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Mixed Odge Structures

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Abstract. These notes start with a summary of cohomology theory (section 2), necessary to understand the background to Hodge theory (section 3) and the polarization (section 4). The Mixed Hodge structure (MHS) is defined as an object of interest in linear algebra (section 5). After developing the spectral sequence of a filtered complex (section 6), canonical MHS are constructed on cohomology of algebraic varieties (sections 7,8).
The lectures in previous courses cover the first three sections. The course on MHS will be presented in a more elementary language than these notes
Lecture 1:Mixed Hodge structure (section 5).
Lecture 2:Polarization (section 4, 5).
Lecture 3:Logarithmic complex (section 6, 7) .
Lecture 4:Spectral sequence and Hodge theory (section 6, 7).
Lecture 5: MHS on cohomology of complex algebraic varieties (section 8).

1. Introduction

Cohomology theory attach to a manifold a group, invariant under homeomorphisms. Introduced first as a topological invariant, it depends only on the topology of the manifold, but it plays a significant role in the differential study of a manifold as shown in DeRham theory. Its importance in algebraic and analytic geometries is reflected in Hodge theory as well in the introduction by J.P. Serre of the cohomology of coherent sheaves. Inspired by the properties of étale cohomology introduced by A. Grothendieck in the study of varieties over fields with positive characteristic, Deligne established the existence of a linear algebraic structure on cohomology of algebraic varieties over fields of characteristic zero, called mixed Hodge structure (MHS); it reflects geometric and topological properties of algebraic varieties, which means it depends on the geometry and not only the topology of such varieties.
This structure proved to be very useful in the interpretation of various geometric invariants classically attached to algebraic varieties, as Picard group or the Jacobian variety. Deformation of geometric structures result into a linear invariant on the cohomlogy of the family of varieties defined by the deformation, called variation of Hodge structure (VHS) by P. Griffiths. More generally a variation of mixed Hodge structure (VMHS) is attached to an algebraic family of varieties not necessarily proper or smooth.

2. Review of cohomology theories

Many techniques have been used to construct homology and cohomology groups. We mention here :*triangulation, singular chains, Čech coverings* and finally *sheaf theory* via injective or acyclic resolutions. In particular De Rham cohomology defined only on differentiable manifolds, is of special interest in our case. We refer to Dieudonné [3] for an historical account of each contribution.
All these constructions give, when they are defined and under mild assumptions on the topological space, isomorphic cohomology groups. Based on their main common properties on the class of spaces defined as finite euclidean simplicial complexes, the *axiomatization of the theory* occurred in the 1940's. The use of *spectral sequences* , introduced by Leray and developed by Serre, appeared to be a powerful tool in computation, see for example [2]. The most general approach since 1955 is due to Grothendieck, first with the systematic use of *injective resolutions* in *abelian categories* [10], second with the introduction with Verdier [13] of the *derived category* and the use of *hypercohomology*, third with the introduction with M. Artin, of a new (Grothendieck) topology and (étale) cohomology theory developed with many colleagues and students (namely P. Deligne, L. Illusie and J.L. Verdier). However, to introduce MHS we may use only flabby resolutions of Godement [8] to construct the weight filtration W on cohomology with coefficients $\mathbb{Q}$. Dolbeault resolutions are of particular interest in the study of the Hodge filtration F, which is an analytic invariant, on the cohomology with coefficients $\mathbb{C}$.

2.1. **Simplicial objects.** Let Δ_n denote the set of integers $\{0,1,\dots,n\}$ and $H_{p,q}$ the set of increasing (large sense) mappings from Δ_p to Δ_q for integers $p,q \geq 0$. We define in this way the simplicial category Δ whose objects are $\Delta_n, n \in \mathbf{N}$, with the natural composition of mappings : $H_{pq} \times H_{qr} \rightarrow H_{pr}$. The semi-simplicial category $\Delta_>$ is obtained when we consider only the strictly increasing morphisms in Δ.
A simplicial (resp. co-simplicial) object of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from Δ to $\mathcal{C}$. For example a simplicial group consists of a family of groups X_i for each $i \in \mathbf{N}$ and of morphisms $X(f)\colon X_q \rightarrow X_p$ for each $f \in H_{p,q}$, compatible with composition. A semi-simplicial (resp. co-semi-simplicial) object of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from $\Delta_>$ to $\mathcal{C}$. We define for $0 \leq i \leq n+1$ the $i-$th face map and denote by

$$\delta_i\colon \Delta_n \rightarrow \Delta_{n+1}, \quad i \notin \delta_i(\Delta_n) := Im\, \delta_i$$

the unique increasing mapping such that $i \notin \delta_i(\Delta_n)$. Such suggestive name corresponds to the topological realization of the simplex where δ_i refers to the basic definition of the boundary. Its image by a functor is also denoted abusively by the same symbol $\delta_i\colon X_{n+1} \rightarrow X_n$ (resp. $\delta_i\colon X_n \rightarrow X_{n+1}$).

2.1.1. *Complexes.* A complex $(Y^{\cdot}, d)$ (resp. chain complex $(Y_{\cdot}, d)$) in an abelian category is given by a sequence of objects (Y^n) and morphisms $d_n : Y^n \rightarrow Y^{n+1}$ (resp. $d_n : Y_n \rightarrow Y_{n-1}$) such that $d_{n+1} \circ d_n = 0$ (resp. $d_{n-1} \circ d_n = 0$) for all integers n. It follows that $Im\, d_n \subset Ker\, d_{n+1}$ (resp. $Im\, d_n \subset Ker\, d_{n-1}$).
Given a complex $(Y^{\cdot}, d)$, we deduce a chain complex $(X_{\cdot}, d)$ if we put $X_n = Y^{-n}$, with the same differential $d_n : X_n \rightarrow X_{n-1}$. Such construction may be inverted.

Definition 2.1. The cohomology of the complex $(Y^\cdot, d)$ is defined for each n by

$$H^n(Y^\cdot, d) = Ker(d\colon Y^n \to Y^{n+1})/Im(d\colon Y^{n-1} \to Y^n).$$

The n^{th} homology of a chain complex $(X., d)$ is defined as

$$H_n(X., d) = ker(d\colon X_n \to X_{n-1})/Im(d\colon X_{n+1} \to X_n).$$

2.1.2. *Chain complex defined by a simplicial object of an abelian category.* Let $X_n, n \in \mathbb{N}$ be a simplicial object of an abelian category, we define the differential of a chain complex

$$d_{n+1}\colon X_{n+1} \to X_n, \qquad d_{n+1} = \Sigma_{i=0}^{n+1}(-1)^i \delta_i, \qquad \text{then } d_n \circ d_{n+1} = 0.$$

A complex may be defined similarly by a cosimplicial object.

2.2. **Triangulation.** [8;I.3.2], [12 ; 3.1] According to Dieudonné [3, p 26], the word triangulation is due to Weyl but the concept can be found in the paper Analysis situs written by Poincaré in 1895 and which is considered as a foundational article of Homotopy and Homology theory.

Definition 2.2. A combinatorial complex K consists of
i) a set K of elements $\{v\}$ called vertices.
ii) a set S of finite non empty subsets $\{s\}$ of K called simplices, such that
1) any set consisting of exactly one vertex is a simplex,
2) any non empty subset of a simplex is a simplex.
iii) A simplex $\{s\}$ is of dimension n if it contains $n+1$ vertices $|s| = n+1$.

Definition 2.3. 1) A singular simplex of dimension n of K is a mapping $\sigma\colon \Delta_n \to K$ whose image is a simplex of K.
2) Let $\Sigma_n(K)$ be the free abelian group generated by the set $S_n(K)$ of all singular simplices of dim n of K (resp. all singular simplices of dim n of K), then $\Sigma.(K)$ is a simplicial group. We denote also by $\Sigma.(K)$ the chain complex associated to it.
3) The homology of K is defined as the homology of the complex $\Sigma.(K)$.

