open cover of the circle.

$$H^0(S^1) \simeq \mathbb{R}, \quad H^0(U) \simeq \mathbb{R}, \quad H^0(V) \simeq \mathbb{R},$$

and

$$H^0(U \cap V) \simeq H^0(A) \oplus H^0(B) \simeq \mathbb{R} \oplus \mathbb{R},$$

represented by constant functions on each connected component.

The Mayer–Vietoris sequence in cohomology gives

$$\begin{array}{llcccc} & & S^1 & U \amalg V & U \cap V & \\ H^1 & \xrightarrow{d^*} & H^1(S^1) & \longrightarrow & 0 & \longrightarrow 0. \\ H^0 & 0 \longrightarrow & \mathbb{R} & \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} & \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} & \end{array}$$

The maps i^* and j^* are given by

$$i^*(a) = (a, a), \qquad j^*(b, c) = (c - b, c - b). \tag{3.6}$$

Thus, $\operatorname{im} j^* \simeq \mathbb{R}$. From the Mayer–Vietoris sequence in cohomology,

$$\begin{aligned} H^1(S^1) &= \operatorname{im} d^* \\ &\simeq \frac{\mathbb{R} \oplus \mathbb{R}}{\ker d^*} = \frac{\mathbb{R} \oplus \mathbb{R}}{\operatorname{im} j^*} \simeq \frac{\mathbb{R} \oplus \mathbb{R}}{\mathbb{R}} \simeq \mathbb{R}. \end{aligned}$$

Problems

3.1. Characterization of a Short Exact Sequence

Show that a sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

of vector spaces and linear maps is exact if and only if

(i) i is injective,
(ii) j is surjective, and

(iii) j induces an isomorphism $B/i(A) \simeq C$.

3.2. Exact Sequences

Prove that

(i) if $0 \to A \to 0$ is an exact sequence of vector spaces, then $A = 0$;

(ii) if $0 \to A \xrightarrow{f} B \to 0$ is an exact sequence of vector spaces, then $f\colon A \xrightarrow{\sim} B$ is a linear isomorphism.

3.3. Kernel and Cokernel of a Linear Map

The *cokernel* $\operatorname{coker} f$ of a linear map $f\colon B \to C$ is by definition the quotient space $C/\operatorname{im} f$. Prove that in an exact sequence

$$0 \to A \to B \overset{f}{\to} C \to D \to 0,$$

$A \simeq \ker f$ and $D \simeq \operatorname{coker} f$.

3.4. Exactness of the Mayer–Vietoris Sequence for Forms

Prove that the Mayer–Vietoris sequence for forms (3.3) is exact at $\mathcal{A}^*(M)$ and at $\mathcal{A}^*(U) \oplus \mathcal{A}^*(V)$.

4. Homotopy Invariance

The homotopy axiom is a powerful tool for computing de Rham cohomology. While homotopy is normally defined in the continuous category, since we are primarily interested in smooth manifolds and smooth maps, our notion of homotopy will be *smooth homotopy.* It differs from the usual homotopy in topology only in that all the maps are assumed to be smooth. In this section we define smooth homotopy, state the homotopy axiom for de Rham cohomology, and compute a few examples.

4.1. **Smooth Homotopy.** Let M and N be manifolds, and I the closed interval $[0,1]$. A map $F\colon M\times I\to N$ is said to be C^∞ if it is C^∞ on a neighborhood of $M\times I$ in $M\times\mathbb{R}$. Two C^∞ maps $f_0, f_1\colon M\to N$ are *(smoothly) homotopic*, written $f_0\sim f_1$, if there is a C^∞ map $F\colon M\times I\to N$ such that

$$F(x,0)=f_0(x) \quad\text{and}\quad F(x,1)=f_1(x)$$

for all $x\in M$; the map F is called a *homotopy* from f_0 to f_1. A homotopy F from f_0 to f_1 can be viewed as a smoothly varying family of maps $\{f_t\colon M\to N \mid t\in\mathbb{R}\}$. We can think of the parameter t as time and a homotopy as an evolution through time of the map $f_0\colon M\to N$.

Example. Straight-line homotopy. Let f and g be C^∞ maps from a manifold M to $\mathbb{R}^n$. Define $F\colon M\times\mathbb{R}\to\mathbb{R}^n$ by

$$\begin{aligned}F(x,t)&=f(x)+t(g(x)-f(x))\\&=(1-t)f(x)+tg(x).\end{aligned}$$

Then F is a homotopy from f to g, called the *straight-line homotopy* from f to g (Figure 4.1).

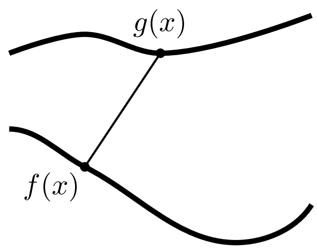

Figure 4.1. Straight-line homotopy.

4.2. **Homotopy Type.** As usual, $\mathbb{1}_M$ denotes the identity map on a manifold M.

Definition 4.1. A map $f\colon M\to N$ is a *homotopy equivalence* if it has a *homotopy inverse*, i.e., a map $g\colon N\to M$ such that $g\circ f$ is homotopic to the identity $\mathbb{1}_M$ on M and $f\circ g$ is homotopic to the identity $\mathbb{1}_N$ on N:

$$g\circ f\sim\mathbb{1}_M \quad\text{and}\quad f\circ g\sim\mathbb{1}_N.$$

In this case we say that M is *homotopy equivalent* to N, or that M and N have the same *homotopy type*.

Example. A diffeomorphism is a homotopy equivalence.

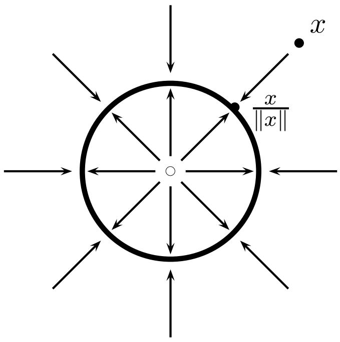

FIGURE 4.2. The punctured plane retracts to the unit circle.

Example 4.2. Homotopy type of the punctured plane. Let $i\colon S^1 \to \mathbb{R}^2 - \{\mathbf{0}\}$ be the inclusion map and let $r\colon \mathbb{R}^2 - \{\mathbf{0}\} \to S^1$ be the map

$$r(x) = \frac{x}{\|x\|}.$$

Then $r \circ i$ is the identity map on S^1.

We claim that

$$i \circ r\colon \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R}^2 - \{\mathbf{0}\}$$

is homotopic to the identity map. Indeed, the line segment from x to $x/\|x\|$ (Figure 4.2) allows us to define the straight-line homotopy

$$F\colon (\mathbb{R}^2 - \{0\}) \times [0,1] \to \mathbb{R}^2 - \{0\},$$
$$F(x,t) = (1-t)x + t\frac{x}{\|x\|}, \quad 0 \le t \le 1.$$

Then $F(x,0) = x = \mathbb{1}(x)$ and $F(x,1) = x/\|x\| = (i \circ r)(x)$. Therefore, $F\colon (\mathbb{R}^2 - \{\mathbf{0}\}) \times \mathbb{R} \to \mathbb{R}^2 - \{\mathbf{0}\}$ provides a homotopy between the identity map on $\mathbb{R}^2 - \{\mathbf{0}\}$ and $i \circ r$ (Figure 4.2). It follows that r and i are homotopy inverse to each other, and $\mathbb{R}^2 - \{\mathbf{0}\}$ and S^1 have the same homotopy type.

