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Introduction to Algebraic Geometry

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Introduction to Algebraic Geometry

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All the rings considered unless specified are commutative rings with unit and the homomorphisms of rings send the unit on the unit.

Most of the text in algebraic is extracted from [2]. Notions of commutative algebra can be learnt in [1] or [3]. The last section can be read in [4].

1 Ringed spaces and Schemes

1.1 Sheaves of abelian groups

Let X be a topological space. A *pre-sheaf of abelian groups* on X is a contravariant functor of the category of open sets of X into the category of Abelian groups.

Example Let X be a topological space. For any open subset U of X, consider the abelian group $\mathcal{T}(U)$ of the continuous functions on U. The correspondence $U \mapsto \mathcal{T}(U)$ defines an abelian group pre-sheaf on X.

Definition Let $\mathcal{F}$ be a pre-sheaf of abelian groups on X. One says that the pre-sheaf $\mathcal{F}$ is a sheaf of abelian groups on X if:

1. if U is an open subset of X, if $(U_i)_{i\in I}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that its restriction $s|U_i = 0$, then $s = 0$.
2. if U is an open subset of X, if $(V_i)_{i\in I}$ is an open covering of U, and if, for each $i \in I$, we have an element $s_i \in \mathcal{F}(V_i)$ such that for each $i, j \in I$, we have the restrictions $s_i|V_i\cap V_j = s_j|V_j\cap V_i$, then there is an element $s \in \mathcal{F}(U)$ such that $s|V_i = s_i$ for any $i \in I$.

Let $\mathcal{F}$ be a pre-sheaf of abelian groups on X. The *stalk* of $\mathcal{F}$ at a point $x \in X$ is the $\varinjlim_{x\in U} \mathcal{F}(U)$.

To any pre-sheaf $\mathcal{F}$ of abelian groups on X one can associate a sheaf $\tilde{\mathcal{F}}$ of abelian groups on X and a morphism $i : \mathcal{F} \to \tilde{\mathcal{F}}$ of pre-sheaves of abelian groups, such that for any morphism of pre-sheaves of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ into a sheaf $\mathcal{G}$, there is a unique morphism of sheaves $\tilde{\varphi} : \tilde{\mathcal{F}} \to \mathcal{G}$, such that:

$$\begin{array}{ccc} \mathcal{F} & \rightarrow & \tilde{\mathcal{F}} \\ \varphi & \searrow & \downarrow \quad \tilde{\varphi} \\ & & \mathcal{G} \end{array}$$

is commutative.

Let X be a topological space. The pre-sheaves (resp. sheaves) of abelian groups on X make a category $\mathcal{P}(X)$ (resp. $\mathcal{S}(X)$ where the morphisms are the morphisms of pre-sheaves (resp. sheaves).

Let $f : X \to Y$ be a continuous map. Let $\mathcal{F}$ be a sheaf of abelian groups on X. The direct image of $\mathcal{F}$ by f is the sheaf $f_*(\mathcal{F})$ on Y defined for an open subset V of Y by:

$$f_*(\mathcal{F})(V) := \mathcal{F}(f^{-1}(V).$$

In this way f_* defines a functor from the category $\mathcal{S}(X)$ of sheaves on X to the category of sheaves on Y.

For any sheaf $\mathcal{G}$ on Y, we define the sheaf $f^{-1}(\mathcal{G})$ by:

$$f^{-1}(\mathcal{G})(U) := \varinjlim_{V \supset f(U)} \mathcal{G}(V)$$

for any open subset of X. Again f^{-1} defines a functor of the category $\mathcal{A}(Y)$ of sheaves on Y to the category of sheaves $\mathcal{A}(X)$.

If X is a subset of Y endow with the induced topology and i is the inclusion of X in Y, then, the sheaf $i^{-1}(\mathcal{G})$ is called the restriction of $\mathcal{G}$ to X. We often denote $i^{-1}(\mathcal{G})$ by $\mathcal{G}|X$. The stalk of $\mathcal{G}|X$ at $x \in X$ is the stalk $\mathcal{G}_x$ of $\mathcal{G}$ at x.

1.2 Ringed spaces

Definition The pair $(X, \mathcal{O}_X)$ of a topological space and a sheaf $\mathcal{O}_X$ with value in the category of commutative rings is called a *ringed space*. It is called a *locally ringed space*, if, for $x \in X$, the stalk $\mathcal{O}_{X,x}$ of $\mathcal{O}_X$ at x is a local ring.

Examples

1. Let X be an complex analytic manifold. Let $\mathcal{O}_X$ be the sheaf of complex analytic functions on X. Since for any open subset U of X the space $\mathcal{O}(U)$ is a ring, the pair $(X, \mathcal{O}_X)$ is a ringed space. The stalk $\mathcal{O}_{X,x}$ for any $x \in X$ is a local ring isomorphic to the local ring of convergent complex power series in n variables, where n is equal to the dimension $\dim_x(X)$ of X at x.

2. Let U be an open subset of the complex affine space $\mathbb{C}^n$. Let $\mathcal{O}_U$ the sheaf of complex analytic functions on U. Since U is a complex analytic manifold, the pair $(U, \mathcal{O}_U)$ is a locally ringed space.

   Let $f_i, i \in I$, be a family of complex analytic functions defined on U. Let $\mathcal{I}$ the sheaf of ideals generated by the family $f_i, i \in I$, in $\mathcal{O}_U$. Let X be the set of zeroes of the family $f_i, i \in I$, in U. The pair $(X, \mathcal{O}_U/\mathcal{I}|X)$ is a locally ringed space called a local analytic space.

3. Let A be a commutative ring. Let $Spec(A)$ be the space of prime ideals of A. Endow $Spec(A)$ with the Zariski topology in which $U \subset Spec(A)$ is open if, there is an ideal $\mathcal{I}$ such that $\mathfrak{P} \in U$ if and only if $\mathcal{I} \not\subset \mathfrak{P}$. The complement is $V(\mathcal{I}) = Spec(A) \setminus U$ is a closed set:

   $$V(\mathcal{I}) = \{\mathfrak{P} \in Spec(A) \,|\, \mathcal{I} \subset \mathfrak{P}\}$$

   For instance, if $a \in A$, the principal ideal (a) generated by a defines a closed set $V((a))$ and an open set $D(a) := Spec(A) \setminus V((a))$. The open sets $D(a)$, for $a \in A$, form a base for the Zariski topology of $Spec(A)$.

   Now, let us define a sheaf $\mathcal{O}_A$ on $Spec(A)$. Let U be an open subset of $Spec(A)$. For any prime ideal $\mathfrak{P} \in U$, we have the localization $A_{\mathfrak{P}}$ of the ring A at the prime ideal $\mathfrak{P}$. An element s belongs to $\mathcal{O}_A(U)$ if it is a map $U \to \coprod_{\mathfrak{P} \in U} A_{\mathfrak{P}}$ such that

   - for $\mathfrak{P} \in U$, $s(\mathfrak{P})$ belongs to $A_{\mathfrak{P}}$;
   - for $\mathfrak{P} \in U$, there are a neighbourhood V of $\mathfrak{P}$ in U and elements a, b of the ring A, such that for $\mathfrak{Q} \in V$, $b \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a/b \in A_{\mathfrak{Q}}$.

   The pair $(Spec(A), \mathcal{O}_A)$ is a locally ringed space.

Definition Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces (resp. locally ringed spaces). A *morphism of ringed spaces* (resp. locally ringed spaces) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, \tilde{f})$ where f is a continuous map from X to Y and $\tilde{f}$ is a morphism of sheaves of rings from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$ (resp. a morphism of sheaves of rings from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$ which induces, for each point $x \in X$ a local ring homomorphism from $\mathcal{O}_{Y,f(x)}$ to $\mathcal{O}_{X,x}$).

An *isomorphism of ringed space* (resp. locally ringed spaces) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, \tilde{f})$ where f is a homeomorphism from X on Y and $\tilde{f}$ is an isomorphism of sheaves of rings from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$ (resp. an isomorphism of sheaves of rings from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$ which induces, for each point $x \in X$ a local ring isomorphism from $\mathcal{O}_{Y,f(x)}$ onto $\mathcal{O}_{X,x}$).

Thus, one can define the category of ringed spaces (resp. locally ringed spaces).