Remark 2.4 (3;I.3.8). Suppose we are given an order on K such that each simplex is totally ordered, then the set of simplices totally ordered is the same as the set of strictly increasing singular simplices $\sigma\colon \Delta_n \to K$ and denote by $\Sigma_n^+(K)$ the subgroup of $\Sigma_n(K)$ generated by such subset of simplices $s \in K$. Then the subcomplex $\Sigma^+.(K)$ of $\Sigma_.(K)$ has the same homology groups as $\Sigma_.(K)$.

2.2.1. *Realization.* The topological realization of Δ_n is the real simplex Σ_n in $\mathbb{R}^{n+1}$ defined as the set of points $t = (t_o, \dots, t_n)$ such that $t_i \geq 0$ and $\Sigma_{i=0}^n t_i = 1$. As well a topological space $R(K)$, called its realization, is defined by K; it consists of real simplices $R(s)$ embedded in some real vector spaces of dimension big enough, one for each simplex s in K, $R(s)$ and $R(s')$ being glued along $R(s \cap s')$ considered as a face of $R(\sigma)$ and $R(\sigma')$. One way of defining this space is given in $[G]$ as follows

Definition 2.5. The realization $R(K)$ of K is the set of mappings $f\colon K \to \mathbb{R}$ such that
i) The set of elements v of K where $f(v) \neq 0$ is a simplex.
ii) $f(v) \geq 0$ for all $v \in K$ and $\Sigma_{v \in K} f(v) = 1$.

The topology on $R(K)$ is defined as follows. For each simplex σ let $R(s)$ denotes the set of elements in $R(K)$ vanishing on $K - s$. It is isomorphic to a real topological simplex in the finite dimensional real vector subspace generated by $R(v)$ for all v

in s in the vector space of all mappings $f\colon K \to \mathbb{R}$. If v is a vertex of s then $R(v)$ is a point in $R(s)$. A subset U of $R(K)$ is open if it induces an open subset $U \cap R(s)$ in $R(s)$ for each simplex s in K.

Definition 2.6. A triangulation of a topological space E is given by a homeomorphism of E onto a realization $R(K)$ of a combinatorial complex K.

The homology of a triangulated space E is defined as the homology of its triangulation K. This notion becomes interesting when one proves the independence of the choice of the triangulation, which is the case [3]. A natural proof is deduced by comparison of homology with singular homology.

2.3. **Singular homology and cohomology.** It is based on the use of continuous mappings defined on the topological simplex Σ_n and generally adapted for an introduction to homology theory, although triangulations are more intuitive.

Definition 2.7. A singular simplex of dimension n of a topological space E is a continuous mapping $\sigma\colon \Sigma_n \to E$.

Let $\Sigma_n(E)$ denote the set of singular simplices of dimension n, and $S_n(E)$ the free abelian group generated by $\Sigma_n(E)$. Then $\Sigma_n(E)$ (resp. $S_n(E)$) define a simplicial set (resp. group) hence a chain complex; while $Hom_{\mathbb{Z}}(S_n(E), \mathbb{Z})$ define a cosimplicial group, hence a complex.

Definition 2.8. i) The homology groups $H_i(E, \mathbb{Z})$ of the chain complex $S_.(E)$ defined by the simplicial complex are called the singular homology groups of E with coefficients in $\mathbb{Z}$.
ii) The cohomology groups $H^i(E, \mathbb{Z})$ of the cochain complex $Hom_{\mathbb{Z}}(S_.(E), \mathbb{Z})$ defined by the cosimplicial group are called the singular cohomology groups of E with coefficients in $\mathbb{Z}$.

When E is triangulable, and this is the case of differentiable manifolds or algebraic varieties, the singular homology coincides with the homology of any triangulation. Homology and Cohomology groups have in general torsion elements. In Hodge theory we are mainly interested in the image of $H^i(E, \mathbb{Z})$ in $H^i(E, \mathbb{Q})$ which is torsion free.

Definition 2.9. A piecewise smooth singular simplex of dimension n of a topological differentiable manifold X is a map $\sigma\colon \Sigma_n \to X$ which extends to a differentiable map on a neighbourhood of Σ_n.

We form the chain complex $S^{ps}_.(E)$ of piecewise smooth integral chains and define its homology denoted by $H^{ps}_i(X, \mathbb{Z})$

Lemma 2.10. *The natural morphism* $H^{ps}_i(X, \mathbb{Z}) \xrightarrow{\sim} H_i(X, \mathbb{Z})$ *is an isomorphism.*

2.3.1. *Poincaré duality.* A general intersection theory is defined on triangulable spaces, in particular compact differentiable manifolds, on the level of chains of triangles. It induces an intersection pairing on homology which is used to prove a general Poincaré duality on oriented topological spaces.

Theorem 2.11 (Poincaré duality)**.** *On a compact oriented n-manifold, the intersection pairing*

$$H_i(X, \mathbb{Z}) \times H_{n-i}(X, \mathbb{Z}) \to \mathbb{H}_0(X, \mathbb{R}) \xrightarrow{degree} \mathbb{Z}$$

is unimodular, that is the induced morphism

$$H_i(X,\mathbb{Z}) \to Hom(H_{n-i}(X,\mathbb{Z}),\mathbb{Z})$$

is surjective and its kernel is the torsion subgroup

We need this duality on compact oriented differentiable manifolds where it can be proved using differential forms. For rational coefficients, it defines a perfect duality

$$H_i(X,\mathbb{Q}) \simeq Hom(H_{n-i}(X,\mathbb{Q}),\mathbb{Q}).$$

2.4. **Čech cohomology.** In this case, we can define cohomology groups with coefficients in any sheaf $\mathcal{F}$ of abelian groups.
Let $\mathcal{U} = (U_i)_{i\in I}$ denotes an open covering of a topological space E and S_p the set of sequences $(i_o,\dots,i_p)$ of length $p+1$ in the set of indices I such that $\cap_{j=0}^{p} U_{i_j} \neq \emptyset$ (Remark that this condition defines a structure of combinatorial complex on I as follows : a finite subset $J \in I$ is a simplex if and only if $\cap_{i\in J} U_i \neq \emptyset$). Define

$$U_s = \cap_{i\in s} U_i,\ s \in S_p \quad ; \quad C^p = C^p(\mathcal{U},\mathcal{F}) = \prod_{s\in S_p} \mathcal{F}(U_s)$$

then $C^\cdot$ is a cosimplicial group in the sense that to a map $f\colon \Delta_p \to \Delta_q$ corresponds covariantly with f; if we denote an element β of C^p as $\beta_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0,\dots,i_p})$, then

$$f^* : C^p \to C^q \ : \ (f^*\alpha)_{i_o\dots i_q} = (\alpha_{i_{f(o)}\dots i_{f(p)}})|_{U_{i_o}\cap\dots\cap U_{i_q}}$$

where we use $|$ for the restriction map of sections on $U_{i_{f(o)}} \cap \dots \cap U_{i_{f(p)}}$ to sections on $U_{i_o} \cap \dots \cap U_{i_q}$. Hence a structure of complex is defined on $C^\cdot$ by the differential

$$d\colon C^p \to C^{p+1}, (d\alpha)_{i_o\dots i_{p+1}} = \Sigma_{j=0}^{p+1}(-1)^j \alpha_{i_o\dots\hat{i}_j\dots i_{p+1}}$$

where we write $\alpha_{i_o\dots\hat{i}_j\dots i_{p+1}}$ for $\delta_j(\alpha)$. The n^{th} cohomology group of $C^\cdot$ is denoted by $\check{H}^n(\mathcal{U},\mathcal{F})$. To obtain a group independent of the covering, one needs to define an inductive limit on the set of all coverings of E using the notion of refinement of coverings, then one put

$$\check{H}^n(X,\mathcal{F}) = \underrightarrow{lim}_{\mathcal{U}} \check{H}^n(\mathcal{U},\mathcal{F})$$

When the space is triangulated, the Čech cohomology groups with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ coincides with simplicial cohomology groups.