Definition 4.3. A manifold is *contractible* if it has the homotopy type of a point.

In this definition, by "the homotopy type of a point" we mean the homotopy type of a set $\{p\}$ whose single element is a point. Such a set is called a *singleton set* or just a *singleton*.

Example 4.4. The Euclidean space $\mathbb{R}^n$ is contractible. Let p be a point in $\mathbb{R}^n$, $i\colon \{p\} \to \mathbb{R}^n$ the inclusion map, and $r\colon \mathbb{R}^n \to \{p\}$ the constant map. Then $r \circ i = \mathbb{1}_{\{p\}}$, the identity map on $\{p\}$. The straight-line homotopy provides a homotopy between the constant map $i \circ r\colon \mathbb{R}^n \to \mathbb{R}^n$ and the identity map on $\mathbb{R}^n$:

$$F(x,t) = (1-t)x + t\,r(x) = (1-t)x + tp.$$

Hence, the Euclidean space $\mathbb{R}^n$ and the set $\{p\}$ have the same homotopy type.

4.3. **Deformation Retractions.** Let S be a submanifold of a manifold M, with $i\colon S \to M$ the inclusion map.

Definition 4.5. A *retraction* from M to S is a map $r\colon M \to S$ that restricts to the identity map on S; in other words, $r \circ i = \mathbb{1}_S$. If there is a retraction from M to S, we say that S is a *retract* of M.

Definition 4.6. A *deformation retraction* from M to S is a map $F\colon M \times \mathbb{R} \to M$ such that for all $x \in M$,

(i) $F(x, 0) = x$,
(ii) there is a retraction $r\colon M \to S$ such that $F(x, 1) = r(x)$,
(iii) for all $s \in S$ and $t \in \mathbb{R}$, $F(s, t) = s$.

If there is a deformation retraction from M to S, we say that S is a *deformation retract* of M.

Setting $f_t(x) = F(x, t)$, we can think of a deformation retraction $F\colon M \times \mathbb{R} \to M$ as a family of maps $f_t\colon M \to M$ such that

(i) f_0 is the identity map on M,
(ii) $f_1(x) = r(x)$ for some retraction $r\colon M \to S$,
(iii) for every t the map $f_t\colon M \to M$ restricts to the identity on S.

We may rephrase Condition (ii) in the definition as follows: there is a retraction $r\colon M \to S$ such that $f_1 = i \circ r$. Thus, a deformation retraction is a homotopy between the identity map $\mathbb{1}_M$ and $i \circ r$ for a retraction $r\colon M \to S$ such that this homotopy leaves S fixed for all time t.

Example. Any point p in a manifold M is a retract of M; simply take a retraction to be the constant map $r\colon M \to \{p\}$.

Example. The map F in Example 4.2 is a deformation retraction from the punctured plane $\mathbb{R}^2 - \{0\}$ to the unit circle S^1. The map F in Example 4.4 is a deformation retraction from $\mathbb{R}^n$ to a singleton $\{p\}$.

Generalizing Example 4.2, we prove the following theorem.

Proposition 4.7. *If $S \subset M$ is a deformation retract of M, then S and M have the same homotopy type.*

Proof. Let $F\colon M \times \mathbb{R} \to M$ be a deformation retraction and let $r(x) = f_1(x) = F(x, 1)$ be the retraction. Because r is a retraction, the composite

$$S \xrightarrow{i} M \xrightarrow{r} S, \quad r \circ i = \mathbb{1}_S,$$

is the identity map on S. By the definition of a deformation retraction, the composite

$$M \xrightarrow{r} S \xrightarrow{i} M$$

is f_1 and the deformation retraction provides a homotopy

$$f_1 = i \circ r \sim f_0 = \mathbb{1}_M.$$

Therefore, $r\colon M \to S$ is a homotopy equivalence, with homotopy inverse $i\colon S \to M$. □

4.4. **The Homotopy Axiom for de Rham Cohomology.** We state here the homotopy axiom and derive a few consequences. For a proof, see [3, Section 28, p. 273].

Theorem 4.8 (Homotopy axiom for de Rham cohomology)**.** *Homotopic maps* $f_0, f_1 \colon M \to N$ *induce the same map* $f_0^* = f_1^* \colon H^*(N) \to H^*(M)$ *in cohomology.*

Corollary 4.9. *If* $f \colon M \to N$ *is a homotopy equivalence, then the induced map in cohomology*

$$f^* \colon H^*(N) \to H^*(M)$$

is an isomorphism.

Proof (of Corollary). Let $g \colon N \to M$ be a homotopy inverse to f. Then

$$g \circ f \sim \mathbb{1}_M, \qquad f \circ g \sim \mathbb{1}_N.$$

By the homotopy axiom,

$$(g \circ f)^* = \mathbb{1}_{H^*(M)}, \qquad (f \circ g)^* = \mathbb{1}_{H^*(N)}.$$

By functoriality,

$$f^* \circ g^* = \mathbb{1}_{H^*(M)}, \qquad g^* \circ f^* = \mathbb{1}_{H^*(N)}.$$

Therefore, f^* is an isomorphism in cohomology. □

Corollary 4.10. *Suppose* S *is a submanifold of a manifold* M *and* F *is a deformation retraction from* M *to* S*. Let* $r \colon M \to S$ *be the retraction* $r(x) = F(x, 1)$*. Then* r *induces an isomorphism in cohomology*

$$r^* \colon H^*(S) \xrightarrow{\sim} H^*(M).$$

Proof. The proof of Proposition 4.7 shows that a retraction $r \colon M \to S$ is a homotopy equivalence. Apply Corollary 4.9. □

Corollary 4.11 (Poincaré lemma)**.** *Since* $\mathbb{R}^n$ *has the homotopy type of a point, the cohomology of* $\mathbb{R}^n$ *is*

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

More generally, any contractible manifold will have the same cohomology as a point. As a consequence, on a contractible manifold a closed form of positive degree is necessary exact.

Example. Cohomology of a punctured plane. For any $p \in \mathbb{R}^2$, the translation $x \mapsto x - p$ is a diffeomorphism of $\mathbb{R}^2 - \{p\}$ with $\mathbb{R}^2 - \{0\}$. Because the punctured plane $\mathbb{R}^2 - \{0\}$ and the circle S^1 have the same homotopy type (Example 4.2), they have isomorphic cohomology. Hence, $H^k(\mathbb{R}^2 - \{p\}) \simeq H^k(S^1)$ for all $k \geq 0$.