1.3 Schemes

Let $(X, \mathcal{O}_X)$ be locally ringed space. Let x be a point of X and U be a neighbourhood of x in X. One can define the pair $(U, \mathcal{O}_X|U)$, where $\mathcal{O}_X|U$ is the restriction of the sheaf $\mathcal{O}_X$ to U. The pair $(U, \mathcal{O}_X|U)$ is also a locally ringed space.

Definition We say that the pair $(X, \mathcal{O}_X)$ is a *scheme* if for any $x \in X$, there is a neighbourhood $U(x)$ of x in X and a ring A, such that the locally ringed space $(U(x), \mathcal{O}_X|U(x))$ is isomorphic to $(Spec(A), \mathcal{O}_A)$ defined in the example above.

The category of schemes is the full subcategory of locally ringed spaces having this property. The objects of this category are schemes and the morphisms are morphisms of schemes considered as locally ringed spaces.

Examples

1. **Affine schemes** Let A be a ring. Let $\mathcal{O}_A$ be the sheaf defined on $Spec(A)$ defined above. The pair $(Spec(A), \mathcal{O}_A)$ is a scheme. Any scheme $(X, \mathcal{O}_X)$ isomorphic to $(Spec(A), \mathcal{O}_A)$ for some ring A is called an affine scheme.

   A morphism of affine schemes $\varphi : (Spec(A), \mathcal{O}_A) \to (Spec(B), \mathcal{O}_B)$ is given by a ring homomorphism $B \to A$.

2. **Open subschemes** Let $(X, \mathcal{O}_X)$ be a scheme. Let U be an open subset of X, endowed with the induced topology. The pair $(U, \mathcal{O}_X|U)$, where $\mathcal{O}_X|U$ is the restriction of $\mathcal{O}_X$ to U, is a scheme.

3. **Glueing of schemes** Let $(X_i, \mathcal{O}_{X_i})$, $i \in I$, be a family of schemes. Suppose for $i \neq j$, we are given an open subset U_{ij} of X_i. Consider the open subschemes $(U_{ij}, \mathcal{O}_{U_{ij}})$. Suppose given for $i \neq j$ an isomorphism of schemes $\varphi_{ij} : (U_{ij}, \mathcal{O}_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{U_{ji}})$ such that

   - for each i, j: $\varphi_{ji} = \varphi_{ij}^{-1}$;
   - for each i, j, k: $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$;
   - for each i, j, k: $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

   There is a scheme $(X, \mathcal{O}_X)$ and morphisms $\Phi_i : (X_i, \mathcal{O}_{X_i}) \to (X, \mathcal{O}_X)$ such that Φ_i is an isomorphism of $(X_i, \mathcal{O}_{X_i})$ onto an open subscheme of $(X, \mathcal{O}_X)$, these open subschemes give a covering of X, for all i, j, $\Phi_i((U_{ij}, \mathcal{O}_{U_{ij}})) = \Phi((X_i, \mathcal{O}_{X_i})) \cap \Phi_j((X_j, \mathcal{O}_{X_j}))$ and $\Phi_i = \Phi_j \circ \varphi_{ij}$ on $(U_{ij}, \mathcal{O}_{U_{ij}})$.

   The scheme $(X, \mathcal{O}_X)$ is said to be obtained by glueing the schemes $(X_i, \mathcal{O}_{X_i})$, $i \in I$, along the isomorphisms φ_{ij}.

   In the case, for all i, j, U_{ij} are empty, the scheme $(X, \mathcal{O}_X)$ is the disjoint union of the $(X_i, \mathcal{O}_{X_i})$, $i \in I$.

4. **Projective schemes** Let $R := \oplus_{n \in \mathbb{Z}} R_n$ be a graded ring and let $R_+ := \oplus_{n>0} R_n$ which is an ideal of R. Consider $Proj(R)$ be the set of homogeneous prime ideals of R which do not contain R_+. Remember that $\mathfrak{P}$ is a prime ideal of R if and only if for any homogeneous element a and b of R, $ab \in \mathfrak{P}$ if $a \in \mathfrak{P}$ or $b \in \mathfrak{P}$. We endow $Proj(R)$ with the Zariski

topology where a closed subset $V(\mathcal{I})$, defined by a homogeneous ideal $\mathcal{I}$, is the set of all the homogeneous prime ideals of R which contain $\mathcal{I}$:

$$V(\mathcal{I}) := \{\mathfrak{P} \in Proj(R) \,|\, \mathcal{I} \subset \mathfrak{P}\}$$

When the sheaf $\mathcal{I}$ is the homogeneous ideal generated by a homogeneous element $a \in R$, we denote:

$$D_+(a) := \{\mathfrak{P} \in Proj(R) \,|\, a \notin \mathfrak{P}\}$$

the open set defined by a. Notice that these open sets of $Proj(R)$ cover $Proj(R)$.

The sheaf $\mathcal{O}_R$ on $Proj(R)$ is defined in the following way. For $\mathfrak{P} \in Proj(R)$ consider the ring $R_{(\mathfrak{P})}$ of elements of degree zero in the localization $S^{-1}R$ with denominators in the multiplicative system S of all homogeneous elements which are not in $\mathfrak{P}$. Let U be an open subset of $Proj(R)$. We define $\mathcal{O}(U)$ as the set of functions $s : U \to \coprod_{\mathfrak{P}\in U} R_{(\mathfrak{P})}$ such that, for $\mathfrak{P} \in U$, the image $s(\mathfrak{P})$ is in $R_{(\mathfrak{P})}$ and, for each $\mathfrak{P} \in U$, there is a neighbourhood V of $\mathfrak{P}$ in U and homogeneous elements a and b of the same degree in R, such that, for all $\mathfrak{Q} \in V$, $b \notin \mathfrak{Q}$ and $s(\mathfrak{Q}) = a/b$ in $R_{(\mathfrak{Q})}$. One can prove that $\mathcal{O}_R$ is a sheaf.

One can show that $(Proj(R), \mathcal{O}_R)$ is a scheme that we call the projective scheme defined by the graded ring R. Furthermore $(D_+(a), \mathcal{O}_R|D_+(a))$ are subschemes of $(Proj(R), \mathcal{O}_R)$ which are isomorphic to $(Spec(R_(a)), \mathcal{O}_{R(a)})$, where $R_{(a)}$ is the subring of elements of degree 0 in the localized ring R_a.

If A is a ring and $A[x_0, \ldots, x_n]$ is the ring of $n+1$ variables over R, the projective scheme $(Proj(A[x_0, \ldots, x_n]), \mathcal{O}_{A[x_0,\ldots,x_n]})$ is the projective n-space $\mathbb{P}^n_A$ over A. In particular, if $A = k$ is an algebraically closed field, it is a scheme whose closed points is a set homeomorphic to the usual projective n-space over k.

A *scheme over the scheme* $(S, \mathcal{O}_S)$ is a scheme morphism $\varphi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$. A morphism of schemes over $(S, \mathcal{O}_S)$, $\varphi : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $\psi : (Y, \mathcal{O}_Y) \to (S, \mathcal{O}_S)$, is a morphism $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $\psi \circ \phi = \varphi$.

When $S = Spec(k)$ where k is a field and $\mathcal{O}_S := \mathcal{O}_k$, a scheme over $(S, \mathcal{O}_S)$ is also called a k-scheme.

2 Varieties and Schemes

2.1 Algebraic sets

Let k be a field. Consider the k-algebra of polynomials $k[X_1, \ldots, X_n]$ of n variables with coefficients in k. Let P be an element of $k[X_1, \ldots, X_n]$. A zero of P is a point $(a_1, \ldots, a_n)$ of the affine space k^n such that $P(a_1, \ldots, a_n) = 0$.

Hilbert Nullstellensatz asserts that if k is algebraically closed and P is not a constant polynomial, P has always a zero. This is why we shall assume in all these lectures that the field is algebraically closed.

Definition An *algebraic set* E of the affine space k^n is the set of all the zeroes of a family $(P_i)_{i \in I}$ of polynomials in $k[X_1, \ldots, X_n]$. Classically the P_i are the equations of E. It is easy to see that any polynomial of the ideal I of $k[X_1, \ldots, X_n]$ generated by the family $(P_i)_{i \in I}$ is an equation of E. In fact, Hilbert Nullstellensatz shows that any polynomial of the root $I(E) = \sqrt{(}I)$ of the ideal I generated by the family $(P_i)_{i \in I}$ is an equation and it gives all the equations of E.