2.5. **Hypercohomology.** Modern theory of sheaves lead to a general theory of cohomology $[G, Gr]$. An abelian category $\mathcal{A}$ is said to have enough injective objects if for any object M there exists a monomorphism $M \to I$ into an injective object I of $\mathcal{A}$. In this case, for any complex $X^\cdot$ of objects of $\mathcal{A}$, bounded to the left, and any functor $\Psi : \mathcal{A} \to \mathcal{B}$ to an abelian category $\mathcal{B}$, which is left exact, Grothendieck defines the hypercohomology $\mathbb{H}^*(\Psi, X^\cdot)$ of Ψ at $X^\cdot$ as follows:
i) *Construct first an injective resolution of* $X^\cdot$, that is a complex $I^\cdot$ of injective objects in $\mathcal{A}$ and a morphism $\Phi\colon X^\cdot \to I^\cdot$ which induces an isomorphism on cohomology objects (then Φ is called a quasi-isomorphism and an isomorphism in the derived category $D^+(\mathcal{A})[D]$).
ii) *Define the hypercohomology as an object in* $\mathcal{B}$

$$\mathbb{H}^*(\Psi, X^\cdot) = H^*(\Psi(I^\cdot))$$

In particular, the previous cohomology groups of a topological space X with coefficients in $\mathbb{Z}$ are isomorphic to the hypercohomology of the functor Γ of global sections of sheaves of abelian groups on X applied to the constant sheaf $\underline{\mathbb{Z}}$ defined by the group $\mathbb{Z}$ on X. The proof is based on the use of adequate resolutions. A sheaf is called acyclic for a functor if its hypercohomology vanishes in degree ≥ 1. Using spectral sequence theory, one can show that all resolutions by acyclic sheaves for the functor of global sections, will give isomorphic cohomology groups. Godement [G] uses a particular type of acyclic sheaves called flabby. Any sheaf has natural resolutions by flabby sheaves and injective sheaves are flabby, so we have the same cohomology objects using either resolutions.

2.6. **DeRham cohomology.** Now let X be a differentiable manifold and let $\mathcal{E}_X$ be the sheaf of $\mathcal{C}^\infty$ real-valued functions on X, $\mathcal{E}_X^p$ the $\mathcal{E}_X$-module of $\mathcal{C}^\infty$ differential p-forms on X and consider the complex $\mathcal{E}_X^*$ with its exterior differential.

Definition 2.12. The cohomology groups of the complex of global sections of $\mathcal{E}_X^*$ are called De Rham cohomology

$$H^i_{DR}(X,\mathbb{R}) := H^i(\Gamma(X,\mathcal{E}_X^*)) := H^i(\mathcal{E}^*(X)) := \frac{Ker(d^i : \mathcal{E}^i(X) \to \mathcal{E}^{i+1}(X)}{Im(d^{i-1} : \mathcal{E}^{i-1}(X) \to \mathcal{E}^i(X)}$$

This definition uses the differentiable structure to introduce the above groups called DeRham realization of the cohomology, as they give isomorphic groups with previous definitions of the cohomology.

2.6.1. *DeRham resolution.* By Poincaré lemma this complex is a resolution of the constant sheaf $\mathbb{R}$ and using "partition of unity" one can prove this is a resolution by fine sheaves.

Lemma 2.13. *The complex of sheaves of differential forms $\mathcal{E}_X^*$ is a fine resolution of the constant sheaf $\mathbb{R}$.*

Since fine resolutions are acyclic for the global section functor, we deduce the isomorphisms

$$H^i(X,\mathbb{R}) \simeq H^i_{DR}(X,\mathbb{R})$$

We define also the cohomology of X with complex coefficients using the global sections of complex valued differential forms

$$H^i(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^i(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} H^i(\mathcal{E}^*(X) \otimes_{\mathbb{R}} \mathbb{C})$$

We note that the cohomology spaces are defined naturally over $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, which provides a natural lattice inside the real vector space and a real structure inside the complex cohomology

$$H^i(X,\mathbb{Z}) \otimes \mathbb{R} \simeq H^i(X,\mathbb{R}), \quad H^i(X,\mathbb{R}) \otimes \mathbb{C} \simeq H^i(X,\mathbb{C})$$

2.6.2. *Comparison with Singular cohomology.* Given a differential form $\omega \in \Gamma(X,\mathcal{E}_X^n)$ and a piecewise smooth singular chain of simplices $\sigma \in S_n^{ps}(X)$ one can define the integral $\int_\sigma \omega$ in $\mathbb{R}$. This integral can be used to construct a map

$$\Gamma(X,\mathcal{E}_X^n) \to S^n_{ps}(X,\mathbb{R}) := Hom_{\mathbb{R}}(S_n^{ps}(X),\mathbb{R})$$

defined by $(\omega,\sigma) = (-1)^{n(n+1)/2} \int_\sigma \omega$.
By Stoke's theorem : $\int_\sigma d\omega = \int_{\partial\sigma} \omega$ where ∂ denotes the boundary map for chains; from which we deduce that the above map extends to a map of complexes.
DeRham's theorem states that the map defined above induces an isomorphism on cohomology groups for each $n \in \mathbb{N}$

Theorem 2.14 (DeRham). *Integration over piecewise smooth singular cycles in a differentiable manifold X, defines an isomorphism of cohomology groups*

$$H^n_{DR}(X,\mathbb{R}) \xrightarrow{\sim} H^n(S^*_{ps}(X,\mathbb{R})) \xrightarrow{\sim} H^n(X,\mathbb{R})$$

compatible with wedge product on DeRham cohomology and cup-product on singular cohomology.

2.6.3. *Poincaré duality.* It can be checked directly on DeRham cohomology

Theorem 2.15 (Poincaré). *The bilinear product*

$$H^{n-i}(X,\mathbb{R}) \otimes H^{n+i}(X,\mathbb{R}) \to H^{2n}(X,\mathbb{R}) \to \mathbb{R} : (\alpha \otimes \beta) \mapsto \alpha \wedge \beta \mapsto \int_X \alpha \wedge \beta$$

defines an isomorphism: $H^{n-i}(X,\mathbb{R}) \simeq Hom(H^{n+i}(X,\mathbb{R}),\mathbb{R})$.

When we identify above the dual space to cohomology with homology, the Poincaré duality is stated as an isomorphism

$$H^{n-i}(X,\mathbb{R}) \simeq H_{n+i}(X,\mathbb{R})$$

Then the cup-product on cohomology corresponds to the topological Intersection theory on the manifold X, such that the duality is stated on homology as follows

$$H_{n-i}(X,\mathbb{R}) \otimes H_{n+i}(X,\mathbb{R}) \xrightarrow{\cap} H_0(X,\mathbb{R}) \xrightarrow{degree} \mathbb{R} : (\alpha \otimes \beta) \mapsto \alpha \cap \beta \mapsto deg\, \alpha \cap \beta$$

3. Harmonic forms

In this section X will denote successively a differentiable , complex analytic, and Kähler compact manifold.

