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Semistable reduction and vanishing theorems revisited, after K.-W. Lan and J. Suh

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Semistable reduction and vanishing theorems revisited, after K.-W. Lan and J. Suh

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In memory of Eckart Viehweg

(12.30.1948 - 1.30.2010)

"We want to draw attention to improvements of vanishing theorems and to properties of direct images of certain sheaves which should be of some interest outside of birational geometry as well."
(E. Viehweg, ICM Berkeley, 1986)

PLAN

1. Review of some classical vanishing theorems

2. Suh's theorem

3. Esnault-Viehweg's cyclic covers revisited

4. Residues

5. Proof of Suh's theorem

6. Applications to Shimura varieties

1. REVIEW OF SOME CLASSICAL VANISHING THEOREMS

(a) Absolute vanishing

THEOREM 1 (Kodaira-Akizuki-Nakano, 1954)

X/k projective, smooth, $\dim(X) = d$, $\mathrm{char}(k) = 0$

L ample line bundle on X. Then :

(1) $\quad H^j(X, L \otimes \Omega^i_{X/k}) = 0 \quad i + j > d$

(2) $\quad H^j(X, L^{-1} \otimes \Omega^i_{X/k}) = 0 \quad i + j < d$

NB. (1) $\Leftrightarrow$ (2) (Serre duality)

THEOREM 2 (Deligne-I, 1987)

k perfect, $\mathrm{char}(k) = p > 0$

X/k projective, smooth, $D \subset X$ sncd

$\dim(X) = d \leq p$

$(X, D)/k$ liftable to $W_2(k)$. Then :

$$\oplus \Omega^i_{X'/k}(\log D')[-i] \xrightarrow{\sim} F_* \Omega^{\cdot}_{X/k}(\log D)$$

in $D(X')$, inducing C^{-1} on $\mathcal{H}^i$

$(F : X \to X'$ relative Frobenius$)$

projection formula $\Rightarrow$

COROLLARY 1 (Raynaud)

If $L =$ line bundle on X, then :

$$\sum_{i+j=n} h^j(X, L \otimes \Omega^i_{X/k}(\log D)) \leq \sum_{i+j=n} h^j(X, L^p \otimes \Omega^i_{X/k}(\log D))$$

COROLLARY 2

If L ample, then :

(1) $\quad H^j(X, L \otimes \Omega^i_{X/k}(\log D)) = 0 \;\; i + j > d$

(2) $\quad H^j(X, L^{-1} \otimes \Omega^i_{X/k}(\log D)) = 0 \;\; i + j < d$

Remarks.

- $\Omega^d_{X/k} = \Omega^d_{X/k}(\log D)(-D) \Rightarrow$ (1), (2) not dual :

(1) dual to

(1') $H^j(X, L^{-1} \otimes \Omega^i_{X/k}(\log D)(-D)) = 0 \quad i+j < d$

(2) dual to

(2') $H^j(X, L \otimes \Omega^i_{X/k}(\log D)(-D)) = 0 \quad i+j > d$

- Cor 2 $\Rightarrow$ (1), (2) hold for L ample,

with no restriction on (X, D) if $\mathrm{char}(k) = 0$

Variants :

Kawamata-Viehweg, Esnault-Viehweg, et al.

Recall :

L nef : $\deg(L|C) \geq 0 \; \forall$ curve $C \subset X$

L big : $\kappa(L) = \dim(X)$

$\kappa(L) = -\infty$ or $\mathrm{tr.deg}(\oplus_{m \geq 0} H^0(X, L^m)) - 1$

(Iitaka dimension of L)

THEOREM 3 (Kawamata-Viehweg, 1982)

X/k projective, smooth, $\dim(X) = d$, $\mathrm{char}(k) = 0$, L nef and big. Then :

$$H^j(X, L^{-1}) = 0, \ \ 0 \le j < d.$$

Remarks.

