The Spread Philosophy in the Study of Algebraic Cycles

Mark L. Green, UCLA

Outline:

Lecture 1: Introduction to Spreads

There are two ways of looking at a smooth projective variety in characteristic 0:
(Geometric) X is a compact Kähler manifold plus a Hodge class, embedded in $\mathbf{CP}^N$ so that the hyperplane bundle H pulls back to a multiple of the Hodge class; or
(Algebraic) X is defined by homogeneous polynomials in $k[x_0, \ldots, x_N]$ for a field k of characteristic zero. We may take k to be the field generated by ratios of coefficients of the defining equations of X, and hence we may take k to be finitely generated over $\mathbf{Q}$. There are algebraic equations with coefficients in $\mathbf{Q}$ that, applied to the coefficients of the defining equations, tell us when X is (not) smooth, irreducible, of dimension n. The crucial additional ingredient to do Hodge theory is an embedding $k \hookrightarrow \mathbf{C}$.

To get a Hodge structure associated to X, we need $k \hookrightarrow \mathbf{C}$. The cohomology groups of X can be computed purely in terms of k, but the integral lattice requires us to have an embedding of k in $\mathbf{C}$.

A field k that is finitely generated/$\mathbf{Q}$ is of the form

$$k = \mathbf{Q}(\alpha_1, \ldots, \alpha_T, \beta_1, \ldots, \beta_A),$$

where $\alpha_1, \ldots, \alpha_T$ are algebraically independent over $\mathbf{Q}$ and $[k : \mathbf{Q}(\alpha_1, \ldots, \alpha_T)] < \infty$. Note $T = \mathrm{tr}\ \deg(k)$, the transcendence degree of k. Alternatively,

$$k \cong \mathbf{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A]/(p_1, \ldots, p_B),$$

where

$$p_1, \ldots, p_B \in \mathbf{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A].$$

Important Idea. *We can make k geometric.*

The idea here is to find a variety S, defined $/\mathbf{Q}$, such that

$$k \cong \mathbf{Q}(S) = \text{field of rational functions of } S.$$

Further, by definition,

$$\mathbf{Q}(S_1) \cong \mathbf{Q}(S_2) \Longleftrightarrow S_1 \text{ birationally equivalent to } S_2.$$

If

$$k \cong \mathbf{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A]/(p_1, \ldots, p_B),$$

then we may take

$$S = \text{projectivization of affine variety in } \mathbf{Q}^{T+A} \text{ def by } p_1, \ldots, p_B.$$

Example 1. *Elliptic curve* $X = \{y^2 = x(x-1)(x-\alpha)\}$

If $\alpha \notin \bar{\mathbf{Q}}$, i.e. α is transcendental, then $k = \mathbf{Q}(\alpha)$ and $S \cong \mathbf{P}^1$. Note S is defined $/\mathbf{Q}$. We can think of X as giving a family

$$\pi\colon \mathcal{X} \to S(\mathbf{C})$$

where

$$X_s = \pi^{-1}(s).$$

Note that

$$X_s \text{ is singular} \iff s \in \{0, 1, \infty\}.$$

Note that the latter is a subvariety of S defined $/\mathbf{Q}$. We have

$$\text{field of definition of } X_s \cong k \text{ for } s \notin \bar{\mathbf{Q}}.$$

The varieties X_{s_1} and X_{s_2} are indistinguishable algebraically if $s_1, s_2 \notin \bar{\mathbf{Q}}$, but we can tell them apart analytically and in general they will have different Hodge structures.

Definition. *For S defined $/\mathbf{Q}$, we will say that $s \in S(\mathbf{C})$ is a* **very general point** *if the Zariski closure over $\mathbf{Q}$ of s is S, i.e. s does not belong to any proper subvariety of S defined $/\mathbf{Q}$.*

We have:

$$\{\text{very general points of } S\} \iff \{\text{embeddings } k \overset{i_s}{\hookrightarrow} \mathbf{C}\}.$$

We get from a very general point of S and a variety X defined $/k$ a variety X_s defined $/\mathbf{C}$. We piece these together to get a family of complex varieties

$$\pi\colon \mathcal{X} \to S,$$

which we call the **spread of X over** k. If X is defined over k by homogeneous polynomials $f_1, \ldots, f_r$ in $k[z_0, \ldots, z_N]$, then expanding out the coefficients of the f_i's in terms of x's and y's, we may take $\mathcal{X}$ to be the projectivization of the variety $/\mathbf{Q}$ defined by $f_1, \ldots f_r, p_1, \ldots, p_B$ in the variables $x_1, \ldots, x_T, y_1, \ldots, y_B, z_0, \ldots, z_N$. We thus have

$$\pi\colon \mathcal{X} \to S,$$

where $\mathcal{X}$, S are defined $/\mathbf{Q}$ and the map π is a map defined $/\mathbf{Q}$. π will be smooth and of maximal rank outside a proper subvariety $\Sigma \subset S$ defined $/\mathbf{Q}$.

The spread of X over k is not unique, but the non-uniqueness can be understood and kept under control.

Example 2. *k a number field*

Here, S consists of a finite set of $[k : \mathbf{Q}]$ points. As a variety, S is defined $/\mathbf{Q}$, although the individual points are defined over a splitting field of k. We get a finite number of complex varieties X_s, corresponding to the $[k : \mathbf{Q}]$ embeddings $k \overset{i_s}{\hookrightarrow} \mathbf{C}$.

Example 3. *Very general points*

Y a projective algebraic variety, irreducible, defined (say) over $\mathbf{Q}$. Let $y \in Y$ be a very general point. Take

$$k = \mathbf{Q}(\text{ratios of coordinates of } y).$$

Then $k \cong \mathbf{Q}(Y)$, so we may take $S = Y$.

Example 4. *Ordered pairs of very general points*

Let X be a variety defined $/\mathbf{Q}$, $X \subseteq \mathbf{P}^N$, $\dim(X) = n$. Let (p, q) be a very general point of $X \times X$. Let

$$k = \mathbf{Q}(\text{ratios of coordinates of } p, \text{ratios of coordinates of } q).$$

Now

$$\text{tr deg}(k) = 2n$$

and we can take $S = X \times X$. The moral of this story is that complicated 0-cycles on X potentially require ever more complicated fields of definition.

Example 5. *Hypersurfaces in $\mathbf{P}^{n+1}$*

Take $F \in \mathbf{C}[z_0, \ldots, z_{n+1}]$, homogeneous of degree d, $X = \{F = 0\}$. Let

$$F = \sum_{|I|=d} a_I z^I$$

using multi-index notation. Assume that $a = (a_I)_{|I|=d}$ is chosen to be a very general point of $\mathbf{P}^{\binom{n+1+d}{d}-1}$. Then the field of definition of X is $\mathbf{Q}(\text{ratios of the } a_I)$, and $S = \mathbf{P}^{\binom{n+1+d}{d}-1}$. Now

$$\pi \colon \mathcal{X} \to S$$

is the universal family of hypersurfaces of degree d.

Essential Observation. *It is usually productive to make use of this geometry.*

To do this, we need constructions that are robust to birational changes.