Example. The central circle of an open Möbius band M is a deformation retract of M (Figure 4.3). Thus, the open Möbius band has the homotopy type of a circle. By the homotopy axiom,

$$H^k(M) = H^k(S^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1, \\ 0 & \text{for } k > 1. \end{cases}$$

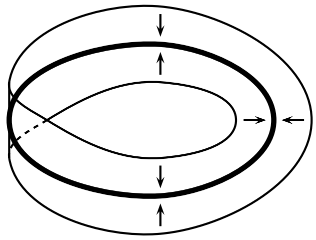

FIGURE 4.3. The Möbius band deformation retracts to its central circle.

4.5. **Computation of de Rham Cohomology.** In this subsection, we discuss the de Rham cohomology of three examples—the sphere, the punctured Euclidean space, and the complex projective space.

Example 4.12. The sphere. Let A be an open band about the equator in the sphere S^n. Let U be the union of the upper hemisphere and A, and V be the union of the lower hemisphere and A. Then $S^n = U \cup V$ and $A = U \cap V$. Using the Mayer–Vietoris sequence and induction, one can compute the de Rham cohomology of S^n to be

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

We leave the details as an exercise (Problem 4.1).

Example 4.13. Punctured Euclidean space. The unit sphere S^{n-1} in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n - \{0\}$ via the deformation retraction

$$F\colon (\mathbb{R}^n - \{0\}) \times I \to \mathbb{R}^n - \{0\}, \qquad F(x,t) = (1-t)x + t\frac{x}{\|x\|}.$$

Therefore, $H^*(\mathbb{R}^n - \{0\}) \simeq H^*(S^{n-1})$.

In the definition of a smooth manifold, if $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$, and smooth maps by holomorphic maps, then the resulting object is called a *complex manifold*. Since homomorphic maps are smooth and $\mathbb{C}^n$ is isomorphic to $\mathbb{R}^{2n}$ as a vector space, a complex manifold of complex dimension n is a smooth manifold of real dimension $2n$. An important example of complex manifold is the complex projective space $\mathbb{C}P^n$, defined in the same way as the real projective space, but with $\mathbb{C}$ instead of $\mathbb{R}$. As a set, $\mathbb{C}P^n$ is the set of all 1-dimensional complex subspaces of the complex vector space $\mathbb{C}^{n+1}$.

Example 4.14. Cohomology of the complex projective line. We will use the Mayer–Vietoris sequence to compute the cohomology of $\mathbb{C}P^1$. The standard atlas $\{U_0, U_1\}$ on $\mathbb{C}P^1$ consists of two open sets $U_i \simeq \mathbb{C} \simeq \mathbb{R}^2$, and their intersection is

$$\begin{aligned} U_0 \cap U_1 &= \{[z^0, z^1] \in \mathbb{C}P^1 \mid z^0 \neq 0 \text{ and } z^1 \neq 0\} \\ &= \{[w, 1] = [z^0/z^1, 1] \in \mathbb{C}P^1 \mid w \neq 0\} \simeq \mathbb{C}^\times, \end{aligned}$$

the set of nonzero complex numbers. Therefore, $U_0 \cap U_1$ has the homotopy type of a circle and the Mayer–Vietoris sequence gives

			$\mathbb{C}P^1$		$U_0 \amalg U_1$		$U_0 \cap U_1$
H^2		$\to$	$H^2(\mathbb{C}P^1)$	$\to$	0	$\to$	0
H^1		$\xrightarrow{d^*}$	$H^1(\mathbb{C}P^1)$	$\to$	0	$\to$	$\mathbb{R}$
H^0	0	$\to$	$\mathbb{R}$	$\xrightarrow{i^*}$	$\mathbb{R} \oplus \mathbb{R}$	$\xrightarrow{j^*}$	$\mathbb{R}$

In the bottom row, elements of H^0 are represented by locally constant functions, i^* is the restriction, and j^* is the difference of restrictions. Thus,

$$i^*(a) = (a, a) \quad \text{and} \quad j^*(u, v) = v - u.$$

It is then clear that j^* is surjective. Since $\ker d^* = \operatorname{im} j^* = \mathbb{R}$, d^* is the zero map. So the H^1 row is

$$0 \to H^1(\mathbb{C}P^1) \to 0 \to \mathbb{R}.$$

Since $H^1(\mathbb{C}P^1)$ is trapped between two zeros, $H^1(\mathbb{C}P^1) = 0$.

From the H^1 and H^2 rows, we get the exact sequence

$$0 \to \mathbb{R} \to H^2(\mathbb{C}P^1) \to 0.$$

By Problem 3.2, $H^2(\mathbb{C}P^1) = \mathbb{R}$. In summary,

$$H^k(\mathbb{C}P^1) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

The same calculation as in the preceding example proves the following proposition.

Proposition 4.15. *In the Mayer–Vietoris sequence, if U, V, and $U \cap V$ are connected and nonempty, then*

(i) *M is connected and*

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0$$

is exact;

(ii) *we may start the Mayer–Vietoris sequence with*

$$0 \to H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{j^*} H^1(U \cap V) \to \cdots.$$

Example 4.16. Cohomology of the complex projective plane. We will again use the Mayer–Vietoris sequence to compute the cohomology of $\mathbb{C}P^2$. As an open cover of $\mathbb{C}P^2$, we take U to be the chart $\{[z^0, z^1, z^2] \in \mathbb{C}P^2 \mid z^2 \neq 0\}$ and V to be the punctured projective plane $\mathbb{C}P^2 - \{[0, 0, 1]\}$. Note that U is diffeomorphic to $\mathbb{C}^2$ via

$$[z^0, z^1, z^2] \mapsto \left(\frac{z^0}{z^2}, \frac{z^1}{z^2} \right),$$
$$[w^0, w^1, 1] \leftarrow\!\shortmid (w^0, w^1).$$

Let $L = \{[z^0, z^1, 0] \in \mathbb{C}P^2\}$. Then L is diffeomorphic to $\mathbb{C}P^1$ and is called a *line at infinity* of $\mathbb{C}P^2$. It is easy to verify that the map $F\colon V \times [0,1] \to V$,

$$F([z^0, z^1, z^2], t) = [z^0, z^1, (1-t)z^2]$$

is a deformation retraction from V to L. By the homotopy axiom (Corollary 4.10), V has the same cohomology as $\mathbb{C}P^1$.

Since the intersection $U \cap V$ is a punctured $\mathbb{C}^2$, it has the homotopy type of S^3 (Example 4.13). By Proposition 4.15, the Mayer–Vietoris sequence for the open cover $\{U, V\}$ then gives

			$\mathbb{C}P^2$		$U \amalg V$ $\sim \mathbb{C}^2 \amalg \mathbb{C}P^1$		$U \cap V$ $\sim S^3$
H^4		$\to$	$H^4(\mathbb{C}P^2)$	$\to$	0	$\to$	0
H^3		$\to$	$H^3(\mathbb{C}P^2)$	$\to$	0	$\to$	$\mathbb{R}$
H^2		$\to$	$H^2(\mathbb{C}P^2)$	$\to$	$\mathbb{R}$	$\to$	0
H^1	0	$\xrightarrow{d^*}$	$H^1(\mathbb{C}P^2)$	$\to$	0	$\to$	0

Thus,

$$H^k(\mathbb{C}P^2) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2, 4, \\ 0 & \text{otherwise.} \end{cases}$$

By induction on n, this same method computes the cohomology of $\mathbb{C}P^n$ to be

$$H^k(\mathbb{C}P^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2, \dots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Problems

4.1. Cohomology of an n-Sphere

Following the indications in Example 4.12, compute the de Rham cohomology of S^n.