3.1. Harmonic forms on a differentiable manifold.

A Riemannian manifold X is endowed with a *scalar product on its tangent bundle* defining a metric on X. A basic result in analysis states that the *cohomology of a compact smooth Riemannian manifold is represented by real harmonic global differential forms* denoted in degree i by $\mathcal{H}^i(X)$. To define harmonic forms, we need to introduce the $\star$*-operator and the Laplacian.*

3.1.1. *Riemannian metric.* A bilinear form g on X is defined at each point x as a product on the tangent space T_x to X at x

$$g_x(,): T_x \otimes_{\mathbb{R}} T_x \to \mathbb{R}$$

where g_x vary smoothly with x, that is $h_{ij}(x) := g_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is differentiable in x, then the product is given as $g_x = \sum_{i,j} h_{ij}(x) dx_i \otimes dx_j$. It is called a metric if moreover the matrix of the the product defined by $h_{ij}(x)$ is positive definite, that is: $g_x(u,u) > 0$ for all $u \neq 0 \in T_x$.
An induced metric is defined on $\Omega^1_x := T^*_x := Hom(T_x,\mathbb{R})$ and on Ω^p_x as follows. Let $e_1,\dots,e_n$ be an orthonormal basis, then on Ω^p_x we define the unique metric such that $e^*_{i_1} \wedge \dots, \wedge e^*_{i_p}$ is an orthonormal basis where $(e^*_i)_{i\in I}$ form a dual basis to $(e_i)_{i\in I}$.

3.1.2. *Volume form, L^2 metric.* Suppose now that the manifold is oriented, then an orthonormal positively oriented basis e_i define a nowhere vanishing section of Ω^n_X called volume section vol satisfying $vol_x := e_1^* \wedge \dots, \wedge e_n^*$.

Exercise 3.1. Let $x_1, \dots, x_n$ denote a local ordered set of coordinates on an open subset U of a covering of X, compatible with the orientation. Prove that

$$\sqrt{d}et(h_{ij})dx_1 \wedge \dots \wedge dx_n$$

defines the global volume form.

We deduce from the volume form a global pairing called the L^2 metric

$$\forall \psi,\, \eta \in \mathcal{E}^i(X), (\psi,\, \eta)_{L^2} = \int_X g_x(\psi(x),\, \eta(x))vol(x)$$

3.1.3. *Laplacian.* We prove the existence of an adjoint $d^* : \mathcal{E}^{i+1}_X \to \mathcal{E}^i_X$ to the differential $d : \mathcal{E}^i_X \to \mathcal{E}^{i+1}_X$, satisfying

$$(d\psi, \eta)_{L^2} = (\psi, d^*\eta)_{L^2}, \quad \forall \psi \in \mathcal{E}^i(X), \eta \in \mathcal{E}^{i+1}(X).$$

The adjoint operator is defined by constructing first an operator $*$

$$\mathcal{E}^i_X \xrightarrow{*} \mathcal{E}^{n-i}_X$$

by requiring at each point $x \in X$

$$(\psi,\, \eta)vol_x = \psi_x \wedge *\eta_x, \quad \forall \psi,\, \eta \in \mathcal{E}^i_{X,x}.$$

The section vol defines an isomorphism $\mathbb{R} \simeq \mathcal{E}^n_X$, inducing the isomorphism $Hom(\mathcal{E}^i_X, \mathbb{R}) \simeq Hom(\mathcal{E}^i_X, \mathcal{E}^n_X) \simeq \mathcal{E}^{n-i}_X$. The definition of $*$ is obtained by composition with the isomorphism $\mathcal{E}^i_X \to Hom(\mathcal{E}^i_X, \mathbb{R})$ defined by the scalar product on $\mathcal{E}^i_X$. We have on $\mathcal{E}^i_X$

$$*^2 = (-1)^{i(n-i)} Id, \quad d^* = (-1)^{n+in+1} * \circ d \circ *.$$

Lemma 3.2. *We have for all $\alpha, \beta \in \mathcal{E}^i(X)$*

$$(\alpha,\, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

Definition 3.3. The Laplacian Δ is defined as

$$\Delta = d^* \circ d + d \circ d^*$$

Harmonic forms are defined as the solutions of the Laplacian

$$\mathcal{H}^i(X) = \{\psi \in \mathcal{E}^i(X) : \Delta(\psi) = 0\}$$

The importance of harmonic forms in Hodge theory stems from the representation of cohomology. However they have the following interesting property.
Consider $\phi \in \mathcal{E}^i(X)$ satisfying $d\phi = 0$ and its cohomology class $\phi + d\psi \in \mathcal{E}^i(X)$. The Laplace equation defines exactly the form in this class with minimal L^2-norm. First note that the class is an affine subspace of the $\mathcal{E}^i(X)$. To show the existence of the solution we introduce the Hilbert space of L^2-forms; then the closure of the affine subspace contains a unique element of minimal distance to 0. A remarkable result in the theory of elliptic differential equations [Warner, Wells] proves the regularity of the solution that is, such element is in fact in the affine space and not only in its completion. The link to Laplace equation is as follows. An element ϕ has minimal norm if

$$\forall t \in \mathbb{R}, \quad \|\phi\|^2_{L^2} \le \|\phi + d\psi\|^2_{L^2}$$

The expansion of the right term in t, $\|\phi\|^2_{L^2} + 2t(\phi, d\psi)_{L^2} + O(t^2)$ shows that the inequality holds if and only if $(\phi, d\psi)_{L^2} = 0$, and equivalently $(d^*\phi, \psi)_{L^2} = 0$ for all ψ, hence $d^*\phi = 0$. That is $d\phi = 0$ (ϕ is closed), $d^*\phi = 0$ (ϕ is co-closed) and $\Delta(\phi) = 0$. Reciprocally if $\Delta(\phi) = 0$, then $0 = (\Delta(\phi), \phi)_{L^2} = (dd^*\phi, \phi)_{L^2} + (d^*d\phi, \phi)_{L^2} = \|d^*\phi\|^2_{L^2} + \|d\phi\|^2_{L^2}$, hence ϕ is closed and co-closed.
A basic result admitted here, is

Theorem 3.4. *On a compact smooth oriented Riemannian manifold each cohomology class is represented by a unique real harmonic global differential form*

$$\mathcal{H}^i(X) \simeq H^i(X, \mathbb{R}).$$

3.2. **Complex manifolds.** An important feature of a complex manifold is the existence of an almost complex structure J on the real tangent bundle.

3.2.1. *Almost complex structure.* Let V be a real vector space. A almost complex structure on V is defined by a linear map $J : V \to V$ satisfying $J^2 = -1$; then dim.V is even and the eigenvalues are i and $-i$ with associated eigenspaces in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$

$$V^+ = \{x - iJx : x \in V\} \subset V_{\mathbb{C}}, \; V^- = \{x + iJx : x \in V\} \subset V_{\mathbb{C}}.$$

Example 3.5. 1- The space $\mathbb{R}^2$ with the action of J defined on the canonical basis by $J(1, 0) := (0, 1)$.
The isomorphism $\varphi : \mathbb{C} \xrightarrow{\sim} \mathbb{R}^2, \varphi(1) = (1, 0), \varphi(i) = (0, 1)$ is compatible with the action of i on $\mathbb{C}$ and the action of J on $\mathbb{R}^2$, hence the action of J amounts to define a complex structure for which J is the multiplication by i.
2- Let V be a complex vector space, then the isomorphism $\varphi : V \to V_{\mathbb{R}}$ with the real underlying space to V transports the action of i to an action J. The image of a complex basis $(e_i)_{i\in[1,n]}$ defines a basis a real basis $\varphi(e_i), J(\varphi(e_i)) = J(i\varphi(e_i), i \in [1, n]$.
Reciprocally, a real vector space W with the action of J corresponds to a structure of complex vector space on V as follows. Let $(e_i)_{i\in[1,n]}$ be a subset of W such that $\{e_i, J(e_i)\}_{i\in[1,n]}$ form a basis of W (first choose a non zero vector e_1 in W and prove that W_1 generated by $e_1, J(e_1)$ is of dimension two, then continue with $e_2 \notin W_1$ and so on). Then there is a complex structure on W for which $e_j, j \in [1, n]$ is a complex basis with the action of $i \in \mathbb{C}$ defined by $i.e_j := J(e_j)$.