One natural idea is to look at the associated **variation of Hodge structure**

$$S - \Sigma \xrightarrow{P} \Gamma/D$$
$$s \mapsto H^r(X_s, \mathbf{C})$$

where D is the Hodge domain (or the appropriate Mumford-Tate domain) and Γ is the group of automorphisms of the integral lattice preserving the intersection pairing. If we have an **algebraic cycle** Z on X, taking spreads yields a cycle $\mathcal{Z}$ on $\mathcal{X}$. Applying Hodge theory to $\mathcal{Z}$ on $\mathcal{X}$ gives invariants of the cycle. Another related situation is **algebraic K-theory**. For example, to study $K_p^{\text{Milnor}}(k)$, the geometry of S can used to construct invariants.

This is, overall, the **spread philosophy**.

Lecture 2: Cycle Class and Spreads

X a smooth projective variety $/k$. By $\Omega^{\bullet}_{X(k)/k}$ we denote the differentials on $X(k)$ over k. The sheaf $\Omega^1_{X(k)/k}$ is defined to be objects of the form $\sum_i f_i dg_i$, where $f_i, g_i \in \mathcal{O}_{X(k)}$ and subject to the rules:
(i) $d(f+g) = df + dg$;
(ii) $d(fg) = fdg + gdf$;
(iii) $d^2 = 0$;
(iv) $dc = 0$ for $c \in k$.
We note that a consequence of (iv) is that $dc = 0$ if $c \in \bar{k}$. We set $\Omega^p_{X(k)/k} = \wedge^p \Omega^1_{X(k)/k}$ and these are made into a complex using d,

$$\mathcal{O}_{X(k)} \xrightarrow{d} \Omega^1_{X(k)/k} \xrightarrow{d} \Omega^2_{X(k)/k} \xrightarrow{d} \cdots$$

which we denote $\Omega^{\bullet}_{X(k)/k}$. By $\Omega^{\geq p}_{X(k)/k}$ we denote the complex

$$\Omega^p_{X(k)/k} \xrightarrow{d} \Omega^{p+1}_{X(k)/k} \xrightarrow{d} \cdots,$$

but indexed to be a subcomplex of $\Omega^{\bullet}_{X(k)/k}$.

Comparison Theorems of Grothendieck. *[Gro] If $k \overset{i_s}{\hookrightarrow} \mathbf{C}$ is a complex embedding of k,*
(1) $\mathbf{H}^r(\Omega^{\bullet}_{X(k)/k}) \otimes_{i_s} \mathbf{C} \cong H^r(X_s, \mathbf{C})$;
(2) $\mathbf{H}^r(\Omega^{\geq p}_{X(k)/k}) \otimes_{i_s} \mathbf{C} \cong F^p H^r(X_s, \mathbf{C})$, where $F^{\bullet} H^r(X_s, \mathbf{C})$ denotes the Hodge filtration;
(3) $H^q(\Omega^p_{X(k)/k}) \otimes_{i_s} \mathbf{C} \cong H^{p,q}(X_s)$.

If in the definition of differentials we instead only require:
(iv') $dc = 0$ for $c \in \mathbf{Q}$
then we get $\Omega^{\bullet}_{X(k)/\mathbf{Q}}$, etc. By $\Omega^1_{k/\mathbf{Q}}$ we denote expressions $\sum_i a_i db_i$, where $a_i, b_i \in k$, subject to the rules (i)-(iii) and (iv'). Note that $\Omega^1_{k/\mathbf{Q}}$ is a k vector space of dimension tr deg(k). If

$$k \cong \mathbf{Q}(x_1, \ldots, x_T)[y_1, \ldots, y_A]/(p_1, \ldots, p_B),$$

then $dx_1, \ldots, dx_T$ give a k basis for $\Omega^1_{k/\mathbf{Q}}$.

There is a natural filtration

$$F^m \Omega^{\bullet}_{X(k)/\mathbf{Q}} = \mathrm{Im}(\Omega^m_{k/\mathbf{Q}} \otimes \Omega^{\bullet - m}_{X(k)/\mathbf{Q}} \to \Omega^{\bullet}_{X(k)/\mathbf{Q}}).$$

The associated graded is

$$Gr^m \Omega^{\bullet}_{X(k)/\mathbf{Q}} = \frac{F^m \Omega^{\bullet}_{X(k)/\mathbf{Q}}}{F^{m+1} \Omega^{\bullet}_{X(k)/\mathbf{Q}}} \cong \Omega^m_{k/\mathbf{Q}} \otimes \Omega^{\bullet - m}_{X(k)/k}.$$

We thus have an exact sequence

$$0 \to Gr^1\Omega^\bullet_{X(k)/\mathbf{Q}} \to \frac{\Omega^\bullet_{X(k)/\mathbf{Q}}}{F^2\Omega^\bullet_{X(k)/\mathbf{Q}}} \to Gr^0\Omega^\bullet_{X(k)/\mathbf{Q}} \to 0.$$

This can be rewritten

$$0 \to \Omega^1_{k/\mathbf{Q}} \otimes \Omega^{\bullet-1}_{X(k)/k} \to \frac{\Omega^\bullet_{X(k)/\mathbf{Q}}}{F^2\Omega^\bullet_{X(k)/\mathbf{Q}}} \to \Omega^\bullet_{X(k)/k} \to 0.$$

From the long exact sequence for (hyper)cohomology, we obtain a map

$$\mathbf{H}^r(\Omega^\bullet_{X(k)/k}) \xrightarrow{\nabla} \Omega^1_{k/\mathbf{Q}} \otimes \mathbf{H}^r(\Omega^\bullet_{X(k)/k});$$

this is the **Gauss-Manin connection**. Similarly, using $F^m\Omega^\bullet_{X(k)/\mathbf{Q}}/F^{m+2}\Omega^\bullet_{X(k)/\mathbf{Q}}$, we get

$$\Omega^m_{k/\mathbf{Q}} \otimes \mathbf{H}^r(\Omega^\bullet_{X(k)/k}) \xrightarrow{\nabla} \Omega^{m+1}_{k/\mathbf{Q}} \otimes \mathbf{H}^r(\Omega^\bullet_{X(k)/k}).$$

We have

$$\nabla^2 = 0 \quad \text{(Integrability of the Gauss} - \text{Manin connection)}.$$

Finally, we note that

$$0 \to \Omega^1_{k/\mathbf{Q}} \otimes \Omega^{\geq p-1}_{X(k)/k} \to \frac{\Omega^{\geq p}_{X(k)/\mathbf{Q}}}{F^2\Omega^{\geq p}_{X(k)/\mathbf{Q}}} \to \Omega^{\geq p}_{X(k)/k} \to 0$$

gives

$$F^p\mathbf{H}^r(\Omega^\bullet_{X(k)/k}) \xrightarrow{\nabla} \Omega^1_{k/\mathbf{Q}} \otimes F^{p-1}\mathbf{H}^r(\Omega^\bullet_{X(k)/k});$$

i.e.

$$\nabla(F^p) \subseteq \Omega^1_{k/\mathbf{Q}} \otimes F^{p-1}$$

which is known as the **infinitesimal period relation** or **Griffiths transversality** (See [Gre]).

Note that all of this takes place in the abstract world of k, without the need to choose an embedding $k \overset{i_s}{\hookrightarrow} \mathbf{C}$. The essential new feature, once we pick i_s, is the **integral lattice**

$$H^r(X_s, \mathbf{Z}) \to \mathbf{H}^r(\Omega^\bullet_{X(k)/k}) \otimes_{i_s} \mathbf{C}.$$

Essential Observation.

