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Shimura varieties

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Shimura Varieties (— 2010 ICTP Summer School on Hodge theory)

In algebraic geometry one has a lot of objects which turn out (by big theorems) to be algebraic but are defined analytically:

- GAGA (proj. varieties, etc.)
- Hodge loci + zero-loci of normal functions (CDK, BP/et al)
- complex tori with a polarization (Kodaira emb. thm. / Θ-fns.)
- Hodge classes (if a certain \$1,000,000 problem could be solved)

and of concern to us in this course:

- modular (locally symmetric) varieties

which can be thought of as the $\Gamma\backslash D$'s for the period maps of certain special VHS's. The fact that they are algebraic is the Baily-Borel theorem (1966).

What one does not know in the Hodge/zero locus setting above is the field of definition — a question related to the existence of Bloch-Beilinson filtrations (to be discussed in Mark's course). For certain cleverly constructed unions of modular varieties, called Shimura varieties, one actually knows the minimal [reflex] field of definition, and also quite a bit about the interplay between "upstairs" and "downstairs" (in $\check{D}$ resp. $\Gamma\backslash D$) fields of definition of subvarieties. My interest in the subject stems from investigating Mumford-Tate domains of Hodge structures, where for example the reflex fields can still be defined even though the

$\Gamma\backslash D$'s are not algebraic varieties in general. Accordingly, I have tried to pack as many Hodge-theoretic punchlines into the exposition as possible.

Of course, Shimura varieties are of <u>central</u> importance from another point of view, that of the Langlands programme. For instance, they provide a major test case for the conjecture (generalizing Shimura–Taniyama) that <u>all motivic L-functions</u> <u>(arising from Galois representations on étale cohomology of varieties/# field)</u> [come from automorphic forms] <u>are automorphic</u> (arising from Hecke eigenforms of adélic algebraic groups). The modern theory is largely due to <u>Deligne/Langlands/Shimura</u>, though many others are implicated in the huge amount of underlying mathematics: e.g.,

- theory of complex multiplication for abelian varieties (Shimura, Taniyama, Weil)
- algebraic groups (Borel, Harish-Chandra)
- class field theory (Artin, Chevalley, Weil; Hensel for p-adics)
- modular varieties (Hilbert, Hecke, Siegel; Baily, Borel, Satake) [compactification]

It seems that much of the impetus, historically, for the study of locally symmetric varieties can be credited to Hilbert's 12th problem generalizing Kronecker's Jugendtraum. Its goal was the construction of abelian extensions* of certain number fields by means of special values of abelian functions in several variables, and it directly underlay the work of Hilbert & (his) students on modular varieties and the theory of CM.

* i.e. algebraic extensions w/ abelian Galois group.

Plan of the course: I. Hermitian symmetric domains — D

II. Locally symmetric varieties — $\Gamma\backslash D$

III. Theory of CM

IV. Shimura varieties ——— $\coprod_i \Gamma_i\backslash D$

V. Field of definition

It's a great pleasure to give these lectures here, and I'll try to take care so that the pleasure is not only mine.

I. Hermitian symmetric domains

A. Algebraic groups (and their properties)

Definition: An algebraic group G / k (= field of char. 0) is an algebraic variety ($= G$) together with morphisms (both /k)

"$\cdot$" : $G \times G \to G$ (multiplication)

$(\cdot)^{-1}$: $G \to G$ (inversion)

and "$\mathbb{1}$" $\in G(k)$ (identity),

subject to rules** which make $G(L)$ into a group for each L/k. [L-rational pts.: $\text{Spec } L \to G$]

($G(\mathbb{R})$ & $G(\mathbb{C})$ have the structure of real & complex Lie groups, in particular.*) G will always be smooth.

Example: $\mathbb{G}_m := \{XY = 1\} \subset \mathbb{A}^2$ (as alg. var.)

$\mathbb{G}_m(k) = k^*$

- G connected $\iff$ $G_{\bar{k}}$ irreducible [extension of scalars: $G_L = G \times_{\text{Spec } k} \text{Spec } L$ for L/k]
- G simple $\iff$ G nonabelian, with no normal connected subgroups $\neq \{1\}, G$.

Example: $k = \mathbb{C}$: SL_n, SO_n, Sp_n, exceptional groups of types $E_{6,7,8}$, F_4, G_2
(A) (B) (D) (C)

$k = \mathbb{R}$: have to worry about real forms ($\cong / \mathbb{C}$, $\not\cong / \mathbb{R}$)

$k = \mathbb{Q}$: all hell breaks loose, as $\mathbb{Q}$-simple $\not\Rightarrow$ $\mathbb{R}$-simple

** Exercise: write out these rules.

* and there are references for Lie groups corresponding to the ones I will introduce for G below (cf. Rotger)

- G (algebraic) **torus** $\iff G_{\bar{k}} \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ (over $\mathbb{C}$: $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$)

Example: inside GL_2, have

$$\begin{aligned} &GL_2 \\ &\cup \\ \mathbb{S} &:= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a^2+b^2 \neq 0 \right\} \\ &\cup \\ \mathbb{U} &:= \left\{ \quad \text{"} \quad \middle|\; a^2+b^2 = 1 \right\} \\ \mathbb{G}_m &\cong \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \neq 0 \right\} \end{aligned}$$

$\mathbb{C}$-pts.	$\mathbb{R}$-pts.
$\mathbb{C}^* \times \mathbb{C}^*$	$\mathbb{C}^*$
$\mathbb{C}^*$	U_1 (or "S^1")
$\mathbb{C}^*$	$\mathbb{R}^*$

different real forms of $GL_1/\mathbb{G}_m$

Example: $G = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ (# field) "Weil restriction" or "restriction of scalars"

is a torus of dimension $[E:\mathbb{Q}]$ with the property that

$$G(\mathbb{Q}) \cong E^*$$

$$G(k) \cong E^* \otimes_{\mathbb{Q}} k \underset{\uparrow}{\cong} (k^*)^{[E:\mathbb{Q}]}$$

$k \supseteq E$ — G *splits* (factors correspond to the different embeddings of $E \hookrightarrow k$)

- G **semisimple** $\iff$ (almost) direct product of simple gps.
- G **reductive** $\iff$ " " " simple gps. + tori

$\iff$ ~~every finite dim'l (linear) representation splits (reducibly) into irreducible factors~~ linear representations are completely reducible.

Example: One finite-dimensional representation is

$$G \xrightarrow{\text{ad}} GL(\mathfrak{g})$$
$$g \longmapsto \{X \mapsto gXg^{-1}\}$$

where $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$

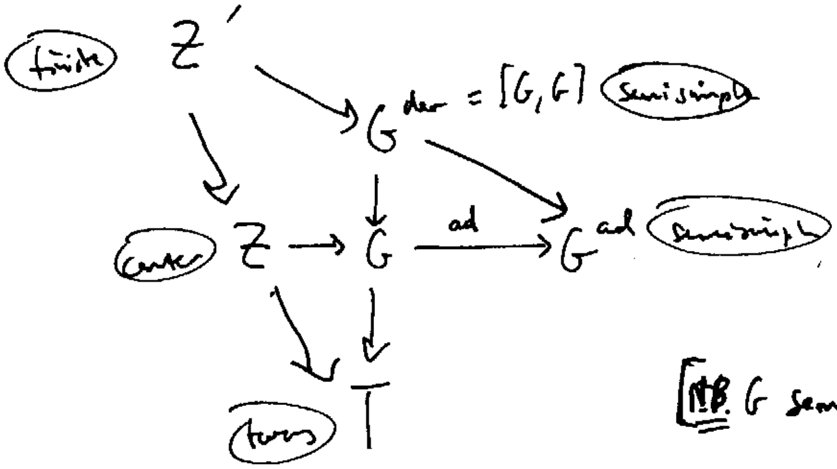
and we are taking the differential (at e) of $\mathrm{Ad}(g) \in \mathrm{Aut}(G)$ (= conj. by g).

For semisimple groups,

- G adjoint $\iff$ ad is injective
- G simply connected $\iff$ any isogeny $G' \twoheadrightarrow G$ w./ G' connected is $\cong$.

Basically, the center is finite; and adjoint $\iff Z(G)$ trivial
simp. conn. $\iff Z(G)$ as large as possible (with given Lie algebra)

For reductive groups, we have short-exact sequences



[N.B. G semisimple $\iff T = \{1\}$]

Finally, let G be a reductive real algebraic group,

$\theta : G \to G$ an involution.

Definition: θ is Cartan $\iff \{ g \in G(\mathbb{C}) \mid g = \theta(\bar{g}) \} =: G^{(\theta)}(\mathbb{R})$ is compact
(compact real form of G)

$\iff \theta = \mathrm{Ad}(C)$ for $C \in G(\mathbb{R})$ with

- $C^2 \in Z(\mathbb{R})$
- $G \hookrightarrow \mathrm{Aut}(V, Q)$ s.t. (Q: symm. bilinear form)

$$Q(\cdot, C(\bar{\cdot})) > 0 \quad \text{on } V_{\mathbb{C}}.$$

[very suggestive!]

These always exist, and

$$G(\mathbb{R}) \text{ compact} \iff \theta = \mathrm{id}.$$

B. Three characterizations of HSD's

1. Hermitian symmetric space of noncompact type (analytic, intrinsic)

(X, g) = connected $\mathbb{C}$-mfld. with Hermitian metric
(or Riemannian manifold with integrable almost complex structure s.t. J acts by isometries)

s.t. $Is(X, g)$ (: holomorphic isometry group = real Lie gp.)

- acts transitively
- contains symmetries s_p $\forall p \in X$ ($s_p^2 = id_X$, p is isolated fixed point)
- $Is(X, g)^+$ (identity connected component) = semisimple adjoint noncompact (real Lie) group

(the noncompactness means (a) Cartan involution projects to "id" in no factor
(b) X has negative sectional curvature)

Bergman metric ↑ ↓ Satake embedding (Harish-Chandra?)

2. Bounded symmetric domain (analytic, extrinsic)

X = connected open subset of $\mathbb{C}^n$ with compact closure

s.t. $Hol(X)$ (group of holomorphic automorphisms) ("real Lie)

- acts transitively
- contains symmetries s_p

([illegible] crucial for any field of defn. questions!)

3. Circle conjugacy class (algebraic)

(up to a "square root", this is what MTD's generalize (more later))

$X = G(\mathbb{R})^+$-conjugacy class of a homomorphism $\phi: \mathbb{U} \to G$ of alg. gps. $/\mathbb{R}$. } pts. of X are $g\phi g^{-1}$

where G = real adjoint (semisimple) algebraic group and

- only $z, 1, z^{-1}$ appear in the rep. $ad \circ \phi$ on $Lie(G)_{\mathbb{C}}$
- $\theta := ad(\phi(-1))$ is Cartan
- $\phi(-1)$ doesn't project to 1 in any simple factor of G

defn. invariant w.r.t. choice of ϕ in ccl.

Under the equivalence of the 3 characterizations

$$Is(X,g)^+ = Hol(X)^+ = G(\mathbb{R})^+ .$$

If $K_p := \mathrm{stab}(p)$ $(p \in X)$, then

$$G(\mathbb{R})^+ / K_p \xrightarrow{\cong} X .$$

<u>(3) → (1)</u>: $p = \phi$

- $K = Z_{G(\mathbb{R})^+}(\phi) \subset G^{(\Theta)}(\mathbb{R})$ $\Longrightarrow$ (a) $\mathfrak{k}_{\mathbb{C}} = 1$-eigenspace of $\mathrm{ad}\,\phi(z)$ in $\mathfrak{g}_{\mathbb{C}}$; (b) K compact (in fact, maximally so)
- (a) $\Rightarrow$ $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$ (eigenvalues 1, z, z^{-1})

  Identifying $\mathfrak{p}^+ \cong \mathfrak{g}_{\mathbb{R}}/\mathfrak{k}$ puts $\mathbb{C}$-structure on $T_\phi X$ for which $d(\phi(z))$ = mult. by z. Using $G(\mathbb{R})^+$ to translate $J := d(\phi(i))$ to all of TX yields an <u>almost</u> complex structure. One way to see this is integrable ($\Rightarrow$ **X $\mathbb{C}$-mfld**) is to embed $X \hookrightarrow \check{X} :=$ Ad $G(\mathbb{C})$ - translates of flag $\begin{cases} F^1 = \mathfrak{p}^+ \\ F^0 = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \end{cases}$ on $\mathfrak{g}_{\mathbb{C}}$

$$\cong G(\mathbb{C})/P^+$$

$$\left(\cong G^{(\Theta)}(\mathbb{R})/K\right) \quad \text{"\underline{compact dual}"}$$

- (b) $\Rightarrow$ $\exists$ K-invariant sym. + def. bilinear form on $T_\phi X$

  $\Rightarrow$ $\exists$ $G(\mathbb{R})^+$ - invariant Riemannian metric g on X commuting with (translates of) J (since $J \in K$)

  ($\Rightarrow$ **g Hermitian**)
- Symmetry at ϕ : **$s_\phi :=$ Ad $\phi(-1)$**
- $\mathrm{ad}(\phi(-1))$ Cartan + doesn't project to 1 in any factor $\Longrightarrow$ **G noncompact** □

(1) → (3) :

⑨

- $Is(X,g)^+$ adjoint + semisimple $\Rightarrow$ (Borel) $= G(\mathbb{R})^+$ for some algebraic $\boxed{G \subset GL(Lie(Is(X,g)^+))}$

[This can only make sense if Is^+ is adjoint hence embeds in $GL(Lie(\cdots))$.
Further, the "+"'s are necessary: if $Is^+ = SO(p,q)^+$, this is not $G(\mathbb{R})$ for algebraic G.]