- nef and big $\Leftrightarrow$ nef and $c_1(L)^d > 0$

- For $D = \emptyset$, KAN vanishing (1), (2) OK if $d \leq 2$ (even in char. $p > 0$, X liftable mod p^2 (Deligne-I)) not OK for $d > 2$

- nef and big $\Rightarrow \exists \nu_0 \geq 0$, effective D, s. t.

$$L^\nu(-D) \text{ ample } \forall \nu \geq \nu_0.$$

THEOREM 4 (Viehweg, Esnault-Viehweg, 1986, 1992)

X/k projective, smooth, $\dim(X) = d$;

$D \subset X$ sncd ; L line bundle on X

Assume :

(*) $\exists$ effective D', $\operatorname{supp}(D') \subset D$, and $\nu_0 \geq 0$

s. t. $L^\nu(-D')$ ample $\forall \nu \geq \nu_0$

Then : If char$(k) = 0$ or k perfect, char$(k) = p > 0$, $d \leq p$, and (X, D), and L lift to $W_2(k)$, then :

(2) $\quad H^j(X, L^{-1} \otimes \Omega^i_X(\log D)) = 0 \ i + j < d.$

Remarks

- Note liftability assumption on L

- ample $\Rightarrow (*) \Rightarrow$ nef and big

- example of $(*)$: $L = \pi^*$(ample),

$\pi : X = \mathsf{Bl}_I(Y) \to Y, \ I.\mathcal{O}_X = \mathcal{O}(-D')$

- (1) may fail (Suh)

(b) Relative vanishing

Grauert-Riemenschneider, Kollár, Esnault-Viehweg, ...

THEOREM 5 (Kollár, 1986)

$\text{char}(k) = 0$, X/k, Y/k projective,

X smooth, $\dim(X) = d$, $\omega_X := \Omega^d_{X/k}$

L ample line bundle on Y

$f : X \to Y$ surjective. Then :

(i) $R^i f_* \omega_X$ torsionfree $\forall i \geq 0$

(ii) $H^j(Y, L \otimes R^i f_* \omega_X) = 0 \ \forall j > 0, \forall i \geq 0$.

f alteration (= generically finite) $\Rightarrow$

$$R^i f_* \omega_X = 0 \; \forall i > 0,$$

and, if f birational, for L nef and big,

$$H^j(Y, L \otimes f_* \omega_X) = 0 \; \forall j > 0$$

(Grauert-Riemenschneider)

The semistable reduction case

$\Rightarrow$ KAN type results

$\mathrm{char}(k)$ arbitrary ; X/k, Y/k proper, smooth

$E = \sum E_i \subset Y$: sncd

$f : X \to Y$; $D := f^{-1}(E)$

Assume : $f : (X, D) \to (Y, E)$ semistable along E

(étale locally on X : $f =$ external product of

copies of $x_1 \cdots x_r = t$)

($\Rightarrow D \subset X =$ ncd, f flat, smooth $/ Y - E$)

$\Omega^{\cdot}_{X/Y}(\log(D/E))$: relative log de Rham complex

$H := \oplus_i R^i f_*(\Omega^{\cdot}_{X/Y}(\log(D/E)))$

$\nabla : H \to \Omega^1_{Y/k}(\log E) \otimes H$: Gauss-Manin connection

$\Omega^{\cdot}_{Y/k}(\log E)(H) := (H \to \Omega^1_{Y/k}(\log E) \otimes H \to \cdots)$:

log DR complex of H, with Hodge filtration

$F^i \Omega^{\cdot}_{Y/k}(\log E)(H) =$

$(F^i H \to F^{i-1} H \otimes \Omega^1_{Y/k}(\log E) \to \cdots)$

(Griffiths transversality)

$$K = \oplus_i \mathrm{gr}^i \Omega^{\cdot}_{Y/k}(\log E)(H)$$

(total) log Kodaira-Spencer complex of H :

$$K = (\mathrm{gr}^{\cdot} H \rightarrow \mathrm{gr}^{\cdot -1} H \otimes \Omega^1_{Y/k}(\log E) \rightarrow \cdots)$$

Note : K is $\mathcal{O}_Y$-linear

THEOREM 6 (I., 1990)

$\dim(Y) = e$, $\dim(X) = d$, k perfect, $\mathsf{char}(k) = p > 0$,

$d < p$, $(X, D) \to (Y, E)$ lifts to $W_2(k)$.