Expressing this map involves transcendentals not already in k. The fields $\mathbf{Q}(\pi) \cong \mathbf{Q}(e)$, where π, e are transcendentals, and thus there is no algebraic construction over the field $k = \mathbf{Q}(x)$ that distinguishes the cohomology of varieties defined over k when we take different embeddings $k \to \mathbf{C}$ taking x to π or e. This is why the integral lattice involves maps transcendental in the elements of k.

Example 1. *Elliptic curves*

We take X to be the projectivization of $y^2 = f(x)$, where $f(x) = x(x-1)(x-\alpha)$, and α is transcendental. $k = \mathbf{Q}(\alpha)$. Differentiating,

$$2ydy = f'(x)dx$$

in $\Omega^1_{X(k)/k}$. If $U_1 = \{y \neq 0\}$ and $U_2 = \{f'(x) \neq 0\}$, then

$$\frac{2dy}{f'(x)} = \frac{dx}{y}$$

and thus we get an element ω of $\mathbf{H}^0(\Omega^{\geq 1}_{X(k)/k})$ that is dx/y in U_1 and $2dy/f'(x)$ in U_2. However, in $\Omega^1_{X(k)/\mathbf{Q}}$, we have

$$2ydy = f'(x)dx - x(x-1)d\alpha$$

on $U_1 \cap U_2$, and thus ω does not lift to $\mathbf{H}^0(\Omega^{\geq 1}_{X(k)/\mathbf{Q}})$. Thus

$$\nabla(\omega) = \frac{x(x-1)}{f'(x)y} d\alpha \in \Omega^1_{k/\mathbf{Q}} \otimes H^1(\mathcal{O}_{X(k)}),$$

where $x(x-1)/f'(x)y$ on $U_1 \cap U_2$ represents a class in $H^1(\mathcal{O}_{X(k)})$.

There is another construction in terms of k that is significant.

Bloch-Quillen Theorem.

$$H^p(\mathcal{K}_p(\mathcal{O}_{X(k)})) \cong CH^p(X(k)),$$

where $\mathcal{K}_p$ denotes the sheaf of K_p's from algebraic K-theory and $CH^p(X(k))$ is cycles on X defined over k modulo rational equivalences defined over k (See [M]).

If we are willing to neglect torsion, we can replace $\mathcal{K}_p$ with the more intuitive $\mathcal{K}_p^{\text{Milnor}}$.

Soulé's Bloch-Quillen Theorem. *[S]*

$$H^p(\mathcal{K}_p^{\text{Milnor}}(\mathcal{O}_{X(k)})) \otimes_{\mathbf{Z}} \mathbf{Q} \cong CH^p(X(k)) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

The description of $\mathcal{K}_p^{\text{Milnor}}(\mathcal{O}_{X(k)})$ proceeds as follows: Regard $\mathcal{O}^*_{X(k)}$ as a $\mathbf{Z}$-module under exponentiation. One takes (locally)a quotient of $\otimes^p_{\mathbf{Z}} \mathcal{O}^*_{X(k)}$, representing $f_1 \otimes \cdots \otimes f_p$ by the symbol$\{f_1, \ldots, f_p\}$. The quotient is defined by the relations generated by:
(**Steinberg relations**): $\{f_1, \ldots, f_p\} = 1$ if $f_i = 1 - f_j$ for some $i \neq j$.

There is now a map

$$\begin{aligned} \mathcal{K}_p^{\text{Milnor}}(\mathcal{O}_{X(k)}) &\longrightarrow \Omega^p_{X(k)/\mathbf{Q}} \\ \{f_1, \ldots, f_p\} &\mapsto \frac{df_1 \wedge df_2 \wedge \cdots \wedge df_p}{f_1 f_2 \cdots f_p}. \end{aligned}$$

We may also regard this as a map

$$\mathcal{K}_p^{\mathrm{Milnor}}(\mathcal{O}_{X(k)}) \longrightarrow \Omega^{\geq p}_{X(k)/\mathbf{Q}}.$$

We thus get maps

$$H^p(\mathcal{K}_p^{\mathrm{Milnor}}(\mathcal{O}_{X(k)})) \to H^p(\Omega^p_{X(k)/\mathbf{Q}})$$

and

$$H^p(\mathcal{K}_p^{\mathrm{Milnor}}(\mathcal{O}_{X(k)})) \to \mathbf{H}^{2p}(\Omega^{\geq p}_{X(k)/\mathbf{Q}}).$$

The shift in index in the last cohomology group is to align it with the indexing for $\Omega^{\bullet}_{X(k)/\mathbf{Q}}$. These are called the **arithmetic cycle class** and were studied by Grothendieck, Srinivas [Sr] and Esnault-Paranjape [E-P].

If we move from differentials over $\mathbf{Q}$ to differentials over k, we obtain the **cycle class map**

$$H^p(\mathcal{K}_p^{\mathrm{Milnor}}(\mathcal{O}_{X(k)})) \xrightarrow{\psi_{X(k)}} F^p\mathbf{H}^{2p}(\Omega^{\bullet}_{X(k)/k}).$$

Note that this is constant under the Gauss-Manin connection, i.e.

$$\nabla \circ \psi_{X(k)} = 0.$$

If we choose a complex embedding of k, we then have a map $\psi_{X(k)} \otimes_{i_s} \mathbf{C}$, which we will denote as ψ_{X_s} which maps

$$H^p(\mathcal{K}_p^{\mathrm{Milnor}}(\mathcal{O}_{X(k)})) \xrightarrow{\psi_{X_s}} F^pH^{2p}(X_s, \mathbf{C}) \cap \mathrm{Im}(H^{2p}(X_s, \mathbf{Q})).$$

Since integral classes are flat under ∇, this is consistent with $\nabla \circ \psi_{X(k)} = 0$. We denote

$$Hg^p(X_s) = F^pH^{2p}(X_s, \mathbf{C}) \cap \mathrm{Im}(H^{2p}(X_s, \mathbf{Z})).$$

Hodge Conjecture.

$$CH^p(X_s(\mathbf{C})) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\psi_{X_s}} Hg^p(X_s) \otimes_z \mathbf{Q}$$

is surjective.

Absolute Hodge Conjecture.[Gro2] Given a class $\xi \in F^p\mathbf{H}^{2p}(\Omega^{\bullet}_{X(k)/k})$, the set of $s \in S$ such that $i_{s*}(\xi) \in Hg^p(X_s)$ is a subvariety of S defined /Q.

Note that the Absolute Hodge Conjecture is weaker than the Hodge Conjecture.

Lecture 3: The Conjectural Filtration on Chow Groups from a Spread Perspective

X a smooth projective variety defined $/k$
$Z^p(X(k))$ = codimension $p$cycles defined over k
$CH^p(X(k)) = Z^p(X(k))/$rational equivalences defined over k
$CH^p(X(k))_{\mathbf{Q}} = CH^p(X(k)) \otimes_{\mathbf{Z}} \mathbf{Q}$
$CH^p(X(k))_{\mathbf{Q},\mathrm{Hom}} = \{Z \in CH^p(X(k))_{\mathbf{Q}} \mid NZ \cong_{\mathrm{Hom}} 0 \text{ for some } N \neq 0\}$
We tensor the Chow group with $\mathbf{Q}$ in order to eliminate torsion phenomena, which tend to be especially difficult–for example, the fact that the Hodge Conjecture is not true over $\mathbf{Z}$.

