

Shimura Varieties (— ²⁰¹⁰ ICTP Summer School on Hodge Theory)

In algebraic geometry one has a lot of objects which turn out (by big theorems) to be algebraic but are defined analytically:

- GAGA (proj. varieties, etc.)
- Hodge loci + zero-loci of normal functions (CBK, BP/et al)
- Complex tori with a polarization (Kodaira emb. thm. / Θ -fun.)
- Hodge classes (of a certain \$1,000,000 problem could be solved)

and of ~~3~~ concern to us in this course:

- modular (locally symmetric) varieties

which can be thought of as the $p|D$'s for the period maps of certain special VHS's. The fact that they are algebraic is the Baily-Borel theorem (1966).

What one does not know in the Hodge/zero locus setting above is the field of definition \rightarrow a question related to the existence of Bloch-Beltrami filtrations (to be discussed in Mark's course).

For certain cleverly constructed varieties of modular varieties, called Shimura varieties, one actually knows the minimal [reflex] field of definition, and also quite a bit about the interplay between "upstairs" and "downstairs" (in \check{D} resp. $p|D$) fields of definition of subvarieties. My interest in the subject stems from investigating Mumford-Tate domains of Hodge structures, when for example the reflex fields can still be defined even though the

$\Gamma \backslash D$'s are not algebraic varieties in general. Accordingly, (2)
I have tried to pack as many Hodge-theoretic punchlines into
the exposition as possible.

Of course, Shimura varieties are of central importance
from another point of view, that of the Langlands programme.

For instance, they provide a major test case for the conjecture
(generalising Shimura-Taniyama) that all motivic L-functions

(arising from Galois representations on étale cohomology of varieties/~~field~~)

~~(arise from automorphic forms)~~
are automorphic (arising from Hecke eigenforms of adelic algebraic

groups). The modern theory is largely due to Deligne/Langlands/

Shimura, though many others are implicated in the huge amount
of underlying mathematics: e.g.,

- theory of complex multiplication for abelian varieties
(Shimura, Taniyama, Weil)
- algebraic groups (Borel, Harish-Chandra)
- class field theory (Artin, Chevalley, Weil; Hensel for p -adics)
- modular varieties (Hilbert, Hecke, Siegel; Borly, Borel, Serre)
(compactification)

It seems that much of the impetus, historically, for the study of
locally symmetric varieties can be credited to Hilbert's 12th problem

generalising Kronecker's Jugendtraum. * Its goal was the construction

of abelian extensions^{*} of certain number fields by means of special
values of abelian functions in several variables, and it directly underlay
the work of Hilbert's 21st students on modular varieties and the theory of CM.

* i.e. algebraic extensions w/ abelian Galois group.

- Plan of the course:
- I. Hermitian symmetric domains $\rightarrow D$
 - II. Locally symmetric varieties $\rightarrow \Gamma \backslash D$
 - III. Theory of CM
 - IV. Shimura varieties $\xrightarrow{\quad} \prod_i \Gamma_i \backslash D_i$
 - V. Field of definition

It's a great pleasure to give these lectures here, and I'll try to take care so that the pleasure is not only mine.

I. Hermitian symmetric domains

A. Algebraic groups (and their properties)

Definition: An algebraic group G/k (= field of char. 0) is an algebraic variety (= G) together with morphisms (both $/k$)

" \cdot " : $G \times G \rightarrow G$ (multiplication)

$(\cdot)^{-1}$: $G \rightarrow G$ (inversion)

and $"1" \in G(k)$ (identity),

L-valued pts:
 $\text{Spec } L \rightarrow G$

subject to rules** which make $G(L)$ into a group for each L/k .

($G(\mathbb{R})$ & $G(\mathbb{C})$ have the structure of real & complex Lie groups, in particular*) G will always be smooth.

Example: $G_m := \{XY=1\} \subset \mathbb{A}^2$ (as alg. var.)

$G_m(k) = k^*$

extension of scalars:
 $G_L = G \times_{\text{Spec } k} \text{Spec } L$
for L/k

- G connected $\Leftrightarrow G_k$ irreducible
- G simple $\Leftrightarrow G$ nonabelian, with no normal connected subgroups $\neq \{1\}, G$.

Example: $k = \mathbb{C}$: SL_n, SO_n, Sp_n , exceptional groups of types $E_6, 7, 8, F_4, G_2$



$k = \mathbb{R}$: have to worry about real forms (\cong / \mathbb{C} vs $\not\cong / \mathbb{R}$)

$k = \mathbb{Q}$: all hell breaks loose, as \mathbb{Q} -simple $\not\cong$ \mathbb{R} -simple

** Exercise: work out these rules.

* and there are refinements for Lie groups corresponding to the ones I will introduce for G later (cf. Remark)

• G (algebraic) torus $\Leftrightarrow G_k \cong G_m \times \dots \times G_m$ (over $\mathbb{C} : \mathbb{C}^* \times \dots \times \mathbb{C}^*$) ⁵

Example: inside GL_2 , have

$$U := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 \neq 0 \right\}$$

$$V := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1 \right\}$$

$$G_m \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0 \right\}$$

\mathbb{C} -pts.	\mathbb{R} -pts.
$\mathbb{C}^* \times \mathbb{C}^*$	\mathbb{C}^*
\mathbb{C}^*	U_1 (= "S ¹ ")
\mathbb{C}^*	\mathbb{R}^*

different real forms of GL_1/G_m

Example: $G = \text{Res}_{E/\mathbb{Q}} G_m$ # field "Weil restriction" or "restriction of scalars"

is a torus of dimension $[E:\mathbb{Q}]$ with the property that

$$G(\mathbb{Q}) \cong E^*$$

$$G(k) \cong E^* \otimes_{\mathbb{Q}} k \cong (k^*)^{[E:\mathbb{Q}]}$$

$k \supseteq E$ G splits (factors correspond to the different embeddings of $E \hookrightarrow k$)

- G semisimple \Leftrightarrow (almost) direct product of simple gcs.
- G reductive \Leftrightarrow " " " " " simple gcs. + tori

\Rightarrow ~~any finite dim. representation splits (is reductive)~~
~~into irreducible factors~~ linear representations are completely reducible.

Example: One finite-dimensional representation is

$$G \xrightarrow{\text{ad}} GL(\mathfrak{g})$$

$$g \longmapsto \{X \mapsto gXg^{-1}\}$$

$$\text{where } \mathfrak{g} = \text{Lie}(G) = T_e G$$

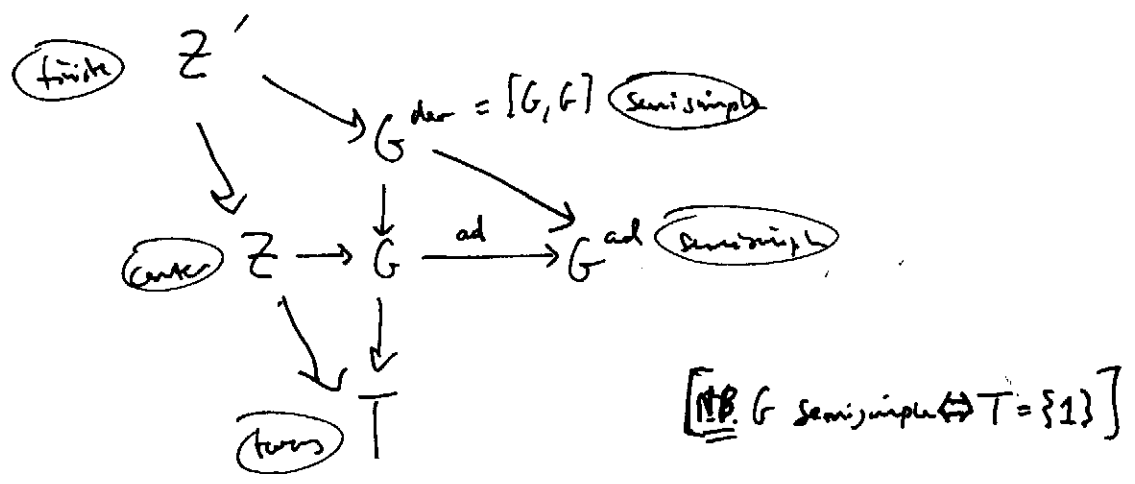
and we are taking the differential (at e) of $\text{Ad}(g) \in \text{Aut}(G)$ (= conj. by g).

For semisimple groups,

- G adjoint \Leftrightarrow ad is injective
- G simply connected \Leftrightarrow any isogeny $G' \rightarrow G$ w./ G' connected is \cong .

Basically, the center is finite; and adjoint $\Leftrightarrow Z(G)$ trivial
simp. conn. $\Leftrightarrow Z(G)$ as large as possible (with given Lie algebra)

For reductive groups, we have short-exact sequences



Finally, let G be a reductive real algebraic group,

$$\theta : G \rightarrow G \text{ an involution.}$$

Definition: θ is Cartan $\Leftrightarrow \{g \in G(\mathbb{R}) \mid g = \theta(\bar{g})\} =: G^{(\theta)}(\mathbb{R})$ is compact
 (compact real form of G)

$$\Leftrightarrow \theta = \text{Ad}(C) \text{ for } C \in G(\mathbb{R}) \text{ with}$$

- $C^2 \in Z(\mathbb{R})$
- $G \hookrightarrow \text{Aut}(V, Q)$ s.t.

$$Q(\cdot, C(\cdot)) > 0 \text{ on } V_C.$$

(very suggestive!)

These always exist, and

$$G(\mathbb{R}) \text{ compact} \Leftrightarrow \theta = \text{id.}$$

B. Three characterizations of HSD's

1. Hermitian symmetric space of noncompact type (analytic, intrinsic)

$(X, g) =$ connected \mathbb{C} -mfld. with Hermitian metric
 (or Riemannian manifold with integrable almost-complex structure s.t. J acts by isometries)

Bergman metric
 Satake embedding
 (Harish-Chandra?)

- s.t. $Is(X, g)$ (holomorphic isometry group = real Lie gp.)
- acts transitively
 - contains symmetries S_p ($S_p^2 = id_X$, p is isolated fixed point)
 - $\underbrace{Is(X, g)^+}_{\text{identity connected component}} =$ semisimple adjoint noncompact (real Lie) group

(the noncompactness means (a) Cartan involution projects to "id" in no factor
 (b) X has negative sectional curvatures)

2. Bounded symmetric domain (analytic, extrinsic)

$X =$ connected open subset of \mathbb{C}^n with compact closure

- s.t. $Hol(X)$ (group of holomorphic automorphisms) (real Lie)
- acts transitively
 - contains symmetries S_p

3. Circle conjugacy class (algebraic)

$X = G(\mathbb{R})^+ -$ conjugacy class of a homomorphism $\phi: \mathbb{U} \rightarrow G$ of alg. gps. / \mathbb{R} . } pts. of X are $g\phi g^{-1}$

~~critical~~ for any field of defn. questions!
 up to a "square root"
 This is what MTD's generalize (more later)

where $G =$ real adjoint ^(semisimple) algebraic group and

defn. invariant w.r.t. choice of ϕ in ccl.

- only $z, 1, z^{-1}$ appear in the rep. $ad \circ \phi$ on $Lie(G)_{\mathbb{C}}$
- $\Theta := ad(\phi(-1))$ is Cartan
- $\phi(-1)$ doesn't project to 1 in any simple factor of G

(1) \rightarrow (3):

- $I_S(X, g)^+$ adjoint + semi-simple \Rightarrow ^{Borel} $= G(\mathbb{R})^+$ for some algebraic $G \subset GL(\text{Lie}(I_S(X, g)^+))$

[This can only make sense if I_S^+ is adjoint hence embeds in $GL(\text{Lie}(\dots))$.

Further, the "+" is necessary: if $I_S^+ = SO(p, q)^+$, this is not $G(\mathbb{R})$ for algebraic G .]

- $p \in X \rightarrow s_p \in \text{Aut}(X)$ with $\begin{cases} p \text{ isolated fixed pt.} \\ s_p^2 = \text{id}_X \end{cases} \Rightarrow ds_p = \text{mult. by } (-1) \text{ on } T_p X$.

Several copies of identity section corr.

In fact, for any $|z|=1$, $\exists!$ isometry $\psi_p(z)$ of (X, g) s.t. (on $T_p X$)
 $d\psi_p(z) = \text{mult. by } z$ (i.e. $a + bJ$)

Since this is a homeomorphism (from U_i) "on $T_p X$ ", the uniqueness means

$$\psi_p : U_i \rightarrow I_S(X, g)^+$$

is also a homeomorphism. It algebraizes to

$$\phi_p : U \rightarrow G \quad (\mathbb{R})$$

- to view X as a ccl, recall that $G(\mathbb{R})^+$ acts transitively and note that for $g \in G(\mathbb{R})^+$ sending $p \rightarrow q$,

$$\phi_q(z) (= g \circ \phi_p(z) \circ g^{-1}) = \text{Ad}(g) \phi_p(z)$$

(again using uniqueness).

- we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus T_p^{1,0} X \oplus T_p^{0,1} X$

$$d\phi_p(z) : z \Rightarrow \begin{matrix} \bar{z} \\ z^{-1} \end{matrix} \text{ since } \phi_p \text{ is real and } T^{1,0} = \overline{T^{0,1}}.$$

" z^{-1} since $z \in U_i$.