Then :

(i) $H = \oplus R^q f_*(\Omega^\cdot_{X/Y}(\log(D/E)))$, $R^j f_* \Omega^i_{X/Y}(\log(D/E))$

locally free of finite type,

$E_1^{ij} = R^j f_* \Omega^i_{X/Y}(\log(D/E)) \Rightarrow R^{i+j} f_* \Omega^\cdot_{X/Y}(\log(D/E))$

degenerates at E_1

(ii)

$$K_{Y_1} \xrightarrow{\sim} F_*\Omega^{\cdot}_{Y/k}(\log E)(H)$$

in $D(Y_1)$

($F : Y \to Y_1 =$ relative Frobenius,

$K = \oplus_i \mathrm{gr}^i \Omega^{\cdot}_{Y/k}(\log E)(H) =$ Kodaira-Spencer com-

plex)

COROLLARY 1

If $L = $ line bundle on Y, then :

$$h^m(Y, L \otimes K) \leq h^m(Y, L^p \otimes K) \ \forall m.$$

COROLLARY 2

L ample. Then :

(1) $H^m(Y, L \otimes K) = 0$ for $m > e$

(2) $H^m(Y, L^{-1} \otimes K) = 0$ for $m < e$

Remarks.

- For char(k) $= 0$, Th. 6 (i) and Cor. 2 hold without restriction of dimension.

- K self-dual up to shift and twist :

$R\mathcal{H}om(K, \Omega_{Y/k}^e[e]) = K[2e](-E)$

$\Rightarrow$ (1) dual to

(1') $H^m(Y, L^{-1}(-E) \otimes K) = 0$ for $m < e$

(2) dual to

(2') $H^m(Y, L(-E) \otimes K) = 0$ for $m > e$

- (2') (for $\mathrm{gr}^d \subset K$) $\Rightarrow H^j(Y, L \otimes R^k f_* \omega_X) = 0$

$\forall j > 0, \forall k \geq 0$ (cf. th. 4 (Kollár))

- In char. 0, local freeness, E_1 degeneration hold for more general "log smooth" maps (Steenbrink (1976), ..., I-Kato-Nakayama (2005))
- Variants for "$F - T$-crystals" (Ogus, 1994)

2. SUH'S THEOREM

Common generalization of th. 4 (Esnault-Viehweg) and (2) of th. 6

THEOREM 7. (Suh, 2010)

$f : (X, D) \to (Y, E)$ as in th. 6 (proper, semistable), $\dim(X) = d$; $\dim(Y) = e$;

k perfect, $\operatorname{char}(k) = p > 0$, $d < p$,

$L =$ line bundle on Y.

Assume :

(*) $\exists$ effective E', $\text{Supp}(E') \subset E$, and $\nu_0 \geq 0$

s. t. $L^\nu(-E')$ ample $\forall \nu \geq \nu_0$,

$(X, D) \rightarrow (Y, E)$, and L lift to $W_2(k)$.

Then :

(2) $H^m(Y, L^{-1} \otimes K) = 0$ for $m < e$

Remarks.

- For $\mathrm{char}(k) = 0$, (2) holds without restriction of dimension

- For $\mathrm{gr}^d \subset K$, th. 7 $\Rightarrow$:

$$H^j(Y, L \otimes R^i f_* \omega_X) = 0 \ \forall j > 0, \forall i \geq 0.$$

Ingredients of proof

- induction on $e = \dim(Y)$ reduces to

vanishing for integral parts of $\mathbf{Q}$-divisors $L^{(i)}$

sitting between L and ample $L^\nu(-E' + E'_{\mathrm{red}})$, $\nu >> 0$

- desired vanishing proved by

Esnault-Viehweg's method :

Frobenius interpolation, using properties of residues,

for H (Gauss-Manin) and the $L^{(i)}$'s (cyclic covers)

3. ESNAULT-VIEHWEG'S CYCLIC COVERS REVISITED

Y/k smooth, $E' = \sum_{1 \leq i \leq r} a_i E_i$, $a_i \geq 0$,

$E = \sum_{1 \leq i \leq r} E_i$ sncd ;

$N \geq 1$ invertible in k ; assume $\mu_N \subset k$

L line bundle on Y s. t. $L^N = \mathcal{O}_Y(E')$.

Esnault-Viehweg : $(Y, E', L, N) \mapsto \mu_N$-cover

$$g : C = C(L, N, E') \to Y$$

ramified along E : C = normalization of Spec$A$,

$A = \mathcal{O}_Y \oplus L^{-1} \oplus \cdots \oplus L^{-(N-1)}$, $L^{-N} = \mathcal{O}_Y(-E') \hookrightarrow \mathcal{O}_Y$.

μ_N acts on C via action of $\mu_N \subset \mathcal{O}^*$ on L

Properties

- g finite, flat, Galois étale $/Y - E$ of group μ_N ;

$C = $ normalization of Y in $C | Y - E$

- Put log structure on Y defined by E. Then :

$\exists$ unique log structure M on C s. t.