Conjectural Filtration. *(See [M]) There is a decreasing filtration*

$$CH^p(X(k))_{\mathbf{Q}} = F^0 CH^p(X(k))_{\mathbf{Q}} \supseteq F^1 CH^p(X(k))_{\mathbf{Q}} \supseteq F^2 CH^p(X(k))_{\mathbf{Q}} \supseteq \cdots$$

with the following properties:
(i) $F^{m_1} CH^{p_1}(X(k))_{\mathbf{Q}} \otimes F^{m_2} CH^{p_2}(X \times Y(k))_{\mathbf{Q}} \to F^{m_1+m_2} CH^{p_1+p_2-\dim(X)}(Y(k))_{\mathbf{Q}}$.
(ii) $F^1 CH^p(X(k)) = CH^p(X(k))_{\mathbf{Q},\mathrm{Hom}}$
(iii) $F^{p+1} CH^p(X(k))_{\mathbf{Q}} = 0$.

An essential feature of this conjectural filtration is that it should be defined in terms of k and not depend on a choice of complex embedding $k \to \mathbf{C}$.

Example 1. $F^1 CH^p(X(k))$

Because Grothendieck identifies

$$\mathbf{H}^r(\Omega^\bullet_{X(k)/k}) \otimes_{i_s} \mathbf{C} \cong H^r(X_s, \mathbf{C}),$$

the condition that $[Z] \in \mathbf{H}^{2p}(\Omega^\bullet_{X(k)/k})$ is zero is equivalent to $i_{s*}[Z] = 0$ in $H^{2p}(X_s, \mathbf{C})$ for any complex embedding $k \overset{i_s}{\hookrightarrow} \mathbf{C}$.

Example 2. $F^2 CH^p(X(k))$

The expectation is that if

$$AJ^p_{X_s,\mathbf{Q}} \colon CH^p(X(k))_{\mathbf{Q},\mathrm{Hom}} \to J^p(X_s) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is the Abel-Jacobi map for X_s tensored with $\mathbf{Q}$, then

$$F^2 CH^p(X(k)) \cong \ker(AJ^p_{X_s,\mathbf{Q}}).$$

The Abel-Jacobi map is highly transcendental, and it is not known that the kernel of the Abel-Jacobi map tensored with $\mathbf{Q}$ is independent of the complex embedding of k.

Let $Gr^m CH^p(X(k))_{\mathbf{Q}} = F^m CH^p(X(k))_{\mathbf{Q}} / F^{m+1} CH^p(X(k))_{\mathbf{Q}}$.

Example 3. *Cycle classes*

We have the cycle class map

$$\psi_{X_s}: Gr^0 CH^p(X(k)) \hookrightarrow H^{2p}(X_s, \mathbf{C}).$$

The Hodge Conjecture says that

$$\mathrm{Im}(\psi_{X_s}) = Hg^p(X_s),$$

the Hodge classes of X_s. Note that the set of Hodge classes is thus conjecturally isomorphically the same for any very general $s \in S$.

Example 4. *Image of the Abel-Jacobi map*

We have

$$AJ^p_{X_s,\mathbf{Q}}: Gr^1 CH^p(X(k)) \hookrightarrow J^p(X_s) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Thus conjecturally, $\mathrm{Im}(AJ^p_{X_s,\mathbf{Q}})$ is isomorphically the same for any very general $s \in S$.

It is thus expected that there should be something nice happening on the Hodge theory side.

Beilinson's Conjectural Formula.

$$Gr^m CH^p(X(k))_{\mathbf{Q}} \cong \mathrm{Ext}^m_{\mathcal{MM}_k}(\mathbf{Q}, H^{2p-m}(X)(p)),$$

where $\mathcal{MM}_k$ means that the extensions are in the category of mixed motives over k (See [R]).

Unfortunately, we do not have an explicit description of what these Ext groups should look like. One explicit consequence of this conjecture is that a cycle $\Gamma \in CH^r(X \times Y(k))$ induces the zero map

$$CH^p(X(k))_{\mathbf{Q}} \xrightarrow{\Gamma_*} Gr^m CH^{p+r-\dim(X)}(Y(k))_{\mathbf{Q}}$$

if the $H^{2\dim(X)-2p+m}(X) \otimes H^{2r+2p-2\dim(X)-m}(Y)$ component of $[\Gamma]$ is zero.

A very different aspect of the conjectures is that the arithmetic properties of k limit the possible graded pieces of the filtration on Chow groups.

Conjecture (Deligne-Bloch-Beilinson).
(i) $Gr^m CH^p(X(k))_{\mathbf{Q}} = 0$ *for* $m > \mathrm{tr\ deg}(k) + 1$.
(ii) In particular, for X defined $/\mathbf{Q}$, $AJ^p_{X,\mathbf{Q}}: CH^p(X(\mathbf{Q}))_{\mathbf{Q},Hom} \to J^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ is injective. (See [R]).

It is thus very natural to try to express the conjectural filtration of Chow groups in terms of spreads.

A first step is to have as good an understanding of the Abel-Jacobi map as possible. There is a nice interpretation of the Abel-Jacobi map as the extension class of an **extension of mixed Hodge structures**. To phrase this in our context, if $Z \in Z^p(X(k))$, we may represent Z by taking smooth varieties Z_i defined over k, maps $f_i\colon Z_i \to X$ defined over k, with

$$Z = \sum_i n_i f_{i*} Z_i.$$

We may construct a complex of differentials $\Omega^\bullet_{(X,|Z|)(k)/k}$ where

$$\Omega^m_{(X,|Z|)(k)/k} = \Omega^m_{X(k)/k} \oplus \bigoplus_i \Omega^{m-1}_{Z_i(k)/k}$$

with the differential

$$d(\omega, \oplus_i \phi_i) = (d\omega, \oplus_i d\phi_i - f_i^* \omega).$$

If $\dim(X) = n$, and thus $\dim(Z_i) = n - p$, we have an exact sequence

$$0 \to \operatorname{coker}(\mathbf{H}^{2n-2p}(\Omega^\bullet_{X(k)/k}) \to \oplus_i \mathbf{H}^{2n-2p}(Z_i)) \to \mathbf{H}^{2n-2p+1}(\Omega^\bullet_{(X,|Z|)(k)/k}) \\ \to \mathbf{H}^{2n-2p+1}(\Omega^\bullet_{X(k)/k}) \to 0.$$