- $p \in X \rightsquigarrow s_p \in Aut(X)$ with $\begin{cases} p \text{ isolated fixed pt.} \\ s_p^2 = id_X \end{cases} \Rightarrow ds_p =$ mult. by (-1) on T_pX.

(left margin: recheck argument involving sectional curv.)

In fact, for any $|z| = 1$, $\exists!$ isometry $u_p(z)$ of (X,g) s.t. (on T_pX)

$$du_p(z) = \text{mult. by } z \quad (\text{i.e. } a + bJ)$$

Since this is a homomorphism (from U_1) "on T_pX", the uniqueness means

$$u_p : U_1 \longrightarrow Is(X,g)^+$$

is also a homomorphism. It algebraizes to

$$\boxed{\phi_p : \mathbb{U} \longrightarrow G \qquad (/\mathbb{R})}.$$

- to view X as a ccl, recall that $G(\mathbb{R})^+$ acts transitively and note that for $g \in G(\mathbb{R})^+$ sending $p \mapsto q$,

$$\boxed{\phi_q(z) \left(= g \circ \phi_p(z) \circ g^{-1}\right) = Ad(g)\, \phi_p(z)}$$

(again using uniqueness).

- we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus T_p^{1,0}X \oplus T_p^{0,1}X$

$$d(\phi_p(z)) : z \Rightarrow \bar{z} \text{ since } \phi_p \text{ is real and } \overline{T^{1,0}} = T^{0,1}.$$

$$\bar{z} = z^{-1} \text{ since } z \in U_1$$

Using uniqueness once more, $Ad(k)\,\phi_p(z) = \phi_p(z)$ for $k \in K_p$

$\Rightarrow Ad\,\phi_p(z)$ acts by id. on $\mathfrak{k} = Lie\,K_p$.

So $\boxed{z, 1, z^{-1} \text{ are the eigenvalues of } ad\,\phi}$

- X has negative sectional curvatures $\Rightarrow$ $\boxed{Ad \circ s_p \text{ is Cartan}}$,

which together w/ X noncompact $\Rightarrow$ $\boxed{s_p \text{ projects to } 1 \text{ in no factor of } G.}$ □

C. Cartan's classification of irreducible HSD's

Let X = irreducible HSD,

G = corresponding simple $\mathbb{R}$-algebraic group

$T \subset G_{\mathbb{C}}$ maximal algebraic torus $/\mathbb{C}$.

The restriction to T

$$T \hookrightarrow G_{\mathbb{C}} \xrightarrow{\text{ad}} GL(\mathfrak{g}_{\mathbb{C}})$$

of the adjoint representation breaks into 1-dimensional eigenspaces on which T acts through characters:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right) \quad \text{where} \quad R \subset \operatorname{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$$

$R = R^+ \amalg R^-$ (with $R^- = -R^+$) are the roots.

$\exists!$
- "basis" $\{\alpha_1, \ldots, \alpha_n\} \subset R$ (simple roots) s.t. each $\alpha \in R^+$ is of the form $\sum m_i \alpha_i$, $m_i \geq 0$.
- "highest" root $\hat{\alpha} \in R^+$ s.t. $\hat{m}_i \geq m_i$ for any other $\alpha \in R^+$

The α_i give the nodes on the Dynkin diagram of G, in which α_i & α_j are connected if they pair nontrivially under a standard inner product (the Killing form). Ex / •—•······• A_n E_6 •—•···•< D_n etc.

Over $\mathbb{C}$, our circle map ϕ defines a cocharacter

$$\mu : \mathbb{G}_m \xrightarrow{(/\mathbb{C})} U \xrightarrow{\phi_{\mathbb{C}}} G_{\mathbb{C}}$$

which has a unique conjugate factoring as

$$\mathbb{G}_m \longrightarrow T \subset G_{\mathbb{C}}$$

in such a way that $\langle \mu, \alpha \rangle \geq 0 \quad \forall \alpha \in R^+$.

(This pairing is defined by $\mathbb{G}_m \xrightarrow{\mu} T \xrightarrow{\alpha} \mathbb{G}_m$, $z \longmapsto z^{\langle \mu, \alpha \rangle}$.)

Now, μ must act through $z, 1, z^{-1}$ $\Longrightarrow$

$$\begin{cases} \langle \mu, \alpha \rangle = 0 \text{ or } 1 & \forall \alpha \in R^+ \\ \text{and } \neq 0 \text{ for some } \alpha \end{cases} \Longrightarrow \text{(look at } \langle \tilde{\alpha}, \mu \rangle \text{)}$$

$$\begin{cases} \langle \mu, \alpha_i \rangle = 1 \text{ for a unique } i, \\ \text{and for this } i,\ \hat{m}_i = 1 \quad (\alpha_i \text{ is special}) \end{cases}.$$

So we have a 1–1 correspondence

Irreducible HSD's $\longleftrightarrow$ special nodes on connected Dynkin diagrams.

and hence a list of the # of distinct $\cong$ classes of irred HSD's corresponding to each simple complex Lie algebra:

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
n	1	1	3	2	1	0	0	0

(nothing of these types — circled around E_8, F_4, G_2)

Examples:

A_n: $X \cong SU(p,q)/S(U_p \times U_q)$ $\qquad p+q = n+1$ (n possibilities)

B_n: $X \cong SO(n,2)^+/SO(n) \times SO(2)$ $\qquad \longleftrightarrow H^2(K3)$

C_n: $X \cong Sp_{2n}(\mathbb{R})/U(n)$ $\qquad$ (Carlson's lecture) $\longleftrightarrow$ wt. 1 HS (level 1)

(my convention: these are $2n \times 2n$ matrices)

$\cong \{ Z \in M_n(\mathbb{C}) \mid Z = {}^tZ, \ \mathrm{Im}(Z) > 0 \}$ $\qquad$ Siegel upper $\frac{1}{2}$-space $\mathfrak{h}_n$

[start w/ clarification of defn 3 (write) of HSD, etc.]

D. Hodge-theoretic interpretation

Let V = $\mathbb{Q}$-vector space

Definition: A Hodge structure (i.e. $\mathbb{Q}$-split MHS) on V is a homomorphism $/\mathbb{R}$

$$\tilde{\varphi} : \mathbb{S} \longrightarrow GL(V)$$

such that (the weight homomorphism)

$$w_{\tilde{\varphi}} : \mathbb{G}_m \hookrightarrow \mathbb{S} \longrightarrow GL(V)$$

is defined $/\mathbb{Q}$.

Associated to $\tilde{\varphi}$ is

$$\mu_{\tilde{\varphi}} : \mathbb{G}_m \longrightarrow GL(V)$$
$$z \longmapsto \tilde{\varphi}_{\mathbb{C}}(z,1).$$

[Note: Recalling that $\mathbb{S}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$, $V^{p,q} \subset V_{\mathbb{C}}$ is the $\begin{cases} z^p w^q \text{ - eigenspace of } \varphi_{\mathbb{C}}(z,w) \\ z^p \text{ - eigenspace of } \mu(z) \end{cases}$, and $w_{\tilde{\varphi}}(r) = \tilde{\varphi}(r,r)$ acts on it by r^{p+q}.]

Fix a weight n, Hodge #'s $\{h^{p,q}\}_{p+q=n}$, and polarization $Q : V \times V \to \mathbb{Q}$; let

- D = period domain parametrizing HS of this type on V, & polarized by Q,
- $t \in \bigoplus_i V^{\otimes k_i} \otimes \check{V}^{\otimes l_i}$ be a (finite) sum of $\mathbb{Q}$-tensors,
- $D_t^+ \subset D$ a connected component of the subset of HS for which these tensors are Hodge ($t \in F^{n(k-l)/2}$),

and

- $M_t \subset GL(V)$ the smallest $\mathbb{Q}$-algebraic subgroup with $M_t(\mathbb{R}) \supset \tilde{\varphi}(\mathbb{S}(\mathbb{R}))$ $\forall\, \tilde{\varphi} \in D_t^+$ (this is reductive).

Then: given any $\tilde{\varphi} \in D_t^+$, the orbit

$$D_t^+ = M_t(\mathbb{R})^+ . \tilde{\varphi} \qquad \text{(action by conjugation)}$$

is called a Mumford-Tate domain. Furthermore, $Ad(\mu_{\tilde{\varphi}}(-1))$ is a Cartan involution (exercise). Now,

the tautological family $\mathcal{V} \to D_t^+$ is a VHS (i.e. the ~~IPR~~ infinitesimal period relation $\mathcal{I} \subset \Omega^\bullet(D_t^+)$ is trivial)

$\Updownarrow$

the induced HS on $Lie(M_t) \subset End(V)$ (at $\tilde{\varphi}$) is of type $(-1,1)+(0,0)+(1,-1)$ (stuff of type (2,2) or worse would violate Griffiths transversality)

$\Updownarrow$

(*) $ad \circ \mu_{\tilde{\varphi}}$ has eigenvalues $z, 1, z^{-1}$ (only),

proving part (a) of

(† we still get all the real * classes of HSD's in the table)

Proposition: (a) A MT domain with trivial IPR (and M_t adjoint) admits the structure of a HSD with† $G/\mathbb{Q}$, and

(b) Conversely (i.e. such HSD's parametrize Hodge structures).

Remarks: (i) (*) $\Rightarrow Ad(\mu_{\tilde{\varphi}}(-1))$ gives a symmetry of D_t^+ at $\tilde{\varphi}$ [, but not conversely: e.g., an example of a MTD with nontrivial IPR but HSD structure is the period domain for HS's of type $(1,0,1,h,1,0,1)$ (weight 6).]

(ii) This doesn't contradict (b), since the same HSD can have different MTD structures.

(iii) The proposition is essentially a theorem of Deligne from 1979.

Proof of (6): Let $X = $ HSD with real circle $\mathbb{U} \xrightarrow{\phi} G$.

(Since a product of MTD's is a MTD, we may assume G $\mathbb{Q}$-simple.)

The composition

$$\begin{array}{c} (z,w) \longmapsto z/w \\ \mathbb{S} \xrightarrow[/\mathbb{R}]{} \mathbb{U} \xrightarrow[\phi]{} G \longrightarrow \mathrm{Aut}(\mathfrak{g}, B) \end{array}$$

$=: \tilde{\varphi}$

$\uparrow$ Killing form: $B(X,Y) = \mathrm{Tr}(\mathrm{ad}X \,\mathrm{ad}Y)$

is a HS on $V = \mathfrak{g}$ polarized by $-B$ (since $\mathrm{Ad}(\phi(-1))$ is Cartan).

The $\mathbb{Q}$-closure of a generic $G(\mathbb{R})$-conjugate is G (since $\phi(-1)$ is nontrivial), G is of the form M_t by Chevalley's theorem (cf Griffiths's lectures), and so $X = G(\mathbb{R})^+ . \tilde{\varphi}$ is a MT domain. The IPR vanishes because $\mathrm{ad} \circ \phi$ has eigenvalues $z, 1, z^{-1}$. $\square$

Example (M. Green): applied to one of the [redacted] HSD's for E_6, this procedure produces a MT domain parametrizing certain HS of type $(h^{2,0}, h^{1,1}, h^{0,2}) = (16, 46, 16)$ — a submanifold D_t^+ of the period domain D for such HS. The IPR $I \subset \Omega^1(D)$ is nontrivial but pulls back to zero on D_t^+. $\square$

[Open problem: find a family of varieties over D_t^+ with this family of HS.*]

The proof of (6) always yields HS's of even weight. Sometimes, by replacing the adjoint representation by a "standard" representation, we can parametrize HS's of odd weight.

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* More generally, it is conjectured that the tautological VHS over every MTD with trivial IPR, comes from algebraic geometry (is "motivic")

Examples: $\mathfrak{h}^4 \cong Sp_8(\mathbb{R})/U(4)$ parametrizes HS's of weight 1 rank 8, or equivalently abelian varieties of dimension 4. There are "two" types of MT subdomains in $\mathfrak{h}^4$:

- corresponding to $\mathrm{End}_{HS}(V)$ $(= \mathrm{End}(A)_{\mathbb{Q}})$ containing a nontrivial fixed subalgebra $\mathcal{E}$ = product of matrix algebras over $\mathbb{Q}$-division algebras of types

Albert classification [all four do occur in $\mathfrak{h}^4$]:

I) totally real field

II) indefinite } quaternion algebra over totally real field

III) definite }

IV) division algebra over a CM field — e.g., just an imaginary quadratic field.

- corresponding to fixed endomorphisms $\mathcal{E}$ + *higher* Hodge tensors. (We think of $\mathrm{End}_{HS}(V) \subset T^{1,1}V$ and the polarization $Q \in T^{0,2}V$; "higher" means [in $T^{k,l}V$] of degree $k+l > 2$.)