An algebraic function on E is by definition the restriction to E of a polynomial function on an affine space which contains E. The algebra of all algebraic functions on E is isomorphic to the quotient algebra:

$$A(E) := \frac{k[X_1, \ldots, X_n]}{I(E)}$$

.

Definition A function φ is regular at a point $x \in E$ if there is an open neighbourhood $U(x)$ of x in E and polynomials f and g, such that g does not vanish on $U(x)$, and for any $y \in U(x)$, we have $\varphi(y) = f(y)/g(y)$. We say that the function φ is regular on E if it is regular at any $x \in E$.

Regular functions on E make a ring isomorphic to $A(E)$.

The points of E correspond to the maximal ideals of $A(E)$. Therefore one can embed E into $Spec(A(E))$. But one can see that $Spec(A(E))$ contains many other points, which are the prime ideals of $A(E))$ which are not maximal.

The sheaf $\mathcal{O}_{A(E))}$ induces the sheaf of algebraic functions on E and the topology of E is the usual Zariski topology of algebraic sets.

An algebraic set is irreducible, if it is not the union of two proper algebraic subsets.

By definition an irreducible algebraic set is an affine variety.

Hilbert finiteness theorem says that any algebraic set is the finite union of affine varieties.

2.2 Projective varieties

Consider the graduated ring $R := k[X_0, \ldots, X_n]$. Consider a homogeneous polynomial f of R. One can speak of zeros of f in the projective space $\mathbb{P}^n$ as lines $m \in \mathbb{P}^n$, so that f vanishes at any point $x \in m$.

One can define *projective algebraic sets* of $\mathbb{P}^n$ as sets of zeros of a set of homogeneous polynomials.

Let X be a projective algebraic set. A function φ is regular at $x \in X$ if there is an open neighbourhood of x in X and homogeneous polynomial f and g of the same degree, such that g does not vanish on $U(x)$ and $\varphi(y) = f(y)/g(y)$.

If X is a projective algebraic function, regular functions on X are constant.

The projective subsets of $\mathbb{P}^n$ define a family of closed subsets of a Topology in $\mathbb{P}^n$ called the Zariski topology of $\mathbb{P}^n$.

A projective set is irreducible if it is not the union of two proper projective sets. If the base field k is algebraically closed, homogeneous prime ideals of R are in bijection with irreducible projective subsets of $\mathbb{P}^n$.

An irreducible projective set is called a projective variety.

2.3 Varieties

Definition A *variety* is a Zariski open subset of an affine variety or of a projective variety.

Notice that a projective space is covered by affine spaces. Let $X_i \in R$ be i-th coordinate. It is a linear form. It defines in the projective space $\mathbb{P}^n$ a hyperplane H_i. Let U_i be the open subset $\mathbb{P}^n \setminus H_i$. There is a map φ_i from U_i to the affine space k^n given by $(a_0, \ldots, a_n) \mapsto (a_0/a_i, \ldots, a_n/a_i)$ where we omit a_i/a_i.

This map is a homeomorphism of U_i onto k^n endowed with the Zariski topology.

Then we obtain that if X is a projective variety (resp. quasi-projective variety) in $\mathbb{P}^n$, X is covered by the open sets $X \cap U_i$ which are homeomorphic with affine (resp. quasi-affine) varieties via the mapping φ_i defined before.

In fact any variety V is covered by affine varieties. For any point x of V, we want to show that there is an open affine subset which contains x. We have seen that V is covered by quasi-affine varieties, so we assume that V is quasi-affine in k^n. Let $W := \overline{V} \setminus V$. We may assume $W \neq \emptyset$. It is a closed set in k^n. Let $\mathcal{I}$ be the ideal of $k[X_1, \ldots, X_n]$ which defines W. Since $x \notin W$, there is a polynomial f in $\mathcal{I}$, such that $f(x) \neq 0$. Let H be the hypersurface of k^n defined by $f = 0$. The space $Y \setminus Y \cap H$ is an open subspace of Y. Also, since $Z \subset H$, $Y \setminus Y \cap H$ is closed in $k^n \setminus H$. Now, $k^n \setminus H$ is an affine variety isomorphic to the hypersurface of k^{n+1} given by $X_{n+1}f = 0$.

2.4 Morphisms

Let X be a quasi-affine variety in k^n. A function $\varphi : X \to k$ is regular at a point $x \in X$ if there is an open neighbourhood U of x in X and polynomials f ang g in $k[X_1, \ldots, X_n]$ such that g is nowhere 0 in U and $\varphi = f/g$ in U. We say φ is regular if it is regular at each point of X.

Now, suppose that X is a quasi-projective variety in $\mathbb{P}^n$. A function $\varphi : X \to k$ is regular at the point $x \in X$, if there is an open neighbourhood U of x in X and homogeneous polynomials f and g of the same degree such that g is nowhere 0 in U and $\varphi = f/g$ on U. We say that φ is regular on X if it is regular at any point of X.

One can define the category of varieties.

Let k be an algebraically closed field. A variety over k is any affine quasi-affine, projective, quasi-projective variety as defined before. A morphism $\phi : X \to Y$ of two varieties is a continuous map such that for any open set $\mathcal{U}$ of Y and any regular function $\varphi : \mathcal{U} \to k$, the function

$$\varphi \circ \phi : \phi^{-1}(\mathcal{U}) \to k$$

is regular.

The composition of two morphisms is a morphism. The identity is a morphism. In particular a morphism ϕ is an isomorphism, if there a morphism $\psi : Y \to X$ such that $\phi \circ \psi = Id_Y$ and $\psi \circ \phi = Id_X$.

k-Varieties define a category that we call $\mathfrak{V}(k)$.

On a variety X, one has the sheaf of regular functions $\mathcal{O}_X$, where for U open subset of X, $\mathcal{O}_X(U)$ is the ring of regular functions on X. For any x in X, the stalk $\mathcal{O}_{X,x}$ is the local ring of germs of regular function at x.

2.5 Rational maps

Definition Let X and Y be varieties over k. A rational map $\psi : X \to Y$ is an equivalence class of pairs (U, ψ_U), where U is a non-empty open subset of X and ψ_U is a morphism from U to Y, and where (U, ψ_U) and (V, ψ_V) are equivalent if ψ_U and ψ_V coincide on $U \cap V$. The rational map is dominant if for some (U, ψ_U), the image ψ_U is dense in Y.

If for some pair (U, ψ_U) of a rational map $\psi : X \to Y$, the image of ψ_U is dense in Y, this is true for any pair (V, ψ_V) of ψ. One can define the category of varieties and dominant rational maps, since one can "compose" rational dominant maps. Notice that a rational map is not a map of X into Y.

Let $\psi : X \to Y$ be a dominant rational map. Let $f \in K(Y)$ be a rational function represented by a regular function f_W defined on the open set W. Since for some pair (U, ψ_U) representing ψ, $\psi(U)$ is dense and non-empty in Y, the set $\psi_U^{-1}(W)$ is a non-empty open set on which $f \circ \psi_U$ induces a regular function. The pair $(\psi_U^{-1}(W), (f \circ \psi_{U|\psi_U^{-1}(W)})$ represents a rational function in $K(X)$. Therefore, we have define a map from the set of dominant rational maps of X into Y into the k-algebras homomorphisms of $K(Y)$ into $K(X)$. For any two varieties X and Y this gives a bijection between the set of dominant rational maps of X into Y and the k-algebras homomorphisms of $K(Y)$ into $K(X)$.

In fact, we have a contravariant equivalence of category between the category of varieties and dominant rational maps and the category of finitely generated field extensions of k.

Using the theorem of the primitive element, one can show that any variety X of dimension r is birational to a hypersurface Y of $\mathbb{P}^{r+1}$.

2.6 Schemes over k

One can also define the category $\mathfrak{S}(k)$ of Schemes over the field k.

Then, there is a functor $F : \mathfrak{V}(k) \to \mathfrak{S}(k)$ from $\mathfrak{V}(k)$ into $\mathfrak{S}(k)$ which is fully faithful. Namely, let V be a k-variety. Let us define the scheme $F(V)$. The topological space $|F(V)|$ underlying $F(V)$ is made of the set of non-empty irreducible closed subsets of V. Since $|F(W)| \subset |F(V)|$ if W is a closed subset of V, the set of closed subsets of $W_1 \cup W_2$ is the union of the sets of closed subsets of W_1 and W_2 and the set of closed subsets of $\cap_{i \in I} W_i$ is the intersection of all the sets of subsets of W_i, $|F(V)|$ is endowed with a topology.