If $Z \equiv_{\mathrm{Hom}} 0$ on X, we can derive from this an exact sequence

$$0 \to k(-(n-p)) \to V \to \mathbf{H}^{2n-2p+1}(\Omega^\bullet_{X(k)/k}) \to 0,$$

where V is a k vector space with a Hodge filtration and a Gauss-Manin connection ∇_V. If we choose a complex embedding of k, and tensor the sequence above $\otimes_{i_s} \mathbf{C}$, we obtain an extension of mixed Hodge structures

$$0 \to \mathbf{Z}(-(n-p)) \xrightarrow{f_s} V_s \xrightarrow{g_s} H^{2n-2p+1}(X_s, \mathbf{C}) \to 0,$$

where the new element added by the complex embedding is the integral lattice. If we pick an integral lifting $\phi_{\mathbf{Z}}\colon H^{2n-2p+1}(X_s, \mathbf{Z}) \to V_{s,\mathbf{Z}}$ and a complex lifting preserving the Hodge filtration, $\phi_{\mathrm{Hodge}}\colon H^{2n-2p+1}(X_s, \mathbf{C}) \to V_s$, then the extension class

$$e_s = f_s^{-1}(\phi_{\mathbf{Z}} - \phi_{\mathrm{Hodge}})$$

with

$$e_s \in \frac{\mathrm{Hom}_{\mathbf{C}}(H^{2n-2p+1}(X_s, \mathbf{C}), \mathbf{C}(-(n-p)))}{F^0 \mathrm{Hom}_{\mathbf{C}}(H^{2n-2p+1}(X_s, \mathbf{C}), \mathbf{C}(-(n-p))) + \mathrm{Hom}_{\mathbf{Z}}(H^{2n-2p+1}(X_s, \mathbf{Z}), \mathbf{Z}(-(n-p)))}.$$

We may rewrite this using Poincaré duality as

$$e_s \in \frac{H^{2p-1}(X_s, \mathbf{C})}{F^p H^{2p-1}(X_s, \mathbf{C}) + H^{2p-1}(X_s, \mathbf{Z})} = J^p(X_s),$$

the **p'th intermediate Jacobian of X_s**. The intermediate Jacobians fit together to give a family $\mathcal{J} \to S$ and $s \mapsto e_s$ gives a section $\nu_Z\colon S \to \mathcal{J}$ of this family, which is called the **normal function associated to the cycle** Z. By an argument analagous to the one used to define the Gauss-Manin connection by looking at the obstruction to lifting to differentials $/\mathbf{Q}$, we get for a local lifting $\tilde{\nu}_Z$ of ν_Z to the variation of Hodge structures $H^{2p-1}(X_s, \mathbf{C})$ that

$$\nabla\tilde{\nu}_Z \in \Omega^1_{S/\mathbf{C}} \otimes F^{p-1}H^{2p-1}(X_s, \mathbf{C})$$

for all s; the fact that we land in $F^{p-1}H^{2p-1}(X_s, \mathbf{C})$ is known as the **infinitesimal relation on normal functions**. The actual value we get depends on how we lift ν_Z, but $\nabla\nu_Z$ gives a well-defined element

$$\delta\nu_Z \in \mathbf{H}^1(\Omega^\bullet_S \otimes F^{p-\bullet}H^{2p-1}(X_s, \mathbf{C}))$$

which is **Griffiths' infinitesimal invariant of the normal function** ν_Z. (See [Gre], Gre2]).

We may also encapsulate the information in the construction above as an extension involving the variation of Hodge structure $\mathcal{H}^{2p-1}_X \to S$ of the form

$$0 \to \mathcal{H}^{2p-1}_X(p) \to \mathcal{V}^* \to \mathbf{Z} \to 0.$$

We may regard such extensions as elements of the group $\mathrm{Ext}^1_S(\mathbf{Z}, \mathcal{H}^{2p-1}_X(p))$, where we must make some technical assumptions about how the families behave over the subvariety Σ where the map $\pi\colon \mathcal{X} \to S$ is not of maximal rank, and thus X_s is singular. Conjecturally, one would expect that

$$Gr^1 CH^p(X(k)) \otimes_{\mathbf{Z}} \mathbf{Q} \hookrightarrow \mathrm{Ext}^1_S(\mathbf{Q}, \mathcal{H}^{2p-1}_X(p))$$

is well-defined and injective.

There are a number of different ways that, conjecturally, produce the conjectural filtration. Two of my favorites are those of Murre and of H. Saito. Let $\dim(X) = n$. The Hodge conjecture says that the **Künneth decomposition of the diagonal** $\Delta \in X \times X$ as $\Delta = \sum_i \pi_i$ where $\pi_i \in Z^n(X \times X)$ and $[\pi_i] \in H^{2n-i}(X, \mathbf{Q}) \otimes H^i(X, \mathbf{Q})$ represents the identity map under Poincaré duality. These induce maps

$$CH^p(X)_{\mathbf{Q}} \xrightarrow{\pi^p_{i*}} CH^p(X)_{\mathbf{Q}}.$$

Now we want for $m \geq 1$

$$F^m CH^p(X)_{\mathbf{Q}} = \cap^{2p}_{i=2p-m+1} \ker(\pi^p_{i*}).$$

This is Murre's definition [M].

The definition of H.Saito (see [J]) generates $F^m CH^p(X(k))_{\mathbf{Q}}$ by taking auxiliary varieties T defined $/k$ and cycles $Z_1 \in CH^{r_1}(X \times T(k))_{\mathrm{hom}}$ and for $i = 2, \ldots, m$ cycles $Z_i \in CH^{r_i}(T(k))_{\mathrm{hom}}$ and looking at

$$p_{X*}(Z_1 \cdot p_T^* Z_2 \cdot p_T^* Z_3 \cdots p_T^* Z_m),$$

where $\sum_i r_i - \dim(T) = p$. Clearly all elements of this form must lie in $F^m CH^p(X(k))_{\mathbf{Q}}$ for any definition satisfying the conditions of the conjectural filtration, but it is not clear that $F^{p+1} CH^p(X(k))_{\mathbf{Q}} = 0$.

Lecture 4: The Case of X defined over $\mathbf{Q}$

This lecture is based on joint work with Phillip Griffiths [G-G].

We now look at the case X a smooth projective connected variety defined $/\mathbf{Q}$, as discussed in a joint paper with Phillip Griffiths. We consider cycles defined over a finitely generated extension k of the rationals. Thus

$$\mathcal{X} \cong X \times S.$$

If $Z \in Z^p(X(k))$, then its spread

$$\mathcal{Z} \in Z^p(X \times S(\mathbf{Q}))$$

is well-defined up to an ambiguity in the form of a cycle

$$\mathcal{W} \in Z^{p-\mathrm{codim}(W)}(X \times W(\mathbf{Q})),$$

where W is a lower-dimensional subvariety of S defined $/\mathbf{Q}$. Cycles rationally equivalent to 0 over k are generated by taking a codimension $p-1$ subvariety $Y \overset{i}{\hookrightarrow} X$ defined $/k$ and $f \in k(Y)$ and taking $i_* \mathrm{div}(f)$. Taking spreads over k, we have $\mathcal{Y} \subset X \times S$ of codimension $p-1$ defined $/\mathbf{Q}$ and $F \in \mathbf{Q}(Y)$. Once again, the ambiguities in this process are supported on a variety of the form $X \times W$, where W is a lower-dimensional subvariety of S defined over $\mathbf{Q}$.