Here are two such examples:

(a) Consider HS's on V with a fixed embedding

$$\mathbb{Q}(i) \hookrightarrow \mathrm{End}(V)$$

such that (writing $\mathrm{Hom}(\mathbb{Q}(i), \mathbb{C}) = \{\eta, \bar{\eta}\}$)

$$V^{1,0} = V^{1,0}_{\eta} \oplus V^{1,0}_{\bar{\eta}} \quad \text{with } \dim V^{1,0}_{\eta} = 1.$$

This yields $D^{+}_{t} \cong SU(1,3)/S(U_1 \times U_3) \cong$ Complex 3-ball, of type A_3.
(CCL) (HSD) (BSD)

> Exercise: show MTG is $U(1,3)$.

(b) [Mumford] constructs a quaternion algebra $\mathcal{Q}$ over a totally real cubic field K, such that $\mathcal{Q} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \oplus \mathbb{H} \oplus M_2(\mathbb{R})$, together with an embedding $\mathcal{Q}^* \hookrightarrow GL_8(\mathbb{Q})$. This yields a $\mathbb{Q}$-simple algebraic group

$$G := \mathrm{Res}_{K/\mathbb{Q}} U_{\mathcal{Q}} \subset GL(V) \quad [\text{actually in } Sp_8]$$

with $G(\mathbb{R}) \cong SU(2)^{\times 2} \times SL_2(\mathbb{R})$, and the $G(\mathbb{R})$-orbit of $\varphi_0 : \mathbb{U} \to G$, $a+ib \mapsto \mathrm{Id}^{\times 2} \times \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ yields a MT domain. Mumford shows that the generic HS it supports has $\mathcal{E} = \mathbb{Q}$ (trivial) so G is cut out by higher Hodge tensors.

II. Locally symmetric varieties

To construct quotients of Hermitian symmetric domains we'll need the basic

<u>Proposition</u>: Let $\mathfrak{X}$ = topological space $\ni x_0$

$\mathcal{G}$ = locally compact group acting on $\mathfrak{X}$

Γ ($\leq \mathcal{G}$) a (discrete) subgroup (no limit pts.)

Assume (i) $K = \mathrm{stab}(x_0)$ is compact, and

(ii) $gK \mapsto gx_0 : \mathcal{G}/K \to \mathfrak{X}$ is a homeomorphism.

Then $\Gamma\backslash\mathfrak{X}$ is Hausdorff.

<u>Proof</u>: nontrivial topology exercise. Writing $\pi : \mathfrak{X} \to {}_{\Gamma}\backslash\mathfrak{X}$, the key points are:

- π^{-1}(compact) is compact
- discrete $\cap$ compact = finite. $\square$

<u>Corollary</u>: Let $X = G(\mathbb{R})^+/K$ HSD

$\Gamma \leq G(\mathbb{R})^+$ discrete & torsion-free.

Then ${}_{\Gamma}\backslash X$ has a unique $\mathbb{C}$-mfld. structure for which π is a local $\cong$.

(If Γ isn't torsion-free then we get an orbifold.)

<u>Examples</u>: (a) $X = \mathfrak{h}^1$, $G = SL_2$ (acting in the standard way)

$\Gamma = \Gamma(N) := \ker\{SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})\}$ with $N > 3$

$\Gamma\backslash X =: Y(N)$ gives the <u>classical modular curves</u>

classifying elliptic curves with <u>marked N-torsion</u>. (an example of a <u>level</u> structure).

(b) $X = \mathfrak{h}^n$, $G = Sp_{2n}$, $\Gamma = Sp_{2n}(\mathbb{Z})$

$\Gamma\backslash X$ = <u>Siegel modular variety</u>

classifying abelian n-folds (w./ fixed polarization)

(c) $X = \underbrace{\mathfrak{h}^1 \times \cdots \times \mathfrak{h}^1}_{n \text{ times}}$, $G = \mathrm{Res}_{F/\mathbb{Q}} SL_2$ for F = totally real field of degree n / $\mathbb{Q}$

(acting through the n embeddings of $F \hookrightarrow \mathbb{R}$)

$\Gamma = SL_2(\mathcal{O}_F)$ ← (or some ideal in $\mathcal{O}_F$)

$\Gamma \backslash X$ is a Hilbert modular variety classifying abelian n-folds with $\mathcal{E} \supseteq F$ (i.e. general member is of Albert type I).

We can view X as a proper MT subdomain of $\mathfrak{h}^n$.

All the Γ's that have come up here are rather special.

Definition: (a) Let G be a $\mathbb{Q}$-algebraic group. $\Gamma \leq G(\mathbb{Q})$ is

<u>arithmetic</u> $\Leftrightarrow$ Γ commensurable with $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$ [commensurable with: $\cap$ of finite index in each]

$\Uparrow$

<u>congruence</u> $\Leftrightarrow$ $\Gamma \supseteq \Gamma(N) := G(\mathbb{Q}) \cap \{ g \in GL_n(\mathbb{Z}) \mid g \equiv_{(N)} \mathrm{Id} \}$ for some N (finite index)

for some embedding $G \hookrightarrow GL_n$

(b) Let H be a connected real Lie group [e.g., $G(\mathbb{R})^+$]

$\Gamma \leq H$ is <u>arithmetic</u> if $\exists$

- $\mathbb{Q}$-algebraic group G,
- an arithmetic $\Gamma_0 \leq G(\mathbb{Q})$, and
- a homomorphism $G(\mathbb{R})^+ \xrightarrow{\rho} H$ (surjective) with compact kernel,

such that $\rho(\Gamma_0 \cap G(\mathbb{R})^+) = \Gamma$.

(This is set up so that Γ will always contain a torsion-free subgroup of finite index.)

<u>Theorem [Baily-Borel, 1966]</u> : Let $X = G(\mathbb{R})^+/K$ be a HSD, $\Gamma < G(\mathbb{R})^+$ a torsion-free arithmetic subgroup.

Then $X(\Gamma) := {}_\Gamma\backslash X$ is canonically a smooth quasi-projective algebraic variety, called a <u>locally symmetric variety</u>.

(If we don't assume torsion-free, we still get a quasi-proj. alg. var., but it is an orbifold (non-smooth) and <u>not</u> called a LSV.)

<u>Idea of proof</u>: Construct a (minimal, highly singular) compactification

$$X(\Gamma)^* := {}_\Gamma\backslash\{X \amalg B\} \qquad (B = \text{"rational boundary components"})$$

and embed it in $\mathbb{P}^N$ (using automorphic forms of sufficiently high weight) as a projective analytic, hence (by GAGA (Chow)) projective algebraic, variety. (The existence of enough automorphic forms to yield an embedding is a convergence question for certain "Poincaré-Eisenstein series".) $\square$

~~[scribbled out]~~

<u>Example</u>: In the modular curve context, $B = \mathbb{P}^1(\mathbb{Q})$ and $X(\Gamma)^* \setminus X(\Gamma)$ is the (finite point set of) <u>cusps</u>. We write $(X(\Gamma) =) Y(N) \subset X(N) (= X(\Gamma)^*)$.

Recalling (from I.D) that X is always a MT domain, we can give a Hodge-theoretic interpretation to the Baily-Borel compactification:

[Cattani-Carlson-Kaplan?]
<u>Proposition</u>: The boundary components B parametrize the possible $\bigoplus_i Gr_i^W H_{lim}$ for VHS into $X(\Gamma)$. (Presumably this is independent of which MTD structure we put on X.)

<u>Idea of proof</u> (Heuristic): Assuming PGL_2 is not a quotient of G, the automorphic forms are Γ-invariant sections of $K_X^{\otimes N}$ for some $N \gg 0$.

K_X ($\cong \Lambda^d \mathfrak{g}^{(-1,1)}$, pointwise) measures the change of the Hodge flag in every direction, so the boundary components parametrized by these

Sections must consist of <u>naive</u> limiting Hodge flags in $\partial\overline{X} = \check{X}$.

In that limit, the relation (projectively) between periods that go to ∞ at different rates (arising from distinct Gr_i^W) is fixed, which means we can't see extension data. On the other hand, since $\exp(zN)$ does not change the $Gr_i^W F^\bullet$, <u>this</u> information is the <u>same</u> for the naive limiting Hodge flag of the LMHS. □

<u>Remark</u>: There are other compactifications with different Hodge-theoretic interpretations:

- AMRT toroidal (smooth) compactifications *
  (capture the entire LMHS)
- Borel-Serre compactification
  (captures Gr_i^W and adjacent extensions, at least in Siegel case).

Later on (e.g. for canonical models) we will need the

<u>Theorem (Borel)</u>: Let V = quasi-projective algebraic variety $/\mathbb{C}$
$X(\Gamma)$ = locally symmetric variety

Then any analytic map $V \to X(\Gamma)$ is algebraic.

<u>Idea of proof</u>: Extend to analytic map $\overline{V} \to X(\Gamma)^*$, use GAGA.

Suppose $X = \mathfrak{h}^1$ and V is a curve; Γ torsion-free $\Rightarrow X(\Gamma) \cong \mathbb{C}\setminus\{\geq 2 \text{ pts.}\}$ (or $\mathbb{P}^1\setminus\{\geq 3 \text{ pts.}\}$)

If a holo. $f: \mathbb{D}^* \to X(\Gamma)$ (punctured disk) does not extend to holo. map $\mathbb{D} \to \mathbb{P}^1$,

then f has an essential singularity at 0; and the Big Picard theorem $\Rightarrow f$ takes all values of $\mathbb{C}$ except possibly one, a contradiction.

The general proof uses $\exists$ of a good compactification $V \subset \overline{V}$ (Hironaka) so that V is locally $\mathbb{D}^{\times k} \times (\mathbb{D}^*)^{\times l}$. □

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* which Kato-Usui generalize (in a sense) to the non-HSD ($X(\Gamma)$ not algebraic) case

III. Complex multiplication

A. CM Abelian varieties

A <u>CM field</u> is a totally imaginary field E possessing an involution $\rho \in \mathrm{Gal}(E/\mathbb{Q}) =: \mathcal{G}_E$ such that $\phi \circ \rho = \bar{\phi}$ for each $\theta \in \mathrm{Hom}(E, \mathbb{C}) =: \mathcal{H}_E$.

[Exercise: E^ρ is then totally real, and $\rho \in Z(\mathcal{G}_E)$.]

Write E^c for a normal closure.

For any decomposition

$$\mathcal{H}_E = \Phi \amalg \bar{\Phi},$$

(pairwise non-conjugate embeddings)

(E, Φ) is a <u>CM type</u>; this is equipped with a <u>reflex field</u>

$$E' := \mathbb{Q}\left(\left\{ \sum_{\phi \in \Phi} \phi(e) \,\middle|\, e \in E \right\} \right) \subset E^c$$

$$= \text{fixed field of } \left\{ \sigma \in \mathcal{G}_{E^c} \,\middle|\, \sigma\tilde{\Phi} = \tilde{\Phi} \right\}$$

(where $\tilde{\Phi} \subset \mathcal{H}_{E^c}$ consists of embeddings restricting [on E] to those in Φ).

Fixing a choice of $\phi_1 \in \Phi$ gives an identification

$$\mathcal{H}_{E^c} \xleftarrow[\text{compose w/}\phi_1]{\cong} \mathcal{G}_{E^c}$$

and a notion of inverse on $\mathcal{H}_{E^c}$. Define the <u>reflex type</u> by

$$\Phi' := \left\{ \tilde{\phi}^{-1}|_{E'} \,\middle|\, \tilde{\phi} \in \tilde{\Phi} \right\},$$

and reflex norm by

$$N_{\Phi'} : (E')^* \longrightarrow E^*$$
$$e' \longmapsto \prod_{\phi' \in \Phi'} \phi'(e') .$$

Examples: (a) All $\mathbb{Q}(\sqrt{-d})$ are CM; in this case $N_{\Phi'}$ is the identity or complex conjugation.

(b) All $\mathbb{Q}(\zeta_n)$ are CM; and if $E/\mathbb{Q}$ is an abelian extension, then $E' = E^c = E$ and $E \subset$ some $\mathbb{Q}(\zeta_n)$.

[For cyclotomic fields we'll write $\phi_j :=$ embedding sending $\zeta_n \mapsto e^{2\pi i j/n}$.]

(c) $(\mathbb{Q}(\zeta_5); \{\phi_1, \phi_2\})$ has reflex $(\mathbb{Q}(\zeta_5); \{\phi_1, \phi_3\})$.

The relation of this to algebraic geometry is the following

Proposition: (a) For a simple Abelian g-fold $A/\mathbb{C}$, TFAE:

(i) MT group of $H^1(A)$ is a torus [not nec. of dim. g: can be "degenerate"]

(ii) $\mathrm{End}(A)_{\mathbb{Q}}$ has (maximal) rank $2g$ $/\mathbb{Q}$ [lots of endomorphisms]

(iii) $\mathrm{End}(A)_{\mathbb{Q}}$ is a CM field

(iv) $A \cong \mathbb{C}^g / \Phi(\mathfrak{a}) =: A_{\mathfrak{a}}^{(E,\Phi)}$ for some CM type and ideal $\mathfrak{a} \subset \mathcal{O}_E$.