There is a natural map $\xi : V \to |F(V)|$ so that $\xi(x) := \{x\}$. One can check that this map gives a bijection between the open subsets of V and the open subsets of $|F(V)|$.

The k-variety V is endowed with a sheaf $\mathcal{O}_V$ of regular functions. So $|F(V)|$ is endowed with the sheaf $\xi_*(\mathcal{O}_V)$.

The pair $(|F(V), \xi_*(\mathcal{O}_V))$ is a scheme over k. Since every variety can be covered by affine varieties, it will be sufficient to prove our assertion when V is an affine variety.

Let suppose that V is an affine variety. Let $A := k[X_1, \dots, X_n]/I(V)$ be the ring of regular functions defined on V. We have a morphism of ringed spaces:

$$\alpha : (V, \mathcal{O}_V) \to (SpecA, \mathcal{O}_A)$$

defined as follows. If $x \in V$, $\alpha(x) = \mathfrak{M}_x$ where $\mathfrak{M}_x$ is the maximal ideal of A defined by x. We have seen that α is a bijection of V with the set of maximal ideals of A and a homeomorphism with its image. For any open subset U of $SpecA$, we define a homomorphism

$$\sigma(U) : \mathcal{O}_A(U) \to \alpha_*(\mathcal{O}_V)(U) = \mathcal{O}_V(\alpha^{-1}(U)).$$

Let $s \in \mathcal{O}_A(U)$. Let $x \in \alpha^{-1}(U)$. The value $\sigma(U)(s)(x)$ is the image of $s(\alpha(x)) \in \mathcal{O}_{A,\alpha(x)}$ in the quotient $\mathcal{O}_{A,\alpha(x)}/\mathfrak{M}_{\alpha(x)} \simeq k$. The function (in x) $\sigma(U)(s)(x)$ is a regular function on $\alpha^{-1}(U)$, and one can prove that $\sigma(U)$ is an isomorphism. Now, the prime ideals of A are in 1-1 correspondence with the irreducible closed subsets of V, therefore it can be shown that the scheme $(Spec(A), \mathcal{O}_A)$ is isomorphic with $(|F(V), \xi_*(\mathcal{O}_V))$.

To give the morphism of $(|F(V), \xi_*(\mathcal{O}_V))$ to $(Spec(k), \mathcal{O}_k)$ we have to define a homomorphism from the field k into $\Gamma(|F(V), \xi_*(\mathcal{O}_V)) = \Gamma(V, \mathcal{O}_V)$. We send $\lambda \in k$ to the constant function λ on V. Therefore, $(|F(V), \xi_*(\mathcal{O}_V))$ is a scheme over k.

To show that the functor is fully faithful it remains to prove that, if V and W are varieties, the natural map:

$$Hom_{\mathfrak{V}(k)}(V, W) \to Hom_{\mathfrak{S}(k)}((|F(V), \xi_*(\mathcal{O}_V)), (|F(W), \xi_*(\mathcal{O}_W)))$$

is an isomorphism.

3 Properties of schemes and their morphisms

3.1 First properties

A scheme is *connected* if its topological space is connected. A scheme is irreducible if its topological space is irreducible.

Definition A scheme $(X, \mathcal{O}_X)$ is reduced if, for any open subset U of X, the ring $\mathcal{O}(U)$ is reduced, i.e. has no nilpotent element.

One can show that a scheme $(X, \mathcal{O}_X)$ is reduced if and only if for any point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

A scheme X is *integral* if , for any open subset U of X, the ring $\mathcal{O}(U)$ is an integral domain.

For example, an affine scheme $(Spec(A), \mathcal{O}_A)$ is irreducible if and only if the nilradical of A is prime. It is reduced if and only if its nilradical is $\{0\}$ and it is integral if and only if A is an integral domain.

A scheme is *locally noetherian* if it can be covered by open affine schemes $(Spec(A_i), \mathcal{O}_{A_i})$ where the rings A_i are noetherian. A scheme is *noetherian*, if it is locally noetherian and quasi-compact.

One can prove that a an affine scheme $(Spec(A), \mathcal{O}_A)$ is noetherian if and only if the ring A is noetherian.

The dimension of a scheme $(X, \mathcal{O}_X)$ is the dimension of the topological space X. It can be proved that the dimension of an affine scheme $(Spec(A), \mathcal{O}_A)$ is the Krull dimension of A.

An *open subscheme* of a scheme $(X, \mathcal{O}_X)$ is a scheme $(U, \mathcal{O}_U)$ whose topological space U is an open subset of the topological space X and the structure sheaf $\mathcal{O}_U$ is isomorphic to the restriction $\mathcal{O}_X|U$.

A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is locally of finite type if there exists a covering of $(Y, \mathcal{O}_Y)$ by open affine subschemes $(Spec(A_i), \mathcal{O}_{A_i})_{i \in I}$ such that, for each i, $f^{-1}(Spec(A_i), \mathcal{O}_{A_i})$ is covered by affine subschemes $(Spec(A_{ij}), \mathcal{O}_{A_{ij}})$, where A_{ij} is a finitely generated A_i-algebra.

The morphism f is of finite type if, furthermore, each $f^{-1}(Spec(A_i), \mathcal{O}_{A_i})$ can be covered by a finite number of $(Spec(A_{ij}), \mathcal{O}_{A_{ij}})$.

An *open immersion* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism which induces an isomorphism of $(X, \mathcal{O}_X)$ with an open subscheme of $(Y, \mathcal{O}_Y)$.

A *closed immersion* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism which induces a homeomorphism of X onto a closed subset of Y and a surjective map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves on Y.

A *closed subscheme* is an equivalence class of closed immersion, where $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ are equivalent if there is an isomorphism $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ such that $g = f \circ i$.

Examples

1. Let $(Y, \mathcal{O}_Y)$ be the affine scheme defined by the ring A. Let X be the closed subspace of $Spec(A)$ defined by the ideal $\mathcal{I}(X)$ of A. Every closed subscheme structure on X is defined by an ideal $\mathfrak{A}$ whose zeros are X.

   If V is an affine variety. It correspond to the scheme $(Spec(A(V)), \mathcal{O}_{A(V)})$. A subvariety W of V is defined by the prime ideal $\mathfrak{P}$ of $A(V)$. The zeros of $\mathfrak{P}^n$, $n \geq 1$, are W, but each $\mathfrak{P}^n$ define a different closed subscheme. For $n = 1$, it is the subscheme which corresponds to the subvariety W. For $n \geq 2$, it corresponds to the n-infinitesimal neighbourhoods of the closed subscheme associated to W.

2. Let $(Y, \mathcal{O}_Y)$ be a scheme. We can see that if X is a closed subset of Y, there are many closed subschemes whose topological space is X. There is one which is "smaller" than any other one, called the *reduced induced closed subscheme* structure.

   Suppose that $(Y, \mathcal{O}_Y)$ is an affine scheme $(Spec(A), \mathcal{O}_A)$. Let X be a closed subset. Let the ideal $\mathcal{I} := \cap_{\mathfrak{P} \in X} \mathfrak{P}$ is the largest ideal which defines X. The reduced induced closed subscheme structure on X is the one defined by this ideal.

   In general the reduced induced closed subscheme structure whose topological space is X is defined by glueing the schemes defined in the affine schemes of a covering of the scheme $(Y, \mathcal{O}_Y)$.

3.2 Properties of morphisms

Let $(S, \mathcal{O}_S)$ be a scheme and $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$, $g : (Y, \mathcal{O}_Y) \to (S, \mathcal{O}_S)$ be schemes over $(S, \mathcal{O}_S)$. We can define the *fibered product* of $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ over $(S, \mathcal{O}_S)$ (in fact the fibered product of f and g). It is a scheme $(X \times_S Y, \mathcal{O}_{X \times_S Y})$ together with morphisms

$$\pi_1 : (X \times_S Y, \mathcal{O}_{X \times_S Y}) \to (X, \mathcal{O}_X)$$

$$\pi_2 : (X \times_S Y, \mathcal{O}_{X \times_S Y}) \to (Y, \mathcal{O}_Y)$$

such that $f \circ \pi_1 = g \circ \pi_2$ and for any pair of morphisms

$$p_1 : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$$

$$p_2 : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$$

such that $f \circ p_1 = g \circ p_2$, there is a unique morphism $\sigma : (Z, \mathcal{O}_Z) \to (X \times_S Y, \mathcal{O}_{X \times_S Y})$ such that $\pi_1 \circ \sigma = p_1$ and $\pi_2 \circ \sigma = p_2$.