At this point, we invoke the conjecture of Deligne-Bloch-Beilinson mentioned in Lecture 3 that for cycles and varieties over $\mathbf{Q}$, the cycle class and the Abel-Jacobi map are a complete set of invariants for cycles modulo rational equivalence, tensored with $\mathbf{Q}$. Thus, if $\mathcal{Z} \in Z^p(X \times S(\mathbf{Q}))$, the invariants are:
(i) $[\mathcal{Z}] \in H^{2p}(X \times S, \mathbf{C})$ and
(ii) If $[\mathcal{Z}] = 0$, then $AJ^p_{X \times S}(\mathcal{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}$.
It follows from this conjecture that $Z = 0$ in $CH^p(X(k))_{\mathbf{Q}}$ if and only if there exists a cycle $\mathcal{W} \in Z^{p-\mathrm{codim}(W)}(X \times W)$ for some lower-dimensional subvariety $W \subset S$ such that:
(i) $[\mathcal{Z} + \mathcal{W}] = 0$ in $H^{2p}(X \times S)$ and
(ii) $AJ_{X \times S, \mathbf{Q}}(\mathcal{Z} + \mathcal{W}) = 0$ in $J^p(X \times S) \otimes_{\mathbf{Z}} \mathbf{Q}$.

The Künneth decomposition of $X \times S$ allows us to write over $\mathbf{Q}$

$$H^{2p}(X \times S) \cong \bigoplus_m H^{2p-m}(X) \otimes H^m(S)$$

and

$$J^p(X \times S) \otimes_{\mathbf{Z}} \mathbf{Q} \cong \bigoplus_m J^p(X \times S)_m$$

where

$$J^p(X \times S)_m = \frac{H^{2p-1-m}(X, \mathbf{C}) \otimes H^m(S, \mathbf{C})}{F^p(H^{2p-1-m}(X, \mathbf{C}) \otimes H^m(S, \mathbf{C})) + H^{2p-1-m}(X, \mathbf{Q}) \otimes H^m(S, \mathbf{Q})}.$$

We will denote the Künneth components of $[\mathcal{Z}]$ as $[\mathcal{Z}]_m$ and the $J^p(X \times S)_m$ component of $AJ_{X \times S}(\mathcal{Z})$ as $AJ_{X \times S}(\mathcal{Z})_m$.

It is important to note that while we need the vanishing of the cycle class in order to define the Abel-Jacobi map, we only need the vanishing of $[\mathcal{Z}]_i$ for $i \leq m+1$ in order to define $AJ_{X \times S}(\mathcal{Z})_m$.

Definition of Filtration on Chow Groups. *$Z \in F^m CH^p(X(k))$ if and only if for some $\mathcal{W}$ as above, $[\mathcal{Z}+\mathcal{W}]_i = 0$ for all $i < m$ and (this is now defined) $AJ_{X\times S}(\mathcal{Z}+\mathcal{W})_i = 0$ for all $i < m-1$.*

In order to understand this definition, it is essential to understand what happens for cycles $\mathcal{W} \in Z^{p-\mathrm{codim}(W)}(X \times W)$. If $r = \mathrm{codim}(W)$, then we have the Gysin map

$$Gy_W^m \colon H^m(W) \to H^{m+2r}(S).$$

This induces a map

$$H^{2p-m}(X) \otimes H^{m-2r}(W) \to H^{2p-m}(X) \otimes H^m(S).$$

We see that

$$[\mathcal{W}]_m \in H^{2p-m}(X) \otimes \mathrm{Im}(Gy_W^{m-2r}).$$

Now $\mathrm{Im}(Gy_W^{m-2r})$ is contained in the largest weight $m-2r$ sub-Hodge structure of $H^m(S)$.

Let H denote the largest weight $m-2r$ sub-Hodge structure of $H^m(S)$. The generalized Hodge conjecture implies that there is a dimension $m-2r$ subvariety V of $S(\mathbf{C})$ such that $\mathrm{Im}(Gy_V^{m-2r}) = H$. Now V might in principle require a finitely generated field of definition L, with $L = \mathbf{Q}(T)$ for some variety T defined $/\mathbf{Q}$. Taking the spread of V over T, with the complex embedding of V in S represented by $t_0 \in T$, we know that there is a lower-dimensional subvariety of T such that away from it, for t in the same connected component of T, $H^{m-2r}(V_t)$ and $H^{m-2r}(V_{t_0})$ have the same image in $H^m(S)$. We may therefore find a point $t_1 \in T(\bar{\mathbf{Q}})$ in the same connected component of $T(\mathbf{C})$ as t_0, such that V_{t_1} is defined over $\bar{\mathbf{Q}}$ and $Gy_{V_{t_1}}^{m-2r}$ has the same image as for V_{t_0}. We take W to be the union of the Galois conjugates of V_{t_1}. Then $H \in \mathrm{Im}(Gy_W^{m-2r})$. It follows that any Hodge class in $H^{2p-m}(X) \otimes H$ is, by the Hodge Conjecture, a $\mathbf{Q}$-multiple of the Hodge class of a cycle $\mathcal{W} \in Z^{p-\mathrm{codim}(W)}(X \times W)$, i.e. an ambiguity. It follows that

$$CH^p(X(k))_{\mathbf{Q}} \to H^{2p-m}(X, \mathbf{Q}) \otimes \frac{H^m(S, \mathbf{Q})}{H^m(S, \mathbf{Q})_{m-2}},$$

where $H^m(S)_{m-2}$ is the largest weight $m-2$ sub-Hodge structure of $H^m(S)$, is well-defined and captures all of the information in the invariant $[\mathcal{Z}]_m$ modulo ambiguities.

We note that if $\dim(S) < m$, then by the Lefschetz theorem $H^m(S) = H^m(S)_{m-2}$, so this invariant vanishes.

We note that a Hodge class in $H^{2p-m}(X) \otimes H^m(S)$ gives us a map of Hodge structures, with a shift, $H^{2n-2p+m}(X) \to H^m(S)$. However, $H^{2n-2p+m}(X) \cong H^{2p-m}(X)(-(n-2p+m))$. If $m > 2p-m$, this implies that the Hodge class actually lies in $H^{2p-m}(X) \otimes H^m(S)_{m-2}$ and thus is an ambiguity. This happens precisely when $m > p$. We thus have that the invariant $[\mathcal{Z}]_m = 0$ modulo ambiguities if $m > p$. An alternative proof is that if X_P is a general $\mathbf{Q}$-linear section of X of dimension $2p-m$, then let $\mathcal{Z}_P = \mathcal{Z} \cap (X_P \times S)$. By the Lefschetz theorem, we have

$$r_P^{2p-m} \colon H^{2p-m}(X) \hookrightarrow H^{2p-m}(X_P)$$

and

$$r_P^{2p-m} \otimes \mathrm{id}_{H^m(S)}([\mathcal{Z}]_m) = [\mathcal{Z}_P]_m.$$

However, if $p > 2p - m$, then of necessity $\mathcal{Z}_P$ projects to a proper $\mathbf{Q}$-subvariety of S, and hence is an ambiguity.

The second argument also shows that taking P so that X_P has dimension $2p - m - 1$, then

$$r_P^{2p-m-1} \otimes \mathrm{id}_{H^m(S)}(AJ^p_{X\times S}(\mathcal{Z})_m) = AJ^p_{X_P\times S}(\mathcal{Z}_P)_m$$

and thus for $p > 2p - m - 1$, $\mathcal{Z}_{\mathcal{P}}$ must project to a proper $\mathbf{Q}$-subvariety of S and hence involves only $H^m(S)_{m-2}$. One can also use the linear section argument to use cycles defined on $X \times W$ for proper $\mathbf{Q}$-subvarieties W of S to kill off portions of the Abel-Jacobi map that involve $H^m(S)_{m-2}$.