$(C, M) \rightarrow (Y, E) = \mu_N$-Kummer étale cover of Y extending $C | Y - E$

locally on Y :

$C \rightarrow Y = $ pull-back of $\mathrm{Spec}\mathbf{Z}[P] \rightarrow \mathrm{Spec}\mathbf{Z}[\mathbf{N}^r]$

where P = saturated amalgamated sum :

$$
\begin{array}{ccc}
\mathbf{N} & \longrightarrow & P \\
\uparrow & & \uparrow \\
\mathbf{N} & \longrightarrow & \mathbf{N}^r
\end{array} \;,
$$

$\mathbf{N} \to \mathbf{N}$ by $x \mapsto Nx$, $\mathbf{N} \to \mathbf{N}^r$ by $x \mapsto (a_1 x, \cdots, a_r x)$.

- μ_N-equivariant decomposition into eigen bundles :

$g_* \mathcal{O}_C = \oplus_{0 \leq i \leq N-1} (L^{(i)})^{-1}$

$L^{(i)} := L^i \otimes \mathcal{O}_Y(-[iE'/N])$, $L^{(1)} = L$

action of μ_N on $L^{(i)}$ via χ^i,

$\chi : \mu_N \hookrightarrow \mathcal{O}^*$ canonical character :

$(L^{(i)})^{-1} = g_* \mathcal{O}_C(\chi^{-i})$.

g log étale $\Rightarrow$

$g^* \Omega^1_{Y/k}(\log E) = \Omega^1_{C/k}(\log M)$

$g_* \Omega^{\cdot}_{C/k}(\log M) = \Omega^{\cdot}_{Y/k}(\log E)(g_* \mathcal{O}_Y)$

$= (g_* \mathcal{O}_C \to \Omega^1_{Y/k}(\log E) \otimes g_* \mathcal{O}_C \to \cdots)$

$g_* \mathcal{O}_C$ has μ_N-equivariant integrable log connection :

$$\nabla = \oplus \nabla_i : \oplus (L^{(i)})^{-1} \to \oplus \Omega^1_{Y/k}(\log E) \otimes (L^{(i)})^{-1},$$

4. RESIDUES

Y/k smooth, $E = \sum_{1 \leq i \leq r} E_i$ sncd

L vector bundle on Y, with log connection

$$\nabla : L \to \Omega^1_{Y/k}(\log E) \otimes L.$$

Recall :

$$0 \to \Omega^1_{Y/k} \to \Omega^1_{Y/k}(\log E) \to \oplus_i \mathcal{O}_{E_i} \to 0$$

Residue of (L, ∇) along E_i :

$$\mathsf{Res}_{E_i}(\nabla) \in \mathsf{End}_{\mathcal{O}_{E_i}}(\mathcal{O}_{E_i} \otimes L)$$

defined as composition

$$\mathsf{Res}_{E_i}(\nabla) : L \to \Omega^1_{Y/k}(\log E) \otimes L \to \mathcal{O}_{E_i} \otimes L.$$

Example 1 : cyclic covers

Y/k smooth, $E' = \sum a_i E_i$, $E = \sum E_i$ sncd,

L /Y, $L^N = \mathcal{O}_Y(E')$ $(N \geq 1)$,

$g : C = C(L, E', N) \to Y$

Esnault-Viehweg cyclic cover

$\nabla_i : (L^{(i)})^{-1} \to \Omega^1_{Y/k}(\log E) \otimes (L^{(i)})^{-1}$

local calculation $\Rightarrow$

Proposition 1.(Esnault-Viehweg)

$$\mathrm{Res}_{E_j}(\nabla_i) = (ia_j/N - [ia_j/N]).Id$$

Example 2 : semistable reduction

$f : (X, D) \to (Y, E)$ as in th. 5 :

X/k, Y/k proper smooth,

D, E sncd, $D = f^{-1}(E)$, semistable reduction

either char$(k) = 0$, or k perfect, char$(k) = p > 0$,

$f : (X, D) \to (Y, E)$ liftable to $W_2(k)$, and dim$(X) < p$

$H = \oplus_i R^i f_* \Omega^{\cdot}_{X/Y}(\log(D/E))$, a vector bundle on Y,

Gauss-Manin connection $\nabla : H \to \Omega^1_{Y/k}(\log E) \otimes H$.

Proposition 2. (Katz, 1970)

$\forall i$, $\mathrm{Res}_{E_i}(\nabla)$ is nilpotent.