3.2.2. *Decomposition into types.* Let V be a complex vector space and $(V_{\mathbb{R}}, J)$ the underlying real vector space with its involution J defined by multiplication by i. Let $W := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ and $W_{\mathbb{C}} := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{C}) \simeq W \otimes \mathbb{C}$, then $Hom_{\mathbb{C}}(V, \mathbb{C})$ embeds into $W_{\mathbb{C}}$ and its image is denoted $W^{1,0}$ while its conjugate with respect to W is the subspace denoted $W^{0,1}$, moreover

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}, \quad \wedge^2 W_{\mathbb{C}} = W^{2,0} \oplus W^{1,1} \oplus W^{0,2}$$

where we define resp. the spaces of forms of type $(2, 0)$, $(1, 1)$, $(0, 2)$ as :

$$W^{2,0} = W^{1,0} \otimes W^{1,0}, W^{1,1} = W^{1,0} \otimes W^{0,1}, W^{0,2} = W^{0,1} \otimes W^{0,1}.$$

3.2.3. *Decomposition into types on the complexified tangent bundle.* The existence of a complex structure on the manifold leads to an almost complex structure on the tangent bundle $T_{X,\mathbb{R}}$ and a decomposition into types of the complexified tangent bundle $T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which splits as a direct sum $T^{1,0}_X \oplus T^{0,1}_X$ where $T^{1,0}_{X,z} = \{u - iJu :$

$u \in T_{X,\mathbb{R},z}\}$ and $\overline{T_X^{1,0}} = T_X^{0,1}$. Precisely, if $z_j = x_j + iy_j, j \in [1,n]$ are local complex coordinates, then $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ in $T_{X,\mathbb{R},z}$. The holomorphic tangent space $T_{X,z}$ embeds isomorphically into $T_X^{1,0}$ generated by $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ of the form $u_j - iJ(u_j)$. From now on we identify the two bundles

$$T_X \xrightarrow{\sim} T_X^{1,0}$$

The dual of the above decomposition of the tangent space is written as $\mathcal{E}_X^1 \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{E}_X^{1,0} \oplus \mathcal{E}_X^{0,1}$ which induces a decomposition of the sheaves of differential forms into types

$$\mathcal{E}_X^i \otimes_{\mathbb{R}} \mathbb{C} \simeq \oplus_{p+q=i} \mathcal{E}_X^{p,q}$$

In terms of complex local coordinates on an open set U, $\phi \in \mathcal{E}^{p,q}(U)$ is written as a linear combination of

$$dz_I \wedge d\overline{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

where $I = \{i_1, \cdots, i_p\}, J = \{j_1, \ cdots, j_q\}$. The differential decomposes as as well $d = \partial + \overline{\partial}$, where $\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$ and $\partial : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q}$ are compatible with the decomposition up to a shift on the bidegree, we deduce from $d^2 = \partial^2 + \overline{\partial}^2 + \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$ that the DeRham complex is the simple complex associated to a double complex

$$(\mathcal{E}_X^* \otimes_{\mathbb{R}} \mathbb{C}, d) \simeq s((\mathcal{E}_X^{*,*}, \partial, \overline{\partial})$$

In particular we recover the notion of a subspace of cohomology type (p,q).

$$H^{p,q}(X) := \{\text{Cohomology classes representable by a closed form of type } (p,q)\}$$

We have $\overline{H^{p,q}(X)} = H^{q,p}(X)$ but it is not always true that $H^i_{DR}(X) = \oplus_{p+q=i} H^{p,q}(X)$. Hodge theory apply to a complex manifold when exactly such decomposition exists.

Remark 3.6. Given a bundle of real vector spaces with differentiable action of J, the problem to lift this action into a complex structure on the bundle is not straightforward. An important result in this direction is the Newlander-Nirenberg theorem.

Example 3.7. 1 - On a compact complex torus $X = \mathbb{C}^n/\Lambda$, a basis of the cohomology is given by the translation-invariant forms, hence with constant coefficients since the tangent space is isomorphic to a direct sum of the structural sheaf $\mathcal{E}_X^0$. Hence the decomposition reduces to the case of $\mathbb{R}^{2n} = \mathbb{C}^n \oplus \overline{\mathbb{C}^n}$.
2 - The case of a compact Riemann surface , reduces to prove

$$H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X).$$

3.2.4. *Holomorphic Poincaré lemma.* On complex manifolds the sheaf of holomorphic forms Ω_X^1 is an important geometric invariant. The holomorphic version of Poincaré lemma resolution lemma shows that the DeRham complex of holomorphic forms Ω_X^* is a resolution of the constant sheaf $\underline{\mathbb{C}}$. However, since the resolution is not acyclic, cohomology spaces are computed only as hypercohomology of the global sections functor, that is after acyclic resolution.

$$H^i(X,\mathbb{C}) \simeq \mathbb{H}^i(X, \Omega_X^*) := R^i\Gamma(X, \Omega_X^*) \simeq H^i(\mathcal{E}^*(X) \otimes_{\mathbb{R}} \mathbb{C})$$

3.2.5. *Dolbeault cohomology.* The complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for each $r \geq 0$, is called a Dolbeault resolution since Dolbeault's result shows it is a resolution of Ω_X^*.

Lemma 3.8. *The Dolbeault complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for $r \geq 0$, is a fine resolution of Ω_X^r, hence*

$$H^i(X, \Omega_X^r) \simeq H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) := H_{\overline{\partial}}^{r,i}(X)$$

The cohomology of the complex of its global sections is called the $\overline{\partial}$ cohomology of X of type (r, i)

$$H_{\overline{\partial}}^{r,i}(X) := H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) = \frac{Ker(\overline{\partial}^i : \mathcal{E}^{r,i}(X) \to \mathcal{E}^{r,i+1}(X))}{Im(\overline{\partial}^{i-1} : \mathcal{E}^{r,i-1}(X) \to \mathcal{E}^{r,i}(X))}$$

A cohomology class of X of type (r, i) defines a $\overline{\partial}$ cohomology class of the same type. It will follows that this map defines an isomorphism whenever Hodge theory apply.

3.3. **Hermitian metric.** A Hermitian product h on a complex manifold X is defined continuously on the tangent space.

3.3.1. *Hermitian metric on a complex vector space.* Let V be a complex vector space and $(V_{\mathbb{R}}, J)$ the underlying real vector space with its involution J defined by multiplication by i. A Hermitian product h on V is a product

$$h : V \times V \to \mathbb{C}, \text{ satisfying } h(u, v) = \overline{h(v, u)}$$

which is biadditive, linear in the first term and antilinear in the second term. Considering the restriction to $V_{\mathbb{R}}$

$$h_{|V_{\mathbb{R}}} = \mathcal{R}eh + i\mathcal{I}mh$$

The product defines a Hermitian metric if moreover the real number $h(u, \overline{u}) > 0$ is positive definite for all $u \neq 0 \in V$, then we deduce a Riemannian structure on $V_{\mathbb{R}}$ defined by $g := \mathcal{R}eh$.

Definition 3.9 (The (1,1) form ω)**.** It is associated to the hermitian metric a real $2-$form $\omega := -\mathcal{I}mh$ on $V_{\mathbb{R}}$, satisfying:
$\omega(Ju, v) = -\omega(u, Jv)$, and $\omega(Ju, Jv) = \omega(u, v)$.

Proof. We have a $2-$form since $\omega(v, u) = -\mathcal{I}mh(v, u) = \mathcal{I}mh(u, v) = -\omega(u, v)$. We check
$\omega(u, Jv) = -\mathcal{I}mh(u, iv) = g(u, v), \omega(Ju, v) = -\mathcal{I}mh(iu, v) = -g(u, v)$
$\omega(Ju, Jv) = -\mathcal{I}mh(iu, iv) = -\mathcal{I}mh(u, v) = \omega(u, v)$. □

Remark 3.10. The hermitian form is entirely determined by g or ω since $g(u, v) = \omega(u, Jv)$ and $\omega(u, v) = -g(u, Jv)$, as one can deduce from the identity: $h(u, iv) = -ih(u, v)$.