Using uniqueness once more, $\text{Ad}(k) \phi_p(z) = \phi_p(z)$ for $k \in K_p$,
 $\Rightarrow \text{Ad} \phi_p(z)$ acts by id. on $\mathfrak{k} = \text{Lie } K_p$.

So $z, 1, z^{-1}$ are the eigenvalues of $\text{ad} \phi$

- X has negative sectional curvatures $\Rightarrow \text{Ad} \circ s_p$ is Cartan,

whom together w./ X noncompact $\Rightarrow s_p$ projects to 1 in no factor of G . □

C. Cartan's classification of irreducible HSD's

Let $X =$ irreducible HSD,

$G =$ corresponding simple \mathbb{R} -algebraic group

$T \subset G_{\mathbb{C}}$ maximal algebraic torus / \mathbb{C} .

The restriction to T

$$T \xrightarrow{\quad} G_{\mathbb{C}} \xrightarrow{\text{ad}} GL(\mathfrak{g}_{\mathbb{C}})$$

of the adjoint representation breaks into 1-dimensional eigenspaces on which T acts through characters:

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha} \right) \quad \text{where} \quad R \subset \text{Hom}(T, \mathbb{C}^{\times}) \cong \mathbb{Z}^n$$

$R^+ \perp R^-$ are the roots.
($-R^+$)

- $\exists!$ "basis" $\{\alpha_1, \dots, \alpha_n\} \subset R$ (simple roots) s.t. each $\alpha \in R^+$ is of the form $\sum m_i \alpha_i, m_i \geq 0$.
- "highest" root $\hat{\alpha} \in R^+$ s.t. $\hat{m}_i \geq m_i$ for any other $\alpha \in R^+$

The α_i give the nodes on the Dynkin diagram of G , in which α_i & α_j are connected if they pair nontrivially under a standard inner product (the Killing form).
Ex / $\begin{matrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$ A_n, D_n etc. E_6

Over \mathbb{C} , our circle map ϕ defines a cocharacter

$$G_m \xrightarrow{\quad \mu \quad} U \xrightarrow{\quad \phi_{\mathbb{C}} \quad} G_{\mathbb{C}}$$

(/ \mathbb{C})

which has a unique conjugate factors as

$$G_m \rightarrow T \subset G_{\mathbb{C}}$$

in such a way that $\langle \mu, \alpha \rangle \geq 0 \quad \forall \alpha \in R^+$

(This pairing is defined by $G_m \xrightarrow{\mu} T \xrightarrow{\alpha} G_m$
 $z \longmapsto z^{\langle \mu, \alpha \rangle}$.)

Now, μ must act through $z, 1, z^{-1}$ \Rightarrow

$$\begin{cases} \langle \mu, \alpha \rangle = 0 \text{ or } 1 \quad \forall \alpha \in R^+ \\ \text{not } \neq 0 \text{ for some } \alpha \end{cases}$$

\Rightarrow
 look at $\langle \mu, \alpha \rangle$

$$\begin{cases} \langle \mu, \alpha_i \rangle = 1 \text{ for a unique } i, \\ \text{and for this } i, \hat{m}_i = 1 \text{ (} \alpha_i \text{ is special)} \end{cases}$$

So we have a 1-1 correspondence

Irreducible HSD's \longleftrightarrow special nodes on connected Dynkin diagrams.

and hence a list of the # of distinct \cong classes of irred HSD's corresponding to each simple complex Lie algebra:

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
n	1	1	3	2	1	0	0	0

grouping of three types (circled around E8, F4, G2)

Examples:

$A_n: X \cong SU(p, q) / S(U_p \times U_q)$

$B_n: X \cong SO(n, 2)^+ / SO(n) \times SO(2)$

$C_n: X \cong \frac{Sp_{2n}(\mathbb{R})}{U(n)}$

$\cong \{ z \in M_n(\mathbb{C}) \mid z = z^t, \text{Im}(z) > 0 \}$ Siegel upper $\frac{1}{2}$ -space

~~with some examples~~
 $ptq = n+1$ (n possibilities)

$\longleftrightarrow H^2(K3)$

(Cartan's lecture) vs. 1 HS (level 1)

my comment: these are $2n \times 2n$ matrices

[Start w./ definition of def 3 of HSD, etc.]

(12)

D. Hodge - Theoretic interpretation

Let $V = \mathbb{Q}$ -vector space

(i.e. \mathbb{Q} -split MKS)

Definition: A Hodge structure on V is a homomorphism \mathbb{R}

$$\tilde{\varphi} : \mathbb{S} \longrightarrow GL(V)$$

such that (the weight homomorphism)

$$w_{\tilde{\varphi}} : \mathbb{G}_m \hookrightarrow \mathbb{S} \longrightarrow GL(V)$$

is defined \mathbb{Q} .

Associated to $\tilde{\varphi}$ is

$$\mu_{\tilde{\varphi}} : \mathbb{G}_m \longrightarrow GL(V)$$

$$z \longmapsto \tilde{\varphi}_{\mathbb{C}}(z, 1).$$

[Note: Recalling that $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$, $V^{p,q} \subset V_{\mathbb{C}}$ is the

$$\begin{cases} z^p w^q - \text{eigenspace of } \varphi_{\mathbb{C}}(z, w) \\ z^p - \text{eigenspace of } \mu(z) \end{cases}$$

, and $w_{\tilde{\varphi}}(r) = \tilde{\varphi}(r, r)$ acts on it by r^{p+q} .

Fix a weight n , Hodge #'s $\{h^{p,q}\}_{p+q=n}$, and

polarization $\mathcal{Q} : V \times V \rightarrow \mathbb{Q}$; let

- $D =$ period domain ~~is~~ parametrizing HS of this type on V , & polarized by \mathcal{Q} ,

- $t \in \bigoplus_i V^{\otimes k_i} \otimes \check{V}^{\otimes l_i}$ be a (finite) sum of \mathcal{Q} -tensors,

- $D_t^+ \subset D$ a connected component of the subset of HS for which these tensors are Hodge ($t \in F^{n(k-l)/2}$),

and

- $M_x \subset GL(V)$ the smallest \mathbb{Q} -algebraic subgroup with $M_x(\mathbb{R}) \supset \tilde{\varphi}(S(\mathbb{R})) \quad \forall \tilde{\varphi} \in D_x^+$ (this is reductive).

Then: given any $\tilde{\varphi} \in D_x^+$, the orbit

$$D_x^+ = M_x(\mathbb{R})^+ \cdot \tilde{\varphi} \quad (\text{action by conjugation})$$

is called a Mumford-Tate domain. Furthermore, $Ad(\mu_{\tilde{\varphi}}(-1))$ is a Cartan involution (exercise). Now,

the topological family $\mathcal{V} \rightarrow D_x^+$ is a VHS (i.e. the IPR is trivial)
(in fundamental period relation of $\mathcal{V} \in \mathcal{H}(D_x^+)$)

the induced H.S. on $Lie(M_x) \subset End(V)$ (at $\tilde{\varphi}$) is of type $(-1, 1) + (0, 0) + (1, -1)$
(stuff of type $(2, 2)$ or worse would violate Griffiths transversality)

(*) $ad \circ \mu_{\tilde{\varphi}}$ has eigenvalues $z, 1, z^{-1}$ (only),

proving part (a) of

† we still get all the real & class of HSD's in the table

Proposition: (a) A MT domain with trivial IPR (and M_x adjoint) admits the structure of a HSD with G/\mathbb{Q} , and (b) Conversely (i.e. such HSD's parametrize Hodge structures).

Remarks: (i) (*) $\Rightarrow Ad(\mu_{\tilde{\varphi}}(-1))$ gives a symmetry of D_x^+ at $\tilde{\varphi}$, but not conversely: e.g., an example of a MTD with nontrivial IPR but HSD structure is the period domain for HS's of type $(1, 0, 1, h, 1, 0, 1)$ (weight 6).

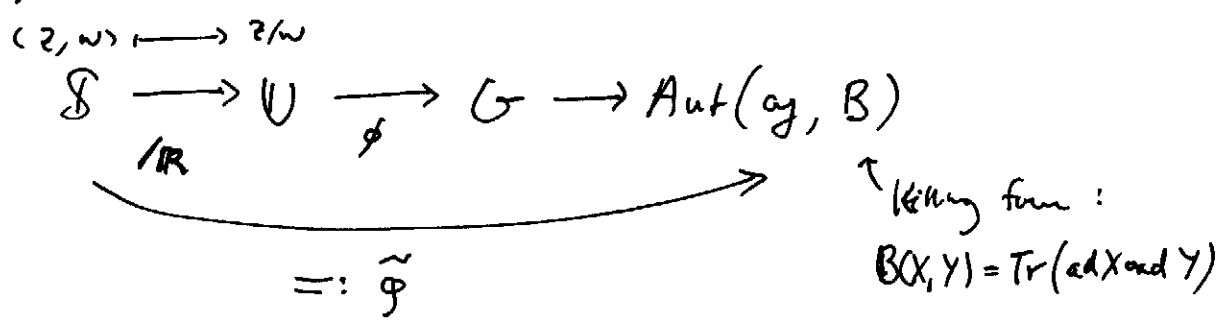
(ii) This doesn't contradict (b), since the same HSD can have different MTD structures.

(iii) The proposition is essentially a theorem of Deligne from 1979.

Proof of (b): Let $X = \text{HSD}$ with real orbifold $V \xrightarrow{\phi} G$.

(Since a product of MTD's is a MTD, we may assume G \mathbb{Q} -simple.)

The composition



is a H/S on $V = o_y$ polarized by $-B$ (since $\text{Ad}(\phi(-1))$ is Cartan).

The \mathbb{Q} -closure of a generic $G(\mathbb{R})$ -conjugate is G (since $\phi(-1)$ is anisotropic),
 G is of the form M_x by Chevalley's theorem (cf. Witt/Klein's lemma),
 and so $X = G(\mathbb{R})^+ \cdot \tilde{\phi}$ is a MT domain. The IPR vanishes because $ad_o\phi$ has eigenvalues $z, 1, z^{-1}$. \square

Example (M. Gross): applied to one of the ~~two~~ HSD's for E_6 ,

this procedure produces a MT domain parametrizing certain HS of type $(h^{2,0}, h^{1,1}, h^{0,2}) = (16, 46, 16)$ — a submanifold D_x^+

of the period domain D for each HS. The IPR $\int \in \Omega^*(D)$ is nontrivial but pulls back to zero on D_x^+ . \square

[Open problem: find a family of varieties over D_x^+ with no family of HS.*]

The proof of (b) always yields HS's of even weight. Sometimes, by replacing the adjoint representation by a "standard" representation, we can parametrize HS's of odd weight.

* More generally, it is conjectured that the topological VHS over every MTD with trivial IPR, comes from algebraic geometry (is "motivic")

Examples: $\mathfrak{h}^4 \cong Sp_3(\mathbb{R})/U(1)$ parameterizes HS's of weight 1 rank 8, or equivalently abelian varieties of dimension 4. There are "two" types of MT subdomains in \mathfrak{h}^4 :

- corresponding to $End_{HS}(V) (= End(A)_{\mathbb{Q}})$ containing a nontrivial fixed subalgebra $\mathcal{E} =$ product of matrix algebras over \mathbb{Q} -division algebras of types

Albert classification

{all four do occur in \mathfrak{h}^4 }	}	I) totally real field
		II) indefinite quaternion algebra over totally real field
		III) definite
		IV) <u>division algebra over a CM field</u> e.g., just an imaginary quadratic field.

- corresponding to fixed endomorphisms \mathcal{E} + higher Hodge tensors. (We think of $End_{HS}(V) \subset T^{1,1}V$ and the polarizations $Q \in T^{0,2}V$; "higher" means [in $T^{k,l}V$] of degree $k+l > 2$.)

Here are two ~~such~~ ^{such} examples:

(a) Consider HS's on V with a fixed embedding

$$\mathbb{Q}(i) \hookrightarrow End(V)$$

such that (writing $Hom(\mathbb{Q}(i), \mathbb{C}) = \{\eta, \bar{\eta}\}$)

$$V^{1,0} = V_{\eta}^{1,0} \oplus V_{\bar{\eta}}^{0,0} \quad \text{with dim } V_{\eta}^{1,0} = 1.$$

This yields $D_{\mathbb{R}}^+ \cong \underbrace{SU(1,3)}_{\text{HSD}} / \underbrace{S(U_1 \times U_3)}_{\text{BSD}} \cong \text{Complex 3-ball}$, of type A_3 .

Exercise: Show MTO is $U(1,3)$.

(b) [Mumford] constructs a quaternion algebra \mathcal{Q} over a totally real cubic field K , such that $\mathcal{Q} \otimes_{\mathbb{Q}} \mathbb{R} \cong H \oplus H \oplus M_2(\mathbb{R})$, together with an embedding

$\mathcal{Q}^* \hookrightarrow GL_{\mathbb{R}}(\mathbb{Q})$. This yields a \mathbb{Q} -simple algebraic group $G := Res_{K/\mathbb{Q}} U_{\mathcal{Q}} \subset GL(V)$ [octahedron in Sp_3]

with $G(\mathbb{R}) \cong SU(2)^{*2} \times SL_2(\mathbb{R})$, and the $G(\mathbb{R})$ -orbit of $\varphi_0: U \rightarrow G$ yields a MT domain. Mumford shows that the general HS if sparse has $\mathcal{E} = \mathbb{Q}$ (trivial) so G is cut out by higher Hodge tensors.