Remarks

● Katz works in char. 0, but char. 0 not used if semistable reduction :

(1) reduce to local statement on Y, with E smooth, $E = V(t)$

(2) calculate $Rf_*\Omega^{\cdot}_{X/Y}$ as $C := f_*\check{\mathcal{C}}(\mathcal{U}, \Omega^{\cdot}_{X/Y})$ for suitable Čech cover $\mathcal{U}$

use Katz-Oda's lifts $\mathrm{Ani}(d)_i$ $(1 \leq i \leq r)$ of $\nabla(d)$ to C

adapted to the D_i's

$(d = t\,d/dt$, "Ani" = elder brother$)$,

$\prod_i \mathrm{Res}_E(\mathrm{Ani}(d)_i) = 0$ on E_1 term of spectral sequence

• alternate proof using Cartier isomorphism (works for

"log smooth, saturated" morphisms)

5. PROOF OF SUH'S THEOREM

Recall : $f : (X, D) \to (Y, E)$, $\dim(X) < p$,

$L^\nu(-E')$ ample $\forall \nu \geq \nu_0$, $E'_{\mathrm{red}} \subset E$

$f : (X, D) \to (Y, E)$, L lift to

$\tilde{f} : (\tilde{X}, \tilde{D}) \to (\tilde{Y}, \tilde{E})$, $\tilde{L}$ / $W_2(k)$

$H = \oplus R^i f_* \Omega^{\cdot}_{X/Y}(\log(D/E))$

$K = $ KS-complex of H

Have to show (2) :

$$H^m(Y, L^{-1} \otimes K) = 0 \text{ for } m < e = \dim(Y)$$

Recall : (2) known if L ample (th. 6 (I.))

Induction on $e = \dim(Y)$; WMA $k = \overline{k}$

Step 1 : Use of a hyperplane section

WMA $L^\nu(-E')$, $L^\nu(-E' + E'_{\text{red}})$ very ample $\forall \nu \geq \nu_0$
(Esnault-Viehweg, uses th. 2, Cor. 2 (D-I-R vanishing))

Choose $s \geq 1$ s. t.

$$N = p^s + 1 > \nu_0,$$

and $N > c_i$ $\forall i$ if $E' = \sum c_i E_i$ ($\Rightarrow [E'/N] = 0$)

Take

$$t \in H^0(\tilde{Y}, \tilde{L}^N(-\tilde{E}'))$$

s. t. $\tilde{Z} := V(t)$ smooth $/W_2(k)$,

$\tilde{E} + \tilde{Z}$, $\tilde{D} + \tilde{f}^{-1}(\tilde{Z})$ sncd $/W_2(k)$

Thus :

$$L^N = \mathcal{O}_Y(E' + Z),$$

$$\Omega^{\cdot}_{X/Y}(\log(D/E)) = \Omega^{\cdot}_{X/Y}(\log((D + f^{-1}Z)/(E + Z))).$$

Local freeness, base change compatibility of
$R^q f_*(\Omega^{\cdot}_{X/Y}(\log(-/-)))$ and $R^j f_*(\Omega^i_{X/Y}(\log(-/-))$ (th.
5) $\Rightarrow$

$0 \to \mathrm{gr}^{\cdot}(\Omega^{\cdot}_{Y/k}(\log E) \otimes H) \to \mathrm{gr}^{\cdot}(\Omega^{\cdot}_{Y/k}(\log E + Z) \otimes H)$
$\to \mathrm{gr}^{\cdot -1}(\Omega^{\cdot}_{Z/k}(\log(E \cap Z)) \otimes H)[-1] \to 0.$

Inductive hypothesis $\Rightarrow$ enough to show :

$(*)_1 \quad H^m(Y, L^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H) = 0$

for $m < e.$

Step 2 : Enters cyclic cover

$$g : C := C(L, E' + Z, N) \rightarrow Y$$

cyclic cover associated with $L^N = \mathcal{O}_Y(E' + Z)$,

$g_* \mathcal{O}_C = \oplus (L^{(i)})^{-1}$, $L^{(i)} = L^i(-[i(E' + Z)/N])$.