Let $W := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ and $W_{\mathbb{C}} := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{C}) \simeq W \otimes \mathbb{C}$, recall the decompositions

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}, \quad \wedge^2 W_{\mathbb{C}} = W^{2,0} \oplus W^{1,1} \oplus W^{0,2}$$

Considering $\wedge_{\mathbb{R}}^2 W \subset \wedge^2 W_{\mathbb{C}}$, we introduce the intersection in $\wedge^2 W_{\mathbb{C}}$

$$W_{\mathbb{R}}^{1,1} := W^{1,1} \cap (\wedge_{\mathbb{R}}^2 W) \subset \wedge^2 W_{\mathbb{R}}$$

Lemma 3.11. *The correspondence: $h \to -\mathcal{I}mh$ is a natural isomorphism between the real vector space of Hermitian forms on $V \times V$ and the space $W^{1,1}_{\mathbb{R}}$ of real forms of type* $(1,1)$.

Definition 3.12. A real form of type $(1,1)$ is positive if the corresponding Hermitian form is positive definite.

Let $\{e_i\}$ of V denote a basis of V. The projections $dz_j, j \in [1,n]$, denote a dual basis of the dual space V^*; then the Hermitian product is written as

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\overline{z_j} \in V^* \otimes_{\mathbb{C}} \overline{V}^* \simeq W^{1,0} \otimes W^{0,1}$$

where $h_{ij} = h(e_i, e_j)$ and $\overline{V}$ is the complex conjugate space to V. Let $H = (h_{ij})$ denotes the matrix with coefficients h_{ij} (equal to its conjugate transpose), then

$$h(u,v) = {}^t\overline{V} H U$$

where U, V are the coordinates of u and v written in columns. For a Hermitian metric, the matrix H is positive definite.
We can always suppose the basis orthonormal, that is $H = Id$. If we write $dz_j = dx_j + i dy_j$, then

$$h = \sum_j dz_j \otimes d\overline{z_j} = \sum_j (dx_j \otimes dx_j + dy_j \otimes dy_j) - i \sum_j (dx_j \otimes dy_j - dy_j \otimes dx_j)$$

where $\sum_j (dx_j \otimes dx_j + dy_j \otimes dy_j)$ defines the Riemannian metric and
$\omega = \sum_j dx_j \wedge dy_j := \sum_j (dx_j \otimes dy_j - dy_j \otimes dx_j) = \frac{i}{2} \sum_j dz_j \otimes d\overline{z_j}$ (note that we avoid the factor $\frac{1}{2}$ in some definition of $\sum_j dx_j \wedge dy_j$).
The volume form $vol \in \wedge^{2n} V_{\mathbb{R}}$, associated to g, can be computed via ω and is equal to $\frac{1}{n!}\omega^n$. We have

$$\omega = \frac{i}{2} \Sigma_j dz_j \wedge d\overline{z}_j = \Sigma_j dx_j \wedge dy_j, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

3.3.2. *Extensions of h to $W^{p,q}$.* Note that the hermitian product h extends to $h^{1,0}$ on $W^{1,0}$ by the condition that the dual basis $\{dz_j\}_{j \in [1,n]}$ to an orthonormal basis $\{e_j\}_{j\in[1,n]}$ of V, is orthonormal for $h^{1,0}$. On $W^{0,1}$ the basis $\{d\overline{z}_j\}_{j\in[1,n]}$ is orthonormal for the extension $h^{0,1}$.
Independently, g on V extends by the condition of orthonormality to W and by linearity to $W \otimes \mathbb{C}$, then we define h^1 on $W_{\mathbb{C}}$ by $h^1(u,v) := g(u, \overline{v})$. Moreover we define $h^{p,q}$ on $W^{p,q} \wedge^p W^{1,0} \otimes \wedge^q W^{0,1}$ and h^k on $\wedge^k W_{\mathbb{C}}$

Lemma 3.13. *The extension h^k of h is equal to $2^k \sum h^{p,q}$ on $\wedge^k W_{\mathbb{C}} \simeq \sum_{p+q=k} W^{p,q}$.*

In fact, $h^{1,0}(dz_j, dz_j) = 1$ while $h^1(dz_j, dz_j) = g(dx_j + i dy_j, dx_j - i dy_j)$
$= g(dx_j, dx_j) + g(dy_j, dy_j) + i(g(dy_j, dx_j) - g(dx_j, dy_j)) = 2$. Similarly $h^1(d\overline{z}_j, d\overline{z}_j) = 2$.

3.3.3. *Hermitian metric on X.* The Hermitian product h on the complex manifold X is defined at each point z as a Hermitian product on the holomorphic tangent space T_z to X at z

$$h_z : T_z \times T_z \to \mathbb{C}$$

moreover h_z vary smoothly with z, that is $h_{ij}(z) := h_z(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ is differentiable in z and $h_{ij}(z) = \overline{h_{ji}(z)}$.

The isomorphism $T_z \xrightarrow{\sim} T_z^{1,0}$. The holomorphic tangent space is generated by $\frac{\partial}{\partial z_i}$ and the isomorphism is defined by

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

in $T_z^{1,0} \subset T_{X,\mathbb{R},z} \otimes \mathbb{C}$. Hence it is equivalent to define the Hermitian product as

$$h_z : T_z^{1,0} \times T_z^{1,0} \to \mathbb{C},\ h_z = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z_j} \in \mathcal{E}_z^{1,0} \otimes_{\mathbb{C}} \mathcal{E}_z^{0,1}$$

The product is called a Hermitian metric if moreover the real number $h_z(u, \overline{u}) > 0$ is positive definite for all $u \neq 0 \in T_z$; equivalently the matrix of the product defined by $h_{ij}(z)$ (equal to its conjugate transpose) is positive definite.
Considering the $\mathbb{R}-$linear isomorphism

$$T_{X,\mathbb{R},z} \xrightarrow{\sim} T_z^{1,0} : \frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}, \frac{\partial}{\partial y_j} = J(\frac{\partial}{\partial x_j}) \mapsto i\frac{\partial}{\partial z_j}$$

with inverse defined by the real part u of $u + iv$, and compatible with the action of J on $T_{X,\mathbb{R},z}$ and i on $T_z^{1,0}$; the induced product is written as

$$h_{z|T_{X,\mathbb{R},z}} = \mathcal{R}eh_z + i\mathcal{I}mh_z$$

We associate to the metric a Riemannian structure on X defined by $g_z := \mathcal{R}eh_z$. Then X of dimension n is viewed as a Riemann manifold of dimension $2n$ with metric g_z. Note that g is defined over the reals since it can be represented by a real matrix over a basis of the real tangent space.
Since h_z is hermitian, we have $g_z(Ju, Jv) = g_z(u, v)$.

3.3.4. *The (1,1) form* ω. We associate also to the metric a real $2-$form $\omega := -\mathcal{I}mH$ of type $(1,1)$ ($\omega \in \mathcal{E}_X^{1,1} \cap \mathcal{E}_X^2$). The volume form $vol \in \mathcal{E}^{2n}$ defined by g on X, can be defined by ω and is equal to $\frac{1}{n!}\omega^n$. We have locally

$$\omega = \frac{i}{2}\Sigma_i dz_i \wedge d\overline{z}_i, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Reciprocally, the metric can be recovered from ω or from g.