We define $[\mathcal{Z}]_m^{\mathrm{red}}$ to be the image of $[\mathcal{Z}]_m$ in $H^{2p-m}(X) \otimes H^m(S)/H^m(S)_{m-2}$ and we define $AJ^p_{X\times S}(\mathcal{Z})^{\mathrm{red}}_m$ to be the image of $AJ^p_{X\times S}(\mathcal{Z})_m$ in the intermediate Jacobian constructed from $H^{2p-m-1}(X) \otimes H^m(S)/H^m(S)_{m-2}$.

Invariants of cycles. *For X defined over $\mathbf{Q}$, a complete set of invariants of $CH^p(X(k))_{\mathbf{Q}}$ are $[\mathcal{Z}]_m^{\mathrm{red}}$ for $0 \leq m \leq p$ and $AJ^p_{X\times S}(\mathcal{Z})^{\mathrm{red}}_m$ for $0 \leq m \leq p-1$. Note that $[\mathcal{Z}]_m^{\mathrm{red}}$ and $AJ^p_{X\times S}(\mathcal{Z})^{\mathrm{red}}_m$ both vanish if $m > \dim(S)$, i.e. $m > \mathrm{tr}\ \deg(k)$. We then get that $F^mCH^p(X(k))_{\mathbf{Q}}$ is defined by the vanishing (tensored with $\mathbf{Q}$) of $[\mathcal{Z}]_i^{\mathrm{red}}$ for $i < m$ and of $AJ^p_{X\times S}(\mathcal{Z})^{\mathrm{red}}_m$ for $i < m-1$. This forces $F^mCH^p(X(k))_{\mathbf{Q}} = 0$ if $m > p$ or if $m >$ $\mathrm{tr}\ \deg(k) + 1$.*

Example 1: $p = 1$

Here $[\mathcal{Z}]_0^{\mathrm{red}}$ is just the cycle class of Z if S is connected, or the various cycle classes coming from different complex embeddings if S is not connected. Next, $[\mathcal{Z}]_1^{\mathrm{red}}$ is in $H^1(X,\mathbf{Q}) \otimes H^1(S,\mathbf{Q})$ and is equivalent to the induced map on cohomology coming from $\mathrm{Alb}(S) \to J^1(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ coming from $s \mapsto AJ^1_X(Z_s)$. If this is zero, then this map is constant on connected components of S, and this allows us to define $AJ^1_{X\times S}(\mathcal{Z})^{\mathrm{red}}_0 \in J^1(X) \otimes_{\mathbf{Z}} H^0(S,\mathbf{Q})$. Note that we do not need to reduce modulo lower weight sub-Hodge structures on S for either of these.

Example 2. $p = 2$

The only really new invariants are $[\mathcal{Z}]_2^{\mathrm{red}}$ and $AJ^p_{X\times S}(\mathcal{Z}])^{\mathrm{red}}_1$, which are the invariants of $Gr^2CH^2(X(k))_{\mathbf{Q}}$. The former is an element of $H^2(X)/Hg^1(X) \otimes H^2(S)/Hg^1(S)$. This invariant was discussed by Voisin, and comes by integrating $2n-2$ forms on X over 2-dimensional families of cycles Z_s. The latter invariant is in

$$\frac{H^2(X,\mathbf{C}) \otimes H^1(S,\mathbf{C})}{F^2(H^2(X,\mathbf{C}) \otimes H^1(S,\mathbf{C})) + H^2(X,\mathbf{Q}) \otimes H^1(S,\mathbf{Q})}.$$

We note that the portion of this coming from $Hg^1(X) \otimes H^1(S)$ can be realized by taking a divisor Y on X representing the Hodge class in $Hg^1(X)$ cross a codimension 1 cycle on S, and hence comes from an ambiguity. Thus $H^2(X)/Hg^1(X) = 0$ implies both invariants of $Gr^2CH^2(X(k))_{\mathbf{Q}}$ are zero. It is worth noting that the geometry of S comes in–if S has no H^1 and no transcendental part of H^2, then $Gr^2CH^2(X(k))_{\mathbf{Q}} = 0$.

Lecture 5: The Tangent Space to Algebraic Cycles

This lecture is based on joint work with Phillip Griffiths [G-G2].

It was noted by Van der Kallen that there is a natural tangent space to algebraic K-theory, and that

$$T\mathcal{K}_p^{\text{Milnor}}(\mathcal{O}_{X(k)}) \cong \Omega^{p-1}_{X(k)/\mathbf{Q}}.$$

The map is

$$\{f_1, \dots, f_p\} \mapsto \frac{\dot{f}_1 df_2 \wedge \cdots \wedge df_p + \cdots + (-1)^{p-1} \dot{f}_p df_1 \wedge \cdots \wedge df_{p-1}}{f_1 f_2 \cdots f_p}.$$

This generalizes the statement for $p = 1$ that

$$T\mathcal{O}^*_{X(k)} \cong \mathcal{O}_{X(k)},$$

but the more exotic differentials over $\mathbf{Q}$ only manifest themselves once we reach $p \geq 2$. Bloch then derived from this the natural formula that

$$TH^p(\mathcal{K}_p(\mathcal{O}_{X(k)})) \cong H^p(\Omega^{p-1}_{X(k)/\mathbf{Q}}),$$

which thus is a formula for $TCH^p(X(k))$.

Example 1. $p = 1$

The formula above reduces to the classical formula

$$TCH^1(X(k)) \cong TH^1(\mathcal{O}^*_{X(k)}) \cong H^1(\mathcal{O}_{X(k)})$$

where the second map is induced by

$$(f_{\alpha\beta}) \mapsto (\frac{\dot{f}_{\alpha\beta}}{f_{\alpha\beta}}).$$

We have a filtration on differentials

$$F^m \Omega^r_{X(k)/\mathbf{Q}} = \mathrm{Im}(\Omega^m_{k/\mathbf{Q}} \otimes_k \Omega^{r-m}_{X(k)/\mathbf{Q}}) \to \Omega^r_{X(k)/\mathbf{Q}}.$$

We thus have

$$Gr^m \Omega^r_{X(k)/\mathbf{Q}} \cong \Omega^m_{k/\mathbf{Q}} \otimes_k \Omega^{r-m}_{X(k)/k}.$$

When X is smooth, there is a spectral sequence that computes the $H^*(\Omega^r_{X(k)/\mathbf{Q}})$ which degenerates at the E_2 term and has

$$E_2^{p,q} = H^p(\Omega^p_{k/\mathbf{Q}} \otimes_k H^{q-p}(\Omega^{r-p}_{X(k)/k}), \nabla).$$

This gives a natural filtration $F^m H^p(\Omega^{p-1}_{X(k)/\mathbf{Q}})$ with

$$Gr^m H^p(\Omega^{p-1}_{X(k)/\mathbf{Q}}) \cong H^m(\Omega^\bullet_{k/\mathbf{Q}} \otimes_k H^p(\Omega^{r-\bullet}_{X(k)/k}), \nabla).$$