II. Locally symmetric varieties

To construct quotients of Hermitian symmetric domains we'll need the basic

Proposition: Let $X =$ topological space $\ni x_0$
 $M =$ locally compact group acting on X
 $\Gamma (\leq M)$ a discrete subgroup (no limit pts.)

Assume (i) $K = \text{stab}(x_0)$ is compact, and
 (ii) $gK \rightarrow gx_0 : M/K \rightarrow X$ is a homeomorphism..

Then $\Gamma \backslash X$ is Hausdorff.

Proof: nontrivial topology exercise. Writing $\pi : X \rightarrow \Gamma \backslash X$, the key points are:

- $\pi^{-1}(\text{compact})$ is compact
- discrete \cap compact = finite.

□

Corollary: Let $X = G(\mathbb{R})^+ / K$ HSD
 $\Gamma \leq G(\mathbb{R})^+$ discrete & torsion-free.

Then $\Gamma \backslash X$ has a unique \mathbb{C} -mfld. structure for which π is a local \cong .
 (If Γ isn't torsion-free then we get an orbifold.)

Examples: (a) $X = \mathbb{H}^1$, $G = \text{SL}_2$ (acting in the standard way)
 $\Gamma = \Gamma(N) := \ker \{ \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \}$ with $N > 3$
 $\Gamma \backslash X =: \mathcal{Y}(N)$ gives the classical modular curves
 classifying elliptic curves with marked N -torsion.

(b) $X = \mathbb{H}^n$, $G = \text{Sp}_{2n}$, $\Gamma = \text{Sp}_{2n}(\mathbb{Z})$ (an example of a lens structure).

$\Gamma \backslash X =$ Siegel modular variety
 classifying abelian n -folds (w./ fixed polarization)

(c) $X = \underbrace{h^2 \times \dots \times h^2}_{n \text{ times}}$, $G = \text{Res}_{F/\mathbb{Q}} SL_2$ for $F =$ totally real field of degree n/\mathbb{Q}
(acting through the n embeddings of $F \hookrightarrow \mathbb{R}$)

$\Gamma = SL_2(\mathcal{O}_F)$
↑ (or some ideal in \mathcal{O}_F)

$\Gamma \backslash X$ is a Hilbert modular variety classifying abelian n -folds with $E \supset F$ (i.e. general member is of Albert type I).

We can view X as a proper MT subdomain of h^n .

All the Γ 's that have come up here are rather special.

n of finite index in each

Definition: (a) Let G be a \mathbb{Q} -algebraic group. $\Gamma \leq G(\mathbb{Q})$ is

$\underbrace{\text{arithmetic}}_{\uparrow} \Leftrightarrow \Gamma \text{ commensurable with } G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$
 $\underbrace{\text{congruence}} \Leftrightarrow \Gamma \supset \Gamma(N) := G(\mathbb{Q}) \cap \{g \in GL_n(\mathbb{Z}) \mid g \equiv \text{Id} \pmod{N}\}$
 (for some N)

for some N considering $G \hookrightarrow GL_n$

(b) Let \mathbb{A} be a connected real Lie group [e.g., $G(\mathbb{R})^+$]

- $\Gamma \leq \mathbb{A}$ is arithmetic of \exists
- \mathbb{Q} -algebraic group G ,
 - an arithmetic $\Gamma_0 \leq G(\mathbb{Q})$, and
 - a homomorphism $G(\mathbb{R})^+ \xrightarrow{\rho} \mathbb{A}$ with compact kernel,

such that $\rho(\Gamma_0 \cap G(\mathbb{R})^+) = \Gamma$.

(This is set up so that Γ will always contain a torsion-free subgroup of finite index.)

Theorem [Baily-Borel, 1966] : Let $X = G(\mathbb{R})^+ / K$ be a HSD,

$\Gamma \triangleleft G(\mathbb{R})^+$ a torsion-free arithmetic subgroup.

Then $X(\Gamma) := \Gamma \backslash X$ is canonically a smooth quasi-projective algebraic variety, called a locally symmetric variety.

(If we don't assume torsion-free, we still get a quasi-proj. alg. var., but it is an orbifold (non-smooth) and not called a LSV.)

Idea of proof: Construct a (minimal, highly singular) compactification

$$X(\Gamma)^* := \Gamma \backslash (X \amalg B) \quad (B = \text{"rational boundary components"})$$

and embed it in \mathbb{P}^N (using automorphic forms of sufficiently high weight) as a projective analytic, hence (by GAGA/Chow) projective algebraic variety. (The existence of enough ~~sections~~ automorphic forms to yield an embedding is a convergence question for certain "Poincaré-Eisenstein series".) \square

~~Locally symmetric varieties~~

Example: In the modular curve context, $B = \mathbb{P}^1(\mathbb{Q})$ and $X(\Gamma)^* \setminus X(\Gamma)$ is the (finite part set of) cusps. We write $(X(\Gamma) =) \gamma(N) \subset X(N) \in X(\Gamma)^*$.

Recalling (from I.D) that X is always a MT domain, we can give a Hodge-theoretic interpretation to the ^{Baily-Borel} compactification:

^[Cartan-Culver-Klein?]
Proposition: The boundary components B parametrize the possible $\bigoplus_i Gr_i^W \cap H_{\text{lim}}$ for VHS into $X(\Gamma)$.
(Presumably this is independent of which MTD structure we put on X .)

Idea of proof: Assuming PGL_2 is not a quotient of G , the automorphic forms are Γ -invariant sections of $K_X^{\otimes N}$ for some $N \gg 0$.

K_X ($\cong \Lambda^d \text{ of } (-1,1)$ _{partials}) measures the change of the Hodge flag in every direction, so the boundary components parametrized by these

Sections must consist of naive limiting Hodge flags in $d\bar{X} = \bar{X}$.

In that limit, the relation (projectively) between periods that go to ∞ at different rates (arising from different G_i^w) is fixed, which means we can't see extension data. On the other hand, since $\exp(2\pi N)$ does not change the $G_i^w F^0$, this information is the same for the naive limiting Hodge flag of the LMMS. \square

Remark: There are other compactifications with different Hodge-theoretic

- interpretations:
- AMRT toroidal (smooth) compactifications*
(Capture the entire LMMS)
 - Borel-Serre compactification
(Captures G_i^w and adjacent extensions, at least in Siegel case)

Later on (e.g. for canonical models) we will need the

Theorem (Borel): Let $V =$ quasi-projective algebraic variety / \mathbb{C}
 $X(\Gamma) =$ locally symmetric variety

Then any analytic map $V \rightarrow X(\Gamma)$ is algebraic.

Idea of proof: Extend to analytic map $\bar{V} \rightarrow X(\Gamma)^*$, use GAGA.

Suppose $X = \mathbb{H}^2$ and V is a curve; Γ torsion-free $\Rightarrow X(\Gamma) \cong \mathbb{C}$ {22 pts.}
(or $\mathbb{P}^1 \setminus \{23 \text{ pts.}\}$)

If a holo. $f: \mathbb{D}^* \rightarrow X(\Gamma)$ does not extend to holo. map $\mathbb{D} \rightarrow \mathbb{P}^1$,
(proceed with)

then f has an essential singularity at 0; and the Big Picard theorem
 $\Rightarrow f$ takes all values of \mathbb{C} except possibly one, a contradiction.

The general proof uses \exists of a good compactification $V \subset \bar{V}$ (Hironaka)
so that V is locally $\mathbb{D}^{*k} \times (\mathbb{D}^*)^{*l}$. \square

* which (Coto-Ulevi generalize (in a sense) to the non-HSD ($X(\Gamma)$ not algebraic) case

III. Complex multiplication

A. CM Abelian varieties

A CM field is a totally imaginary field E possessing an involution $\rho \in \text{Gal}(E/\mathbb{Q}) =: \mathcal{M}_E$ such that $\phi \circ \rho = \bar{\phi}$ for each $\phi \in \text{Hom}(E, \mathbb{C}) =: \mathcal{K}_E$.

[Exercise: E^ρ is then totally real, and $\rho \in \mathbb{Z}(\mathcal{M}_E)$.]

Write E^c for a normal closure.

For any decomposition $\mathcal{K}_E = \mathbb{C} \oplus \bar{\mathbb{C}}$ (possibly non-conjugate embeddings)

$$\mathcal{K}_E = \mathbb{C} \oplus \bar{\mathbb{C}},$$

(E, \mathbb{C}) is a CM type; this is equipped with a reflex field

$$E' := \mathbb{Q} \left(\left\{ \sum_{\phi \in \mathbb{C}} \phi(e) \mid e \in E \right\} \right) \subset E^c$$

$$= \text{fixed field of } \left\{ \sigma \in \mathcal{M}_{E^c} \mid \sigma \tilde{\mathbb{C}} = \tilde{\mathbb{C}} \right\}$$

(where $\tilde{\mathbb{C}} \subset \mathcal{K}_{E^c}$ consists of embeddings restricting (on E) to those in \mathbb{C}).

Fixing a choice of $\phi_1 \in \mathbb{C}$ gives an identification

$$\mathcal{K}_{E^c} \xrightarrow[\substack{\cong \\ \text{compose} \\ \text{w/ } \phi_1}]{\cong} \mathcal{M}_{E^c}$$

and a notion of inverse on \mathcal{K}_{E^c} . Define the reflex type by

$$\mathbb{C}' := \left\{ \tilde{\phi}^{-1} \Big|_{E'} \mid \tilde{\phi} \in \tilde{\mathbb{C}} \right\},$$

and reflex norm by

$$N_{\Phi'} : (E')^* \rightarrow E^*$$

$$e' \mapsto \prod_{\phi' \in \Phi'} \phi'(e')$$

- Examples:
- (a) All $\mathbb{Q}(\sqrt{-d})$ are CM; in this case $N_{\Phi'}$ is the identity or complex conjugation.
 - (b) All $\mathbb{Q}(\zeta_n)$ are CM; and if E/\mathbb{Q} is an abelian extension, then $E' = E^c = E$ and $E \subset$ some $\mathbb{Q}(\zeta_n)$.
 [For cyclotomic fields we'll write $\phi_j :=$ embedding sending $\zeta_n \mapsto \zeta_n^j$.]
 - (c) $(\mathbb{Q}(\zeta_5); \{\phi_1, \phi_2\})$ has reflex $(\mathbb{Q}(\zeta_5); \{\phi_3, \phi_4\})$.

The relation of this to algebraic geometry is the following

- Proposition:
- (a) For a simple Abelian g -fold A/\mathbb{C} , TFAE:
 - (i) MT group of $H^1(A)$ is a torus [not nec. of dim. g : can be "degenerate"]
 - (ii) $\text{End}(A)_{\mathbb{Q}}$ has (maximal) rank $2g/\mathbb{Q}$ [lots of endomorphisms]
 - (iii) $\text{End}(A)_{\mathbb{C}}$ is a CM field
 - (iv) $A \cong \mathbb{C}^g / \Phi(\alpha) =: A_{\alpha}^{(E, \Phi)}$ for some CM type and ideal $\alpha \subset \mathcal{O}_E$.
 - (b) Furthermore, any complex torus of the form $A_{\alpha}^{(E, \Phi)}$ is algebraic.

Example: $\Phi(\alpha)$ means $\left\{ \begin{pmatrix} \phi_1(\alpha) \\ \vdots \\ \phi_g(\alpha) \end{pmatrix} \mid \alpha \in \alpha \right\}$ ($= 2g$ -letter);

for $E(\alpha)$ above

$$A_{\alpha}^{(E, \Phi)} = \mathbb{C}^2 / \mathbb{Z} \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix}, \begin{pmatrix} e^{2\pi i/5} \\ e^{-2\pi i/5} \end{pmatrix}, \begin{pmatrix} e^{4\pi i/5} \\ e^{-4\pi i/5} \end{pmatrix}, \begin{pmatrix} e^{6\pi i/5} \\ e^{-6\pi i/5} \end{pmatrix} \right\rangle$$

The interesting points are: when does the CM field come from, and why is $A_{\alpha}^{(E, \Phi)}$ algebraic?

Pf. of (i) \Rightarrow (iii): $V = H^1(A)$ is spanned by some Q . (22)

Set $\mathcal{E} := \text{End}_{H^1(A)}(H^1(A)) = \underbrace{\left(\underbrace{\mathcal{Z}_{GL(V)}(M)}_{\mathbb{Q}\text{-algebraic group, contains a maximal torus } T} \right) (\mathbb{Q}) \cup \{0\}}$

Since T commutes with M and is maximal, $T \supset M$.

We deduce that $\bullet T(\mathbb{Q}) \cup \{0\} =: \mathbb{E} \left(\begin{smallmatrix} \text{cons } \mathcal{E} \\ ? \end{smallmatrix} \right)$ is a field not a general equality!

$\bullet V = 1$ -dimensional vector space / \mathbb{E} , & finally that

$\bullet \mathbb{E}$ is actually all of \mathcal{E} .