Example 3.14. 1) The Hermitian metric on $\mathbb{C}^n$ is defined by $h := \sum_{i=1}^n dz_i \otimes d\overline{z_j}$ with induced metric the standard metric on $\mathbb{R}^{2n}$

$$H(z, w) = \sum_{i=1}^n z_i \overline{w}_i$$

where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. By writing $z_k = x_k + iy_k$, it is possible to consider $\mathbb{C}^n = \mathbb{R}^{2n}$, in which case $\mathcal{R}eH$ is the Euclidean inner product and $\mathcal{I}mH$ is a nondegenerate alternating bilinear form, i.e., a symplectic form.
Explicitly, in $\mathbb{C}^2$, the standard Hermitian form is expressed as

$$\begin{aligned} H((z_{1,1}, z_{1,2}), (z_{2,1}, z_{2,2})) =& x_{1,1}x_{2,1} + x_{1,2}x_{2,2} + y_{1,1}y_{2,1} + y_{1,2}y_{2,2} \\ &+ i(x_{2,1}y_{1,1} - x_{1,1}y_{2,1} + x_{2,2}y_{1,2} - x_{1,2}y_{2,2}). \end{aligned}$$

2) If $\Lambda \subset \mathbb{C}^n$ is a full lattice, then the above metric induces a Hermitian metric on the torus $\frac{\mathbb{C}^n}{\Lambda}$.
3) The *Fubini -Study metric on the projective space* $\mathbb{P}^n$. Let $\pi : \mathbb{C}^n - 0 \to \mathbb{P}^n$ be the projection and consider a section $Z : U \to \mathbb{C}^n - 0$ of π on an open subset U, then the form $\omega = \frac{i}{2\pi}\partial\overline{\partial} log ||Z||^2$ is a $(1,1)$ form, globally defined on $\mathbb{P}^n$, since we can check

that forms on two open subsets glue together on the intersection. For the section $Z = (1, w_1, \dots, w_n)$, $\omega = \frac{i}{2\pi}\sum_{i=1}^{n} dw_i \wedge d\overline{w_i} > 0$ is a real form (takes real value on real tangent vectors) and positive that is associated to a positive Hermitian form. Hence the projective space admits a Hermitian metric.

3.4. **Harmonic forms on compact complex manifolds.** In this section, a study of the $\overline{\partial}$ operator (similar to the previous study of d) will lead to the representation of Dolbeault cohomology by $\overline{\partial}$ harmonic forms as stated in Hodge theorem. We choose a Hermitian metric h with associated $(1,1)$ form ω with volume form $vol = \frac{1}{n!}\omega^n$. Then we define the L^2-inner product on $\mathcal{E}_X^{p,q}$ using the underlying Riemannian metric g and the adjoint operator to $\overline{\partial}$

$$\overline{\partial}^* : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q-1}$$

defined as $\overline{\partial}^* = - * \partial *$ satisfying

$$(\overline{\partial}^* \psi, \eta)_{L^2} = (\psi, \overline{\partial}\eta)_{L^2}$$

so to introduce the $\overline{\partial}-$Laplacian

Definition 3.15. The $\overline{\partial}-$Laplacian is defined as

$$\Delta_{\overline{\partial}} = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}$$

Harmonic forms of type (p, q) are defined as the solutions of the $\overline{\partial}-$Laplacian

$$\mathcal{H}^{p,q}(X) = \{\psi \in \mathcal{E}^{p,q}(X) \otimes \mathbb{C} : \Delta_{\overline{\partial}}(\psi) = 0\}.$$

We define similarly $\partial * = - * \overline{\partial} *$ and a Laplacian

$$\Delta_{\partial} = \partial \circ \partial^* + \partial^* \circ \partial$$

A basic result is

Theorem 3.16 (Hodge theorem)**.** *On a compact complex Hermitian manifold each cohomology class of type (p, q) is represented by a unique $\overline{\partial}-$harmonic global complex differential form of type (p, q)*

$$\mathcal{H}^{p,q}(X) \simeq H_{\overline{\partial}}^{p,q}(X)$$

moreover the space of $\overline{\partial}-$harmonic forms is finite dimensional (hence the space of $\overline{\partial}-$cohomology also).

Corollary 3.17.

$$H^q(X, \Omega_X^p) \simeq \mathcal{H}^{p,q}(X)$$

Corollary 3.18 (Serre duality)**.** The trace map is an isomorphism

$$Tr : H^n(X, \Omega_X^n) \xrightarrow{\sim} H^{2n}(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{C},\ \omega \mapsto \frac{1}{(2i\pi)^n}\int_X \omega$$

and the pairing is

$$H^p(X, \Omega_X^n) \otimes H^{n-p}(X, \Omega_X^n) \to H^n(X, \Omega_X^n)$$

is nondegenerate.

Notice that we introduce a factor $\frac{1}{(2i\pi)^n}$ of the integral to fit later with the fact that $H^n(X, \Omega_X^n)$ is a Hodge structure of weight $2n$.

3.5. **Kähler manifolds.** In general the exterior product of harmonic forms is not harmonic neither the restriction of harmonic forms to a submanifold is harmonic for the induced metric. The $\overline{\partial}-$Laplacian $\Delta_{\overline{\partial}}$ and the Laplacian Δ_d are not related in general. The theory will be more natural if we add the Kähler condition on the metric, most importantly the components of harmonic forms into type will be harmonic.

Definition 3.19. The Hermitian metric is Kähler if its associated $(1,1)$ form ω is closed: $d\omega = 0$.

Example 3.20. 1) A Hermitian metric on a compact Riemann surface is Kähler since $d\omega$ of degree 3 must vanish.
2) The Enclidean metric on a compact complex torus is Kähler.
3) The projective space with its canonical metric is Kähler.
4) The restriction of a Kähler metric on a submanifold is Kähler with associated $(1,1)$ form induced from the associated $(1,1)$ form on the ambient manifold.
5) The product of two Kähler manifolds is Kähler.

Corollary 3.21. 1) The fundamental class of a subvariety $Y \subset X$ is non zero in $H^*_{DR}(X,\mathbb{C})$. 2) The even Betti numbers $b_{2q}(X)$ are positive. Moreover

$$H^0(X,\Omega^q_X) \hookrightarrow H^q_{DR}(X,\mathbb{C})$$

We have in this case the following important relations of the Laplacians

Lemma 3.22.

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}$$

Consider the projections

$$\Pi^{p,q} : \mathcal{E}^r(X) \otimes \mathbb{C} \to \mathcal{E}^{p,q}(X)$$

Corollary 3.23. The projection on the (p,q) component of an harmonic form is harmonic and we have a natural decomposition

$$\mathcal{H}^r(X) \otimes \mathbb{C} = \oplus_{p+q=r} \mathcal{H}^{p,q}(X),\ \mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$$

One prove first that the Laplacian Δ_d commutes with the projection $\Pi^{p,q}$, hence the (p,q) components of an harmonic form, are harmonic. Since Δ_d is real, we deduce the conjugation property.

3.5.1. *Hodge decomposition.* Recall the definition of the type of cohomology

$$H^{p,q}(X) = \frac{Z_d^{p+q}(X)}{d\mathcal{E}^*(X) \cap Z_d^{p+q}(X)} \text{ where } Z_d^{p+q}(X) = Ker\, d \cap \mathcal{E}^{p,q}(X)$$

Theorem 3.24 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. There is an isomorphism of cohomology classes of type (p,q) with harmonic forms of the same type*

$$H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X).$$

which results into a decomposition of the complex cohomology spaces into a direct sum of complex subspaces

$$H^i(X,\mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}}(X)$$

called Hodge decomposition.

Corollary 3.25. There exists a Hodge decomposition on the cohomology of a smooth complex projective variety.

3.5.2. *Applications to Hodge theory.* We remark first the isomorphisms

$$H^{p,q}(X) \simeq H^{p,q}_{\overline{\partial}}(X) \simeq H^q(X, \Omega^p_X)$$

which shows in particular for $q = 0$

$$H^{p,0}(X) \simeq H^0(X, \Omega^p_X)$$

1) The holomorphic forms are closed and harmonic for any Kähler metric.
2) There are no non zero global holomorphic forms on $\mathbb{P}^n$ and more generally

$$H^{p,q}_{\overline{\partial}}(\mathbb{P}^n) \simeq H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = \begin{cases} 0, & \text{if } p \neq q \\ \mathbb{C} & \text{if } p = q \end{cases}$$

Exercise 3.26 (Holomorphic invariants in cohomology)**.** Any holomorphic $p-$form on a compact Kähler manifold is closed and it is 0 iff it is exact. Equivalently, there is a natural injective morphism: global holomorphic $p-$forms $\to H^p_{DR}$ with image $H^{p,0}$.