4.2. Cohomology of $\mathbb{C}P^n$

As in Example 4.16, calculate the cohomology of $\mathbb{C}P^n$.

5. Presheaves and Čech Cohomology

5.1. **Presheaves.** The functor $\mathcal{A}^*(\ )$ that assigns to every open set U on a manifold the vector space of C^∞ forms on U is an example of a *presheaf*. By definition a *presheaf* $\mathcal{F}$ on a topological space X is a function that assigns to every open set U in X an abelian group $\mathcal{F}(U)$ and to every inclusion of open sets $i_U^V\colon V \to U$ a group homomorphism, called the *restriction* from U to V,

$$\mathcal{F}(i_U^V) := \rho_V^U\colon \mathcal{F}(U) \to \mathcal{F}(V),$$

satisfying the following properties:

(i) (identity) $\rho_U^U =$ identity map on $\mathcal{F}(U)$;

(ii) (transitivity) if $W \subset V \subset U$, then $\rho_W^V \circ \rho_V^U = \rho_W^U$.

We refer to elements of $\mathcal{F}(U)$ as *sections* of $\mathcal{F}$ over U.

If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on X, a *morphism* $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves is a collection of group homomorphisms $f_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set U in X, that commute with the restrictions:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{F}(U) \\ {\scriptstyle \rho_V^U}\downarrow & & \downarrow{\scriptstyle \rho_V^U} \\ \mathcal{F}(V) & \xrightarrow[f_V]{} & \mathcal{G}(V). \end{array} \tag{5.1}$$

If we write $\omega|_U$ for $\rho_V^U(\omega)$, then the diagram (5.1) is equivalent to $f_V(\omega|_V) = f_U(\omega)|_V$ for all $\omega \in \mathcal{F}(U)$.

For any topological space X, let $\mathrm{Open}(X)$ be the category whose objects are open subsets of X and for any two open subsets U, V of X,

$$\mathrm{Mor}(V,U) = \begin{cases} \{\text{inclusion } i_U^V\colon V \to U\} & \text{if } V \subset U, \\ \varnothing & \text{otherwise.} \end{cases}$$

In functorial language, a presheaf is simply a contravariant functor from the category $\mathrm{Open}(X)$ to the category of abelian groups, and a homomorphism of presheaves is a *natural transformation* from the functor $\mathcal{F}$ to the functor $\mathcal{G}$. What we have defined are presheaves of abelian groups; it is possible to define similarly presheaves of vector spaces, algebras, and indeed objects in any category.

If G is an abelian group, we define the *presheaf of locally constant G-valued functions on X* to be the presheaf $\underline{G}$ that associates to every open set U in X the group

$$\underline{G}(U) = \{\text{locally constant functions } f\colon U \to G\}$$

and to every inclusion of open sets $V \subset U$, the restriction $\rho_V^U\colon \underline{G}(U) \to \underline{G}(V)$ of locally constant functions.

5.2. **Čech Cohomology of an Open Cover.** Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathrm{A}}$ be an open cover of the topological space X indexed by an ordered set A, and $\mathcal{F}$ a presheaf of abelian groups on X. To simplify the notation, we will write the $(p+1)$-fold intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ as $U_{\alpha_0 \ldots \alpha_p}$. Define the group

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0 \ldots \alpha_p}).$$

An element ω of $C^p(\mathfrak{U}, \mathcal{F})$ is called a *p-cochain on $\mathfrak{U}$ with values in the presheaf $\mathcal{F}$*; it is a function that assigns to each $(p+1)$-fold intersection $U_{\alpha_0 \ldots \alpha_p}$ an element $\omega_{\alpha_0 \ldots \alpha_p} \in \mathcal{F}(U_{\alpha_0 \ldots \alpha_p})$. We will write $\omega = (\omega_{\alpha_0 \ldots \alpha_p})$, where the subscript ranges over all $\alpha_0 < \cdots < \alpha_p$. Define the *Čech coboundary operator*

$$\delta = \delta_p \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$$

to be the alternating sum

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+1}},$$

where on the right-hand side the restriction of $\omega_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+1}}$ from $U_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+1}}$ to $U_{\alpha_0 \ldots \alpha_{p+1}}$ is suppressed.

Proposition 5.1. *If δ is the Čech coboundary operator, then $\delta^2 = 0$.*

Proof. Basically this is true because in $(\delta^2\omega)_{\alpha_0 \ldots \alpha_{p+2}}$, we omit two indices α_i, α_j twice with opposite signs. To be precise,

$$\begin{aligned}(\delta^2\omega)_{\alpha_0 \ldots \alpha_{p+2}} &= \sum (-1)^i (\delta\omega)_{\alpha_0 \ldots \hat{\alpha_i} \ldots \alpha_{p+2}} \\ &= \sum_{j<i} (-1)^i (-1)^j \omega_{\alpha_0 \ldots \hat{\alpha_j} \ldots \hat{\alpha_i} \ldots \alpha_{p+2}} \\ &\quad + \sum_{j>i} (-1)^i (-1)^{j-1} \omega_{\alpha_0 \ldots \hat{\alpha_i} \ldots \hat{\alpha_j} \ldots \alpha_{p+2}} \\ &= 0.\end{aligned}$$

□

Convention. Up until now the indices in $\omega_{\alpha_0 \ldots \alpha_p}$ are all in increasing order $\alpha_0 < \cdots \alpha_p$. More generally, we will allow indices in any order, even with repetitions, subject to the convention that when two indices are interchanged, the Čech component becomes its negative:

$$\omega_{\ldots\alpha\ldots\beta\ldots} = -\omega_{\ldots\beta\ldots\alpha\ldots}.$$

In particular, a component $\omega_{\ldots\alpha\ldots\alpha\ldots}$ with repeated indices is 0.

It follows from Proposition 5.1 that $C^*(\mathfrak{U}, \mathcal{F}) := \bigoplus_{p=0}^{\infty} C^p(\mathfrak{U}, \mathcal{F})$ is a cochain complex with differential δ. In fact, one can extend p to all integers by setting $C^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p < 0$. The cohomology of the complex $(C^*(\mathfrak{U}, \mathcal{F}), \delta)$,

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = \frac{\ker \delta_p}{\operatorname{im} \delta_{p-1}} = \frac{\{p\text{-cocylces}\}}{\{p\text{-coboundaries}\}},$$

is called the *Čech cohomology* of the open cover $\mathfrak{U}$ with values in the presheaf $\mathcal{F}$.