M diagonalizes with respect to a Hadamard basis

$$\omega_1, \dots, \omega_g; \bar{\omega}_1, \dots, \bar{\omega}_g$$

such that $\sqrt{-1} Q(\omega_i, \bar{\omega}_j) = \delta_{ij}$. The maximal torus $\subset GL(V)$

this basis defines, centralizes M hence must be T .

Now write $\mathcal{H}_{\mathbb{E}} = \{\phi_1, \dots, \phi_{2g}\}$, $\mathbb{E} = \mathbb{Q}(\xi)$ and

$$m_{\xi}(\lambda) = \prod_{i=1}^{2g} (\lambda - \phi_i(\xi)) \quad \text{for the minimal polynomial of } \xi, \text{ hence } \gamma(\xi).$$

Up to reordering, we \therefore have

$$[\gamma(\xi)]_{\omega} = \text{diag}(\{\phi_i(\xi)\}_{i=1}^{2g}).$$

Since $\gamma(\xi) \in GL(V_{\mathbb{Q}})$ (a fortiori $\in GL(V_{\mathbb{R}})$), and $\phi_j(\xi)$ determines ϕ_j ,

$$\omega_{itg} = \bar{\omega}_i \Rightarrow \phi_{itg} = \bar{\phi}_i.$$

Define the Rechts Involution $\iota: \mathcal{E} \rightarrow \mathcal{E}$ by

$$\boxed{Q(\iota^t v, w) = Q(v, \iota w) \quad \forall v, w \in V.}$$

This produces $\rho := \gamma^{-1} \circ \iota \circ \gamma \in \mathcal{H}_{\mathbb{E}}$, and we compute

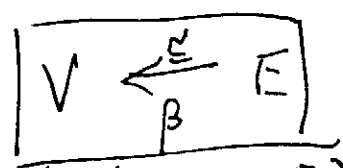
$$\begin{aligned} \phi_{itg}(e) Q(\omega_i, \omega_{itg}) &= Q(\omega_i, \gamma(e) \omega_{itg}) = Q(\gamma(e)^t \omega_i, \omega_{itg}) \\ &= Q(\gamma(\rho(e)) \omega_i, \omega_{itg}) = \phi_i(\rho(e)) Q(\omega_i, \omega_{itg}) \end{aligned}$$

$$\Rightarrow \phi_i \circ \rho = \bar{\phi}_i.$$

□

Proof of (b): We have the following construction of $H^1(A)$:

Let $V = 2g - \dim V$ \mathbb{Q} -vector space with identification



Inducing (via multiplication in E)

$$\gamma: E \hookrightarrow \text{End}_{\mathbb{Q}}(V).$$

Moreover, \exists basis $\omega = \{\omega_1, \dots, \omega_g; \bar{\omega}_1, \dots, \bar{\omega}_g\}$ of $V_{\mathbb{C}}$ with

respect to which $[\gamma_{\mathbb{C}}(e)]_{\omega} = \text{diag}\{\phi_1(e), \dots, \phi_g(e); \bar{\phi}_1(e), \dots, \bar{\phi}_g(e)\}$,

and we set $V'^0 := \mathbb{C}\langle \omega_1, \dots, \omega_g \rangle$ (so that $\gamma(e) \in \text{End}_{\mathbb{R}}(V)$). This gives $V \cong_{\mathbb{R}} H^1(A)$.

Now, $\exists \xi \in E$ s.t. $\forall i \phi_i(\xi) > 0$ ($i=1, \dots, g$) and we can put

$$\boxed{Q(\beta(e), \beta(\bar{e})) := \text{Tr}_{E/\mathbb{Q}}(\xi \cdot e \cdot \beta(\bar{e})) : V \times V \rightarrow \mathbb{Q}}$$

Over \mathbb{C} , this becomes

$$[Q]_{\omega} = \begin{pmatrix} 0 & \phi_1(\xi) & & \\ & & \ddots & \\ & & & \phi_g(\xi) \\ \bar{\phi}_1(\xi) & & & 0 \\ \vdots & & & \\ \bar{\phi}_g(\xi) & & & \end{pmatrix} \quad \square$$

Remark: $N_{\mathbb{F}/\mathbb{Q}}$ algebraizes to a homomorphism of algebraic groups

$$N_{\mathbb{F}/\mathbb{Q}}: \text{Res}_{E/\mathbb{Q}} G_m \rightarrow \text{Res}_{E/\mathbb{Q}} G_m$$

which gives $N_{\mathbb{F}/\mathbb{Q}}$ on the \mathbb{Q} -points, and the MG group

$$M_{H^1(A)} \cong \text{im}(N_{\mathbb{F}/\mathbb{Q}}) \quad \square$$

Let E be a CM field with $[E:\mathbb{Q}] = 2g$. In algebraic number theory we have

$\mathcal{I}(E) =$ monoid of nonzero ideals in \mathcal{O}_E

$\mathcal{Q}(E) =$ fractional ideals (of form eI , $e \in E^* \ \& \ I \in \mathcal{I}(E)$)

$\mathcal{P}(E) =$ principal fractional ideals (of form $(e) := e \cdot \mathcal{O}_E$, $e \in E^*$).

The (abelian!) ideal class group

$$Cl(E) := \frac{I(E)}{P(E)}$$

or more precisely the class number

$$h_E := |Cl(E)|,$$

expresses (if $\neq 1$) the failure of \mathcal{O}_E to be a PID (f to have unique factorization). Each class $\tau \in Cl(E)$ has a representative $I \in I(E)$ with norm \leq the Minkowski bound, which $\Rightarrow h_E < \infty$.

Now let

$$Ab(\mathcal{O}_E, \Phi) := \frac{\{A_{\alpha} \mid \alpha \in \mathcal{O}_E\}}{\text{isomorphism}}$$

abelian g -fold
 $\nu/\mathcal{O}_E \subset \text{End}(A)$
acting on $T_0 A$
through Φ . They
are all isogenous.

and note that for any $c \in E^*$

$$A_{\alpha} \xrightarrow{\text{mult. by } c} A_{c\alpha}$$

Hence we get a bijection

$$Cl(E) \xrightarrow{\cong} Ab(\mathcal{O}_E, \Phi)$$

$$[\alpha] \longleftrightarrow A_{\alpha}$$

(class of variety)

Example: The class # of $\mathbb{Q}(\sqrt{-5})$ is 2. Representatives of $Ab(\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}, \Phi)$ are the elliptic curves $\mathbb{C}/\mathbb{Z}\langle 2, 1+\sqrt{-5} \rangle$ and $\mathbb{C}/\mathbb{Z}\langle 1, \sqrt{-5} \rangle$. \square

The key point now is to notice that for $A \in Ab(\mathcal{O}_E, \Phi)$ and $\sigma \in \text{Aut}(E)$,

$$\sigma^* A \in Ab(\mathcal{O}_E, \sigma^* \Phi)$$

since endomorphisms are given by α 's. cycles (and so the internal structure of E is left unchanged). From the definition of E' we see that

$$\underline{\text{Gal}(\mathbb{C}/E')} \text{ acts on } Ab(\mathcal{O}_E, \Phi)$$

which suggests that the individual abelian varieties should

be defined over an extension of E' of degree $h_{E'}$. (25)
 (This isn't exactly true if $\text{Aut}(K) \neq \{\text{id}\}$, but the argument does prove $K/\bar{\mathbb{Q}}$.)

B. Class field theory

In fact, it gets much better: not only is there a distinguished field extension H_L/L of degree h_L (for any # field); in fact

$$(*) \quad \text{Gal}(H_L/L) \xleftarrow{\cong} \mathcal{O}(L)$$

So that (to quote Chevalley) " L contains within itself the elements of its own transcendence".

Idea of the construction ^{of (*)}: let \tilde{L}/L be an extension of degree d which is

- abelian: Galois with $\text{Gal}(\tilde{L}/L)$ abelian
- unramified: for each prime $\mathfrak{p} \in \mathcal{O}(L)$, $\mathfrak{p} \cdot \mathcal{O}_{\tilde{L}} = \prod_{i=1}^r \mathfrak{P}_i$ power = 1
 for some $r | d$.

The image of (degree- $N(\mathfrak{p})$ extension of finite fields)

$$\text{Gal} \left(\underbrace{(\mathcal{O}_{\tilde{L}}/\mathfrak{P}_i)/(\mathcal{O}_L/\mathfrak{p})}_{|:| = N(\mathfrak{p})} \right) \xleftarrow{\cong} \{ \sigma \in \text{Gal}(\tilde{L}/L) \mid \sigma \mathfrak{P}_i = \mathfrak{P}_i \} \subset \text{Gal}(\tilde{L}/L)$$

↑ generated by

$$\left\{ x \mapsto x^{N(\mathfrak{p})} \pmod{\mathfrak{P}_i} \right\}$$

$$\longleftarrow \text{=: Frob}_{\mathfrak{p}}$$

is (as the notation suggests) independent of i , yielding a map from

$$\left\{ \text{prime ideals of } L \right\} \longrightarrow \text{Gal}(\tilde{L}/L).$$

Taking \tilde{L} to be

$$H_L := \text{maximal unramified abelian extension of } L \quad (\text{"Hilbert class field"})$$

This leads ~~to~~ (eventually) to (*). □

More generally, given $I \in \mathcal{I}(K)$ we have

$$\mathcal{I}(I) = \frac{\text{fractional ideals prime to } I}{\text{principal " " with generator } \equiv 1 \pmod{I}} \quad \left(\begin{array}{l} \text{"ray class"} \\ \text{group mod } I \end{array} \right)$$

$L_I =$ (approximately*) the maximal abelian ext. in which primes dividing I are allowed to ramify ("ray class field mod I ")

\Rightarrow finite/L, and H_L , with equality when $I = \mathcal{O}_L$

and an isomorphism

$$\text{Gal}(L_I/L) \cong \mathcal{I}(I).$$

Example: for $L = \mathbb{Q}$, $L_{(n)} = \mathbb{Q}(\zeta_n)$.

□

To deal with

$L^{ab} :=$ maximal abelian extension of L ($\subset \bar{\mathbb{Q}}$),

which is infinite / L , we have to introduce the adèles.

In studying abelian varieties one considers (for $l \in \mathbb{Z}$ prime) the finite groups of l -torsion points $A[l]$; multiplication by l gives maps

$$\dots \rightarrow A[l^{n+1}] \rightarrow A[l^n] \rightarrow \dots$$

If we do the same thing on the unit circle $S^1 \subset \mathbb{C}^*$, we get

$$\dots \rightarrow S^1[l^{n+1}] \xrightarrow{\cdot l} S^1[l^n] \rightarrow \dots$$

$$\downarrow \cong \quad \quad \quad \downarrow \cong$$

$$\rightarrow \mathbb{Z}/l^{n+1}\mathbb{Z} \xrightarrow{\text{natural map}} \mathbb{Z}/l^n\mathbb{Z} \rightarrow$$

and one can define the l -adic integers by the inverse limit**

$$\mathbb{Z}_l := \varprojlim_n \mathbb{Z}/l^n\mathbb{Z}$$

** on elements of this is (by definition) an infinite sequence of elements in the $\mathbb{Z}/l^n\mathbb{Z}$ mapping to each other.

* more precisely, the maximal abelian ext. in which all primes $\equiv 1 \pmod{I}$ split completely.

There is the natural inclusion

$$\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell,$$

and

$$\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Elements of \mathbb{Q}_ℓ can be written as power series $\sum_{i \geq n} a_i \ell^i$ (for some $n \in \mathbb{Z}, \geq 0$ if one wants an element of \mathbb{Z}_ℓ). \mathbb{Q}_ℓ can also be thought of as the completion of \mathbb{Q} with respect to the ~~metric~~ metric given by

$$d(x, y) = \frac{1}{\ell^n} \text{ if } x - y = \ell^n \left(\frac{a}{b}\right) \text{ [with } a, b \text{ relatively prime to } \ell\text{]},$$

and the resulting topology on \mathbb{Z}_ℓ makes

$$U_n(d) := \{d + \lambda \ell^n \mid \lambda \in \mathbb{Z}_\ell\}$$

into "the open disk about $d \in \mathbb{Z}_\ell$ of radius $\frac{1}{\ell^n}$ ". \mathbb{Z}_ℓ itself is compact and totally disconnected.

Now set

$$\hat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/\ell^n \mathbb{Z} = \prod_{\ell} \mathbb{Z}_\ell$$

Chinese Remainder Thm.

The finite adèles appear naturally as

$$A_f := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \prod_{\ell}' \mathbb{Q}_\ell,$$

where \prod' means the ∞ -types with all but finitely many entries in \mathbb{Z}_ℓ . The "full" adèles are constructed by writing

$$A_{\mathbb{Z}} := \mathbb{R} \times \hat{\mathbb{Z}}$$

$$A_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} A_{\mathbb{Z}} = \mathbb{R} \times A_f,$$

which generalizes for a # field L to

$$A_L = L \otimes_{\mathbb{Q}} A_{\mathbb{Q}} = \mathbb{R}^{[L:\mathbb{Q}]} \times \prod_{\ell \in \mathcal{S}(L)}' L_{\ell}$$

* \prod' means all but finitely many entries in $(\mathbb{Q}_\ell)_{\mathfrak{p}}$.