(b) Furthermore, any complex torus of the form $A_{\mathfrak{a}}^{(E,\Phi)}$ is algebraic.

Example: $\Phi(\mathfrak{a})$ means $\left\{ \begin{pmatrix} \phi_1(a) \\ \vdots \\ \phi_g(a) \end{pmatrix} \,\middle|\, a \in \mathfrak{a} \right\}$ $(= 2g\text{-lattice})$;

for Ex (c) above

$$A_{\mathcal{O}_{\mathbb{Q}(\zeta_5)}} = \mathbb{C}^2 \Big/ \mathbb{Z}\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} e^{2\pi i/5} \\ e^{4\pi i/5} \end{pmatrix}, \begin{pmatrix} e^{4\pi i/5} \\ e^{8\pi i/5} \end{pmatrix}, \begin{pmatrix} e^{6\pi i/5} \\ e^{2\pi i/5} \end{pmatrix} \right\rangle$$

The interesting points are: where does the CM field come from, and why is $A^{(E,\Phi)}$ polarized?

Pf. of (i) ⇒ (iii): $V = H^1(A)$ is polarized by some Q.

Set $\mathcal{E} := \mathrm{End}_{HS}(H^1(A)) = \left(Z_{GL(V)}(M)\right)(\mathbb{Q}) \cup \{0\}$

($Z_{GL(V)}(M)$: $\mathbb{Q}$-algebraic group, contains a maximal torus T)

Since T commutes with M and is maximal, $T \supset M$. (not in general an equality!)

One deduces that
- $T(\mathbb{Q}) \cup \{0\} =: E \left(\underset{\eta}{\hookrightarrow} \mathcal{E}\right)$ is a field
- V = 1-dimensional vector space $/E$, & finally that
- E is actually all of $\mathcal{E}$.

M diagonalizes with respect to a Hodge basis

$$\omega_1, \ldots, \omega_g \,;\, \overline{\omega_1}, \ldots, \overline{\omega_g}$$

such that $\sqrt{-1}\, Q(\omega_i, \overline{\omega_j}) = \delta_{ij}$. The maximal torus $\subset GL(V)$ this basis defines, centralizes M hence must be T.

Now write $\mathcal{H}_E = \{\phi_1, \ldots, \phi_{2g}\}$, $E = \mathbb{Q}(\xi)$ and

$$m_\xi(\lambda) = \prod_{i=1}^{2g} (\lambda - \phi_i(\xi))$$

for the minimal polynomial of ξ, hence $\eta(\xi)$.

Up to reordering, we $\therefore$ have

$$[\eta(\xi)]_\omega = \mathrm{diag}\left(\{\phi_i(\xi)\}_{i=1}^{2g}\right).$$

Since $\eta(\xi) \in GL(V_{(\mathbb{Q})})$ (a fortiori $\in GL(V_{\mathbb{R}})$), and $\phi_j(\xi)$ determines ϕ_j,

$$\omega_{i+g} = \overline{\omega_i} \implies \phi_{i+g} = \overline{\phi_i}.$$

Define the Rosati Involution $\dagger : \mathcal{E} \to \mathcal{E}$ by

$$\boxed{Q(\epsilon^\dagger v, w) = Q(v, \epsilon w) \quad \forall v, w \in V.}$$

This produces $\rho := \eta^{-1} \circ \dagger \circ \eta \in \mathcal{G}_E$, and we compute

$$\phi_{i+g}(e)\, Q(\omega_i, \omega_{i+g}) = Q(\omega_i, \eta(e)\,\omega_{i+g}) = Q(\eta(e)^\dagger \omega_i, \omega_{i+g})$$

$$= Q(\eta(\rho(e))\,\omega_i, \omega_{i+g}) = \phi_i(\rho(e))\, Q(\omega_i, \omega_{i+g})$$

$$\implies \phi_i \circ \rho = \overline{\phi_i}. \qquad \square$$

<u>Proof of (b)</u>: We have the following construction of $H^1(A)$:

let V = $2g$-dim'l $\mathbb{Q}$-vector space with identification

$$\boxed{V \xleftarrow[\beta]{\cong} E}$$

inducing (via multiplication in E)

$$\eta : E \hookrightarrow \mathrm{End}_{\mathbb{Q}}(V).$$

Moreover, $\exists$ basis $\omega = \{\omega_1, \ldots, \omega_g ; \bar{\omega}_1, \ldots, \bar{\omega}_g\}$ of $V_{\mathbb{C}}$ with respect to which $[\eta_{\mathbb{C}}(e)]_\omega = \mathrm{diag}\{\phi_1(e), \ldots, \phi_g(e) ; \overline{\phi_1}(e), \ldots, \overline{\phi_g}(e)\}$, and we set $V^{1,0} := \mathbb{C}\langle\omega_1, \ldots, \omega_g\rangle$ (so that $\eta(e) \in \mathrm{End}_{HS}(V)$). This gives $V \underset{HS}{\cong} H^1(A)$.

Now, $\exists\, \xi \in E$ s.t. $\sqrt{-1}\,\phi_i(\xi) > 0$ $(i = 1, \ldots, g)$ and we can put

$$\boxed{Q(\beta(e), \beta(\tilde{e})) := \mathrm{Tr}_{E/\mathbb{Q}}(\xi \cdot e \cdot \rho(\tilde{e})) \quad : V \times V \to \mathbb{Q}}$$

Over $\mathbb{C}$, this becomes

$$[Q]_\omega = \begin{pmatrix} 0 & \begin{matrix} \phi_1(\xi) & & \\ & \ddots & \\ & & \phi_g(\xi) \end{matrix} \\ \begin{matrix} \overline{\phi_1}(\xi) & & \\ & \ddots & \\ & & \overline{\phi_g}(\xi) \end{matrix} & 0 \end{pmatrix}. \qquad \square$$

<u>Remark</u>: $N_{\Phi'}$ algebraizes to a homomorphism of algebraic groups

$$\mathcal{N}_{\Phi'} : \mathrm{Res}_{E'/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

which gives $N_{\Phi'}$ on the $\mathbb{Q}$-points, and the MT group

$$M_{H^1(A)} \cong \mathrm{im}(\mathcal{N}_{\Phi'}). \qquad \square$$

Let E be a CM field with $[E:\mathbb{Q}] = 2g$. In algebraic number theory we have

$\mathcal{J}(E)$ = monoid of <u>nonzero</u> ideals in $\mathcal{O}_E$

$\cap$

$\mathcal{I}(E)$ = fractional ideals (of form eI, $e \in E^*$ & $I \in \mathcal{J}(E)$)

$\cup$

$\mathcal{P}(E)$ = principal fractional ideals (of form $(e) := e \cdot \mathcal{O}_E$, $e \in E^*$).

The (abelian!) <u>ideal class group</u>

$$Cl(E) := \frac{\mathcal{I}(E)}{\mathcal{P}(E)},$$

or more precisely the <u>class number</u>

$$h_E := |Cl(E)|,$$

expresses (if $\neq 1$) the failure of $\mathcal{O}_E$ to be a PID ($\nLeftrightarrow$ to have unique factorization). Each class $\tau \in Cl(E)$ has a representative $I \in \mathcal{I}(E)$ with norm $\leq$ the <u>Minkowski bound</u>, which $\Rightarrow h_E < \infty$.

Now let

$$Ab(\mathcal{O}_E, \Phi) := \frac{\{A_{\mathfrak{a}}^{(E,\Phi)} \mid \mathfrak{a} \in \mathcal{I}(E)\}}{\text{isomorphism}}$$

← abelian g-folds w/ $\mathcal{O}_E \subset End(A)$ acting on $T_0 A$ through Φ. They are all <u>isogenous</u>.

and note that for any $c \in E^*$

$$A_{\mathfrak{a}} \xrightarrow[\cong]{\text{mult. by } c} A_{c\mathfrak{a}}.$$

Hence we get a bijection

$$Cl(E) \xrightarrow{\cong} Ab(\mathcal{O}_E, \Phi)$$
$$[\mathfrak{a}] \longmapsto A_{\mathfrak{a}}.$$

<u>Example</u>: The class # of $\mathbb{Q}(\sqrt{-5})$ is 2. Representatives of $Ab(\mathcal{O}_{\mathbb{Q}(\sqrt{-5})}, \phi)$ (↑ 1 class of embedding) are the elliptic curves $\mathbb{C}/\mathbb{Z}\langle 2, 1+\sqrt{-5}\rangle$ and $\mathbb{C}/\mathbb{Z}\langle 1, \sqrt{-5}\rangle$. □

The key point now is to notice that for $A \in Ab(\mathcal{O}_E, \Phi)$ and $\sigma \in Aut(\mathbb{C})$,

$$^{\sigma}A \in Ab(\mathcal{O}_E, \sigma\Phi)$$

since endomorphisms are given by alg. cycles (and so the internal structure of E is left unchanged). From the definition of E' we see that

<u>$Gal(\mathbb{C}/E')$ acts on $Ab(\mathcal{O}_E, \Phi)$</u>

which suggests that the individual abelian varieties should

be defined over an extension of E' of degree $h_{E'}$.
(This isn't exactly true if $\mathrm{Aut}(A) \neq \{id\}$, but the argument does prove $A/\overline{\mathbb{Q}}$.)

B. Class field theory

In fact, it gets much better: not only is there a distinguished field extension H_L/L of degree h_L (for any # field); in fact

$$(*) \qquad \mathrm{Gal}(H_L/L) \xleftarrow{\cong} Cl(L)$$

so that (to quote Chevalley) "L contains within itself the elements of its own transcendence".

Idea of the construction (of (*)): let $\tilde{L}/L$ be an extension of degree d which is

- abelian: Galois with $\mathrm{Gal}(\tilde{L}/L)$ abelian
- unramified: for each prime $\mathfrak{p} \in \mathcal{I}(L)$, $\mathfrak{p} \cdot \mathcal{O}_{\tilde{L}} = \prod_{i=1}^{r} \mathfrak{P}_i$ (power $= 1$) for some $r \mid d$.

The image of

$$\mathrm{Gal}\left((\mathcal{O}_{\tilde{L}}/\mathfrak{P}_i) / (\mathcal{O}_L/\mathfrak{p}) \right) \xleftarrow{\cong} \{ \sigma \in \mathrm{Gal}(\tilde{L}/L) \mid \sigma \mathfrak{P}_i = \mathfrak{P}_i \} \subset \mathrm{Gal}(\tilde{L}/L)$$

(degree-$N(\mathfrak{p})^{d/r}$ extension of finite fields), $|\cdot| = N(\mathfrak{p})$

generated by

$$\left\{ \begin{array}{c} \alpha \mapsto \alpha^{N(\mathfrak{p})} \\ (\mathrm{mod}\ \mathfrak{P}_i) \end{array} \right\} \longmapsto \quad =: \mathrm{Frob}_{\mathfrak{p}}$$

is (as the notation suggests) independent of i, yielding a map from

$$\left\{ \begin{array}{c} \text{prime ideals} \\ \text{of } L \end{array} \right\} \longrightarrow \mathrm{Gal}(\tilde{L}/L).$$

Taking $\tilde{L}$ to be

$H_L :=$ maximal unramified abelian extension of L ("Hilbert class field"),

this leads (eventually) to (*). $\square$

More generally, given $I \in \mathcal{I}(K)$ we have

$$Cl(I) = \frac{\text{fractional ideals prime to } I}{\text{principal " " with generator} \equiv 1 \pmod I}$$ ("ray class group mod I")

L_I = (approximately*) the maximal abelian ext. in which primes dividing I are allowed to ramify ("ray class field mod I")

($\supseteq H_L$, finite$/L$, and with equality when $I = \mathcal{O}_L$)

and an isomorphism

$$Gal(L_I/L) \xleftarrow{\simeq} Cl(I).$$

__Example__: for $L = \mathbb{Q}$, $L_{(n)} = \mathbb{Q}(\zeta_n)$. □

To deal with

$$L^{ab} := \text{maximal abelian extension of } L \quad (\subset \bar{\mathbb{Q}}),$$

which is infinite $/L$, we have to introduce the adèles.

In studying abelian varieties one considers (for $\ell \in \mathbb{Z}$ prime) the finite groups of ℓ-torsion points $A[\ell]$; multiplication by ℓ gives maps

$$\cdots \to A[\ell^{n+1}] \to A[\ell^n] \to \cdots$$

If we do the same thing on the unit circle $S^1 \subset \mathbb{C}^*$, we get

$$\cdots \to S^1[\ell^{n+1}] \xrightarrow{\cdot \ell} S^1[\ell^n] \to \cdots$$

$$\to \mathbb{Z}/\ell^{n+1}\mathbb{Z} \xrightarrow[\text{natural map}]{} \mathbb{Z}/\ell^n\mathbb{Z} \to$$

(with $\cong$ between the rows) and one can define the ℓ-adic integers by the inverse limit**

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}.$$

** an element of this is (by definition) an infinite sequence of elements in the $\mathbb{Z}/\ell^n\mathbb{Z}$ mapping to each other.

* more precisely, the maximal abelian ext. in which all primes $\equiv 1 \bmod I$ split completely.