5.3. **The Direct Limit.** To define the Čech cohomology groups of a topological space, we introduce in this section an algebraic construction called the *direct limit* of a direct system of abelian groups.

A *directed set* is a set I with a binary relation $<$ satisfying

(i) (reflexivity) $a < a$ for all $a \in I$;
(ii) (transitivity) if $a < b$ and $b < c$, then $a < c$;
(iii) (upper bound) for an $a, b \in I$, there is an element $c \in I$, called an *upper bound* such that $a < c$ and $b < c$.

We often write $b > a$ if $a < b$.

On a topological space X, an open cover $\mathfrak{V} = \{V_\beta\}_{\beta\in \mathrm{B}}$ *refines* an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in \mathrm{A}}$ if every V_β is a subset of some U_α. If $\mathfrak{V}$ refines $\mathfrak{U}$, we also say that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ or that $\mathfrak{U}$ is *refined* by $\mathfrak{V}$. Note that $\mathfrak{V}$ refines $\mathfrak{U}$ if and only if there is a map $\phi\colon \mathrm{B} \to \mathrm{A}$ (in general not unique), called a *refinement map*, such that for every $\beta \in \mathrm{B}$, $V_\beta \subset U_{\phi(\beta)}$. We write $\mathfrak{U} \prec \mathfrak{V}$ to mean "$\mathfrak{U}$ is refined by $\mathfrak{V}$."

Example. Let V be a proper open set in a topological space X. The two open covers $\mathfrak{U} = \{X\}$ and $\mathfrak{V} = \{X, V\}$ refine each other, but $\mathfrak{U} \neq \mathfrak{V}$.

This example shows that the relation of refinement $\prec$ is not antisymmetric, so it is not a partial order. However, it is clearly reflexive and transitive. Any two open covers $\mathfrak{U} = \{U_\alpha\}_{\alpha\in \mathrm{A}}$ and $\mathfrak{V} = \{V_\beta\}_{\beta\in \mathrm{B}}$ of a topological space X have a common refinement $\{U_\alpha \cap V_\beta\}_{(\alpha,\beta)\in \mathrm{A}\times\mathrm{B}}$. Thus, the refinement relation $\prec$ makes the set of all open covers of X into a directed set.

A *direct system of groups* is a collection of groups $\{G_i\}_{i\in I}$ indexed by a directed set I such that for any pair $a < b$ in I there is a group homomorphism $f^a_b\colon G_a \to G_b$ satisfying for all $a, b, c \in I$,

(i) $f^a_a =$ identity;
(ii) $f^a_c = f^b_c \circ f^a_b$ if $a < b < c$.

On the disjoint union $\amalg_i G_i$ we introduce an equivalence relation $\sim$ by decreeing two elements g_a in G_a and g_b in G_b to be equivalent if for some upper bound c of a and b, we have $f^a_c(g_a) = f^b_c(g_b)$ in G_c. The *direct limit* of the direct system, denoted by $\varinjlim_{i\in I} G_i$, is the quotient of the disjoint union $\amalg_i G_i$ by the equivalence relation $\sim$; in other words, two elements of $\amalg_i G_i$ represent the same element in the direct limit if they are "eventually equal." We make the direct limit $\varinjlim G_i$ into a group by defining $[g_a] + [g_b] = [f^a_c(g_a) + f^b_c(g_b)]$, where c is an upper bound of a and b and $[g_a]$ is the equivalence class of g_a. It is easy to check that the direct limit $\varinjlim G_i$ is indeed a group; moreover, if all the groups G_i are abelian, so is their direct limit. Instead of groups, one can obviously also consider direct systems of modules, rings, algebras, and so on.

Example. Fix a point p in a manifold M and let I be the directed set consisting of all neighborhoods of p in M, with $<$ being reverse inclusion: $U < V$ if and only if $V \subset U$. Let $C^\infty(U)$ be the ring of C^∞ functions on U. Then $\{C^\infty(U)\}_{U\in I}$ is a direct system of rings and its direct limit $\varinjlim_{p\in U} C^\infty(U)$ is precisely the ring of germs of C^∞ functions at p.

Example. Stalks of a presheaf. If $\mathcal{F}$ is a presheaf of abelian groups on a topological space X and p is a point in X, then $\{\mathcal{F}(U)\}_{p\in U}$ is a direct system of abelian groups. The direct $\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U)$ is called the *stalk* of $\mathcal{F}$ at p. An element of the stalk $\mathcal{F}_p$ is a germ of sections at p.

A morphism of presheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ induces a morphism of stalks $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ by sending the germ of a section $s \in \mathcal{F}(U)$ to the germ of the section $\varphi(s) \in \mathcal{G}(U)$.

5.4. **Čech Cohomology of a Topological Space.** Let $\mathcal{F}$ be a presheaf on the topological space X. Suppose the open cover $\mathfrak{V} = \{V_\beta\}_{\beta\in \mathrm{B}}$ of X is a refinement of the open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathrm{A}}$ with refinement map $\phi \colon \mathrm{B} \to \mathrm{A}$. Then there is an induced group homomorphism

$$\begin{aligned} &\phi^\# \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{V}, \mathcal{F}), \\ &(\phi^\# \omega)_{\beta_0 \dots \beta_p} = \omega_{\phi(\beta_0)\dots\phi(\beta_p)}|_{V_{\beta_1 \dots \beta_p}} \text{ for } \omega \in C^p(\mathfrak{U}, \mathcal{F}). \end{aligned}$$

On the right-hand side, we usually omit the restriction.

Lemma 5.2. *The induced group homomorphism $\phi^\#$ is a cochain map, i.e., it commutes with the coboundary operator δ.*

Proof. For $\omega \in C^p(\mathfrak{U}, \mathcal{F})$,

$$\begin{aligned} (\delta \phi^\# \omega)_{\beta_0 \dots \beta_{p+1}} &= \sum (-1)^i (\phi^\# \omega)_{\beta_0 \dots \widehat{\beta_i} \dots \beta_{p+1}} \\ &= \sum (-1)^i \omega_{\phi(\beta_0) \dots \widehat{\phi(\beta_i)} \dots \phi(\beta_{p+1})}. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\phi^\# \delta \omega)_{\beta_0 \dots \beta_{p+1}} &= (\delta \omega)_{\phi(\beta_0) \dots \phi(\beta_{p+1})} \\ &= (-1)^i \omega_{\phi(\beta_0) \dots \widehat{\phi(\beta_i)} \dots \phi(\beta_{p+1})}. \end{aligned}$$

□

A standard method for showing that two cochain maps $f, g \colon (\mathcal{A}, d) \to (\mathcal{B}, d)$ induce the same map in cohomology is to find a linear map $K \colon \mathcal{A}^k \to \mathcal{B}^{k-1}$ of degree -1 such that

$$f - g = d \circ K + K \circ d,$$

for on the right-hand side $d \circ K + K \circ d$ maps cocycles to coboundaries and induces the zero map in cohomology. Such a map K is called a *cochain homotopy* between f and g, and f and g are said to be *cochain homotopic.*

Suppose $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathrm{A}}$ is an open cover of the topological space X and $\mathfrak{V} = \{V_\beta\}_{\beta \in \mathrm{B}}$ is a refinement of $\mathfrak{U}$, with two refinement maps ϕ and $\psi \colon \mathrm{B} \to \mathrm{A}$. The following lemma shows that the induced cochain maps $\phi^\#$ and $\psi^\# \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{V}, \mathcal{F})$ are cochain homotopic.