For a \mathbb{Q} -algebraic group G , we can define

$$G(A_f) := \prod' G(\mathbb{Q}_p)$$

$$G(A_{\mathbb{Q}}) := G(\mathbb{R}) \times G(A_f)$$

\prod' : all but finitely many entries in $G(\mathbb{Q}_p)$, for some embedding $G \hookrightarrow GL_n$

with generalizations to $A_{L(f)}$. The idèles

$$A_{(f)}^{\times} = \mathbb{C}_m(A_{(f)})$$

$$A_{L(f)}^{\times} = (R_{\mathbb{Q}/\mathbb{R}} \otimes_{\mathbb{Q}} G_m)(A_{(f)}) = G_m(A_{L(f)})$$

were historically defined first of well introduced "adèle" (also a girl's name, which was intentional) as a contraction of "additive idèle".

The norm $N_{L/\mathbb{Q}}$ and reflex norm $N_{\mathbb{Q}/L}$ extend to maps of idèles, using the formulation of these maps as morphisms of \mathbb{Q} -algebraic groups.

Now returning to $S^1 \subset \mathbb{C}^*$, let S be an n th root of 1 and $a = (a_n) \in \hat{\mathbb{Z}}$; then

$$s^a := s^{a_n} \text{ defines an action of } \hat{\mathbb{Z}}$$

on the torsion points of S^1 (which generate \mathbb{Q}^{ab}).

The cyclotomic character

$$\chi: \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^{\times} \cong \mathbb{Q}^{\times} / A_{\mathbb{Q},f}^{\times}$$

is defined by $\sigma(S) := S^{\chi(\sigma)}$

and we can think of it as providing a "continuous envelope" ^{*}

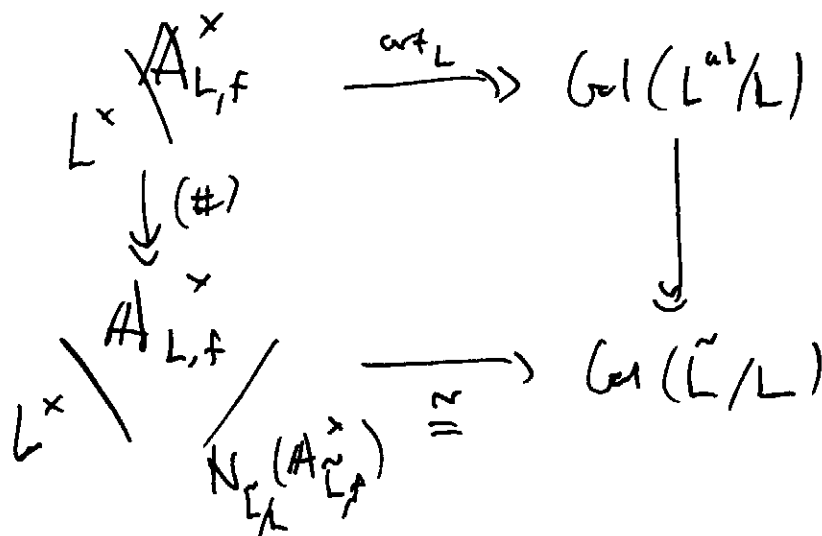
* I use this term because the automorphisms of \mathbb{C} other than complex conj. induce highly discontinuous (non-measurable!) maps on the complex points of a variety/ \mathbb{Q} . But if one specifies a set of points it is sometimes poss. to produce a continuous (non-continuous/algebraic) automorphism action in the same way as those σ !

for the action of a given σ on any finite order of torsion. (29)

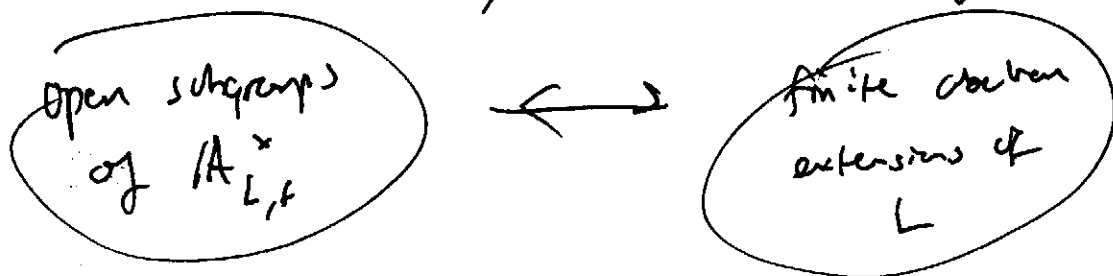
The Artin reciprocity map is simply its inverse

$$\text{art}_Q := \chi^{-1}$$

for \mathbb{Q} , but generalizes to (assuming L now totally imaginary)



If $\tilde{L} = L_f$ then the double-coset turns out to be $\mathcal{O}(I)$, so this recovers the earlier maps for ray class fields. (For $\tilde{L} = H_L$, we can replace $N_{L^x/L^x}(\dots)$ by $\hat{\mathcal{O}}_L$.) The correspondence between



is the essence of class field theory. Also note that $(\#)$ gives compatible maps to all the class groups of L , so that $A_{L,f}^x$ acts on them.

C. Main Theorem of CM

Now let's bring the adèles to bear upon abelian varieties. Taking the product of the Tak modules

$$T_{\mathbb{Q}} A := \varprojlim_n A[\mathbb{Z}^n]$$

(rank- $2g$ \mathbb{Z} -module)

(30)

of an abelian g -fold yields

$$T_f A := \prod_{\mathbb{Z}} T_{\mathbb{Q}} A, \quad V_f A := T_f A \otimes_{\mathbb{Z}} \mathbb{Q}$$

(= rank- $2g$ \mathbb{A}_f -module)

with (for example)

$$\text{Aut}(V_f A) \cong \text{GL}_{2g}(\mathbb{A}_f).$$

The main theorem* is basically a detailed description of the action of $\text{Gal}(\mathbb{C}/E')$ on $\text{Ab}(\mathcal{O}_E, \Phi)$ and the torsion points of the (finitely many \cong classes of) abelian varieties it classifies.

Theorem: Given $A_{[\text{Gal}]} \in \text{Ab}(\mathcal{O}_E, \Phi)$, $\sigma \in \text{Gal}(\mathbb{C}/E')$.

For any $a \in A_{E',f}^{\times}$ with $\text{art}_{E'}(a) = \sigma|_{(E')^{\text{ab}}}$:

(a) ${}^{\sigma} A_{[\text{Gal}]} \cong A_{\underbrace{N_{\mathbb{Q}/\mathbb{Z}}(a)}_{\text{depends only on } \sigma|_{H_{E'}}}} \cdot [a]$ (where $N_{\mathbb{Q}/\mathbb{Z}}(a) \in A_{E',f}^{\times}$), and

(b) $\exists!$ E -linear isogeny $\alpha: A_{[\text{Gal}]} \rightarrow {}^{\sigma} A_{[\text{Gal}]}$ s.t.

$$\alpha(N_{\mathbb{Q}/\mathbb{Z}}(a) \cdot x) = \sigma x \quad \forall x \in V_f A$$

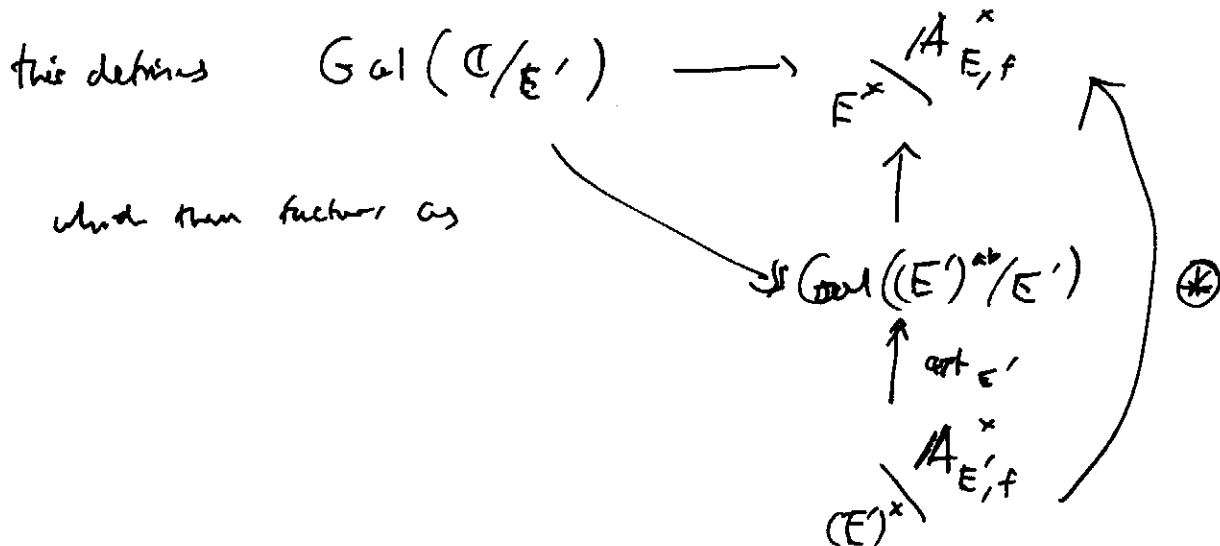
Idea: ${}^{\sigma} A \in \text{Ab}(\mathcal{O}_E, \Phi) \Rightarrow \exists (E\text{-linear})$ isogeny $\alpha: A \rightarrow {}^{\sigma} A$

of (b): $V_f A$ is free of rank 1 / $\mathbb{A}_{E',f}$.

* often credited to Shimura & Taniyama, though Weil played a huge role.
 As Milne puts it: "When (Weil) arrived at the famous Tokyo-Milne conference in ~~1955~~ (1955) planning to speak about CM, he was disconcerted to find ^{that} 'few young Japanese mathematicians, Shimura & Taniyama, were planning to speak about the same topic."

- the composition $V_f(A) \xrightarrow{\sigma} V_f(\bar{A}) \xrightarrow{V_f(\alpha)^{-1}} V_f(A)$ is $A_{E,f}$ -linear, (31)
 so = mult. by $s \in A_{E,f}^*$.

- s is independent (up to E^*) of the choice of α , and so



- A is defined / # field k ; the Shimura - Taniyama computation of the prime decompositions of the elements of $E \cong \text{End}(A)_{\mathbb{Q}}$ reducing to various Frobenius maps (in residue fields of k) then shows that the ~~vertical~~ ^{vertical} map \circledast is $N_{\mathbb{Q}/\mathbb{F}}$. Hence

$$N_{\mathbb{Q}/\mathbb{F}}(a) = s = V_f(\alpha)^{-1} \circ \sigma$$

which gives the formula in (b). □

So what does (b) mean? Like the cyclotomic character, we get a very nice interpretation when we restrict to the action on m -torsion points of A for any fixed $m \in \mathbb{N}$:

Corollary: $\exists!$ E -linear isogeny $\alpha_m: A \rightarrow \bar{A}$ such that

$$\alpha_m(x) = \bar{x} \quad \forall x \in A[m].$$

This is the second appearance of a "continuous envelope" for the action of automorphisms of \mathbb{C} on special points, ~~which will appear in the next lecture.~~

IV. Shimura varieties

A. Three key adelic lemmas

Besides the main theorem of CM, there is another (related) connection between the class field theory described in III.B and abelian varieties. The tower of ray class groups associated to the ideals of a CM field E can be expressed as

$$(*) \quad E^x \backslash \mathbb{A}_{E,f}^x / \mathcal{O}_Z \cong \underbrace{T(\mathbb{A}_f)}_{T(\mathcal{O})} \backslash \mathcal{U}_I$$

where

$$\mathcal{U}_I := \left\{ (a_p) \in \mathbb{A}_{E,f}^x \mid \begin{array}{l} a_p \in (\mathcal{O}_E)_{\mathfrak{p}} \ (\forall \mathfrak{p}) \\ a_p \equiv 1 \pmod{\mathfrak{p}^{ord_{\mathfrak{p}} I}} \end{array} \right\}$$

$\mathfrak{p} \in \mathcal{O}(E)$
prime for the finitely many \mathfrak{p} dividing I

is a compact open subgroup of $\mathbb{A}_{E,f}^x = T(\mathbb{A}_f)$ and

$$T = \text{Res}_{E/\mathbb{Q}} G_m$$

(*) may be seen as parametrizing abelian varieties with CM by a type (E, Φ) and having fixed level structure — which “refines” the set parametrized by $\text{Cl}(E)$. Shimura varieties give a way of extending this story to more general abelian varieties with other endomorphism / Hodge-tensor structures, as well as the other families of HS parametrized by HSD’s.

The first fundamental result we will need is

Lemma 1: For T any \mathbb{Q} -algebraic torus, and $K_f \in T(\mathbb{A}_f)$ any open subgroup, $T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / K_f$ is finite.

Sketch: This follows (from the definition of compactness) if we can show $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)$ is compact. For any # field, this is closed in $T(F) \backslash T(\mathbb{A}_{F,f})$, and for some F over which T splits the latter is $(F^\times \backslash \mathbb{A}_{F,f}^\times)^{\dim(T)}$. Finally, by the Minkowski bound

and $\hat{\mathcal{O}}_F$ is compact (like $\hat{\mathbb{Z}}$), so $F^\times \backslash \mathbb{A}_{F,f}^\times$ is compact. \square

For a very different class of \mathbb{Q} -algebraic groups, we have the contrasting

Lemma 2: Suppose G/\mathbb{Q} is semisimple and simply connected, of noncompact type; then

- (a) [Strong approximation] $G(\mathbb{Q}) \subseteq G(\mathbb{A}_f)$ is dense.
- (b) For any open $K_f \subseteq G(\mathbb{A}_f)$, $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K_f$
 (\Rightarrow the double coset $G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K_f$ is finite).