There is the natural inclusion

$$\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell ,$$

and

$$\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathbb{Q}_\ell .$$

Elements of $\mathbb{Q}_\ell$ can be written as power series $\sum_{i \geq n} a_i \ell^i$ (for some $n \in \mathbb{Z}$, ≥ 0 if one wants an element of $\mathbb{Z}_\ell$). $\mathbb{Q}_\ell$ can also be thought of as the completion of $\mathbb{Q}$ with respect to the ~~[illegible]~~ metric given by

$$d(x,y) = \frac{1}{\ell^n} \quad \text{if} \quad x - y = \ell^n \left(\frac{a}{b}\right) \quad [\text{with } a, b \text{ relatively prime to } p],$$

and the resulting topology on $\mathbb{Z}_\ell$ makes

$$U_n(\alpha) := \{\alpha + \lambda \ell^n \mid \lambda \in \mathbb{Z}_\ell\}$$

into "the open disk about $\alpha \in \mathbb{Z}_\ell$ of radius $\frac{1}{\ell^n}$". $\mathbb{Z}_\ell$ itself is compact and totally disconnected.

Now set

$$\hat{\mathbb{Z}} := \varprojlim_{n \in \mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \underset{\substack{\uparrow \\ \text{Chinese Remainder Thm.}}}{=} \prod_\ell \mathbb{Z}_\ell .$$

The <u>finite adèles</u> appear naturally as

$$\mathbb{A}_f := \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = \prod_\ell{}' \mathbb{Q}_\ell ,$$

where $\prod'$ means the ∞-tuples with all but finitely many entries in $\mathbb{Z}_\ell$. The "full" <u>adèles</u> are constructed by writing

$$\mathbb{A}_{\mathbb{Z}} := \mathbb{R} \times \hat{\mathbb{Z}}$$

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{A}_{\mathbb{Z}} = \mathbb{R} \times \mathbb{A}_f ,$$

which generalizes for a # field L to*

$$\mathbb{A}_L = L \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}} = \mathbb{R}^{[L:\mathbb{Q}]} \times \underbrace{\prod_{\mathfrak{p} \in \mathcal{S}(L)}{}' L_{\mathfrak{p}}}_{= \mathbb{A}_{L,f}} .$$

* $\prod'$ means all but finitely many entries in $(\mathcal{O}_L)_{\mathfrak{p}}$.

For a $\mathbb{Q}$-algebraic group G, we can define

$$G(\mathbb{A}_f) := \prod{}' G(\mathbb{Q}_\ell)$$

$$G(\mathbb{A}_\mathbb{Q}) := G(\mathbb{R}) \times G(\mathbb{A}_f)$$

($\prod'$: all but finitely many entries in $G(\mathbb{Z}_\ell)$, for some embedding $G \hookrightarrow GL_N$)

with generalizations to $\mathbb{A}_{L,(f)}$. The idèles

$$\mathbb{A}_{(f)}^\times = \mathbb{G}_m(\mathbb{A}_{(f)})$$

$$\mathbb{A}_{L,(f)}^\times = (\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m)(\mathbb{A}_{(f)}) = \mathbb{G}_m(\mathbb{A}_{L,(f)})$$

were historically defined first; Weil introduced "adèle" (also a girls' name, which was intentional) as a contraction of "additive idèle". The usual norm $N_{\tilde{L}/L}$ and reflex norm N_{Φ}, extend to maps of idèles, using the formulation of these maps as morphisms of $\mathbb{Q}$-algebraic groups.

Now returning to $S^1 \subset \mathbb{C}^*$, let ζ be an Nth root of 1 and $a = (a_n) \in \hat{\mathbb{Z}}$; then

<u>$\zeta^a := \zeta^{a_N}$ defines an action of $\hat{\mathbb{Z}}$ on the torsion points of S^1</u> (which generate $\mathbb{Q}^{ab}$).

The cyclotomic character

$$\chi : \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^\times \cong \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q},f}^\times$$

is defined by $\sigma(\zeta) := \zeta^{\chi(\sigma)}$,

and we can think of it as providing a "continuous envelope"*

* I use this term because the automorphisms of $\mathbb{C}$ other than complex conj. induce highly discontinuous (non-measurable!) maps on the complex points of a variety/$\mathbb{Q}$. But if one specifies a set of points it is sometimes poss. to produce a continuous (even analytic/algebraic) automorphism acting in the same way on those pts!

for the action of a given σ on any finite order of torsion.

The Artin reciprocity map is simply its inverse

$$\mathrm{art}_{\mathbb{Q}} := \chi^{-1}$$

for $\mathbb{Q}$, but generalizes to (assuming L now totally imaginary)

$$\begin{array}{ccc}
L^{\times}\backslash \mathbb{A}^{\times}_{L,f} & \xrightarrow{\ \mathrm{art}_L\ } & \mathrm{Gal}(L^{ab}/L) \\
\downarrow (\#) & & \downarrow \\
L^{\times}\backslash \mathbb{A}^{\times}_{L,f} / N_{\tilde{L}/L}(\mathbb{A}^{\times}_{\tilde{L},f}) & \xrightarrow{\ \cong\ } & \mathrm{Gal}(\tilde{L}/L)
\end{array}$$

If $\tilde{L} = L_I$ then the double-coset turns out to be $Cl(I)$, so this recovers the earlier maps for ray class fields. (For $\tilde{L} = H_L$, we can replace $N_{\tilde{L}/L}(\cdots)$ by $\hat{O}_L$.) The correspondence between

$$\left\{\begin{array}{c}\text{open subgroups} \\ \text{of } \mathbb{A}^{\times}_{L,f}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{finite abelian} \\ \text{extensions of} \\ L\end{array}\right\}$$

is the essence of class field theory. Also note that $(\#)$ gives compatible maps to all the class groups of L, so that $\mathbb{A}^{\times}_{L,f}$ acts on them.

C. Main Theorem of CM

Now let's bring the adeles to bear upon abelian varieties. Taking the product of the Tate modules

$$T_\ell A := \varprojlim_n A[\ell^n] \qquad (\text{free rank-}2g\ \mathbb{Z}_\ell\text{-module})$$

of an abelian g-fold yields

$$T_f A := \prod_\ell T_\ell A \quad , \quad V_f A := T_f A \otimes_{\mathbb{Z}} \mathbb{Q}$$
$$(= \text{free rank-}2g\ \mathbb{A}_f\text{-module}) .$$

with (for example)

$$\mathrm{Aut}(V_f A) \cong GL_{2g}(\mathbb{A}_f) .$$

The main theorem* is basically a detailed description of the action of $\mathrm{Gal}(\mathbb{C}/E')$ on $\mathcal{Ab}(\mathcal{O}_E, \Phi)_{\cong}$ and the torsion points of the (finitely many $\cong$ classes of) abelian varieties it classifies.

__Theorem__: Given $A_{[\mathfrak{a}]} \in \mathcal{Ab}(\mathcal{O}_E, \Phi)$, $\sigma \in \mathrm{Gal}(\mathbb{C}/E')$.
For any $a \in \mathbb{A}^\times_{E',f}$ with $\mathrm{art}_{E'}(a) = \sigma|_{(E')^{ab}}$:

(a) ${}^\sigma A_{[\mathfrak{a}]} \cong A_{\underbrace{N_{\Phi'}(a).[\mathfrak{a}]}_{\text{depends only on } \sigma|_{H_{E'}}}}$ (where $N_{\Phi'}(a) \in \mathbb{A}^\times_{E,f}$), and

(b) $\exists !$ E-linear isogeny $\alpha : A_{[\mathfrak{a}]} \to {}^\sigma A_{[\mathfrak{a}]}$ s.t.

$$\alpha(N_{\Phi'}(a).x) = {}^\sigma x \qquad \underline{\forall x \in V_f A} .$$

Idea of (b): • ${}^\sigma A \in \mathcal{Ab}(\mathcal{O}_E, \Phi) \implies \exists$ (E-linear) isogeny $\alpha : A \to {}^\sigma A$.
• $V_f A$ is free of rank 1 / $\mathbb{A}_{E,f}$.

---

* often credited to Shimura & Taniyama, though Weil played a huge rôle.
As Milne puts it: "When [Weil] arrived at the famous Tokyo-Nikko conference in [redacted] 1955 planning to speak about CM, he was disconcerted to find that two young Japanese mathematicians, Shimura & Taniyama, were planning to speak about the same topic."

- the composition $V_f(A) \xrightarrow{\sigma} V_f({}^\sigma A) \xrightarrow{V_f(\alpha)^{-1}} V_f(A)$ is $\mathbb{A}_{E,f}$-linear, so $=$ mult. by $s \in \mathbb{A}_{E,f}^\times$.
- s is independent (up to $E^\times$) of the choice of α, and so this defines $\mathrm{Gal}(\mathbb{C}/E') \longrightarrow E^\times \backslash \mathbb{A}_{E,f}^\times$

which then factors as

$$\begin{array}{ccc}
\mathrm{Gal}(\mathbb{C}/E') & \longrightarrow & E^\times \backslash \mathbb{A}_{E,f}^\times \\
 & \searrow & \uparrow \\
 & & \mathrm{Gal}((E')^{ab}/E') \\
 & & \uparrow \mathrm{art}_{E'} \\
 & & (E')^\times \backslash \mathbb{A}_{E',f}^\times
\end{array}
\quad \circledast$$

(where $\circledast$ is the curved arrow $(E')^\times \backslash \mathbb{A}_{E',f}^\times \to E^\times \backslash \mathbb{A}_{E,f}^\times$)

- A is defined / # field k; the Shimura – Taniyama computation of the prime decompositions of the elements of $E \cong \mathrm{End}(A)_\mathbb{Q}$ reducing to various Frobenius maps (in residue fields of k) then shows that the vertical map $\circledast$ is $N_{\Phi'}$. Hence

$$N_{\Phi'}(a) = s = V_f(\alpha)^{-1} \circ \sigma$$

which gives the formula in (b). $\square$

So what does (b) mean? Like the cyclotomic character, we get a very nice interpretation when we restrict to the action on m-torsion points of A for any fixed $m \in \mathbb{N}$:

<u>Corollary</u>: $\exists!$ E-linear isogeny $\alpha_m : A \to {}^\sigma A$ such that

$$\alpha_m(x) = {}^\sigma x \qquad \forall x \in A[m].$$

This is the second appearance of a "continuous envelope" for the action of automorphisms of $\mathbb{C}$ on special points, ~~a theme which will reappear in later.~~

IV. Shimura varieties

A. Three key adélic lemmas

Besides the main theorem of CM, there is another (related) connection between the class field theory described in III.B and abelian varieties. The tower of ray class groups associated to the ideals of a CM field E can be expressed as

(*)
$$E^\times \backslash \mathbb{A}_{E,f}^\times / \mathcal{U}_I \cong {}_{T(\mathbb{Q})}\backslash^{T(\mathbb{A}_f)} / \mathcal{U}_I$$

where
$$\mathcal{U}_I := \left\{ (a_\mathfrak{p}) \in \mathbb{A}_{E,f}^\times \;\middle|\; \begin{array}{l} a_\mathfrak{p} \in (\mathcal{O}_E)_\mathfrak{p} \quad (\forall \mathfrak{p}) \\ a_\mathfrak{p} \equiv 1 \mod \mathfrak{p}^{\mathrm{ord}_\mathfrak{p} I} \end{array} \right\}$$

($\mathfrak{p} \in \mathcal{I}(E)$ prime; the congruence condition is for the finitely many $\mathfrak{p}$ dividing I)

is a compact open subgroup of $\mathbb{A}_{E,f}^\times = T(\mathbb{A}_f)$ and
$$T = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m .$$

(*) may be seen as parametrizing abelian varieties with CM by a type (E, Φ) and having fixed level structure — which "refines" the set parametrized by $Cl(E)$. Shimura varieties give a way of extending this story to more general abelian varieties with other endomorphism / Hodge-tensor structures, as well as the other families of HS's parametrized by HSD's.

The first fundamental result we will need is

<u>Lemma 1</u>: For T any $\mathbb{Q}$-algebraic torus, and $K_f \subset T(\mathbb{A}_f)$ any open subgroup, $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_f$ is finite.

<u>Sketch</u>: This follows (from the definition of compactness) if we can show $T(\mathbb{Q})\backslash T(\mathbb{A}_f)$ is compact. For any # field, this is closed in $T(F)\backslash T(\mathbb{A}_{F,f})$, and for some F over which T splits the latter is $(F^\times\backslash \mathbb{A}_{F,f}^\times)^{\dim(T)}$. Finally, by the Minkowski bound

$$F^\times\backslash \mathbb{A}_{F,f}^\times/\hat{\mathcal{O}}_F = Cl(F) \text{ is finite,}$$

and $\hat{\mathcal{O}}_F$ is compact (like $\hat{\mathbb{Z}}$), so $F^\times\backslash\mathbb{A}_{F,f}^\times$ is compact. ☐

For a very different class of $\mathbb{Q}$-algebraic groups, we have the contrasting

<u>Lemma 2</u>: Suppose $G/\mathbb{Q}$ is semisimple and simply connected, of noncompact type*; then

(a) [Strong approximation] $G(\mathbb{Q}) \subseteq G(\mathbb{A}_f)$ is dense.