The fibered product is unique up to isomorphism.

Let $f : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ and $g : (S', \mathcal{O}_{S'}) \to (S, \mathcal{O}_S)$. The projection $(X \times_S S', \mathcal{O}_{X \times_S S'}) \to (S', \mathcal{O}_{S'})$ is called the morphism obtained from f by *base change* g.

In the case $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ and $(S, \mathcal{O}_S)$ are affine schemes given by the rings A, B and C, the morphisms f and g are given by the ring homomorphisms $\tilde{f} : C \to A$ and $\tilde{g} : C \to B$. Therefore, A and B can be considered through respectively $\tilde{f}$ and $\tilde{g}$ as C-algebras. The fibered product in this case is an affine scheme $(Spec(A \otimes_C B), \mathcal{O}_{A \otimes_C B})$ given by tensor product $A \otimes_C B$. The construction of the fibered product proceeds by glueing schemes.

When $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of schemes and $y \in Y$. Let $k(y)$ the residue field of y, i.e. the quotient $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$. We have a natural morphism $i : (Spec(k(y)), \mathcal{O}_{k(y)}) \to (Y, \mathcal{O}_Y)$ which gives the point $y \in Y$. The fibered product of f and i is the fiber of f over y:

$$(X_y, \mathcal{O}_{X_y}) = (X, \mathcal{O}_X) \times_{(Y, \mathcal{O}_Y)} (Spec(k(y)), \mathcal{O}_{k(y)})$$

Let $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. The diagonal morphism of f is the unique morphism $(X, \mathcal{O}_X) \to (X \times_Y X, \mathcal{O}_{X \times_Y X})$ such that the composition with the morphisms π_1 or π_2 from $(X \times_Y X, \mathcal{O}_{X \times_Y X})$ to $(X, \mathcal{O}_X)$ is Id_X.

We say that the morphism f is *separated* if its diagonal morphism is a closed immersion. We say that the scheme $(X, \mathcal{O}_X)$ is separated if its natural morphism to $(Spec(\mathbb{Z}), \mathcal{O}_{\mathbb{Z}})$ is separated.

Any morphism of affine schemes is separated. An arbitrary morphism is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.

In the case all the schemes are noetherian:

1. Open and closed immersions are separated.
2. A composition of two separated morphisms is separated.
3. Separated morphisms are stable under base change.
4. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $f' : (X', \mathcal{O}_{X'}) \to (Y', \mathcal{O}_{Y'})$ are morphisms over a scheme $(S, \mathcal{O}_S)$ and are separated, the product
$$f \times f' : (X \times_S X', \mathcal{O}_{X \times_S X'}) \to (Y \times_S Y', \mathcal{O}_{Y \times_S Y'})$$
is separated.
5. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of schemes and $g \circ f$ is separated, then f is separated.
6. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is separated if Y can be covered by open subsets V_i, $i \in I$, such that the morphisms $(f^{-1}(V_i), \mathcal{O}_X|f^{-1}(V_i)) \to (V_i, \mathcal{O}_Y|V_i)$ induced by f are separated for each $i \in I$.

A morphism is *proper* if it is separated, of finite type, and universally closed.

A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is universally closed if it is closed and, for any morphism $g : (Y', \mathcal{O}_{Y'}) \to (Y, \mathcal{O}_Y)$, the morphism $f' : (X', \mathcal{O}_{X'}) \to (Y', \mathcal{O}_{Y'})$ obtained from f by base change g is closed.

As above, if we only deal with noetherian schemes, we have:

1. Closed immersions are proper.
2. A composition of two proper morphisms is proper.
3. Proper morphisms are stable under base change.
4. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $f' : (X', \mathcal{O}_{X'}) \to (Y', \mathcal{O}_{Y'})$ are morphisms over a scheme $(S, \mathcal{O}_S)$ and are proper, the product
$$f \times f' : (X \times_S X', \mathcal{O}_{X \times_S X'}) \to (Y \times_S Y', \mathcal{O}_{Y \times_S Y'})$$
is proper.
5. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of schemes and $g \circ f$ is proper, then , if g is separated, f is proper.
6. A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is proper if Y can be covered by open subsets V_i, such that the morphisms $(f^{-1}(V_i), \mathcal{O}_X|f^{-1}(V_i)) \to (V_i, \mathcal{O}_Y|V_i)$ induced by f are proper for each index i.

4 Projective schemes and morphisms

4.1 Projective space over a ring

We have defined for any graded ring R the scheme $(Proj(R), \mathcal{O}_R)$ which is the projective scheme defined by the graded ring R.

When $R := A[X_0, \dots, X_n]$, the corresponding scheme is the *projective scheme* $\mathbb{P}^n_A$ which has a natural morphism to the affine scheme defined by the ring A. It is the n-projective space over the ring A. If $A \to B$ is a ring homomorphism, we have an affine scheme morphism:

$$(Spec(B), \mathcal{O}_B) \to (Spec(A), \mathcal{O}_A),$$

and one can check that $\mathbb{P}^n_B \simeq \mathbb{P}^n_A \times_{Spec(A)} Spec(B)$. In particular, $\mathbb{P}^n_A \simeq \mathbb{P}^n_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} Spec(A)$.

For any scheme $(Y, \mathcal{O}_Y)$, we define the scheme $\mathbb{P}^n_Y$ to be the fiber product $\mathbb{P}^n_{\mathbb{Z}} \times_{Spec(\mathbb{Z})} Y$.

4.2 Projective morphisms and schemes

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. We say that it is *projective* if it factors through a closed immersion $(X, \mathcal{O}_X) \to \mathbb{P}^n_Y$ composed with the projection $\mathbb{P}^n_Y \to Y$.

The morphism f is *quasi-projective* if it factors through an open immersion $(X, \mathcal{O}_X) \to (X', \mathcal{O}_{X'})$ composed with a projective morphism $(X', \mathcal{O}_{X'}) \to (Y, \mathcal{O}_Y)$.

One can prove that a projective morphism of noetherian schemes is proper.

A quasi-projective of morphism of noetherian schemes is of finite type and separated.

A scheme $(Y, \mathcal{O}_Y)$ over the affine scheme defined by A is projective if it is isomorphic to a closed subscheme of the A-projective space $\mathbb{P}^r_A$. A scheme $(Y, \mathcal{O}_Y)$ over the affine scheme defined by A is projective if and only if $(Y, \mathcal{O}_Y)$ is isomorphic to the projective scheme defined by the graded ring R, where $R_0 = A$ and R is finitely generated by R_1 as R_0-algebra.

4.3 Varieties and projective schemes

We have given a fully faithful functor F from the category $\mathfrak{V}(k)$ of k-varieties into the category of k-schemes $\mathfrak{S}(k)$.

The schemes in the image of the functor F are integral, separated schemes of finite type over k. The image of the set of projective varieties is the set of projective integral schemes.

We define an *abstract variety* as an integral separated scheme of finite type over an algebraically closed field k. If it is proper over k we say that the abstract variety is *complete*.

An abstract variety of dimension one is called a *curve*, an abstract variety of dimension two is called a *surface*.

One can prove:

1. Every complete curve is projective.
2. There exists non-projective complete surfaces.

5 Sheaves of Modules

5.1 Ringed spaces

Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{M}$ of abelian groups on X, such that, for every open subset $U \subset X$, the abelian group $\mathcal{M}(U)$ is a module over the ring $\mathcal{O}(U)$ and, for each inclusion of open sets $V \subset U$, the restriction homorphism $\mathcal{M}(U) \to \mathcal{M}(V)$ is compatible with the modules structures and the restriction ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

A morphism $\mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_X$-modules is a morphism of sheaves such that, for any open subset U of X, $\mathcal{M}(U) \to \mathcal{N}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules.

The $\mathcal{O}_X$-modules over the ringed space $(X, \mathcal{O}_X)$ is an abelian category.

The tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheaf associated to the pre-sheaf given, for any open subset U of X by:

$$U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U)$$

A sheaf of $\mathcal{O}_X$-modules is free if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. It is locally free, if the space X can be covered by open subsets U, such that the restriction of the sheaf of $\mathcal{O}_X$-modules to each U is free on $(U, \mathcal{O}_X|U)$. The rank of a locally free module on a open set where it is free is the number of copies of the structure sheaf needed. If X is connected, the rank of a locally free module is the same everywhere. A locally sheaf of rank one is also called an invertible sheaf.

A sheaf of ideals on $(X, \mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules which is a subsheaf $\mathcal{I}$ of $\mathcal{O}_X$, i.e. for any open subset U, $\mathcal{I}(U)$ is an ideal of $\mathcal{O}(U)$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed space. Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. We have defined the direct image $f_*(\mathcal{M})$ of the sheaf $\mathcal{M}$. It is a $f_*(\mathcal{O}_X)$-module. Since f defines a ring homomorphism $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$, the sheaf $f_*(\mathcal{M})$ is also a $\mathcal{O}_Y$-module called the direct image of $\mathcal{M}$ by f.

Similarly if $\mathcal{N}$ is a sheaf of $\mathcal{O}_Y$-modules, we have defined $f^{-1}\mathcal{N}$. It is a $f^{-1}\mathcal{O}_Y$-module. We have a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. The inverse image $f^*\mathcal{N}$ of $\mathcal{N}$ by f is:

$$f^{-1}\mathcal{N} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

which is naturally a $\mathcal{O}_X$-module.

One can show that f_* and f^* define adjoint functors between the category of $\mathcal{O}_X$-modules and the category of $\mathcal{O}_Y$-modules, i.e. for any $\mathcal{O}_X$-module $\mathcal{M}$ and any $\mathcal{O}_Y$-module $\mathcal{N}$, we have a natural isomorphism:

$$Hom_{\mathcal{O}_X}(f^*\mathcal{N}, \mathcal{M}) \simeq Hom_{\mathcal{O}_Y}(\mathcal{N}, f_*\mathcal{M})$$

5.2 Affine schemes

In the case of schemes let us consider first affine schemes $(Spec(A), \mathcal{O}_A)$.

Any A-module M defines a $\mathcal{O}_A$-module $\tilde{M}$. For each prime ideal $\mathfrak{P}$ of A, let $M_{\mathfrak{P}}$ be the localization of M at $\mathfrak{P}$. For any open subset U of $Spec(A)$, the $\mathcal{O}(U)$-module $\tilde{M}(U)$ is the set of functions

$$s : U \to \coprod_{\mathfrak{P} \in U} M_{\mathfrak{P}}$$

such that, for each $\mathfrak{P} \in U$, $s(\mathfrak{P})$ befongs to $M_{\mathfrak{P}}$ and for each $\mathfrak{Q} \in U$ there is a neighbourhood V of $\mathfrak{Q}$ in U and elements $m \in M$ and $a \in A$ such that for each $\mathfrak{R} \in V$, $a \notin \mathfrak{R}$ and $s(\mathfrak{R}) = m/a$. Using the obvious restriction maps, $\tilde{M}$ is a sheaf of $\mathcal{O}_A$-modules.

We can prove:

1. For each $\mathfrak{P} \in Spec(A)$, the stalk of $\tilde{M}$ at $\mathfrak{P}$ is isomorphic to the localization $M_{\mathfrak{P}}$.
2. For any $a \in A$, A_a-module $\tilde{M}(D(a))$ over the open set $D(a)$ of prime ideals $\mathfrak{P}$ which do not contain a is isomorphic to the module M_a. In particular $\Gamma(Spec(A), \tilde{M}) = M$.
3. The map $M \mapsto \tilde{M}$ is a faithfull functor from the category of A-module into the category of $\mathcal{O}_A$-modules which is exact, i.e. the natural homomorphism:
$$Hom_A(M, N) \to Hom_{\mathcal{O}_A}(\tilde{M}, \tilde{N})$$
is an isomorphism and any exact sequence:
$$0 \to M' \to M \to M'' \to 0$$
gives an exact sequence of $\mathcal{O}_A$-modules:
$$0 \to \tilde{M'} \to \tilde{M} \to \tilde{M''} \to 0.$$
4. If M and N are A-modules, $(M \otimes_A N)\tilde{} = \tilde{M} \otimes_{\mathcal{O}_A} \tilde{N}$.
5. For any family of A-modules $(M_i)_{i \in I}$, we have $(\oplus_{i \in I} M_i)\tilde{} = \oplus_{i \in I} \tilde{M}_i$.
6. Let $f : (Spec(B), \mathcal{O}_B) \to (Spec(A), \mathcal{O}_A)$ be a morphism of affine schemes, for any B-module N, we have $f_* \tilde{N} = \tilde{N}_{(A)}$, where $N_{(A)}$ is the module N considered as A-module through the homomorphism $A \to B$ defined by f.
7. For any A-module M, $f^* \tilde{M} = (M \otimes_A B)\tilde{}$.

These sheaves of the form $\tilde{M}$ on an affine scheme will be the model for quasi-coherent sheaves.

We say that a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is a *quasi-coherent sheaf* on the scheme $(X, \mathcal{O}_X)$ if X is covered by open affine subsets U_i, such that $(U_i, \mathcal{O}_X|U_i)$ is affine and isomorphic to $(Spec(A_i), \mathcal{O}_{A_i})$ and $\mathcal{M}|U_i$ is isomorphic to $\tilde{M}_i$ for some A_i-module M_i. We say that $\mathcal{M}$ is *coherent* if each M_i is a finitely generated A_i-module.

We have:

1. On any scheme $(X, \mathcal{O}_X)$, the structure sheaf $\mathcal{O}_X$ is coherent.
2. If $(X, \mathcal{O}_X)$ is an affine scheme defined by a ring A, if $(Y, \mathcal{O}_Y)$ is the closed subscheme defined by the ideal $\mathcal{I}$ of A, and $i : Y \to X$ is the inclusion, then $i_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_X$-module, isomorphic to $(A/\mathcal{I})\tilde{}$.

3. Let A be a ring. The correspondence $M \mapsto \tilde{M}$ gives an equivalence of category between the category of A-modules and the category of quasi-coherent $\mathcal{O}_A$-modules. If, furthermore A is a noetherian ring, it gives an equivalence of category between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_A$-modules.

4. Let $(X, \mathcal{O}_X)$ be a scheme. The category of quasi-coherent $\mathcal{O}_X$-modules on X is an abelian category. If $(X, \mathcal{O}_X)$ is a noetherian scheme, the category of coherent $\mathcal{O}_X$-modules on X is abelian.

5. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. If $\mathcal{G}$ is a quasi-coherent $\mathcal{O}_Y$-module, then the inverse image $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module. If both $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, if $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module, then the inverse image $f^*\mathcal{G}$ is a coherent $\mathcal{O}_X$-module.

5.3 Projective schemes

Let R be a graded ring and M a graded R-module. We define the sheaf associated to M, denoted by $\tilde{M}$ on the projective scheme $(Proj(R), \mathcal{O}_R)$. For each homogeneous prime $\mathfrak{P}$ in $Proj(R)$, $M_{(\mathfrak{P})}$ is the the abelian group of elements of degree 0 in the localization $S^{-1}M$ of M with denominators in the multiplicative set S of elements of R not contained in $\mathfrak{P}$. Let U be an open subset of $Proj(R)$. The group $\tilde{M}(U)$ is given by the set of functions $s : U \to \coprod_{\mathfrak{P} \in U} M_{(\mathfrak{P})}$ such that, for any $\mathfrak{Q} \in U$, there exist an open neighbourhood V of $\mathfrak{Q}$ in U and homogeneous elements $m \in M$ and $a \in R$ of the same degree, such that, for every $\mathfrak{Q}$ in V, $a \notin \mathfrak{Q}$, and $s(\mathfrak{Q}) = m/a$. With the obvious restriction maps, $\tilde{M}$ is a sheaf on $Proj(R)$. It is also an $\mathcal{O}_R$-module.

One can prove:

1. The stalk $\tilde{M}_{\mathfrak{P}}$ of $\tilde{M}$ at $\mathfrak{P} \in Proj(R)$ is $M_{(\mathfrak{P})}$.

2. For any $a \in R$ of degree ≥ 1, the restriction $\tilde{M}|D_+(a)$ is isomorphic to $(M_{(a)})\tilde{}$, considering $D_+(a)$ isomorphic to the affine scheme defined by $R_{(a)}$, where $M_{(a)}$ is the group of elements of degree 0 in M_a.