$L^{(1)} = L$

$L^{(N-1)} = L^{N-1}(-E' + E'_{\text{red}})$

Recall : $N - 1 = p^s \geq \nu_0$,

$L^{(p^s)} = L^{p^s}(-E' + E'_{\mathsf{red}})$ ample

$\Rightarrow$ we know (th. 5) that, for $m < e$,

$$H^m(Y, (L^{(p^s)})^{-1} \otimes \mathsf{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H)) = 0$$

Want to show :

$(*)_1 \quad H^m(Y, L^{-1} \otimes \mathsf{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H)) = 0$

Will show by descending induction $(i = s, \cdots, 0)$

$(*)_i \quad H^m(Y, (L^{(p^i)})^{-1} \otimes \mathsf{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H)) = 0$

Step 3 : Frobenius interpolation

$(*)_{i+1} \Rightarrow (*)_i$ follows from

Key lemma.

For $0 < a < pa < N$, $m \geq 0$,

$\dim H^m(Y, (L^{(a)})^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H))$

$\leq \dim H^m(Y, (L^{(pa)})^{-1} \otimes \mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H)).$

Proof.

$F : Y \to Y_1$: relative Frobenius ($/k$)

th. 5 $\Rightarrow$

$F_*(\Omega^{\cdot}_{Y/k}(\log(E + Z)) \otimes H)$

$= \mathrm{gr}(\Omega^{\cdot}_{Y_1/k}(\log(E_1 + Z_1)) \otimes H_1)$

$\Rightarrow$ (projection formula)

$H^m(Y_1, (L_1^{(a)})^{-1} \otimes K_1) = H^m(Y, F^*((L_1^{(a)})^{-1}) \otimes \Omega \otimes H)$

$(K_1 := \mathrm{gr}(\Omega^{\cdot}_{Y_1/k}(\log(E_1 + Z_1)) \otimes H_1),$

$\Omega := \Omega^{\cdot}_{Y/k}(\log(E + Z))$ for short)

Key point

The inclusion :

$$F^*((L_1^{(a)})^{-1}) = L^{-pa}(p[a(E' + Z)/N]) \hookrightarrow (L^{(pa)})^{-1}$$

(i) is compatible with connections $1 \otimes d_{Y/k}$ and ∇_{pa}

(ii) induces quasi-isomorphism

$$F^*((L_1^{(a)})^{-1}) \otimes \Omega \otimes H \to (L^{(pa)})^{-1} \otimes \Omega \otimes H$$

Key point $\Rightarrow$

$$H^m(Y, F^*((L_1^{(a)})^{-1}) \otimes \Omega \otimes H)$$

$$\xrightarrow{\sim} H^m(Y, (L^{(pa)})^{-1} \otimes \Omega \otimes H)$$

$\Rightarrow$ key lemma

(as

$$\dim H^m(Y, (L^{(pa)})^{-1} \otimes \Omega \otimes H) \leq \dim H^m(Y, (L^{(pa)})^{-1} \otimes$$

$$\text{gr}(\Omega^{\cdot}_{Y/k}(\log(E+Z)) \otimes H))$$

Proof of key point

(i) (Esnault-Viehweg) : seen on Frobenius diagram :

$$
\begin{array}{ccccc}
C_1 & \longleftarrow & C' & \overset{F}{\longleftarrow} & C \\
{\scriptstyle g}\big\downarrow & & {\scriptstyle g'}\big\downarrow & & \swarrow{\scriptstyle g} \\
Y_1 & \underset{F}{\longleftarrow} & Y & &
\end{array}
$$

with log étale vertical maps :

inclusion $= (g'_* \mathcal{O}_{C'}(\chi^{-pa}) \hookrightarrow g_* \mathcal{O}_C(\chi^{-pa}))$

(ii) (core of the proof) :

$$F^*((L_1^{(a)})^{-1}) \otimes H = (L^{(pa)})^{-1}(-B) \otimes H$$

$$\hookrightarrow (L^{(pa)})^{-1} \otimes H,$$

$$(B = [pa(E' + Z)/N] - p[a(E' + Z)/N] = \sum b_i E_i,$$

$$b_i = [pac_i/N] - p[ac_i/N], \ E' = \sum c_i E_i)$$

Look at residues :

$$R_i = \mathrm{Res}_{E_i}(H) \text{ nilpotent (Prop. 2) (Katz)}$$

$$S_i := \mathrm{Res}_{E_i}((L^{(pa)})^{-1} \otimes H) = -b_i \otimes Id + Id \otimes R_i$$