(b) For any open $K_f \subseteq G(\mathbb{A}_f)$, $G(\mathbb{A}_f) = G(\mathbb{Q})\cdot K_f$

($\Rightarrow$ the double coset $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_f$ is trivial).

<u>note</u>: now double cosets are essential (as we are in the nonabelian setting!)

<u>Sketch of (a)⇒(b)</u>: Given $(\gamma_\ell) \in G(\mathbb{A}_f)$, $U := (\gamma_\ell)\cdot K_f$ is an open subset of $G(\mathbb{A}_f)$ hence (by (a)) $\exists\, g \in U \cap G(\mathbb{Q})$. Clearly $g = (\gamma_\ell)\cdot k$ for some k, and so $(\gamma_\ell) = g\cdot k^{-1}$. ☐

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* i.e. none of its simple almost-direct $\mathbb{Q}$-factors G_i have $G_i(\mathbb{R})$ compact

Basic non-example: $\mathbb{G}_m$, which is of course reductive but not semisimple. If (b) held, then the ray class groups of $\mathbb{Q}$ ($\cong (\mathbb{Z}/\ell\mathbb{Z})^*$) would be trivial. But even more directly, were $\mathbb{Q}^*$ dense in $\mathbb{A}_f^*$, there would be $q \in \mathbb{Q}^*$ close to any $(a_\ell) \in \prod \mathbb{Z}_\ell^*$. This forces [the image in $\mathbb{A}_f^*$ of] q to lie in $\prod \mathbb{Z}_\ell^*$, which means for each ℓ that (in lowest terms) the numerator + denominator of q are prime to ℓ. So $q = \pm 1$, a contradiction. □

Finally, for a general $\mathbb{Q}$-algebraic group G, we have

Lemma 3: The congruence subgroups of $G(\mathbb{Q})$ are precisely the $\underbrace{K_f \cap G(\mathbb{Q})}_{\cap \text{ in } G(\mathbb{A}_f)}$ for compact open $K_f \leq G(\mathbb{A}_f)$.

Sketch*: For $N \in \mathbb{N}$ the

$$K(N) := \left\{ (g_\ell) \in G(\mathbb{A}_f) \;\middle|\; \begin{array}{l} g_\ell \in G(\mathbb{Z}_\ell) \quad (\forall \ell) \\ \underbrace{g_\ell \equiv \mathbb{I} \mod \ell^{\mathrm{ord}_\ell(N)}}_{\text{finite set of } \ell\text{'s}} \end{array} \right\}$$

are compact open in $G(\mathbb{A}_f)$, and

$$K(N) \cap G(\mathbb{Q}) = \left\{ g \in G(\mathbb{Z}) \;\middle|\; \begin{array}{l} g \equiv \mathbb{I} \mod \ell^{\mathrm{ord}_\ell(N)} \\ \text{for each } \ell \mid N \end{array} \right\}$$

$$= \left\{ g \in G(\mathbb{Z}) \;\middle|\; g \equiv \mathbb{I} \mod N \right\} = \Gamma(N).$$

In fact, the $K(N)$ are a basis of open subsets $\ni \mathbb{I}$. So any compact open K_f contains some $K(N)$, and

---

* again one has to be careful about the $\mathbb{Z}$ (hence $\mathbb{Z}_\ell$) structure arising from a choice of embedding $G \hookrightarrow GL_N$.

$$K_f \cap G(\mathbb{Q}) \big/ K(N) \cap G(\mathbb{Q}) \;\leq\; K_f \big/ K(N)$$

is a discrete subgroup of a compact set, and $\therefore$ finite. $\square$

In some sense, K_f is itself the congruence condition.

B. Shimura data

A $\begin{cases} \text{Shimura datum} \\ \text{connected S.d.} \end{cases}$ is a pair $\left(G, \begin{cases} \tilde{X} \\ X \end{cases}\right)$

consisting of • $G = \begin{cases} \text{reductive} \\ \text{semisimple} \end{cases}$ algebraic group $/\mathbb{Q}$

• $\begin{cases} \tilde{X} \\ X \end{cases} = \begin{cases} G(\mathbb{R})\text{-} \\ G^{ad}(\mathbb{R})^{+}\text{-} \end{cases}$ ccl of homomorphisms $\tilde{\varphi} : \mathbb{S} \to \begin{cases} G_{\mathbb{R}} \\ G_{\mathbb{R}}^{ad} \end{cases}$

satisfying the axioms

SV1: Only $z/\bar{z}$, 1, $\bar{z}/z$ occur as eigenvalues of $\mathrm{ad} \circ \tilde{\varphi} : \mathbb{S} \to GL(\mathrm{Lie}\,(G^{ad})_{\mathbb{C}})$.

SV2: $\mathrm{Ad}(\tilde{\varphi}(i)) \in \mathrm{Aut}(G^{ad})$ is Cartan

SV3: G^{ad} has no $\mathbb{Q}$-factor on which the projection of every $\tilde{\varphi} \in \overset{(\sim)}{X}$ is trivial

SV4: The weight homomorphism $\mathbb{G}_m \hookrightarrow \mathbb{S} \xrightarrow{\tilde{\varphi}} G_{\mathbb{R}}$ (composite $w_{\tilde{\varphi}}$) is defined $/\mathbb{Q}$

SV5: $\left(Z^{\circ} \big/ w_{\tilde{\varphi}}(\mathbb{G}_m) \right)(\mathbb{R})$ is compact [sometimes weakened — cf. Milne Chap. V]

SV6: Z° splits over a CM field.

In the "connected" case, SV 5–6 are trivial, while SV1–3 already imply

- X = HSD (in precise sense of (3) in I.B) [SV1]
- G of noncompact type, but with $\ker(G(\mathbb{R})^+ \to \mathrm{Hol}(X)^+)$ compact
  - [SV3] (noncompact type); [SV2] (kernel compact)
  - (map defined by the action on the ccl)

SV4–6 are sometimes omitted (for example, canonical models exist for SV's without them), but are satisfied in the context of Hodge theory, and so we include them. Indeed, a MTD $\tilde{X}$ for PHS with generic MT group G, produces a S.d. satisfying

SV4: $\mathbb{Q}$–HS !!

SV5: G MT group $\Rightarrow$ $G/\mathbb{G}_m$ contains a $G(\mathbb{R})^+$-ccl of anisotropic maximal real tori; these contain $Z^\circ_{\mathbb{R}}/\mathbb{G}_m$.

SV6: Since $G/\mathbb{Q}$, the ccl contains tori defined $/\mathbb{Q}$; and so $X \ni \tilde{\varphi}$ factoring through some such rational $\mathbb{T}$. This defines a polarized CM-HS, with MT group a $\mathbb{Q}$-torus $\mathbb{T}_0 \leq \mathbb{T}$ split over a CM field (cf. lecture III). If $\mathbb{T}_0 \not\supseteq Z^\circ$, then the projection of $\tilde{\varphi}$ to some $\mathbb{Q}$-factor of Z° is trivial and then it is trivial for all its conjugates, contradicting SV3 (or: the MT group is then smaller). Hence $\mathbb{T}_0 \supseteq Z^\circ$ and Z° splits over the CM field. □

Any S.d. produces a connected S.d. by

- replacing G by G^{der} (which has the same G^{ad})
- replacing $\tilde{X}$ by a connected component X (which we may view as a $G^{ad}(\mathbb{R})^+$-ccl of homomorphisms $\mathrm{ad}\circ\tilde{\varphi}$),

and so

$\tilde{X}$ is a finite union of HSD's.

As in the "Hodge domains" described by Griffiths, for any faithful representation $\rho: G \hookrightarrow GL(V)$ $\tilde{X}$ is realized as a MT domain (but with trivial IPR) parametrizing the HS's $\rho \circ \tilde{\varphi}$. I don't know whether SV1–6 $\Rightarrow$ G (and not some subgroup surjecting onto G^{ad}) is the MT group.

Now given a connected Shimura datum (G, X), we add one more ingredient: let

$\Gamma \leq G^{ad}(\mathbb{Q})^+$ be a torsion-free arithmetic subgroup, with inverse image in $G(\mathbb{Q})^+$ a congruence subgroup.

Its image $\bar{\Gamma}$ in $Hol(X)^+$ is

(i) [torsion-free +] arithmetic: since $\ker(G(\mathbb{R})^+ \twoheadrightarrow Hol(X)^+)$ compact

(ii) isomorphic to Γ: $\Gamma \cap \ker$ = discrete $\cap$ compact = finite hence torsion (and there is no torsion).

~~(iii) torsion-free~~

We may write

$$X(\Gamma) := \Gamma \backslash X \underset{(ii)}{=} \bar{\Gamma} \backslash X \quad ;$$

and $\bar{\Gamma}\backslash X$ is a locally symmetric variety by (i) and Baily-Borel.

By Borel's theorem,

(*) $\Gamma \geq \Gamma' \implies X(\Gamma') \twoheadrightarrow X(\Gamma)$ is algebraic.

Definition: The connected SV (Shimura variety) associated with (G, X, Γ) is

$$Sh^{\circ}_{\Gamma}(G, X) := X(\Gamma) .$$

Remark: Every $X(\Gamma)$ is covered by an $X(\Gamma')$ with Γ' the image of a congruence subgroup of $G(\mathbb{Q})^+$. If one works with "sufficiently small" congruence subgroups of $G(\mathbb{Q})$, then

- they belong to $G(\mathbb{Q})^+$
- they have torsion-free image in $G^{ad}(\mathbb{Q})^+$
- congruence $\Rightarrow$ arithmetic.

This will be tacit in what follows.

C. The adelic reformulation

Consider a connected S.d. (X, G) with G simply connected. Let $K_f \leq G(\mathbb{A}_f)$ be a ("sufficiently small") compact open subgroup and (Lemma 3) $\Gamma = G(\mathbb{Q}) \cap K_f$ the corresponding subgroup of $G(\mathbb{Q})$; replacing the earlier notation we write

$$Sh^{\circ}_{K_f}(G, X)$$

for the associated locally symmetric variety.

Proposition: $(\Gamma\backslash X =)\ Sh^{\circ}_K(G, X) \underset{\text{homeo.}}{\cong} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K_f$, except $X = G^{ad}(\mathbb{R})^+/K_{\mathbb{R}}$ (circled $X \times G(\mathbb{A}_f)/K_f$ annotated: closure $G(\mathbb{A}_{\mathbb{Q}})/K = K_{\mathbb{R}} \times K_f$),

where $g \cdot (\tilde{\varphi}, a) \cdot k := (\underbrace{g \cdot \tilde{\varphi}}_{g\tilde{\varphi}g^{-1}}, gak)$.

The first part of the Theorem below says (a) doesn't contribute: the indexing of the components is "entirely arithmetic". First, some

<u>Notation</u>: $X :=$ a connected component of $\tilde{X}$

$G(\mathbb{R})_+ :=$ preimage of $G^{ad}(\mathbb{R})^+$ in $G(\mathbb{R})$

(recall from I.A)

$$\begin{array}{ccc} Z & \overset{\text{isogeny}}{\dashrightarrow} & \\ \downarrow & & \\ G & \xrightarrow{\nu} & T \end{array}$$ (maximal ~~abelian~~ quotient of G)

$T(\mathbb{R})^\dagger := \mathrm{Im}(Z(\mathbb{R}) \to T(\mathbb{R}))$

$Y := T(\mathbb{Q})^\dagger \backslash T(\mathbb{Q})$

<u>Theorem</u>: (i) $G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f) / K_f \xrightarrow{\cong} G(\mathbb{Q}) \backslash \tilde{X} \times G(\mathbb{A}_f) / K_f$

(ii) The map $G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f) / K_f$

$$\downarrow$$

$$\mathcal{C} := G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / K_f \underset{\nu}{\overset{\cong}{\longrightarrow}} T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / \nu(K_f) \cong T(\mathbb{Q}) \backslash Y \times T(\mathbb{A}_f) / \nu(K_f)$$

(the isomorphism $\xrightarrow{\cong}$ via ν and the last one: for G^{der} <u>simply</u> connected)

"indexes" the connected components. (Henceforth "$\mathcal{C}$" denotes a set of representatives in $G(\mathbb{A}_f)$.)

(iii) $Sh_{K_f}(G, \tilde{X}) \cong \coprod_{g \in \mathcal{C}} \Gamma_g \backslash X$, a <u>finite</u> union.

<u>Proof</u>: <u>Exercise</u>: • preimage of $[1] \in \mathcal{C}$ is $G(\mathbb{Q})_+ \backslash X \times G(\mathbb{A}_f)_+ / K_f$ $\left(\cong Sh_{K_f}(G^{der}, X) ? \right)$

• check that $(\gamma . \tilde{\varphi}, g) \equiv (\tilde{\varphi}, g)$ for $\gamma \in \Gamma_g$.

That $|\mathcal{C}| < \infty$ is by Lemma 1! The rest is in Milne Ch. 5. $\square$

These arise as C in the Theorem and also from CM-Hodge structures (MT group = T).