Lemma 5.3. *Define $K \colon C^p(\mathfrak{U}, \mathcal{F}) \to C^{p-1}(\mathfrak{V}, \mathcal{F})$ by*

$$(K\omega)_{\beta_0 \dots \beta_{p-1}} = \sum (-1)^i \omega_{\phi(\alpha_0) \dots \phi(\beta_i) \psi(\beta_i) \dots \psi(\beta_{p-1})}.$$

Then

$$\psi^\# - \phi^\# = \delta K + K \delta.$$

Proof. The proof is a straightforward but long and delicate verification of the definitions. We leave it as an exercise. □

It follows that $\phi^\#$ and $\psi^\#$ induce the same homomorphism in cohomology

$$(\phi^\#)^* = (\psi^\#)^* \colon \check{H}^*(\mathfrak{U}, \mathcal{F}) \to \check{H}^*(\mathfrak{V}, \mathcal{F}).$$

Thus, if $\mathfrak{U} < \mathfrak{V}$, then any refinement map for $\mathfrak{V}$ as a refinement of $\mathfrak{U}$ induces a group homomorphism in cohomology

$$\rho^{\mathfrak{U}}_{\mathfrak{V}} = (\phi^\#)^* \colon \check{H}^*(\mathfrak{U}, \mathcal{F}) \to \check{H}^*(\mathfrak{V}, \mathcal{F}),$$

which is independent of the refinement map. This makes the collection $\{\check{H}^*(\mathfrak{U}, \mathcal{F})\}_{\mathfrak{U}}$ of cohomology groups into a direct system of groups indexed by the directed set of all open covers of X. The direct limit of this direct system

$$\check{H}^*(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^*(\mathfrak{U}, \mathcal{F})$$

is the *Čech cohomology of the topological space X with values in the presheaf $\mathcal{F}$*.

5.5. **Cohomology with Coefficients in the Presheaf of C^∞ q-Forms.** To show the vanishing of the Čech cohomology with coefficients in the presheaf $\mathcal{A}^q$ of C^∞ q-forms, we will find a cochain homotopy K between the identity map $\mathbb{1}\colon C^*(\mathfrak{U}, \mathcal{A}^q) \to C^*(\mathfrak{U}, \mathcal{A}^q)$ and the zero map.

Proposition 5.4. *Let $\mathcal{A}^q$ be the presheaf of C^∞ q-forms on a manifold M. Then the Čech cohomology $\check{H}^k(M, \mathcal{A}^q) = 0$ for all $k > 0$.*

Proof. Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of M and let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. For $k \geq 1$, define $K\colon C^k(\mathfrak{U}, \mathcal{A}^q) \to C^{k-1}(\mathfrak{U}, \mathcal{A}^q)$ by

$$(K\omega)_{\alpha_0 \ldots \alpha_{k-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0 \ldots \alpha_{k-1}}.$$

Then

$$\begin{aligned}(\delta K\omega)_{\alpha_0 \ldots \alpha_k} &= \sum_{i=0}^{k} (-1)^i (K\omega)_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_k} \\ &= \sum_{i=0}^{k} \sum_\alpha (-1)^i \rho_\alpha \omega_{\alpha\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_k}\end{aligned}$$

and

$$\begin{aligned}(K\delta\omega)_{\alpha_0 \ldots \alpha_k} &= \sum_\alpha \rho_\alpha (\delta\omega)_{\alpha\alpha_0 \ldots \alpha_k} \\ &= \sum_\alpha \rho_\alpha \omega_{\alpha_0 \ldots \alpha_k} + \sum_\alpha \sum_{i=0}^{k} (-1)^{i+1} \rho_\alpha \omega_{\alpha\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_k}.\end{aligned}$$

Hence,

$$((\delta K + K\delta)\omega)_{\alpha_0 \ldots \alpha_k} = \left(\sum_\alpha \rho_\alpha\right) \omega_{\alpha_0 \ldots \alpha_k} = \omega_{\alpha_0 \ldots \alpha_k}.$$

So for $k \geq 1$,

$$\delta \circ K + K \circ \delta = \mathbb{1}\colon C^k(\mathfrak{U}, \mathcal{A}^q) \to C^k(\mathfrak{U}, \mathcal{A}^q). \tag{5.2}$$

By the discussion preceding this proposition, $H^k(\mathfrak{U}, \mathcal{A}^q) = 0$ for $k \geq 1$. Since this is true for all open covers $\mathfrak{U}$ of the manifold M, $H^k(M, \mathcal{A}^q) = 0$ for $k \geq 1$. □

When $k = 0$, the equality (5.2) does not hold. Indeed,

$$\begin{aligned}\check{H}^0(M, \mathcal{A}^q) &= \ker \delta\colon \prod_i \mathcal{A}^q(U_i) \to \prod_{i,j} \mathcal{A}^q(U_{ij}) \\ &= \{C^\infty \text{ } q\text{-forms on } M\}.\end{aligned}$$

Problems

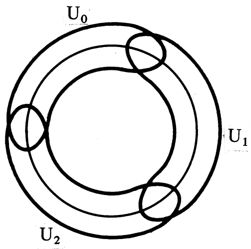

FIGURE 5.1. An open cover of the circle.

5.1. Čech Cohomology of an Open Cover
Let $\mathfrak{U} = \{U_0, U_1, U_2\}$ be the good cover of the circle in Figure 5.1. Suppose $\mathcal{F}$ is a presheaf on S^1 that associates to every nonempty intersection of $\mathfrak{U}$ the group $\mathbb{Z}$, with restriction homomorphisms:

$$\begin{aligned} \rho_{01}^0 &= \rho_{01}^1 = 1, \\ \rho_{12}^1 &= \rho_{12}^2 = 1, \\ \rho_{02}^2 &= 1, \quad \rho_{02}^2 = 1, \end{aligned}$$

where ρ_{ij}^i the the restriction from U_i to U_{ij}. Compute $\check{H}^*(\mathfrak{U}, \mathcal{F})$. (*Hint*: The answer is not $H^0 = 0$ and $H^1 = 0$.)

6. Sheaves and the Čech–de Rham isomorphism

In this section we introduce the concept of a sheaf and use an acyclic resolution of the constant sheaf $\underline{\mathbb{R}}$ to prove an isomorphism between Čech cohomology with coefficients in the constant sheaf $\underline{\mathbb{R}}$ and de Rham cohomology.