3. $\tilde{M}$ is a quasi-coherent $\mathcal{O}_R$-module. If R is noetherian and M finitely generated over R, $\tilde{M}$ is a coherent $\mathcal{O}_R$-module.

Let R be a graded module. Let $(X, \mathcal{O}_X) = (Proj(R), \mathcal{O}_R)$. For any $n \in \mathbb{Z}$, we define $\mathcal{O}_X(n)$ as $(R(n))\tilde{}$.

The sheaf $\mathcal{O}_X(1)$ is called the twisting sheaf of Serre.

For any sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules, we denote by $\mathcal{M}(n)$ the twisted sheaf $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Then, if we assume that R is generated by R_1 as R_0-algebra:

1. The sheaf $\mathcal{O}_X(n)$ is invertible.

2. $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) \simeq \mathcal{O}(n+m)$

Let R be a graduated ring. Let $(X, \mathcal{O}_X) = (Proj(R), \mathcal{O}_R)$. Let $\mathcal{M}$ a sheaf of $\mathcal{O}_X$-modules. We have an abelian group $\Gamma_*(\mathcal{M}) := \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{M}(n))$. We endow $\Gamma_*(\mathcal{M})$ with a strucure of graded R-module as follows: any $s \in R_d$ determines naturally a global section $s \in \Gamma(X, \mathcal{O}(d))$, then, for any element $m \in \Gamma(X, \mathcal{M}(n))$, we define $s.m \in \Gamma(X, \mathcal{M}(n+d)$, by taking the tensor product $m \otimes s$ and by using the isomorphism $\mathcal{M}(n) \otimes_{\mathcal{O}_X} \mathcal{O}(d) \simeq \mathcal{M}(n+d)$.

If $R := A[X_0, \ldots, X_n]$ is the polynomial ring with coefficients in the ring A (for $n \geq 1$), with $X = Proj(R)$, then, $\Gamma_*(\mathcal{O}_X) \simeq R$.

Let R be a graded ring finitely generated by R_1 as R_0-algebra. Let $X = Proj(R)$ and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Then we have a natural isomorphism:

$$(\Gamma_*(\mathcal{M}))\tilde{} \simeq \mathcal{M}$$

Let A be a ring. Let $(Y, \mathcal{O}_Y)$ be a closed subscheme of $\mathbb{P}^n_A$. Then there is a homogeneous ideal $I \subset R = A[X_0, \ldots, X_n]$, such that $(Y, \mathcal{O}_Y)$ is the closed subscheme defined by I.

For any scheme $(Y, \mathcal{O}_Y)$, we define the twisting sheaf $\mathcal{O}(1)$ on $\mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ as $f^*\mathcal{O}(1)$, where $f : \mathbb{P}^n_Y \to \mathbb{P}^n_{\mathbb{Z}}$ is the natural map.

For any scheme $(X, \mathcal{O}_X)$ over $(Y, \mathcal{O}_Y)$, an invertible sheaf $\mathcal{L}$ is very ample if there is a morphism $i : (X, \mathcal{O}_X) \to \mathbb{P}^n_Y$ for some n which induces an isomorphism on an open subscheme of a closed scheme of $\mathbb{P}^n_Y$ such that $\mathcal{L}$ is isomorphic to $i^*\mathcal{O}(1)$.

6 Differential forms

6.1 Kähler Differentials

Let A be a ring and B be an A-algebra. Let M be a B-module.

An A-derivation of B into M is a map $d : B \to M$, such that:

1. the map d is additive;
2. for $b, b' \in B$, $d(bb') = bd(b') + b'd(b)$;
3. for all $a \in A$, $d(a.1_B) = 0$.

The module of relative differential forms of B over A is a B-module $\Omega_{B|A}$, together with a A-derivation $d : B \to \Omega_{B|A}$ satisfying the following universal property: for any B-module M and any A-derivation $d' : B \to M$, there is a unique B-module homomorphism $h : \Omega_{B|A} \to M$ such that $h \circ d = d'$.

Such a module of relative differential forms of B over A is unique up to isomorphism. In fact:

Let $\delta : B \otimes_A B \to B$ be the diagonal homomorphism defined by $\delta(b \otimes b') = bb'$. Let I be the kernel of δ. Consider $B \otimes_A B$ as a B-module by multiplication on the right. Then , the quotient I/I^2 is also a B-module. We have a map $d : B \to I/I^2$ given by $d(b) = [1 \otimes b - b \otimes 1]$, where $[1 \otimes b - b \otimes 1]$ is the class of $1 \otimes b - b \otimes 1$ in I/I^2. Then the pair $(I/I^2, d)$ is a module of relative differential forms of B over A.

We have: if A' and B are A-algebras, define $B' := B \otimes_A A'$; then, $\Omega_{B'|A'} \simeq \Omega_{B|A} \otimes_B B'$. Furthermore if T is a multiplicative system of B, then $\Omega_{T^{-1}B|A} \simeq T^{-1}\Omega_{B|A}$.

If $B := A[X_1, \ldots, X_n]$ is the polynomial ring with coefficients in A, the module $\Omega_{B|A}$ of relative differential forms of B over A is the free B-module of rank n generated by $dX_1, \ldots, dX_n$.

(First exact sequence) Let $A \to B \to C$ be homomorphisms of rings. There is a natural exact sequence of C-modules:

$$\Omega_{B|A} \otimes_B C \to \Omega_{C|A} \to \Omega_{B|A} \to 0.$$

(Second exact sequence) Let B be an A-algebra. Let $\mathcal{I}$ be an ideal of B and C be the quotient $B/\mathcal{I}$. We have a natural exact sequence:

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{B|A} \otimes_B C \to \Omega_{C|A} \to 0$$

6.2 Sheaves of Differentials

Let $f : (X, \mathcal{O}) \to (Y, \mathcal{O}_Y)$ be a separated morphism of schemes. Consider the diagonal morphism $\Delta : (X, \mathcal{O}_X) \to (X \times_Y X, \mathcal{O}_{X \times_Y X})$. The image of Δ is a closed subscheme. Let $\mathcal{I}$ the ideal sheaf which defines the image of Δ. The sheaf of relative differentials of $(X, \mathcal{O})$ over $(Y, \mathcal{O}_Y)$ is the sheaf $\Omega_{X|Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

In the case $(X, \mathcal{O})$ and $(Y, \mathcal{O}_Y)$ are affine schemes defined by the rings B and A and the morphism is given by a ring homomorphism $A \to B$, it is easy to see that $\Omega_{X|Y} = (\Omega_{B|A})\tilde{}$.

The module of differentials behaves well by base change. Namely, let $f : (X, \mathcal{O}) \to (Y, \mathcal{O}_Y)$ be a separated morphism of schemes. Let $g : (Y', \mathcal{O}_{Y'}) \to (Y, \mathcal{O}_Y)$ be another morphism. Consider the base extension of f by g:

$$f' : (X', \mathcal{O}_{X'}) = (X \times_Y Y', \mathcal{O}_{X \times_Y Y'}) \to (Y', \mathcal{O}_{Y'})$$

Then the relative differentials $\Omega_{X'|Y'} \simeq (g')^* \Omega_{X|Y}$, where $g' : (X \times_Y Y', \mathcal{O}_{X \times_Y Y'}) \to (X, \mathcal{O}_X)$ is the other projection.

Notice that if the schemes are affine schemes with $X = Spec(A)$, $Y = Spec(B)$, $Y' = Spec(B')$, this is the consequence of the algebraic result $\Omega_{B'|A'} \simeq \Omega_{B|A} \otimes_B B'$ given above.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $g : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ are morphisms of schemes. There is an exact sequence of sheaves on X:

$$f^*\Omega_{Y|Z} \to \Omega_{X|Z} \to \Omega_{X|Y} \to 0.$$

It is consequence of the first exact sequence considered above.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes and let Z be a closed subscheme of $(X, \mathcal{O}_X)$ defined by the ideal sheaf $\mathcal{I}$. There is an exact sequence of sheaves:

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X|Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{X|Z} \to 0.$$

This is consequence of the second exact sequence above.

As consequence of the algebraic settings we also have that, if $(X, \mathcal{O}_X)$ is the relative affine space k^n_Y, the sheaf $\Omega_{X|Y}$ is a free $\mathcal{O}_X$-module of rank n generated by the global sections $dX_1, \dots, dX_n$, where $X_1, \dots, X_n$ are coordinates of k^n.