(2) Siegel modular variety

V = $\mathbb{Q}$-vector space
ψ = nondegenerate alternating form $/\mathbb{Q}$ } (V, ψ) $\mathbb{Q}$-symplectic space

$G := GSp(V, \psi) = \{ g \in GL(V) \mid \psi(gu, gv) = \chi(g)\psi(u,v) \quad \forall u,v \in V$ and some $\chi(g) \in \mathbb{Q}^* \}$

[Exercise: $\chi : G \to \mathbb{G}_m$ defines a character.]

let $X^{\pm} := \{ J \in Sp(V,\psi)(\mathbb{R}) \mid J^2 = -I, \ \psi(u, Jv)$ is $\pm$-definite$\}$

(= positive & negative symplectic complex structures)

and put

$\tilde{X} := X^+ \amalg X^-$ regarded as homomorphisms via $\tilde{\varphi}(a+bi) := a + bJ$ for $a+bi \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$.

Then $G(\mathbb{R})$ acts transitively on $\tilde{X}$, and the datum $(G, \tilde{X})$ satisfies SV1-6;

for any compact open $K_f \subseteq G(\mathbb{A}_f)$ the attached SV is a <u>Siegel modular variety</u>.

Now consider the set

$$\mathcal{M}_{K_f} := \left\{ (A, a, \eta) \;\middle|\; \begin{array}{l} A \text{ abelian variety } /\mathbb{C} \\ \pm a \text{ polarization of } H_1(A,\mathbb{Q}) \\ \eta : V_{\mathbb{A}_f} \to V_f(A) \ (\cong H_1(A, \mathbb{A}_f)) \text{ isomorphism} \\ \quad \text{sending } \psi \mapsto a \cdot a \ (a \in \mathbb{A}_f) \end{array} \right\}$$

$\cong$

where an isomorphism

$$(A, Q, \eta) \xrightarrow{\simeq} (A', Q', \eta')$$

is an isogeny $f: A \to A'$ sending $\Psi \mapsto q \cdot Q$ $(q \in \mathbb{Q}^*)$

such that for some $k \in K_f$

$$\begin{array}{ccc} V_{\mathbb{A}_f} & \xrightarrow{\eta} & V_f(A) \\ \downarrow \cdot k & & \downarrow f \\ V_{\mathbb{A}_f} & \xrightarrow{\eta'} & V_f(A') \end{array}$$

commutes.

$\mathcal{M}_{K_f}$ is a moduli space for polarized abelian varieties with K_f-level structure. Write $\tilde{\varphi}_A$ for the HS on $H_1(A)$, and choose an $\simeq$ $\alpha: H_1(A, \mathbb{Q}) \to V$ sending Ψ to Q (up to $\mathbb{Q}^*$).

<u>Proposition</u>: The (well-defined) map

$$\mathcal{M}_{K_f} \longrightarrow Sh_{K_f}(G, \tilde{X})$$

induced by

$$(A, Q, \eta) \longmapsto (\alpha \circ \varphi_A \circ \alpha^{-1}, \alpha \circ \eta)$$

is a bijection.

③ Shimura varieties of PEL type

(V, Ψ) symplectic $(B, *)$-module, i.e.

- (V, Ψ) symplectic space $/\mathbb{Q}$
- $(B, *)$ simple $\mathbb{Q}$-algebra with positive involution $*$ $\left(\mathrm{tr}_{B\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}(b^* b) > 0\right)$

- V is a B-module and $\psi(bu, v) = \psi(u, b^* v)$.

We put $G = \mathrm{Aut}_B(V) \cap GSp(V, \psi)$, which is of generalized SL, Sp, or SO type (related to the Albert classification) according to the structure of $(B_{\mathbb{Q}}, *)$. (Basically, G is cut out of GSp by fixing tensors in $T^{1,1}V$.) The (canonical) associated ccl $\tilde{X}$ completes this to a Shimura datum, and the associated SV's parametrize (P)olarized abelian varieties with (E)ndomorphism and (L)evel structure (essentially a union of quotients of MT domains cut out by $\mathcal{E}$).

(4) Shimura varieties of Hodge type

G is cut out of GSp by fixing tensors of all degrees.

Remark:
In both (3) & (4) X is a subdomain of a Siegel domain, so "of Hodge type" excludes the type $D^{\mathbb{H}}$/E Hermitian symmetric domains which still do yield SV's parametrizing equivalence classes of HS's. So the last example is more general still ———

(5) Mumford-Tate groups/domains w./vanishing IPR

In notation from I.D, $G = M_t$, $\tilde{X} = M_t(\mathbb{R}).\tilde{\varphi} =: D_t$ (for $MT(\tilde{\varphi}) = M_t$) ($X$ will be D_t^+), $K_f \leq M_t(\mathbb{A}_f)$ will produce Shimura varieties parametrizing higher weight HS's, under the assumption that D_t has trivial IPR.

(under "higher weight HS's": Γ-equivalence classes of)

In addition to the examples in I.D, one prototypical example is the MT group (essentially a $U(1,n)$) and domain for HS of weight 3 and type $(1, n, n, 1)$ with endomorphisms by an imaginary quadratic field, in such a way that the two eigenspaces are $V^{3,0} \oplus V^{2,1}$ and $V^{1,2} \oplus V^{3,0}$. This has vanishing IPR and yields a Shimura variety.

—//—.

V. Fields of definition

Let D be a period domain for HS polarized by Q with fixed Hodge #'s. The compact dual of a MT domain $D_M = M(\mathbb{R}).\varphi \subseteq D$ is the $M(\mathbb{C})$-orbit of the attached filtration F_φ,

$$\check{D}_M = M(\mathbb{C}).F_\varphi .$$

It is a connected component of the "MT Noether-Lefschetz locus" cut out of $\check{D}$ by the criterion of (a Hodge flag $\in \check{D}$) having MT group contained in M,

$$\check{D}_M \subset \check{NL}_M \subset \check{D}.$$

Now, $\check{NL}_M$ is cut out by $\mathbb{Q}$-tensors hence defined$/\mathbb{Q}$, but its components ($M(\mathbb{C})$-orbits) are permuted by the action of $\mathrm{Aut}(\mathbb{C})$. The fixed field of the subgroup of $\mathrm{Aut}(\mathbb{C})$ preserving $\check{D}_M$, is considered its field of definition; this is defined regardless of the vanishing of the IPR (or D_M being Hermitian symmetric). What is interesting in the Shimura variety case, is that this field has meaning "downstairs", for $\Gamma \backslash D_M$ — even though the upstairs-downstairs correspondence is highly transcendental.

A. Reflex field of a Shimura datum

Let $(G, \tilde{X})$ be a Shimura datum; we start by repeating the definition just alluded to in this context. Recall that any $\tilde{\varphi} \in \tilde{X}$ determines a complex cocharacter of G by $z \mapsto \tilde{\varphi}_{\mathbb{C}}(z,1) =: \mu_{\tilde{\varphi}}(z)$. (Complex cocharacters are themselves more general and essentially correspond to points of the compact dual.)

Write, for any subfield $k \subset \mathbb{C}$,

$$\mathcal{C}(k) = G(k) \backslash \mathrm{Hom}_k(\mathbb{G}_m, G_k)$$

for the set of $G(k)$-conjugacy classes of k-cocharacters. ~~[crossed out]~~ $\mathrm{Gal}(k/\mathbb{Q})$ acts on $\mathcal{C}(k)$, since $\mathbb{G}_m$ & G are $\mathbb{Q}$-algebraic groups.

The element

$$c(\tilde{X}) := [\mu_{\tilde{\varphi}}] \in \mathcal{C}(\mathbb{C})$$

is independent of the choice of $\tilde{\varphi} \in \tilde{X}$.

__Definition__: $E(G, \tilde{X})$ is the fixed field of the subgroup of $\mathrm{Aut}(\mathbb{C})$ fixing $c(\tilde{X})$ as an element of $\mathcal{C}(\mathbb{C})$.

__Examples__:

(1) A = abelian variety of CM type (E, Φ), E' = associated reflex field (lecture III)
$\tilde{\varphi}$ the HS on $H^1(A)$, $T = M_{\tilde{\varphi}} \subset \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$.

Then $\mu_{\tilde{\varphi}}(z)$ multiplies $H^{1,0}$ by z and $H^{0,1}$ by 1
($H^{1,0}$: Φ-eigenspaces for E; $H^{0,1}$: $\bar{\Phi}$-eigenspaces for E)

Clearly $\mathrm{Ad}(\sigma)\mu_{\tilde{\varphi}}(z)$ (for $\sigma \in \mathrm{Aut}(\mathbb{C})$) multiplies the $\sigma\bar{\Phi}$-eigenspaces

by z, while $T(\mathbb{C})$ acts trivially on $\mathrm{Hom}_{\mathbb{C}}(\mathbb{G}_m, T_{\mathbb{C}})$.
Consequently, σ fixes $c(\{\tilde{\varphi}\}) \iff \sigma$ fixes $\bar{\Phi}$, and so

$$\underline{E(T, \{\tilde{\varphi}\}) = E'}.$$

(2) For an inclusion $(G', \tilde{X}') \hookrightarrow (G, \tilde{X})$, one has $E(G', \tilde{X}') \supset E(G, \tilde{X})$. Every $\tilde{X}$ has $\tilde{\varphi}$ factoring through rational tori, which are then CM-HS. The torus then splits over a CM field, and so $E(G, \tilde{X})$ is always contained in a CM field. (In fact, it is always either CM or totally real.)

(3) For the $\begin{cases} \text{Siegel case} \\ \text{PEL} \end{cases}$, $E(G, \tilde{X}) = \begin{cases} \mathbb{Q} \\ \mathbb{Q}\left(\{\mathrm{Tr}(b|_{T_{\bar{x}} t_0(A)})\}_{b \in B}\right) \end{cases}$ □

Let T be a $\mathbb{Q}$-algebraic torus, μ a cocharacter defined over a finite extension $K/\mathbb{Q}$. Denote by

$$r(T, \mu) : \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m \longrightarrow T$$

the homomorphism given on rational points by

$$K^* \longrightarrow T(\mathbb{Q})$$
$$k \longmapsto \prod_{\phi \in \mathrm{Hom}(K, \bar{\mathbb{Q}})} \phi(\mu(k)).$$

As in Example (2), every $(G, \tilde{X})$ contains a CM-pair $(T, \{\tilde{\varphi}\})$. Let $E(\tilde{\varphi}) :=$ field of definition of $\mu_{\tilde{\varphi}}$ $(= E(T, \{\tilde{\varphi}\}))$. The map

$$r(T, \mu_{\tilde{\varphi}}) : \mathrm{Res}_{E(\tilde{\varphi})/\mathbb{Q}} \mathbb{G}_m \longrightarrow T$$

yields on $\mathbb{A}_{\mathbb{Q}}$-points

$$\mathbb{A}^\times_{E(\tilde{\varphi})} \xrightarrow[r(T,\mu_{\tilde{\varphi}})]{} T(\mathbb{A}_\mathbb{Q}) \xrightarrow[\text{project}]{} T(\mathbb{A}_f).$$

(the composite arrow $\mathbb{A}^\times_{E(\tilde{\varphi})} \to T(\mathbb{A}_f)$ is denoted $=: r_{\tilde{\varphi}}$)

Ex/ For Example (4) above, $E(\tilde{\varphi}) = E'$ and $T = \mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_m$

$\Rightarrow$ the $r(T,\mu_{\tilde{\varphi}})$ part of this map is the adelicized

reflex norm $\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{N_{\Phi'}(\mathbb{A}_\mathbb{Q}) : \mathbb{A}^\times_{E'} \longrightarrow \mathbb{A}^\times_E}$.

(This is not a trivial calculation.)

B. Canonical Models

The Shimura varieties we have been discussing, i.e. $Sh_{K_f}(G, \tilde{X})$, are finite disjoint unions of locally symmetric varieties and hence algebraic varieties defined *a priori* $/\mathbb{C}$. More generally, if $\mathcal{Y}$ is any variety $/\mathbb{C}$ and $k \subset \mathbb{C}$ is a subfield, a <u>model of $\mathcal{Y}$ over k</u> is

- a variety $\mathcal{Y}_0 / k$, together with
- an isomorphism $\mathcal{Y}_{0,\mathbb{C}} \xrightarrow[\cong]{\theta} \mathcal{Y}$.

For general algebraic varieties, it is <u>not</u> true that two models over the same field k are necessarily isomorphic <u>over that field</u>. But if we impose a condition on how

$\mathrm{Gal}(\mathbb{C}/E_1')$ acts on a dense set of points on any model, then the composite isomorphism $\mathcal{Y}_{0,\mathbb{C}} \xrightarrow[\cong]{\Theta} \mathcal{Y} \xrightarrow[\cong]{\tilde{\Theta}^{-1}} \tilde{\mathcal{Y}}_{0,\mathbb{C}}$ is forced to be $\mathrm{Gal}(\mathbb{C}/E_1')$-equivariant, making $\mathcal{Y}_0$ and $\tilde{\mathcal{Y}}_0$ isomorphic over E_1'. Repeating this criterion for more point sets and $\{E_i'\}$'s means $\mathcal{Y}_0$ and $\tilde{\mathcal{Y}}_0$ will be $\cong$ over $\bigcap_i E_i'$.