(Prop. 1)(Esnault-Viehweg)

$$b_i \neq 0 \Rightarrow 0 < b_i < p \Rightarrow S_i \text{ invertible}$$

$\Rightarrow$ (by Esnault-Viehweg's lemma below)

$$\Omega(-B_i) \otimes (L^{(pa)})^{-1} \otimes H \rightarrow \Omega \otimes (L^{(pa)})^{-1} \otimes H$$

= quasi-isomorphism

$$\Rightarrow F^*((L_1^{(a)})^{-1}) \otimes \Omega \otimes H \rightarrow (L^{(pa)})^{-1} \otimes \Omega \otimes H$$

= quasi-isomorphism

Lemma (Esnault-Viehweg)

X/k smooth, $D = D_1 + \cdots + D_r$ ncd on X,

$\nabla : V \rightarrow \Omega^1_{X/k}(\log D) \otimes V$

vector bundle with integrable log connection .

Assume :

$\mathrm{Res}_{D_1}(\nabla) : V \otimes \mathcal{O}_{D_1} \rightarrow V \otimes \mathcal{O}_{D_1} =$ isomorphism.

Then, for $a \geq 0$:

$\Omega^{\cdot}_{X/k}(\log D)(-aD_1) \otimes V \rightarrow \Omega^{\cdot}_{X/k}(\log D) \otimes V$

$=$ quasi-isomorphism.

Variants and generalizations

(needed for applications to Shimura varieties)

$f : (X, M) \to (Y, E)$ proper, log smooth, integral,

Assume f, L liftable to $W(k)$, L satisfying (*), and :

(a) Hdg $\Rightarrow$ DR(f) degenerates at E_1,

E_1^{ij} loc. free of f. t., base change compatible

(b) $\mathrm{Res}_{E_i}(\nabla)(H)$ nilpotent ($H : Rf_*(\Omega^{\cdot}_{X/Y}(\log(M/E))$

(c) $\mathrm{gr}(\Omega^{\cdot}_{Y/k}(\log E)(\mathcal{H}^q))_1 \xrightarrow{\sim} F_*\Omega^{\cdot}_{Y/k}(\log E)(\mathcal{H}^q))$,

$F : Y \to Y_1$, $\mathcal{H}^q = R^q f_* \Omega^{\cdot}_{X/Y}(\log M/E)$, $q + e < p$

Then : Suh's vanishing (2) holds for $H = \mathcal{H}^q$.

6. APPLICATIONS TO SHIMURA VARIETIES, AFTER K.-W. LAN AND J. SUH

6.1. (Rough) goal :

Given Shimura variety, PEL type, dimension d

$$(\mathrm{Sh}_H \otimes_{F_0} \mathbf{C})^{\mathsf{an}} = G(\mathbf{Q}) \backslash \mathcal{X} \times G(\mathbf{A}^f)/H$$

with corresponding moduli space $M = M_H$,

$V = V_\mu$ a Betti $\mathbf{Z}$-local system on $M_{\mathbf{C}}$

(or ℓ-adic or de Rham variants on integral models)

associated with irreducible representation of G

of highest weight μ

(e. g. $V = Sym^k R^1 f_* \mathbf{Z}$, f univ. ell. curve)

and good prime p (unramifiedness),

with suitable restrictions on :

level H (neat, prime to p)

weight μ (regularity, smallness)

(e. g. Sym^k, $\mu = k$, $k + 1 < p$)

get vanishing and p-torsionfreeness

- $H^i(M_{\mathbf{C}}, V) = 0$ for $i \neq e$ ($e = \dim M_{\mathbf{C}}$)

- $H^e(M_{\mathbf{C}}, V)$ p-torsionfree

(and DR, ℓ-adic variants)

6.2. The geometric set-up

Given integral PEL datum $D = (\mathcal{B}, *, L, \langle, \rangle, h_0)$,

$h_0 : \mathbf{C} \to \mathsf{End}_{\mathcal{B} \otimes_{\mathbf{Z}} \mathbf{R}}(L \otimes_{\mathbf{Z}} \mathbf{R})$

with associated reductive group G, reflex field F_0,

good prime p, (unramified in $\mathcal{B}$, $\langle, \rangle \otimes \mathbf{Z}_p$ self-dual)

neat, prime to p level H

get smooth, quasi-projective moduli scheme

$$M_H / S_0$$

($S_0 =$ localization at p of ring of integers of F_0)

($M_H = \{A/S + \text{PEL structure of type } (D, H)\}$)

and compactifications : minimal (Satake-Baily-Borel),
toroidal (Chai-Faltings et al.)