Example 2. $p = 2$

There are two graded pieces to $H^2(\Omega^1_{X(k)/\mathbf{Q}})$:
$Gr^0H^2(\Omega^1_{X(k)/\mathbf{Q}}) = \ker(H^2(\Omega^1_{X(k)/k}) \xrightarrow{\nabla} \Omega^1_{k/\mathbf{Q}} \otimes_k H^3(\mathcal{O}_{X(k)}))$;
$Gr^1H^2(\Omega^1_{X(k)/\mathbf{Q}}) = \operatorname{coker}(H^1(\Omega^1_{X(k)/k}) \xrightarrow{\nabla} \Omega^1_{k/\mathbf{Q}} \otimes H^2(\mathcal{O}_{X(k)}))$.
We know that geometrically $/\mathbf{C}$, the Generalized Hodge Conjecture predicts that the image of AJ^2_X is precisely the part of $J^2(X)$ constructed from $H^3(X, \mathbf{C})_1$, i.e. the maximal weight 1 sub-Hodge structure of $H^3(X, \mathbf{C})$. The Absolute Hodge Conjecture implies that this is contained in the image of $\ker(H^2(\Omega^1_{X(k)/k}) \xrightarrow{\nabla} \Omega^1_{k/\mathbf{Q}} \otimes_k H^3(\mathcal{O}_{X(k)})) \otimes_{i_s} \mathbf{C}$. However, one does not expect that the two coincide, and indeed the tangent space to Chow groups is correct only formally and not geometrically. Note that $Gr^1H^2(\Omega^1_{X(k)/\mathbf{Q}})$ has dimension over k that grows linearly with the transcendence degree of k once $H^{2,0}(X) \neq 0$, which is in line with our expectations from Roitman's Theorem. This example was discussed by Esnault-Paranjape [E-P].

Geometrically, there are two problems with the tangent space formula for Chow groups. First order tangent vectors may fail to be part of an actual geometric family, and first order tangents to rational equivalences may fail to be part of an actual geometric family of rational equivalences. In order to understand this phenomenon better, we need to lift from tangent spaces to Chow groups to obtaining a formula for tangent spaces to algebraic cycles. The strategy adopted for doing this is to look at a kind of Zariski tangent space.

Example 3. *0-cycles on an algebraic curve*

Let X be an irreducible algebraic curve defined $/k$. If $\{U_\alpha\}$ is a k-Zariski cover of X, then a 0-cycle is defined by giving non-zero k-rational functions r_α on U_α whose ratios on overlaps $U_\alpha \cap U_\beta$ belong to $\mathcal{O}^*_{X(k)}$. Now $\dot{r}_\alpha / r_\alpha$ describes a tangent vector. We may define the sheaf of **principal parts** $\mathcal{PP}_{X(k)}$ to be given as the additive sheaf $\mathcal{M}_{X(k)}/\mathcal{O}_{X(k)}$, where $\mathcal{M}_{X(k)}$ is the sheaf of germs of k-rational functions. Then

$$TZ^1(X(k)) \cong H^0(X, \mathcal{PP}_{X(k)}).$$

From the exact sequence

$$0 \to \mathcal{O}_{X(k)} \to \mathcal{M}_{X(k)} \to \mathcal{PP}_{X(k)} \to 0$$

we get a natural map

$$H^0(X, \mathcal{PP}_{X(k)}) \to H^1(\mathcal{O}_{X(k)})$$

which we may think of as a map

$$TZ^1(X(k)) \to TCH^1(X(k)).$$

Because $\mathcal{M}_{X(k)}$ is flasque, this map is surjective. We may think of the exact sequence above as the tangent exact sequence to

$$0 \to \mathcal{O}^*_{X(k)} \to \mathcal{M}^*_{X(k)} \to \mathcal{D}_{X(k)} \to 0,$$

where $\mathcal{D}_{X(k)}$ is the sheaf of k-divisors on X.

What is of course missing in this discussion is the **exponential sheaf sequence**

$$0 \to \mathbf{Z} \to \mathcal{O}_{X(\mathbf{C})} \to \mathcal{O}^*_{X(\mathbf{C})} \to 0.$$

This leaves the algebraic category in two ways–we need to use the classical topology rather than the Zariski topology, and we need to use the transcendental function $f \mapsto \exp(2\pi i f)$. Once we have this, we get the exact sequence

$$0 \to J^1(X(\mathbf{C})) \to CH^1(X(\mathbf{C})) \to Hg^1(X(\mathbf{C})) \to 0$$

which completely solves the problem of describing $CH^1(X(\mathbf{C}))$ in the analytic category. One of the enduringly nice features of taking derivatives is that transcendental maps in algebraic geometry frequently have algebraic derivatives.

Once we pass to higher codimension, we no longer have the exponential sheaf sequence, and we do not really know what the right transcendental functions to invoke are–in some cases, these turn out to involve **polylogarithms**.

Example 4. $K_2^{\mathrm{Milnor}}(k)$

Here we have

$$TK_2^{\mathrm{Milnor}}(k) \cong \Omega^1_{k/\mathbf{Q}}.$$

Unfortunately, this is deceptively simple. Consider, for example, the family of elements

$$t \mapsto \{a, t\},$$

where $a \in k$. The derivative is $da/a \in \Omega^1_{k/\mathbf{Q}}$. If $a \in \bar{\mathbf{Q}}$, then $da = 0$ and thus the derivative of this map vanishes identically. However, it is known that this map is actually constant if $\mathrm{tr}\deg(k) > 0$ only when a is a root of unity. This is an example of what we call a **null curve**, one whose formal derivative is identically zero but which is not constant. The problem is that $TK_2^{\mathrm{Milnor}}(k)$ is really a quotient of two tangent spaces–to the space of possible products of Steinberg symbols, and to the space of Steinberg relations. If we are unable to integrate a tangent vector in the space of Steinberg relations up to a geometric family of Steinberg relations, we might expect this to produce a null curve.

Example 5. *0-cycles on a surface*

This turns out to already embody many of the complexities of tangent spaces to cycles. The answer is

$$TZ^2(X(k)) \cong \oplus_{|Z|} H^2_{|Z|}(\Omega^1_{X(k)/\mathbf{Q}}),$$

where the sum is over supports of irreducible codimension 2 k-subvarieties of X. This involves local cohomology. There is a natural map

$$H^2_{|Z|}(\Omega^1_{X(k)/\mathbf{Q}}) \to H^2(\Omega^1_{X(k)/\mathbf{Q}})$$

which we may interpret as giving us a map

$$TZ^2(X(k)) \to TCH^2(X(k)).$$

These are quite complicated objects. For example, if we work over $\mathbf{C}$, then the tangent space to 0-cycles supported at a point $x \in X$ is

$$TZ^2(X(\mathbf{C}))_x \cong \lim_{|Z|=x} Ext^2_{\mathcal{O}_{X(\mathbf{C})}}(\mathcal{O}_Z, \Omega^1_{X(\mathbf{C})/\mathbf{Q}}).$$

An example of the simplest family where the distinction between differentials over k and differentials over $\mathbf{Q}$ comes in is the family

$$Z(t) = \mathrm{Var}(x^2 - \alpha y^2, xy - t),$$

where $\alpha \in \mathbf{C}^*$ is transcendental.

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