For the dense sets of points:

Definition 1: (a) A point $\tilde{\varphi} \in \tilde{X}$ is a CM point $\iff$ $\exists$ a ($\mathbb{Q}$-algebraic, minimal) torus $\mathbb{T} \subset G$ s.t. $\tilde{\varphi}(\mathbb{S}(\mathbb{R})) \subset \mathbb{T}(\mathbb{R})$.

(b) $(\mathbb{T}, \{\tilde{\varphi}\})$ is then a ~~special~~ CM pair in $(G, \tilde{X})$.

~~To get density in $\tilde{X}$, orbits of φ by $G(\mathbb{Q})$; in $Sh_{K_f}(G,\tilde{X})$, points of $[(\tilde{\varphi}, a)]$~~ (Such a $\tilde{\varphi}$ exists since in a $\mathbb{Q}$-algebraic group every conj. of maximal real tori contains one defined $/\mathbb{Q}$; to get density in $\tilde{X}$ look at the orbit $G(\mathbb{Q}).\tilde{\varphi}$; to get density, more importantly, in $Sh_{K_f}(G,\tilde{X})$, look at the sets $[(\tilde{\varphi}, a)]$.)

For the condition on Galois action:

~~Recall $E(\varphi)$ ... $E(\varphi)$ ... $\mathrm{Gal}(E(\varphi)^{ab}/E(\varphi))$~~.

For any CM field E', recall the Artin reciprocity map

$$\mathrm{art}_{E'} : \mathbb{A}_{E'}^{\times} \twoheadrightarrow \mathrm{Gal}(E'^{ab}/E').$$

Definition 2: A model $M_{K_f}(G,\tilde{X})$ of $Sh_{K_f}(G,\tilde{X})$ over $E(G,\tilde{X})$ is canonical iff $\left(\overset{\Theta}{\curvearrowright}\right)$

for every
- CM $(T, \tilde{\varphi}) \subset (G, \tilde{X})$
- $a \in G(\mathbb{A}_f)$
- $\sigma \in \mathrm{Gal}(E(\tilde{\varphi})^{ab}/E(\tilde{\varphi}))$
- $s \in \mathrm{art}^{-1}_{E(\tilde{\varphi})}(\sigma) \subset \mathbb{A}^{\times}_{E(\varphi)}$,

$\theta^{-1}[(\tilde{\varphi}, a)]$ is a point defined $/E(\tilde{\varphi})^{ab}$, and

$$(\#) \qquad \sigma . \theta^{-1}[(\tilde{\varphi}, a)] = \theta^{-1}[(\tilde{\varphi}, r_{\tilde{\varphi}}(s) a)] .$$

($r_{\tilde{\varphi}}$: essentially a reflex norm) $\square$

~~Remark: This action on CM points turns out to force the following action on $\pi_0(Sh_{K_f}(G, \tilde{X})) \cong T(\mathbb{Q})^{+} \backslash T(\mathbb{A}_f)/\nu(K_f)$: for any $\tilde{\varphi} \in \tilde{X}$, put (recall $G \twoheadrightarrow T$ maximal abelian quotient)~~

~~$$r = r(T, \nu \circ \mu_{\tilde{\varphi}}) : \mathbb{A}^{\times}_{E(G,\tilde{X})} \to T(\mathbb{A}_{\mathbb{Q}}) ;$$~~

~~then for $\sigma \in \mathrm{Gal}(E(G,\tilde{X})^{ab}/E(G,\tilde{X}))$ and $s \in \mathrm{art}^{-1}_{E(G,\tilde{X})}\sigma$ we get~~

~~$$\sigma . [y, a] = [r(s)_{\infty} . y , r(s)_f . a] .$$~~

The uniqueness of the canonical model is clear from the argument above — if one exists — since we can take the E_i' to be various $E(\tilde{\varphi})$ for CM $\tilde{\varphi}$, whose intersections are known to give $E(G, \tilde{X})$.

To see how existence might come about for SV's of Hodge type (it is known for all SV's), first note that by

- Baily–Borel,

$\mathcal{Y} := Sh_K(G, \tilde{X})$ is a variety $/\mathbb{C}$. Now we know that

- Sh_K is a moduli space for certain abelian varieties, say $\mathcal{A} \to \mathcal{Y}$. Let $E = E(G, \tilde{X})$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$.

Given $P \in \mathcal{Y}(\mathbb{C})$ we have an equivalence class $[A_P]$ of abelian varieties, and we define a map

$$({}^{\sigma}\mathcal{Y})(\mathbb{C}) \to \mathcal{Y}(\mathbb{C})$$

by

$$\sigma(P) \longmapsto [{}^{\sigma}A_P].$$

That ${}^{\sigma}A_P$ is still "in the family $\mathcal{A}$" follows from

- definition of the reflex field
- Deligne's theorem (that the Hodge tensors determining $\mathcal{A}$ are absolute).

That these maps produce regular (iso)morphisms

$$f_\sigma : {}^{\sigma}\mathcal{Y} \to \mathcal{Y}$$

boils down to

- Borel's theorem.

Now $\mathcal{Y}$ has (for free) a model $\mathcal{Y}_0$ over some L f.g./E, and using

- $|\mathrm{Aut}(Sh_K)| < \infty$

we may deduce that for σ' fixing L

$$\begin{array}{ccc} {}^{\sigma'}\mathcal{Y} & \xrightarrow{f_{\sigma'}} & \mathcal{Y} \\ {}^{\sigma'}\theta \nwarrow & & \nearrow \theta \\ & \mathcal{Y}_0 & \end{array}$$

commutes. At this point it makes sense to spread $\mathcal{Y}_0$ out over E — i.e. take all $\mathrm{Gal}(\mathbb{C}/E)$-conjugates, viewed as a variety via $\mathcal{Y}_0 \to \operatorname{Spec} L \to \operatorname{Spec} E$.

The diagram

$$\begin{array}{ccc} {}^{\sigma}\mathcal{Y} & \xrightarrow[\cong]{f_\sigma} & \mathcal{Y} \\ {}^{\sigma}\theta \uparrow \cong & & \cong \uparrow \theta \\ {}^{\sigma}\mathcal{Y}_0 & & \mathcal{Y}_0 \end{array}$$

shows that the spread is constant; extending it over a quasi-projective base shows $\mathcal{Y}$ has a model defined over a <u>finite</u> extension of E.

(To get all the way down to E requires some serious descent theory.) Finally, that the action of $\mathrm{Aut}(\mathbb{C}/E)$ on the resulting model implied by the $\{f_\sigma\}$ satisfies (#) (hence yields a <u>canonical</u> model), is true by

- the main theorem of CM.

(In fact, (#) is precisely encoding how Galois conjugation acts on various $\mathcal{A}b(E,\overline{\Phi})$ + level structure.)

So the three key points are:

→ the <u>entire</u> theory is used in the construction of canonical models

→ $Sh_{K_f}(G,\tilde{X})$ is defined over $E(G,\tilde{X})$ independently of K_f

→ the field of definition of a connected component $Sh_{K_f}(G,\tilde{X})^+$ is contained in $E(G,\tilde{X})^{ab}$ and gets larger as K_f shrinks (and the # of connected components increases).

C. Connected components and VHS

Assume G^{der} is simply connected.

The action on CM points imposed by (#) turns out to force the following action on $\pi_0(Sh_{K_f}(G,\tilde{X})) \cong T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/\nu(K_f)$, where $G \overset{\nu}{\twoheadrightarrow} T$ is the maximal abelian quotient:

For <u>any</u> $\tilde{\varphi} \in \tilde{X}$, put

$$r = r(T, \nu\circ\mu_{\tilde{\varphi}}) : \mathbb{A}^{\times}_{E(G,\tilde{X})} \to T(\mathbb{A}_{\mathbb{Q}}) ;$$

then for $\sigma \in \mathrm{Gal}(E(G,\tilde{X})^{ab}/E(G,\tilde{X}))$ and $s \in \mathrm{art}^{-1}_{E(G,\tilde{X})}\sigma$ we get

$$\sigma.[y,a] = [r(s)_\infty . y,\ r(s)_f . a].$$

Assume for simplicity ψ is trivial. Writing $E' := E(G,\tilde{X})$ and $\mathbb{E}$ for the field of definition of the component $\mathcal{S} := Sh_{K_f}(G,\tilde{X})^+$ over $[1;1]$, we have [the finite abelian extension of E']

$$\mathbb{E} = \text{fixed field of } art_{E'}\left(r_f^{-1}\left(T(\mathbb{Q})\cdot \nu(K_f)\right)\right).$$

That is, by virtue of the ~~[illegible]~~ theory of canonical models we can essentially write down a minimal field of definition of the locally symmetric variety $\mathcal{S}$.

Example: $(G,\tilde{X}) = (T,\{\tilde{\varphi}\})$ associated to an abelian variety with CM by E, so that E' is the reflex field (and $\mathbb{E}$ the field of definition of the point it lies over in a relevant Siegel modular variety).

Let $K_f = \mathcal{U}_I$ for $I \in \mathcal{I}(E)$ and consider the diagram

$$\begin{array}{ccccc}
\mathbb{A}_{E',f}^{\times} & \xrightarrow{N_{\Phi'}(=r_f)} & \mathbb{A}_{E,f}^{\times} & \longrightarrow & E^{\times}\backslash \mathbb{A}_{E,f}^{\times}/\mathcal{U}_I \\
\downarrow art_{E'} & & \downarrow art_E & & \cong \downarrow \overline{art}_E \\
Gal((E')^{ab}/E') & \xrightarrow{N_{\Phi'}} & Gal(E^{ab}/E) & \twoheadrightarrow & Gal(E_I/E)
\end{array}$$

↑ this exists (N is continuous) in such a way that the l.h. square commutes

↑ ray class field mod I

We get that

$$\mathbb{E} = ff\left(\mathrm{art}_{E'}\left(\mathcal{N}_{\Phi'}^{-1}(E^{\times}\mathcal{U}_I)\right)\right) = ff\left(\eta_{\Phi'}^{-1}\left(\mathrm{Gal}(E^{ab}/E_I)\right)\right).$$

In case the CM abelian variety is an elliptic curve, $\mathcal{N}_{\Phi'}$ and $\eta_{\Phi'}$ are essentially the identity (and $E' = E$), so

$$\mathbb{E} = ff\left(\mathrm{Gal}(E^{ab}/E_I)\right) = E_I.$$

It is a well-known result that, for example, the j-invariant of a CM elliptic curve generates (over the imaginary quadratic field E) its Hilbert class field $E_{(1)}$. We also see that the fields of definition of CM points in $X(N)$ (the modular curve) are ray class fields mod N. □

An application to VHS? Let $\mathbb{V} \to \mathcal{S}$ be a VHS with reference HS V_s over $s \in \mathcal{S}$. The underlying local system $\mathbb{V}$ produces a monodromy representation

$$\rho : \pi_1(\mathcal{S}) \longrightarrow GL(V_s),$$

and we denote $\rho(\pi_1(\mathcal{S})) =: \Gamma_0$ with geometric monodromy group

$\Pi :=$ identity component of $\mathbb{Q}$-Zariski closure of Γ_0.

Moreover, $\mathbb{V}$ has a MT group M; and we make the following two crucial assumptions:

- $\Pi = M^{der}$
- D_M has vanishing IPR.

In particular, this means that the quotient of D_M by a congruence subgroup is a connected component of a Shimura variety, and that Π is as big as it can be.

For any compact open $K_f \subseteq M(\mathbb{A}_f)$ such that $\Gamma := K_f \cap M(\mathbb{Q}) \supseteq \Gamma_0$, $\mathcal{V}$ gives a period (analytic) mapping

$$\Psi^{an}_{K_f} : \mathcal{S}^{an}_{\mathbb{C}} \longrightarrow \Gamma \backslash D_M \cong \left(Sh_{K_f}(M, D_M)^+ \otimes_{\mathbb{E}} \mathbb{C} \right)^{an}.$$

Call this $\mathbb{E}(K_f)$

This morphism is algebraic by Borel's theorem, and has minimal field of definition (trivially) bounded below by the field of definition $\mathbb{E}(K_f)$ of $Sh_{K_f}(M, D_M)^+$.

The period mapping which gives the most information about $\mathcal{V}$, is the one attached to the smallest congruence subgroup $\Gamma \subset M(\mathbb{Q})$ containing Γ_0. Taking then the largest K_f with $K_f \cap M(\mathbb{Q}) =$ this Γ, minimizes the resulting $\mathbb{E}(K_f)$. It is this last field which it seems natural to consider as the "reflex field of a VHS" — an "expected lower bound" for the field of definition of a period mapping of $\mathcal{V}$. Furthermore, if $\mathcal{V}$ arises (motivically) from $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$, then assuming Deligne's absolute Hodge conjecture, the $\bar{\mathbb{Q}}$-spread of π produces a period mapping into D_M modulo a larger Γ, and our "reflex field of V" may be an upper bound for the minimal field of definition of this period map.

At any rate, the relations between fields of definition of

- varieties X_s,
- transcendental period points in D_M, &
- equivalence classes of period points in $\Gamma \backslash D_M$,

and hence between spreads of

- families of varieties,
- VHS, &
- period mappings,

is very rich and our suggested definition may be just one useful tool (& only in the Shimura case where the $IPR = 0$).

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