Note: Now double cosets are essential (as we are in the noncompact setting!)

Sketch of (a) \Rightarrow (b): Given $(\gamma_f) \in G(\mathbb{A}_f)$, $U := (\gamma_f) \cdot K_f$ is an open subset of $G(\mathbb{A}_f)$ hence (by (a)) $\exists g \in U \cap G(\mathbb{Q})$. Clearly $g = (\gamma_f) \cdot k$ for some k , and so $(\gamma_f) = g \cdot k^{-1}$. \square

\leftarrow i.e. none of its simple almost-direct factors G_i have $G_i(\mathbb{R})$ compact

Basic non-example: G_m , which is of course reductive but not semisimple. If (b) held, then the ray class groups of \mathbb{Q} ($\cong (\mathbb{Z}/l\mathbb{Z})^*$) would be trivial. But even more directly, were \mathbb{Q}^* dense in A_f^* , there would be $q \in \mathbb{Q}^*$ close to any $(g_l) \in \prod \mathbb{Z}_l^*$. This forces (the image in A_f^* of) q to lie in $\prod \mathbb{Z}_l^*$, which means for each l that (in lowest terms) the numerator + denominator of q are prime to l . So $q = \pm 1$, a contradiction. \square

Finally, for a general \mathbb{Q} -algebraic group G , we have

Lemma 3: The congruence subgroups of $G(\mathbb{Q})$ are precisely the $\underbrace{K_f \cap G(\mathbb{Q})}_{n \text{ in } G(\mathbb{A}_f)}$ for compact open $K_f \leq G(\mathbb{A}_f)$.

Sketch*: For $N \in \mathbb{N}$ the

$$K(N) := \left\{ (g_l) \in G(\mathbb{A}_f) \mid \underbrace{\left. \begin{array}{l} g_l \in G(\mathbb{Z}_l) \quad (\forall l) \\ g_l \equiv \mathbb{I} \pmod{l^{\text{ord}_l(N)}} \end{array} \right\}}_{\text{finite set of } l\text{'s}} \right\}$$

are compact open in $G(\mathbb{A}_f)$, and

$$\begin{aligned} K(N) \cap G(\mathbb{Q}) &= \left\{ g \in G(\mathbb{Z}) \mid g \equiv \mathbb{I} \pmod{l^{\text{ord}_l(N)}} \text{ for each } l \in \mathbb{N} \right\} \\ &= \left\{ g \in G(\mathbb{Z}) \mid g \equiv \mathbb{I} \pmod{N} \right\} = \Gamma(N). \end{aligned}$$

In fact, the $K(N)$ are a basis of open subsets $\ni \mathbb{I}$. So any compact open K_f contains some $K(N)$, and

* again one has to be careful about the \mathbb{Z} (hence \mathbb{Z}_l) structure arising from a choice of embedding $G \hookrightarrow GL_n$.

$$K_f \cap G(\mathbb{Q}) / K(N) \cap G(\mathbb{Q}) \subseteq K_f / K(N)$$

is a discrete subgroup of a compact set, and \therefore finite. \square

In some sense, K_f is itself the congruence condition.

B. Shimura data

A $\left\{ \begin{array}{l} \text{Shimura datum} \\ \text{connected s.d.} \end{array} \right.$ is a pair $(G, \left\{ \begin{array}{l} \tilde{X} \\ X \end{array} \right\})$

consisting of $\bullet G = \left\{ \begin{array}{l} \text{reductive} \\ \text{semisimple} \end{array} \right.$ algebraic group / \mathbb{Q}

$\bullet \left\{ \begin{array}{l} \tilde{X} \\ X \end{array} \right. = \left\{ \begin{array}{l} G(\mathbb{R})^- \\ G^{ad}(\mathbb{R})^+ \end{array} \right.$ ccl of homeomorphisms $\tilde{\varphi}: \mathbb{I} \rightarrow \left\{ \begin{array}{l} G_{\mathbb{R}} \\ G_{\mathbb{R}}^{ad} \end{array} \right.$

satisfying the axioms

SV1: only $z/\bar{z}, 1, \bar{z}/z$ occur as eigenvalues of $\text{Ad} \circ \tilde{\varphi}: \mathbb{I} \rightarrow GL(\text{Lie}(G^{ad})_{\mathbb{C}})$.

SV2: $\text{Ad}(\tilde{\varphi}(i)) \in \text{Aut}(G^{ad})$ is Cartan

SV3: G^{ad} has no \mathbb{Q} -factor on which the projection of every $\tilde{\varphi} \in X^{(n)}$ is trivial

SV4: The weight homomorphism $G_m \xrightarrow{w_{\tilde{\varphi}}} \mathbb{I} \xrightarrow{\tilde{\varphi}} G_{\mathbb{R}}$ is defined / \mathbb{Q}

SV5: $(\mathbb{Z}^{\circ} / w_{\tilde{\varphi}}(\mathbb{Z}^{\circ}))(\mathbb{R})$ is compact [sometimes weakened - cf. Milne Chap. IV]

SV6: \mathbb{Z}° splits over a CM field.

In the "connected" case, SV5-6 are trivial, while

SV1-3 already imply

• $X = \text{HSD}$ (in precise sense of (3) in I.B) [SV1]

• G of noncompact type, but with $\ker(G(\mathbb{R})^+ \rightarrow \text{Hol}(X)^+)$ compact [SV2]

map defined by the action on the cell

SV4-6 are sometimes omitted (for example, canonical models exist for SV's without them), but are satisfied in the context of Hodge theory, and so we include them. Indeed, a MTD \tilde{X} for PHS with generic MT group G , produces a S.d. satisfying

SV4: \mathbb{Q} -HS!!

SV5: G MT group $\Rightarrow G/G_m$ contains a $G(\mathbb{R})^+$ -cell of anisotropic maximal real tori; these contain \mathbb{Z}^0/G_m .

SV6: Since G/\mathbb{Q} , the cell contains tori defined $/\mathbb{Q}$; and so $X \ni \tilde{\varphi}$ factoring through some such rational Π . This defines a polarized CM-HS, with MT group a \mathbb{Q} -torus $\Pi_0 \leq \Pi$ split over a CM field (cf. lecture III). If $\Pi_0 \not\geq \mathbb{Z}^0$, then the projection of $\tilde{\varphi}$ to some \mathbb{Q} -factor of \mathbb{Z}^0 is trivial and then it is trivial for all its conjugates, contradicting SV3 (or: the MT group is then smaller). Hence $\Pi_0 \geq \mathbb{Z}^0$ and \mathbb{Z}^0 splits over the CM field. □

Any S.d. produces a connected S.d. by

• replacing G by G^{der} (which has the same G^{ad})

• replacing \tilde{X} by a connected component X

(which we may view as a $G^{\text{ad}}(\mathbb{R})^+$ -cell of homomorphisms $\text{ad} \circ \tilde{\varphi}$),

and so

\tilde{X} is a finite union of HSD's.

As in the "Hodge domains" described by Griffiths, for any faithful representation $\rho: G \hookrightarrow GL(V)$ \tilde{X} is realized as a MT domain (but with trivial IPR) parametrizing the HS's $\rho \circ \tilde{\rho}$. I don't know whether $SV1-6 \Rightarrow G$ (and not some subgroup surjecting onto G^{ad}) is the MT group.

Now given a connected Shimura domain (G, X) , we add one more ingredient: let

$\Gamma \leq G^{ad}(\mathbb{Q})^+$ be a torsion-free arithmetic subgroup, with inverse image in $G(\mathbb{Q})^+$ a congruence subgroup.

Its image $\bar{\Gamma}$ in $Hol(X)^+$ is

- (i) [torsion-free] arithmetic: since $\ker(G(\mathbb{R})^+ \rightarrow Hol(X)^+)$ compact
- (ii) isomorphic to Γ : $\Gamma \cap \ker = \text{discrete} \cap \text{compact} = \text{finite}$ hence torsion (and there is no torsion).

~~(iii) torsion-free~~

We may write

$$X(\Gamma) := \Gamma \backslash X \stackrel{(ii)}{=} \bar{\Gamma} \backslash X$$

and $\bar{\Gamma} \backslash X$ is a locally symmetric variety by (i) and Baily-Borel.

By Borel's theorem,

(*) $\Gamma \geq \Gamma' \Rightarrow X(\Gamma') \twoheadrightarrow X(\Gamma)$ is algebraic.

Definition: The connected S.V. ^(Shimura variety) associated with (G, X, Γ) is

$$Sh_{\Gamma}^{\circ}(G, X) := X(\Gamma)$$

Remark: Every $X(\Gamma)$ is covered by $X(\Gamma')$ with Γ' the image of a congruence subgroup of $G(\mathbb{Q})^+$. If one works with

"sufficiently small" congruence subgroups of $G(\mathbb{Q})^+$, then

- they belong to $G(\mathbb{Q})^+$
- the home torsion-free image in $G^{ad}(\mathbb{Q})^+$
- congruence \Rightarrow arithmetic.

This will be tacit in what follows.

C. The adelic reformulation

Consider a connected S.d. (X, G) with G simply connected.

Let $K_f \leq G(\mathbb{A}_f)$ be a ("sufficiently small") compact open subgroup and (Lemma 3) $\Gamma = G(\mathbb{Q}) \cap K_f$ the corresponding subgroup of $G(\mathbb{Q})$; replacing the earlier notation we write

$$Sh_{K_f}^{\circ}(G, X)$$

for the associated locally symmetric variety.

Proposition: $(\Gamma \backslash X) \cong Sh_{K_f}^{\circ}(G, X) \cong_{\text{homeo.}} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K_f$

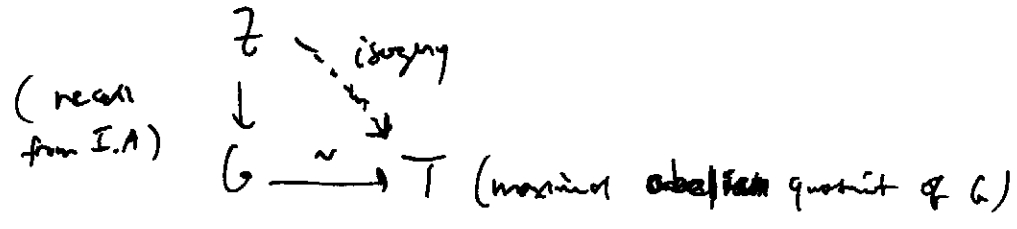
class
 $G(\mathbb{A}_f) / K = K_{\mathbb{Q}} \times K_f$,
except $X = G^{ad}(\mathbb{Q}) \backslash K_{\mathbb{R}}$

where $g \cdot (\tilde{\varphi}, a) \cdot k := (\underbrace{g \cdot \tilde{\varphi}}_{g \tilde{\varphi} g^{-1}}, g a k)$

The first part of the Theorem below says (a) doesn't

~~depend~~ contribute: the indexing of the components is "entirely arbitrary". First, some

Notation: $X :=$ a connected component of \tilde{X}
 $G(\mathbb{R})_+ :=$ preimage of $G^{der}(\mathbb{R})^+$ in $G(\mathbb{R})$



$$T(\mathbb{R})^\dagger := \text{Im}(Z(\mathbb{R}) \rightarrow T(\mathbb{R}))$$

$$Y := T(\mathbb{Q})^\dagger \setminus T(\mathbb{Q})$$

Theorem: (i) $G(\mathbb{Q})_+ \setminus X \times G(A_f) / K_f \cong G(\mathbb{Q}) \setminus \tilde{X} \times G(A_f) / K_f$

(ii) The map $G(\mathbb{Q})_+ \setminus X \times G(A_f) / K_f \rightarrow \dots$ for G^{der} simply connected

$$C := G(\mathbb{Q})_+ \setminus G(A_f) / K_f \xrightarrow[\sim]{\nu} T(\mathbb{Q}) \setminus T(A_f) / \nu(K_f) \cong T(\mathbb{Q}) \setminus Y \times T(A_f) / \nu(K_f)$$

"indexes" the connected components. (Henceforth "C" denotes a set of representatives in $G(A_f)$.)

(iii) $Sh_{K_f}(G, \tilde{X}) \cong \coprod_{g \in C} \Gamma_g \setminus X$, a finite union.

Proof: Exercise: • preimage of $\Gamma \in C$ is $G(\mathbb{Q})_+ \setminus X \times G(A_f)_+ / K_f \cong Sh_{K_f}(G^{der}, X)$
 • check that $(\gamma \cdot \tilde{p}, g) \equiv (\tilde{p}, g)$ for $\gamma \in \Gamma_g$.

That $|C| < \infty$ is by lemma 1! The rest is in Milne Ch.5. □

These arise as \mathcal{C} in the Theorem and also from CM-Hodge structures (MT group = T).