$$
\begin{array}{ccc}
A & \subset & A^{\mathrm{tor}} \ , \\
\downarrow & & \downarrow \\
M_H & \subset & M_{H,\Sigma}^{\mathrm{tor}} \\
& & \downarrow \pi \\
& & M_H^{\mathrm{min}}
\end{array}
$$

$Y = M_{H,\Sigma}^{\mathrm{tor}}$ proper, smooth $/S_0$,

$E = M_{H,\Sigma}^{\mathrm{tor}} - M_H$ sncd$/S_0$

A universal abelian scheme,

A^{tor} toroidal compactification of A

basic automorphic line bundle on Y

$$\omega := \det(e^*\Omega^1_{\tilde{A}/Y})$$

($\tilde{A}$ semi-abelian extension of A, acts on A^{tor})

ω not ample in general

($= \pi^*$(ample line bundle on M^{min}),

$\pi : M^{\mathsf{tor}} \to M^{\mathsf{min}} =$ normalized blow-up of I,

$I.\mathcal{O}_Y = \mathcal{O}_Y(-E')$, $E'_{\mathsf{red}} \subset E$)

but satisfies Esnault-Viehweg condition (*) :

$\exists \nu_0 \geq 0$ s. t. $\omega^\nu(-E')$ ample $\forall \nu \geq \nu_0$

final adjustments :

- replace M, M^{tor}

by schematic closure of Sh_H $(\hookrightarrow M \otimes F_0)$ in M, M^{tor},

- pull-back to suitable $S = \mathsf{Spec}\, W(k)/S_0$,

k perfect, char(k) = p

- keep same notations : $A^{\mathrm{tor}} \to M^{\mathrm{tor}}$, $E \subset M^{\mathrm{tor}}$.

6.3. The compact case

Assume $(\mathrm{Sh}_H \otimes_{F_0} \mathbf{C})^{\mathsf{an}}$ compact.

Then :

- $Y = M = M^{\mathsf{tor}}$ projective, smooth $/S$, $E = \emptyset$

- $f : A = A^{\mathsf{tor}} \to Y = $ abelian scheme

- ω ample

vanishing th. 6 (I.) applied to $f_n \otimes k$,

for $f_n : A^n \to Y$ $(n \geq 1)$ and $L = \omega \Rightarrow$

THEOREM 8 (K.-W. Lan, J. Suh, 2010)

$\mathcal{V} =$ flat bundle $/Y = M_H$ associated with irreducible representation $G \to GL(V)$, highest weight μ assumed to be p-small, sufficiently regular.

Then :

- $H^i_{dR}(Y, \breve{\mathcal{V}}) = 0$ for $i \neq e$, $(e = \dim(Y/S) = \dim(\mathsf{Sh}_H))$
- $H^e_{dR}(Y, \breve{\mathcal{V}})$ free, f. t. $/\mathcal{O}_S$.

Remark. Conditions on μ independent of H ; "small"

includes $|\mu| + e < p$

Using C_{cris}, gives :

COROLLARY (Lan-Suh)

$\mathcal{V}_{\mathbf{C}} =$ lisse $\mathbf{Z}$-sheaf on $Y_{\mathbf{C}}$ associated with μ

Then :

(i) $H^i(Y_{\mathbf{C}}, \check{\mathcal{V}}_{\mathbf{C}}) = 0$ for $i \neq e$,

(ii) $H^e(Y_{\mathbf{C}}, \check{\mathcal{V}}_{\mathbf{C}})$ p-torsion free

NB. (i) $\Rightarrow$ Faltings's theorem (1982)

6.4. The non-compact case

Apply variants of vanishing th. 7 (Suh)

to $f_n : X \to Y$,

(suitable log smooth compactification of $A^n \to Y$)

and suitable root of ω.

Get analogues of Th. 8 and corollary for

interior cohomology $\mathrm{Im} H_c^i \to H^i$.

Note : In general, no semistable model f_n exists.)

But variants of vanishing (2) apply :

• log smooth, integral models f_n exist (Lan),

local freeness of $\mathcal{H}^*_{DR}$, etc. OK (Lan)

(Chai-Faltings in Siegel case)

$\Rightarrow$ (a) OK

• Res$(\nabla)(H)$ nilpotent (Lan-Suh) $\Rightarrow$ (b) OK

• Ogus's th. on F-T-crystals

$\Rightarrow$ decomposition (c) OK