6.1. **Sheaves.** The stalk of a presheaf at a point embodies in it the local character of the presheaf about the point. However, in general, there is no relation between the global sections and the stalks of a presheaf. For example, if G is an abelian group and $\mathcal{F}$ is the presheaf on a topological space X defined by $\mathcal{F}(X) = G$ and $\mathcal{F}(U) = 0$ for all $U \neq X$, then all the stalks $\mathcal{F}_p$ vanish but $\mathcal{F}$ is not the zero presheaf.

A *sheaf* is a presheaf with two additional properties, which link the global and local sections of the presheaf. In practice, most of the presheaves we encounter are sheaves.

Definition 6.1. A *sheaf* $\mathcal{F}$ of abelian groups on a topological space X is a presheaf satisfying two additional conditions for any open set $U \subset X$ and any open cover $\{U_i\}$ of U:

(i) (uniqueness) if $s \in \mathcal{F}(U)$ is a section such that $s|_{U_i} = 0$ for all i, then $s = 0$ on U;

(ii) (patching-up) if $\{s_i \in \mathcal{F}(U_i)\}$ is a collection of sections such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Consider the sequence of maps

$$0 \to \mathcal{F}(U) \xrightarrow{r} \prod_i \mathcal{F}(U_i) \xrightarrow{\delta} \prod_{i,j} \mathcal{F}(U_i \cap U_j), \tag{6.1}$$

where r is the restriction $r(\omega) = (\omega|_{U_i})$ and δ is the Čech coboundary operator

$$(\delta\omega)_{ij} = \omega_j - \omega_i.$$

Then the two sheaf conditions (i) and (ii) are equivalent to the exactness of the sequence (6.1), i.e., the map r is injective and $\ker \delta = \operatorname{im} r$.

Example. For any open subset U of a topological space X, let $\mathcal{F}(U)$ be the abelian group of constant real-valued functions on U. If $V \subset U$, let $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ be the restriction of functions. Then $\mathcal{F}$ is a presheaf on X. The presheaf $\mathcal{F}$ satisfies the uniqueness condition but not the patching-up condition of a sheaf: if U_1 and U_2 are disjoint open sets in X, and $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ have different values, then there is no constant function s on $U_1 \cup U_2$ that restricts to s_1 on U_1 and to s_2 on U_2.

Example. Let $\underline{\mathbb{R}}$ be the presheaf on a topological space X that associates to every open set $U \subset X$ the abelian group $\underline{\mathbb{R}}(U)$ consisting of all locally constant functions on U. Then $\underline{\mathbb{R}}$ is a presheaf that is also a sheaf.

Example. The presheaf $\mathcal{A}^k$ on a manifold that assigns to each open set U the abelian group of C^∞ k-forms on U is a sheaf.

Example. The presheaf $\mathcal{Z}^k$ on a manifold that associates to each open set U the abelian group of closed C^∞ k-forms on U is a sheaf.

6.2. **Čech Cohomology in Degree Zero.** The two defining properties of a sheaf $\mathcal{F}$ on a space X allow us to identify the zeroth cohomology $\check{H}^0(X, \mathcal{F})$ with its space of global sections.

Proposition 6.2. *If $\mathcal{F}$ is a sheaf on a topological space X, then $\check{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathrm{A}}$ be an open cover of X. In terms of Čech cochain groups, the sequence (6.1) assumes the form

$$0 \to \mathcal{F}(X) \stackrel{r}{\to} C^0(\mathfrak{U}, \mathcal{F}) \stackrel{\delta}{\to} C^1(\mathfrak{U}, \mathcal{F}).$$

By the exactness of this sequence,

$$\check{H}^0(\mathfrak{U}, \mathcal{F}) = \ker \delta = \operatorname{im} r \simeq \mathcal{F}(X).$$

If $\mathfrak{V}$ is an open cover of X that refines $\mathfrak{U}$, then there is a commutative diagram

$$\begin{array}{ccc} \check{H}^0(\mathfrak{U}, \mathcal{F}) & \xrightarrow{\sim} & \mathcal{F}(X) \\ {\scriptstyle \rho^U_V}\downarrow & & \| \\ \check{H}^0(\mathfrak{V}, \mathcal{F}) & \xrightarrow{\sim} & \mathcal{F}(X). \end{array}$$

Taking the direct limit, we obtain $\check{H}^0(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, \mathcal{F}) \simeq \mathcal{F}(X)$. □

6.3. **Sheaf Associated to a Presheaf.** Let $\mathcal{F}$ be a presheaf on a topological space X. For any open set $U \subset X$, we call a function $s \colon U \to \coprod_{p \in U} \mathcal{F}_p$ for which $s(p) \in \mathcal{F}_p$ for all $p \in U$ a *section* of $\coprod_{p \in U} \mathcal{F}_p$ over U. If $t \in \mathcal{F}(U)$ and $p \in U$, we let t_p be the germ of t at p in the stalk $\mathcal{F}_p$. A section of $\coprod_{p \in U} \mathcal{F}_p$ is said to be *locally given by sections of* $\mathcal{F}$ is for every $p \in U$, there are a neighborhood V of p contained in U and a section $t \in \mathcal{F}(V)$ such that for every $q \in V$, $s(q) = t_q \in \mathcal{F}_p$.

Define

$$\mathcal{F}^+(U) = \{\text{sections } s \colon U \to \coprod_{p \in U} \mathcal{F}_p \text{ locally given by sections of } \mathcal{F}\}.$$

Then $\mathcal{F}^+$ is easily seen to be a sheaf, called the *sheafification* of $\mathcal{F}$ or the *sheaf associated to the presheaf* $\mathcal{F}$. There is an obvious map $\theta \colon \mathcal{F} \to \mathcal{F}^+$ that sends a section $s \in \mathcal{F}(U)$ to the function $p \mapsto s_p \in \mathcal{F}_p$, $p \in U$.

Proposition 6.3. *For every sheaf $\mathcal{G}$ and every presheaf morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$, there is a unique sheaf morphism $\varphi^+ \colon \mathcal{F}^+ \to \mathcal{G}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{F}^+ & & \\ {\scriptstyle \theta}\uparrow & \searrow^{\varphi^+} & \\ \mathcal{F} & \xrightarrow[\varphi]{} & \mathcal{G} \end{array} \tag{6.2}$$

commutes.

Proof. The proof is straightforward and is left as an exercise. □

6.4. **Sheaf Morphisms.** A *morphism* of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is by definition a morphism of presheaves. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the *presheaf kernel*

$$U \mapsto \ker(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U))$$

is a sheaf, called the *kernel* of φ and written $\ker\varphi$. The *presheaf image*

$$U \mapsto \operatorname{im}(\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)),$$

however, is not always a sheaf. The *image* of φ, denoted $\operatorname{im}\varphi$, is defined to be the sheaf associated to the presheaf image of φ.

A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is said to be *injective* if $\ker\varphi = 0$, and *surjective* if $\operatorname{im}\varphi = \mathcal{G}$.

Proposition 6.4. (i) *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective if and only if the stalk map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is injective for every p.*
(ii) *A morphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is surjective if and only if the stalk map $\varphi_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective for every p.*

PROOF. Exercise. □

In this proposition, while (i) is true also for morphism of presheaves, (ii) is not necessarily so. It is the truth of (ii) that makes sheaves so much more useful than presheaves.