② Siegel modular variety

$V = \mathbb{Q}$ -vector space

$\psi =$ nondegenerate alternating form / \mathbb{Q}

(V, ψ) \mathbb{Q} -symplectic space

$$G := \mathbb{G}Sp(V, \psi) = \left\{ g \in GL(V) \mid \psi(gu, gv) = \chi(g) \psi(u, v) \quad \forall u, v \in V \text{ and same } \chi(g) \in \mathbb{Q}^* \right\}$$

[Exercise: $\chi: G \rightarrow \mathbb{G}_m$ defines a character.]

$$\text{let } X^\pm := \left\{ J \in Sp(V, \psi)(\mathbb{R}) \mid \begin{array}{l} \text{ ~~} J^2 = -I \text{ } \end{array}, \psi(u, Jv) \text{ is } \pm\text{-definite} \right\}~~$$

(= positive & negative symplectic ^{complex structures})

and put

$$\tilde{X} := X^+ \amalg X^- \quad \text{regarded as homomorphism ^{via} } \tilde{\rho}(a+bi) := a+bJ \text{ for } a+bi \in \mathbb{C}^* = \mathbb{S}(\mathbb{R}).$$

Then $G(\mathbb{R})$ acts transitively on \tilde{X} , and the datum (G, \tilde{X}) satisfies SV1-6;

for any compact open $K_f \leq G(\mathbb{A}_f)$ the attached SV is a Siegel modular variety.

Now consider the set

$$\mathcal{M}_{K_f} := \left\{ (A, \mathcal{Q}, \eta) \mid \begin{array}{l} A \text{ abelian variety / } \mathbb{C} \\ \pm \mathbb{Q} \text{ polarization of } H_1(A, \mathbb{Q}) \\ \eta: V_{\mathbb{A}_f} \rightarrow V_f(A) (\cong H_1(A, \mathbb{A}_f)) \text{ isomorphism} \\ \text{sending } \psi \mapsto a \cdot \mathcal{Q} \quad (a \in \mathbb{A}_f^*) \end{array} \right\}$$

\cong

where an isomorphism

$$(A, Q, \gamma) \xrightarrow{\cong} (A', Q', \gamma')$$

is an isogeny $f: A \rightarrow A'$ sending $\Psi \mapsto q \cdot Q$ ($q \in \mathbb{Q}^*$)

such that for some $k \in K_f$

$$\begin{array}{ccc}
 V_{A_f} & \xrightarrow{\gamma} & V_f(A) \\
 \downarrow \cdot k & & \downarrow f \\
 V_{A_f} & \xrightarrow{\gamma'} & V_f(A')
 \end{array}
 \quad \text{commutes.}$$

M_{K_f} is a moduli space for polarized abelian varieties with K_f -level structure. Write $\tilde{\varphi}_A$ for the HS on $H_1(A)$, and choose an $\alpha: H_1(A, \mathbb{Q}) \rightarrow V$ sending Ψ to Q (up to \mathbb{Q}^*).

Proposition: The (well-defined) map

$$M_{K_f} \xrightarrow{\cong} Sh_{K_f}(G, \tilde{X})$$

induced by

$$(A, Q, \gamma) \longleftrightarrow (\alpha \circ \tilde{\varphi}_A \circ \alpha^{-1}, \alpha \circ \gamma)$$

is a bijection.

③ Shimura varieties of PEL type

(V, ψ) symplectic $(B, *)$ -module, i.e.

- (V, ψ) symplectic space / \mathbb{Q}
- $(B, *)$ simple \mathbb{Q} -algebra with positive involution $*$
 $(\text{Tr}_{B \otimes_{\mathbb{Q}} \mathbb{R} / \mathbb{R}}(b^* b) > 0)$

• V is a B -module and $\Psi(bu, v) = \Psi(u, b^*v)$. (49)

We put $G = \text{Aut}_B(V) \cap \text{GSp}(V, \Psi)$, which is of generalized Sh , Sp , or SO type (related to the Albert classification) according to the structure of $(B, *)$. (Basically, G is cut out of GSp by fixing tensors in $T^{\otimes 1}V$.) The (canonical) associated ccd \tilde{X} complexes this to a Shimura domain, and the associated SV's parametrize \mathbb{P} -divided abelian varieties with \mathbb{E} -endomorphism and \mathbb{L} -level structure (essentially, a union of \mathbb{Q} -quotients of MT domains cut out by \mathbb{E}).

(4) Shimura varieties of Hodge type

G is cut out of GSp by fixing tensors of all degrees.

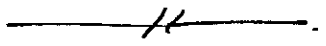
Remark:

In both (3) & (4) X is a subdomain of a fixed domain, so "of Hodge type" excludes the type \mathbb{D}/\mathbb{E} Hermitian symmetric domains which still do yield SV's parametrizing equivalence classes of HS's. So the last example is more general still.

(5) Mumford-Tate groups / domains w./-vanishing IPR

In notation from I.D, $G = M_x$, $\tilde{X} = M_x(\mathbb{R})$, $\tilde{\varphi} =: D_x$
 (for $\text{MT}(\tilde{\varphi}) = M_x$)
 (X will be D_x^+), $K_f \leq M_x(\mathbb{A}_f)$ with product Shimura varieties parametrizing higher weight HS's, under the assumption that D_x has trivial IPR.
 \mathbb{R} -equivalence classes of

In addition to the examples in I.D, one prototypical example is the MT group (essentially a $U(1, n)$) and domain for HS of weight 3 and type $(1, n, n, 1)$ with endomorphisms by an imaginary quadratic field, in such a way that the two eigenspaces are $V^{3,0} \oplus V^{2,1}$ and $V^{1,2} \oplus V^{3,0}$. This has vanishing IPR and yields a Shimura variety.



V. Fields of definition

Let D be a period domain for HS polarized by \mathbb{Q} with fixed Hodge #'s. The compact dual of a MT domain $D_M = M(\mathbb{R}) \cdot \varphi \subseteq D$ is the $M(\mathbb{C})$ -orbit of the attached filtration F_φ ,

$$\check{D}_M = M(\mathbb{C}) \cdot F_\varphi.$$

It is a connected component of the "MT Noether-Lefschetz locus" cut out of \check{D} by the criterion of (a Hodge flag $\in \check{D}$) having MT group contained in M ,

$$\check{D}_M \subset \check{N}L_M \subset \check{D}.$$

Now, $\check{N}L_M$ is cut out by \mathbb{Q} -tensors hence defined / \mathbb{Q} , but its components ($M(\mathbb{C})$ -orbits) are permuted by the action of $\text{Aut}(\mathbb{C})$. The fixed field of the subgroup of $\text{Aut}(\mathbb{C})$ preserving \check{D}_M , is considered its field of definition; this is defined regardless of the vanishing of the IPR (or D_M being Hermitian symmetric). What is interesting in the Shimura variety case, is that this field has many "downstairs", for $\rho \in \rho_M$ — even though the upstairs-downstairs correspondence is highly transcendental.

A. Reflex field of a Shimura datum

Let (G, \tilde{X}) be a Shimura datum; we start by repeating the definition just alluded to in this context. Recall that any $\tilde{\varphi} \in \tilde{X}$ determines a complex cocharacter of G by $z \mapsto \tilde{\varphi}_{\mathbb{C}}(z, 1) =: \mu_{\tilde{\varphi}}(z)$. (Complex cocharacters are themselves more general and essentially correspond to points of the compact dual.)

Write, for any subfield $k \subset \mathbb{C}$,

$$\mathcal{P}(k) = G(k) \backslash \text{Hom}_k(G_{\mathbb{R}}, G_k)$$

for the set of $G(k)$ -conjugacy classes of k -cocharacters. ~~is~~

~~isomorphic to~~ $\text{Gal}(k/\mathbb{Q})$ acts on $\mathcal{P}(k)$, since $G_{\mathbb{R}}$ & G are \mathbb{Q} -algebraic groups.

The element

$$c(\tilde{X}) := [\mu_{\tilde{\varphi}}] \in \mathcal{P}(\mathbb{C})$$

is independent of the choice of $\tilde{\varphi} \in \tilde{X}$.

Definition: $E(G, \tilde{X})$ is the fixed field of the subgroup of $\text{Aut}(\mathbb{C})$ fixing $c(\tilde{X})$ as an element of $\mathcal{P}(\mathbb{C})$.

Examples:

(1) A = abelian variety of CM type (E, Φ) , E' = associated reflex field (Lecture III)
 $\tilde{\varphi}$ the HS on $H^1(A)$, $T = M_{\tilde{\varphi}} \subset \text{Res}_{E/\mathbb{Q}} G_m$.

Then $\mu_{\tilde{\varphi}}(z)$ multiplies $\underbrace{H^{1,0}}_{\Phi\text{-eigenspaces for } E}$ by z and $\underbrace{H^{0,1}}_{\bar{\Phi}\text{-eigenspaces for } E}$ by \bar{z}

(clearly $\text{Ad}(\sigma) \mu_{\tilde{\varphi}}(z)$ (for $\sigma \in \text{Aut}(\mathbb{C})$) multiplies the $\sigma\bar{\Phi}$ -eigenspaces

by z , while $T(\mathbb{C})$ acts trivially on $\text{Hom}_{\mathbb{C}}(G_m, T_{\mathbb{C}})$.

Consequently, σ fixes $c(\{\tilde{\varphi}\}) \Leftrightarrow \sigma$ fixes \mathbb{D} , and so

$$\underline{E(T, \{\tilde{\varphi}\}) = E'}$$

(2) For an inclusion $(G', \tilde{X}') \hookrightarrow (G, \tilde{X})$, one has $E(G', \tilde{X}') \supseteq E(G, \tilde{X})$. Every \tilde{X} has $\tilde{\varphi}$ factoring through relevant tori, which are then CM-HS. The torus then splits over \mathbb{Q} a CM field, and so $E(G, \tilde{X})$ is always contained in a CM field. (In fact, it is always either CM or totally real.)

(3) For the $\begin{cases} \text{Siegel case} \\ \text{PGL} \end{cases}$, $E(G, \tilde{X}) = \begin{cases} \mathbb{Q} \\ \mathbb{Q} \left(\left\{ \text{Tr}(\perp |_{T_{\mathbb{R}} \otimes_{\mathbb{Q}}(A)} \right\}_{s \in S} \right) \end{cases}$. □

Let T be a \mathbb{Q} -algebraic torus, μ a character defined over a finite extension K/\mathbb{Q} . Define by

$$r(T, \mu) : \text{Res}_{K/\mathbb{Q}} G_m \rightarrow T$$

the homomorphism given on rational points by

$$\begin{aligned} K^* &\rightarrow T(\mathbb{Q}) \\ k &\mapsto \prod_{\varphi \in \text{Hom}(K, \bar{\mathbb{Q}})} \varphi(\mu(k)) \end{aligned}$$

As in Example (2), every (G, \tilde{X}) contains a CM-pair $(T, \{\tilde{\varphi}\})$.

Let $E(\tilde{\varphi}) :=$ field of definition of $\mu_{\tilde{\varphi}} (= E(T, \{\tilde{\varphi}\}))$. The

map
$$r(T, \mu_{\tilde{\varphi}}) : \text{Res}_{E(\tilde{\varphi})/\mathbb{Q}} G_m \rightarrow T$$

yields on $A_{\mathbb{Q}}$ - points

$$A_{E(\tilde{\varphi})}^{\times} \xrightarrow{r(T, \mu_{\tilde{\varphi}})} T(A_{\mathbb{Q}}) \xrightarrow{\text{forget}} T(A_f)$$

$\underbrace{\hspace{15em}}_{=: r_{\tilde{\varphi}}}$

(49)

Ex / For Example (4) above, $E(\tilde{\varphi}) = E'$ and $T = \text{Res}_{E/\mathbb{Q}} G_m$

\Rightarrow the $r(T, \mu_{\tilde{\varphi}})$ part of this map is the adelicized

reflex norm $\mathcal{N}_{\mathbb{Q}, E'}(A_{\mathbb{Q}}) : A_{E'}^{\times} \rightarrow A_E^{\times}$

(This is not a trivial calculation.)

B. Canonical Models

The Shimura varieties we have been discussing, i.e. $Sh_{K_f}(G, X)$, are finite disjoint unions of locally symmetric varieties and hence algebraic varieties defined a priori / \mathbb{C} . More generally,

if Y is any variety / \mathbb{C} and $k \subset \mathbb{C}$ is a subfield, a model of Y over k is

- a variety Y_0 / k , together with
- an isomorphism $Y_0, \mathbb{C} \xrightarrow[\cong]{\theta} Y$.

~~The~~ For general algebraic varieties, it is not true that two models over the same field k are necessarily isomorphic over that field. But if we impose a condition on how

$\text{Gal}(\mathbb{C}/E, \cdot)$ acts on a chosen set of points on any model,

then the composite isomorphism $\gamma_{0, \mathbb{C}} \xrightarrow{\cong} \gamma \xrightarrow{\cong} \tilde{\gamma}_{0, \mathbb{C}}$

is forced to be $\text{Gal}(\mathbb{C}/E, \cdot)$ -equivariant, making γ_0 and $\tilde{\gamma}_0$ isomorphic over E' . Repeating this criterion for more point sets and $\{E_i'\}$'s means γ_0 and $\tilde{\gamma}_0$ will be \cong over $\bigcap_i E_i'$.

For the dense sets of points:

Definition 1: (a) A point $\tilde{q} \in \tilde{X}$ is a CM point \Leftrightarrow

\exists a (\mathbb{Q} -algebraic) ^{minimal,} torus $T \subset G$ s.t. $\tilde{q}(T(\mathbb{R})) \subset T(\mathbb{R})$.