Let A be a ring. Let $(Y, \mathcal{O}_Y)$ be the affine scheme defined by A. Let $(X, \mathcal{O}_X)$ be the relative projective space $\mathbb{P}^n_Y$. Then, we have the exact sequence of sheaves:

$$0 \to \Omega_{X|Y} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

7 Non-singular varieties

7.1 Definition

Let $Y \subset k^n$ be an affine variety. Let $f_1, \dots, f_s$ be generators of the ideal $I(Y)$ of Y in $k[X_1, \dots, X_n]$. The affine variety Y of dimension r is non-singular at the point x if the rank of the matrix $(\partial f_i/\partial X_j)_{1\le i\le s, 1\le j\le n}$ is $n - r$. The variety Y is non-singular if it is non-singular at every point of Y.

It can be proved that the affine variety Y is non-singular at the point $x \in Y$ if and only if the local ring $\mathcal{O}_{Y,x}$ is a regular local ring.

In general, for any variety Y, Y is non-singular at the point x if the local ring $\mathcal{O}_{Y,x}$ is a regular local ring. The variety Y is non-singular if it is non-singular at every point. The variety Y is singular if it is not non-singular.

Let Y be a variety. The set $Sing(Y)$ of singular points of Y is a proper closed subset.

Since we have defined an abstract variety to be an integral separated scheme of finite type over an algebraically closed field, we shall say that an abstract variety is non-singular if all its local rings are regular local rings. Considering the functor $F : \mathfrak{V}(k) \to \mathfrak{S}(k)$ of the category of "classical k-varieties" into the category of schemes over k, the image of a non-singular variety in the "classical" sense is a non-singular abstract variety, because any local ring at a point of the abstract variety is the localization of a local ring at a prime ideal and the localization of a regular local ring at a prime ideal is a regular local ring.

7.2 Basic properties of non-singular points

Then, if $(X, \mathcal{O}_X)$ is an irreducible separated scheme of finite type over an algebraically closed field k, the sheaf of differentials $\Omega_{X|k}$ is a locally free sheaf of rank $n = \dim X$ if and only if $(X, \mathcal{O}_X)$ is a non-singular variety.

As a consequence, if $(X, \mathcal{O}_X)$ is a variety over k, then there is a open dense subset U of X which is non-singular. To prove this fact it is enough to prove that the fiber of $\Omega_{X|k}$ over the generic point $\{0\}$ is $\Omega_{K|k}$. One shows that it is a K-vector space of dimension n. Then in a neighbourhood U of the generic point $\Omega_{X|k}$ is locally free of rank n. So, the open set U must be non-singular.

(Bertini theorem) Let $(X, \mathcal{O}_X)$ be a non-singular closed subvariety of the projective space $\mathbb{P}^n_k$ over an algebraically closed filed. Then, there exists a hyperplane H not containing X, such that $H \cap X$ is non-singular. Furthermore the set of hyperplane having this property forms an open dense subset of the space of all hyperplanes.

7.3 Some invariants

The tangent sheaf of a non-singular variety $(X, \mathcal{O}_X)$ is the sheaf $\mathcal{H}om(\Omega X|k, \mathcal{O}_X)$. It is a locally free sheaf of rank $n = \dim(X)$. The canonical sheaf is the sheaf $\omega_X := \wedge^n \Omega_{X|k}$, the nth exterior power of the sheaf of differentials. It is an invertible sheaf on $(X, \mathcal{O}_X)$.

If the variety $(X, \mathcal{O}_X)$ is projective and non-singular, the geometric genus of $(X, \mathcal{O}_X)$ is

$$p_g(X) := \dim_k \Gamma(X, \omega_X).$$

Then if X and X' are birationally equivalent non-singular projective varieties over k, then they have equal geometric genus.

Let $(Y, \mathcal{O}_Y)$ be a non-singular closed subvariety of the non-singular variety $(X, \mathcal{O}_X)$ defined by the sheaf of ideals $\mathcal{I}$. The local free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the conormal sheaf $\mathcal{N}_{Y|X}$ of Y. Its dual $\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{N}_{Y|X}, \mathcal{O}_Y)$ is the normal sheaf of Y in X. It is locally free of rank $r = \operatorname{codim}(Y, X)$.

Let $(Y, \mathcal{O}_Y)$ be a non-singular closed subvariety of codimension r in the non-singular variety $(X, \mathcal{O}_X)$. Then:

$$\omega_Y \simeq \omega_X \otimes_{\mathcal{O}_X} \wedge^r \mathcal{N}_{Y|X}.$$

In the case $r = 1$, Y can be considered as a divisor. Let $\mathcal{L}$ be the associated invertible sheaf on $(X, \mathcal{O}_X)$. Then, $\omega_Y \simeq \omega_X \otimes_{\mathcal{O}_X} \mathcal{L} \otimes_{\mathcal{O}_X} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. This last assertion comes from the fact $\mathcal{I} \simeq \mathcal{L}^{-1}$. Since $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, we have $\mathcal{N}_{Y|X} \simeq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$.

8 Complex case

8.1 Analytic spaces

Above we have defined local analytic spaces. As we have defined schemes, we may define complex analytic spaces.

We say that the locally ringed space $(X, \mathcal{O}_X)$ is a complex analytic space if X is Hausdorff and, for any point $x \in X$, there is an open neighbourhood $U(x)$, such that $(U(x), \mathcal{O}_X|U(x))$ is a local analytic space. We say that the complex analytic space $(X, \mathcal{O}_X)$ is reduced if, for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a reduced ring.

A morphism of complex analytic spaces is a morphism of the corresponding locally ringed spaces. Therefore, we can define the category of complex analytic spaces $\mathfrak{A}$.

We say that the complex analytic space $(X, \mathcal{O}_X)$ is non-singular if, for any point $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular. A non-singular complex analytic space is a complex analytic manifold.

8.2 Complex algebraic varieties

One can associate a reduced complex analytic space to any complex algebraic variety. We have a functor Φ from the category of complex algebraic varieties $\mathfrak{V}(\mathbb{C})$ into the category of complex analytic spaces $\mathfrak{A}$. We shall write $\Phi(X) = (X^{an}, \mathcal{O}_X^{an})$. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.

The properties of this correspondence are the following:

1. The completion of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,x}^{(an)}$ are equal.
2. We have an inclusion $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,x}^{(an)}$ and the quotient $\mathcal{O}_{X,x}^{(an)}/\mathcal{O}_{X,x}$ is $\mathcal{O}_{X,x}$-flat.
3. Let $f : X \to Y$ be a regular application between algebraic varieties, the closure of $f(X)$ and the Zariski closure of $f(X)$ coincide.

If $\mathcal{F}$ is an $\mathcal{O}_X$-module, one can associate a sheaf on $X^{(an)}$ in the following manner: we have a continuous map given by the identity $i : X^{(an)} \to X$; let $\mathcal{F}'$ be the reciprocal sheaf on $X^{(an)}$ by i; then, $\mathcal{F}^{(an)}$ is given by $\mathcal{F}' \otimes_{\mathcal{O}'_X} \mathcal{O}_X^{(an)}$. Then we have:

1. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on the variety X, the sheaf $\mathcal{F}^{(an)}$ is an $\mathcal{O}^{(an)}{}_X$-module.
2. If $\mathcal{F}$ is coherent, the analytic sheaf $\mathcal{F}^{(an)}$ is coherent.
3. If X is a projective variety and $\mathcal{F}$ a coherent module on X, the k-th cohomology of $\mathcal{F}$ is isomorphic to the k-th cohomology of $\mathcal{F}^{(an)}$.

4. Let$X$ be a projective variety and $\mathcal{F}$, $\mathcal{G}$ be coherent modules on X, any analytic morphism of $\mathcal{F}^{(an)}$ into $\mathcal{G}^{(an)}$ is defined by one and only one algebraic morphism of $\mathcal{F}$ into $\mathcal{G}$.

5. Let$X$ be a projective variety. Let $\mathcal{M}$ be an analytic coherent sheaf on $X^{(an)}$. There is an algebraic coherent sheaf $\mathcal{F}$ on X which is unique up to isomorphism, such that $\mathcal{F}^{(an)}$ is isomorphic to $\mathcal{M}$.

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