6.5. **Exact Sequences of Sheaves.** A sequence of sheaves

$$\cdots \longrightarrow \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \mathcal{F}_3 \xrightarrow{d_3} \cdots$$

is said to be *exact* at $\mathcal{F}_k$ if $\operatorname{im} d_{k-1} = \ker d_k$. An exact sequence of sheaves of the form

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

is said to be *short-exact*.

Theorem 6.5. *A short exact sequence of sheaves*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

on a topological space X gives rise to a long exact sequence in cohomology

$$\cdots \longrightarrow \check{H}^{k-1}(X,\mathcal{G}) \xrightarrow{d^*} \check{H}^k(X,\mathcal{E}) \longrightarrow \check{H}^k(X,\mathcal{F}) \longrightarrow \check{H}^k(X,\mathcal{G}) \xrightarrow{d^*} \check{H}^{k+1}(X,\mathcal{E}) \longrightarrow \cdots.$$

PROOF. See [4]. □

6.6. **The Čech–de Rham Isomorphism.** Let $\underline{\mathbb{R}}$ be the sheaf of locally constant functions with valued in $\mathbb{R}$ and let $\mathcal{A}^k$ be the sheaf of C^∞ k-forms on a manifold M. For every open set U in M, the exterior derivative $d\colon \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U)$ induces a morphism of sheaves $d\colon \mathcal{A}^k \to \mathcal{A}^{k+1}$.

Proposition 6.6. *On any manifold M of dimension n, the sequence of sheaves*

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n \to 0 \tag{6.3}$$

is exact.

PROOF. Exactness at $\mathcal{A}^0$ is equivalent to the exactness of the sequence of stalk maps $\underline{\mathbb{R}}_p \to \mathcal{A}^0_p \xrightarrow{d} \mathcal{A}^1_p$ for all $p \in M$. Fix a point $p \in M$. Suppose $[f] \in \mathcal{A}^0_p$ is the germ of a C^∞ function (U, f) such that $d[f] = [0]$ in $\mathcal{A}^1_p$. Then there is a neighborhood $V \subset U$ of p on which $df \equiv 0$. Hence, f is locally constant on V and $[f] \in \underline{\mathbb{R}}_p$. Conversely, if $[f] \in \underline{\mathbb{R}}_p$, then $d[f] = 0$. This proves the exactness of the sequence (6.3) at $\mathcal{A}^0$.

Next, suppose $[\omega] \in \mathcal{A}^k_p$ is the germ of a C^∞ k-form on some neighborhood of p such that $d[\omega] \in \mathcal{A}^{k+1}_p$. This means there is a neighborhood V of p on which $d\omega \equiv 0$. By making V smaller, we may assume that V is contractible. By the Poincaré lemma, ω is exact on V, say $\omega = d\tau$ for some $\tau \in \mathcal{A}^{k-1}(V)$. Hence, $[\omega] = d([\tau])$. □

Let $\mathcal{Z}^k$ be the sheaf of closed C^∞ k-forms on a manifold M. Then $\mathcal{Z}^k = \ker(d\colon \mathcal{A}^k \to \mathcal{A}^{k+1})$ and by the exactness of (6.3), $\mathcal{Z}^k = \operatorname{im}(d\colon \mathcal{A}^{k-1} \to \mathcal{A}^k)$. The long exact sequence (6.3) can be broken up into a collection of short-exact sequences:

$$0 \to \underline{\mathbb{R}} \to \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \to 0, \tag{1}$$

$$0 \to \mathcal{Z}^1 \to \mathcal{A}^1 \xrightarrow{d} \mathcal{Z}^2 \to 0, \tag{2}$$

$$\vdots$$

$$0 \to \mathcal{Z}^{k-1} \to \mathcal{A}^{k-1} \xrightarrow{d} \mathcal{Z}^k \to 0. \tag{k}$$

By the long exact sequence in cohomology of a short exact sequence of sheaves, we get from (1) the exact sequence

$$\check{H}^{k-1}(M, \mathcal{A}^0) \to \check{H}^{k-1}(M, \mathcal{Z}^1) \to \check{H}^k(M, \underline{\mathbb{R}}) \to \check{H}^k(M, \mathcal{A}^0).$$

Since $\mathcal{A}^0$ is acyclic, $\check{H}^{k-1}(\mathcal{A}^0) = \check{H}^k(\mathcal{A}^0) = 0$ for $k > 1$. Thus,

$$\check{H}^k(M, \underline{\mathbb{R}}) \simeq \check{H}^{k-1}(M, \mathcal{Z}^1).$$

By the same argument, (2) gives

$$\check{H}^{k-1}(M, \mathcal{Z}^1) \simeq \check{H}^{k-2}(M, \mathcal{Z}^2)$$

for $k > 2$. Continuing in this way, we get

$$\check{H}^k(M, \underline{\mathbb{R}}) \simeq \check{H}^{k-1}(M, \mathcal{Z}^1) \simeq \check{H}^{k-2}(M, \mathcal{Z}^2) \simeq \cdots \simeq \check{H}^1(M, \mathcal{Z}^{k-1}).$$

From the final short exact sequence (k), wet get

$$\check{H}^0(M, \mathcal{A}^{k-1}) \xrightarrow{d} \check{H}^0(M, \mathcal{Z}^k) \to \check{H}^1(M, \mathcal{Z}^{k-1}) \to 0.$$

Hence,

$$\check{H}^1(M, \mathcal{Z}^{k-1}) \simeq \frac{\check{H}^0(M, \mathcal{Z}^k)}{\operatorname{im} d} \simeq \frac{\mathcal{Z}^k(M)}{d(\mathcal{A}^{k-1}(M))} \simeq \check{H}^k(M).$$

Putting all the isomorphisms together gives

$$\check{H}^k(M, \underline{\mathbb{R}}) \simeq H^k(M).$$

Thus, the Čech cohomology of a manifold M with coefficients in the sheaf of locally constant real-valued functions is isomorphic to the de Rham cohomology of M.

In general, an exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \cdots$$

on a topological space X is called a *resolution* of the sheaf $\mathcal{A}$. The resolution is *acyclic* if each sheaf $\mathcal{F}^q$ is acyclic on X, i.e., $\check{H}^k(X, \mathcal{F}^q) = 0$ for all $k > 0$. Exactly the same proof as that of the Čech–de Rham isomorphism proves the following theorem.

Theorem 6.7. *Let*

$$0 \to \mathcal{A} \to \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \xrightarrow{d} \mathcal{F}^2 \xrightarrow{d} \cdots$$

be an acyclic resolution of the sheaf $\mathcal{A}$ *on a topological space* X*. Then there is an isomorphism*

$$\check{H}^k(X, \mathcal{A}) \simeq \frac{\ker d : \mathcal{F}^k(X) \to \mathcal{F}^{k+1}(X)}{\operatorname{im} d : \mathcal{F}^{k-1}(X) \to \mathcal{F}^k(X)}.$$

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