(b) $(\Pi, \{\tilde{q}\})$ is then a CM pair in (G, \tilde{X}) .

~~To get density in \tilde{X} , look at $G(\mathbb{R})$ in $\text{Sh}_k(G, \tilde{X})$,~~

~~for \tilde{q} , look at $T(\mathbb{R})$.~~ (Such a \tilde{q} exists since in a \mathbb{Q} -algebraic group every set of ^{maximal} 'real' tori contains one defined / \mathbb{Q} ;

to get density in \tilde{X} look at the orbit $G(\mathbb{R}) \cdot \tilde{q}$; to get density, more importantly, in $\text{Sh}_{k_f}(G, \tilde{X})$, look at the set $\{(\tilde{q}, a)\}$.)

For the condition on Galois action:

~~For any CM field E' , recall the Artin reciprocity map~~

For any CM field E' , recall the Artin reciprocity map

$$\text{Art}_{E'} : \mathbb{A}_{E'}^\times \rightarrow \text{Gal}(E'^{\text{ab}}/E')$$

Definition 2: A model $M_{k_f}(G, \tilde{X})$ of $\text{Sh}_{k_f}(G, \tilde{X})$ over $E(G, \tilde{X})$

is canonical iff $(\begin{matrix} \bullet \\ \downarrow \\ \emptyset \end{matrix})$

- for any
- $(M, (\pi, \tilde{\varphi})) \subset (G, \tilde{X})$
 - $a \in G(A_F)$
 - $\sigma \in \text{Gal}(E(\tilde{\varphi})^{cb}/E(\tilde{\varphi}))$
 - $s \in \text{Art}_{E(\tilde{\varphi})}^{-1}(\sigma) \subset A_{E(\tilde{\varphi})}^*$

$\theta^{-1}[(\tilde{\varphi}, a)]$ is a point defined $/ E(\tilde{\varphi})^{cb}$, and

(#) $\sigma \cdot \theta^{-1}[(\tilde{\varphi}, a)] = \theta^{-1}[(\tilde{\varphi}, r_{\tilde{\varphi}}(s) a)]$ □

↑
(essentially a reflex norm)

~~Remark: This action on CM points turns out to have the following action on $\text{Sh}_{K_F}(G, \tilde{X}) \cong (T(\mathbb{Q}) \backslash Y \times T(A_F) \backslash U(K_F))$:~~

~~for any $\tilde{\varphi} \in \tilde{X}$, put $r = r(T, U, \mu) : A_{E(G, \tilde{X})}^* \rightarrow T(A_{\mathbb{Q}})$; (kernel G has T maximal abelian quotient)~~

~~then for $\sigma \in \text{Gal}(E(G, \tilde{X})^{cb}/E(G, \tilde{X}))$ and $s \in \text{Art}_{E(G, \tilde{X})}^{-1}(\sigma)$ we get~~

~~$\sigma \cdot [y, a] = [r(s) \cdot y, r(s) \cdot a]$.~~

The uniqueness of the canonical model is clear from the argument above — if one exists — since we can take the E_i to be various $E(\tilde{\varphi})$ for $(m, \tilde{\varphi})$, whose intersections are known to give $E(G, \tilde{X})$.

To see how existence might come about for SV's of Hodge type (it is known for all SV's), first note that by

- Baily-Borel,

$\mathcal{M}_g := \text{Sh}_K(G, \tilde{X})$ is a variety $/ \mathbb{C}$.

Now we know that

- Sh_K is a moduli space for certain abelian varieties,

say $A \rightarrow Y$. Let $E = E(\theta, \tilde{x})$ and $\sigma \in \text{Aut}(C/E)$.

Given $P \in Y(C)$ we have an equivalence class $[A_P]$ of cubic vertices, and we define a map

$$\begin{aligned} (\sigma Y)(C) &\rightarrow Y(C) \\ \text{by } \sigma(P) &\longmapsto [\sigma A_P]. \end{aligned}$$

That σA_P is still "in the family A " follows from

- definition of the reflex field
- Deligne's theorem (that the Hodge tensors determining A are absolute).

That these maps produce regular (iso)morphisms

$$f_\sigma : \sigma Y \rightarrow Y$$

boils down to

- Borel's theorem.

Now Y has (for free) a model ~~over~~ Y_0 over some L f.g. $/E$, and using

- $|\text{Aut}(Sh_k)| < \infty$

we may deduce that for σ' fixing L

$$\begin{array}{ccc} \sigma'_1 Y & \xrightarrow{\sigma'_1} & Y \\ \sigma'_1 \uparrow & & \uparrow \theta \\ \theta & & \theta \\ & & Y_0 \end{array}$$

commutes. At this point

it makes sense to spread Y_0 out over E — i.e. take all $\text{Gal}(C/E)$ -conjugates, viewed as a variety via $Y_0 \rightarrow \text{Spec } L \rightarrow \text{Spec } E$.

The diagram

$$\begin{array}{ccc} \sigma Y & \xrightarrow{f_\sigma} & Y \\ \sigma \uparrow \cong & & \cong \uparrow \theta \\ \theta & & \theta \\ \sigma Y_0 & & Y_0 \end{array}$$

Shows that the spread is constant; extending it over a quasi-projective base

~~shows~~ Y has a model defined over a finite extension of E .

(To get all the way down to \mathbb{F} requires some serious descent theory.) Finally, that the action of $\text{Aut}(\mathbb{C}/\mathbb{F})$ on the resulting model implied by the $\{f_\sigma\}$ satisfies (#) (hence yields a conical model), is true by

- the main theorem of CM.

(In fact, (#) is precisely encoding how Galois conjugation acts on various $\text{Ab}(E, \Phi) + \text{level structure}$.)

So the three key points are:

- the entire theory is used in the construction of conical models
- $\text{Sh}_{K_f}(G, \tilde{X})$ is defined over $\mathbb{F}(G, \tilde{X})$ independently of K_f
- the field of definition of a connected component $\text{Sh}_{K_f}(G, \tilde{X})^+$ is contained in $\mathbb{F}(G, \tilde{X})^{ab}$ and gets larger as K_f shrinks (and the # of connected components increases).

C. Connected Components and VHS

Assume G_{der} is simply connected.

The action on CM points imposed by (#) turns out to

force the following action on $\pi_0(\text{Sh}_{K_f}(G, \tilde{X})) \cong T(\mathbb{Q}) \backslash Y \times T(\mathbb{A}_f) / \nu(K_f$, where $G \twoheadrightarrow T$ is the maximal abelian quotient:

For any $\tilde{g} \in \tilde{X}$, put

$$r = r(T, \nu \circ \mu_{\tilde{g}}) : \mathbb{A}_{\mathbb{F}(G, \tilde{X})}^{\times} \rightarrow T(\mathbb{A}_{\mathbb{Q}});$$

then for $\sigma \in \text{Gal}(\mathbb{F}(G, \tilde{X})^{cb} / \mathbb{F}(G, \tilde{X}))$ and $s \in \text{ar}_{\mathbb{F}(G, \tilde{X})}^{-1} \sigma$ we get

$$\sigma \cdot [y, a] = [r(s)_{\infty} \cdot y, r(s)_f \cdot a].$$

Assume for simplicity χ is trivial. Writing $E' := E(G, \tilde{X})$ (54)
 and \mathbb{F} for the field of definition of the component $S :=$
 $\text{Sh}_{K_f}(G, \tilde{X})^+$ over $[1; 1]$, we have [the finite abelian extension
 of E']

$$\mathbb{F} = \text{fixed field of } \text{act}_{E'} \left(r_f^{-1} \left(T(\text{Gal}(\tilde{X}/K_f)) \right) \right).$$

That is, by virtue of the ~~theory~~ ^{theory} of central models
 we can essentially work down a minimal field of definition of
 the locally symmetric variety S .

Example: $(G, \tilde{X}) = (\Pi_{\tilde{X}} \tilde{\varphi})$ associated to an abelian
 variety with CM by \mathbb{F} , so that E' is the
 reflex field (and \mathbb{F} the field of definition of the
 point it lies over in a relevant Siegel modular variety).

Let $K_f = \mathcal{O}_{\mathbb{I}}$ for $\mathbb{I} \in \mathcal{L}(E)$ and consider the diagram

$$\begin{array}{ccccc}
 A_{E', \mathbb{F}}^{\times} & \xrightarrow{\mathcal{N}_{\mathbb{F}/\mathbb{F}} (= r_f)} & A_{E, \mathbb{F}}^{\times} & \longrightarrow & E^{\times} \setminus A_{E, \mathbb{F}}^{\times} / \mathcal{O}_{\mathbb{I}} \\
 \downarrow \text{act}_{E'} & & \downarrow \text{act}_E & & \cong \downarrow \overline{\text{act}}_E \\
 \text{Gal}((E')^{\text{ab}}/E') & \xrightarrow{\text{act}_{\mathbb{F}}} & \text{Gal}(E^{\text{ab}}/E) & \longrightarrow & \text{Gal}(\mathbb{I}/E) \\
 & & \text{act}_{\mathbb{F}} & & \uparrow \text{ray class field mod } \mathbb{I} \\
 & & \text{this exists (if } \mathbb{F} \text{ is continuous)} & & \\
 & & \text{in such a way that the l.h. space connects} & &
 \end{array}$$

We get that

$$\mathbb{E} = \text{ff}(\text{art}_{\mathbb{E}'}(\mathcal{N}_{\mathbb{Q}}^{-1}(\mathbb{E}^{\times} \mathcal{U}_{\mathbb{I}}))) = \text{ff}(\mathcal{N}_{\mathbb{Q}}^{-1}(\text{Gal}(\mathbb{E}^{\times} / \mathbb{E}_{\mathbb{I}})))$$

In case the CM abelian variety is an elliptic curve, $\mathcal{N}_{\mathbb{Q}}$ and $\mathcal{N}_{\mathbb{Z}}$ are essentially the identity (and $\mathbb{E}' = \mathbb{E}$), so

$$\mathbb{E} = \text{ff}(\text{Gal}(\mathbb{E}^{\times} / \mathbb{E}_{\mathbb{I}})) = \mathbb{E}_{\mathbb{I}}$$

It is a well-known result that, for example, the j -invariant of a CM elliptic curve generates (over the imaginary quadratic field $\mathbb{Q}(E)$) its Hilbert class field $\mathbb{E}_{(E)}$. We also see that the fields of definition of CM points in $X(N)$ (the modular curve) are ray class fields mod N . □

An application to VHS? Let $V \rightarrow S$ be a VHS with reference HS V_s over $s \in S$. The underlying local system V produces a monodromy representation

$$\rho : \pi_1(S) \rightarrow \text{GL}(V_s)$$

and we denote $\rho(\pi_1(S)) =: \Gamma_0$ with geometric monodromy group

$\overline{\Gamma} :=$ identity component of \mathbb{Q} -Zariski closure of Γ_0 .

Moreover, V has a MT group M ; and we make the following two crucial assumptions:

- $\bar{\Gamma} = M^{\text{der}}$
- D_M has vanishing IPR.

In particular, this means that the quotient of D_M by a congruence subgroup is a connected component of a Shimura variety, and that $\bar{\Gamma}$ is as big as it can be.

For any compact open $K_f \subseteq M(\mathbb{A}_f)$ such that

$\Gamma := K_f \cap M(\mathbb{Q}) \supseteq \Gamma_0$, V gives a period (analytic) mapping

$$\bar{\Psi}_{K_f}^{\text{an}} : \mathcal{S}_{\mathbb{C}}^{\text{an}} \rightarrow \Gamma \backslash D_M \cong \left(\text{Sh}_{K_f}(M, D_M)^+ \otimes_{\mathbb{E}} \mathbb{C} \right)^{\text{an}}$$

$\mathbb{E} = \mathbb{E}(K_f)$

This morphism is algebraic by Borel's theorem, and has minimal field of definition (trivially) bounded below by the field of definition $\mathbb{E}(K_f)$ of $\text{Sh}_{K_f}(M, D_M)^+$.

The period mapping which gives the most information about

V , is the one attached to the smallest congruence subgroup

$\Gamma \subset M(\mathbb{Q})$ containing Γ_0 . Taking then the largest K_f

with $K_f \cap M(\mathbb{Q}) = \Gamma$, minimizes the resulting $\mathbb{E}(K_f)$.

It is this last field which it seems natural to consider

as the "reflex field of a VHS" — an "expected lower bound"

for the field of definition of a period mapping of V . Furthermore,

if V arises (motrically) from $X \xrightarrow{\pi} \mathcal{S}$, then assuming Deligne's

Absolute Hodge conjecture, the $\bar{\mathbb{Q}}$ -spread of π produces a period mapping into D_M modulo a larger Γ , and our "reflex field of V " may be an upper bound for the minimal field of definition of this period map.

At any rate, the relations between fields of definition

- of
- varieties X_S ,
- transcendental period points \mathcal{P} in D_M ,
- equivalence classes of period points in $\Gamma \backslash D_M$,

and hence between spreads of

- families of varieties,
- VHS,
- period mappings,

is very rich and our suggested definition may be just an ~~essential~~ useful tool (if only in the Shimura case where the LPR = 0).

—————//—————