Proof. Let ϕ be holomorphic of type $p, 0$, then $\overline{\partial}\phi = 0$ and $\overline{\partial}^* \phi$ of type $p, -1$ must vanish, hence the Laplacian $\Delta_{\overline{\partial}}\phi = 0$, hence on Kähler manifold $\Delta_d \phi = 0$ and $d\phi = 0$.
Since ϕ is harmonic, it is orthogonal to the space of exact forms, hence if it is exact it must be 0. □

Remark 3.27. i) The form zdw on $\mathbb{C}^2$ is not closed.
ii) The spaces $H^{p,q}$ in H^{p+q}_{DR} are isomorphic to the holomorphic invariant $H^q(X, \Omega^p)$, but for a family the embedding of $H^{p,q}(X_t)$ into DeRham cohomology is not holomorphic.

Exercise 3.28 (Riemann Surface)**.** Let $\overline{C}$ be a compact Riemann surface, and let $C = \overline{C} - x_1, \dots, x_m$ be the open surface with m points in $\overline{C}$ deleted. Consider the long exact sequence

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \oplus_{i=1}^{i=m} \mathbb{Z} \to H^2(\overline{C}, \mathbb{Z}) \simeq \mathbb{Z} \to H^2(C, \mathbb{Z}) = 0$$

then

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m} \mathbb{Z} \to \mathbb{Z})$$

presents $H^1(C, \mathbb{Z})$ as an extension, with weight $W_1 H^1(C, \mathbb{Z}) = H^1(\overline{C}, \mathbb{Z})$ and $W_2 H^1(C, \mathbb{Z}) = H^1(C, \mathbb{Z})$.
The Hodge filtration is defined by the residue

$$0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(log\{x_1, \dots, x_m\}) \to \mathcal{O}_{\{x_1, \dots, x_m\}} \to 0$$

$F^0 H^1(C, \mathbb{C}) = H^1(C, \mathbb{C})$,
$F^1 H^1(C, \mathbb{C}) = H^0(\overline{C}, \Omega^1_{\overline{C}}(log\{x_1, \dots, x_m\})$.
$F^2 H^1(C, \mathbb{C}) = 0$. That is an extension of pure HS

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathrm{Ker}\,(\oplus_{i=1}^{i=m} \mathbb{Z}(-1) \to H^2(\overline{C}) \to 0$$

.

Exercise 3.29 (Algebraic cycles). Let X be a compact Kähler manifold and $\omega := -\mathcal{I}mH$ its real 2−form of type $(1,1)$ ($\omega \in \mathcal{E}^{1,1}_X \cap \mathcal{E}^2_X$). The volume form $vol \in \mathcal{E}^{2n}$ defined by g on X, can be defined by ω and is equal to $\frac{1}{n!}\omega^n$. We have locally

$$\omega = \frac{i}{2}\Sigma_i dz_i \wedge d\overline{z}_i, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Lemma 3.30. *For a manifold X compact and Kähler $H^{p,p}(X,\mathbb{Z}) \neq 0$ for $0 \leq p \leq$ dim.M.*

In fact the integral of the volume form $\int_X \omega^n > 0$. It follows that the cohomology class $\omega^n \neq 0 \in H^{2n}(X,\mathbb{C})$, hence the cohomology class $\omega^p \neq 0 \in H^{p,p}(X,\mathbb{C})$ since its cup product with ω^{n-p} is not zero.

3.6. **Cohomology class of a subvariety.** A compact oriented manifold X of real dimension m is triangulated and the sum of the oriented simplices of dim. m define a cycle class $[X] \in H_m(X,\mathbb{Z})$. The Poincaré dual of $[X]$ is called a cohomlology fundamental $[X] \in H^0(X,\mathbb{R})$. In particular, if X is a complex manifold of dim. n, then it has a class $[X] \in H_{2n}(X,\mathbb{Z})$. Let Z be a complex submanifold of codim. p in X and $i : Z \hookrightarrow X$ the embedding, then $i_*[Z] \in H_{2(n-p)}(X,\mathbb{Z})$ corresponds, by rational Poincaré duality, to a class in $H^{2p}(X,\mathbb{Z})$. Let

$$H^{p,p}(X,\mathbb{Z}) := \{a \in H^{2p}(X,\mathbb{Z}),\ \mathrm{Im}\,(a) \in H^{p,p}(X,\mathbb{C})\}$$

We define similarly the rational version by taking the image $H^{p,p}(X,\mathbb{Z})$ in the rational cohomology and by Poincaré duality its homology version the image $H_{p,p}(X,\mathbb{Z})$ in the rational homology.

Definition 3.31. For a complex compact manifold X the cohomology class $[\eta_Z] \in H^{p,p}(X,\mathbb{Z})$ of a complex submanifold Z of codim p is defined by the following relation

$$\forall \omega \in \mathcal{E}^{n-p,n-p}(X), \int_X \omega \wedge \eta_Z = \int_{i_*[Z]} \omega.$$

Lemma 3.32. *For a manifold X compact and Kähler the rational cohomology class of a complex submanifold Z of codim p is a non zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{Z})$ for $0 \leq p \leq$ dim X.*

Proof. For a compact Kähler manifold X, let ω be a Kähler form, then the integral on the homology class $[Z]$ of the restriction $i^*\omega$ is positive since it is a Kähler form on Z

$$\int_{i_*[Z]} \wedge^{n-p}\omega = \int_{[Z]} i^* \wedge^{n-p} \omega > 0.$$

□

Lemma 3.33. *For a projective smooth variety X, the fundamental class of an irreducible subvariety Z of codimension p is well defined and it is a non zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{Z})$ for $0 \leq p \leq dimX$.*

The easiest proof is to use a desingularisation $\pi : Z' \to Z$ and define $[Z] = \pi_*[Z'] \in H_{n-p,n-p}(X,\mathbb{Z})$, then take its Poincaré dual in $H^{p,p}(X,\mathbb{Z})$.

Definition 3.34. An $r-$cycle of an algebraic variety X is a formal finite linear combination $\Sigma_{i\in[1,h]} m_i Z_i$ of irreducible subvarieties Z_i with integer coefficients m_i. The group of $r-$cycles is denoted by $\mathcal{Z}_r(X)$.

The class of irreducible subvarieties extend into a linear morphism

$$cl : \mathcal{Z}_r(X) \to H^{r,r}(X, \mathbb{Z})$$

defined modulo linear equivalence of cycles. The cohomology cycles in the image of the cycle class map are called algebraic. Respectively we define the map

$$cl_{\mathbb{Q}} : \mathcal{Z}_r(X) \otimes \mathbb{Q} \to H^{r,r}(X, \mathbb{Q})$$

3.6.1. *Hodge conjecture.* Is the map $cl_{\mathbb{Q}}$ surjective?

4. Polarized Hodge structure

4.1. **Hodge structure.** Having the above properties of cohomology of Kähler manifolds in mind, it appeared to be rewarding to introduce this notion as a formal structure in linear algebra

Definition 4.1. A Hodge structure (HS) of weight n is defined by the data
i)A free abelian group $H_{\mathbb{Z}}$ (the lattice)
ii) A decomposition by complex subspaces

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=n} H^{p,q} \text{ satisfying } \quad H^{p,q} = \overline{H^{q,p}}$$

Remark 4.2. The conjugation on $H_{\mathbb{C}}$ makes sense with respect to $H_{\mathbb{Z}}$. A subspace $V \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $\overline{V} = V$ has a real structure, that is $V = (V \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. In particular $H^{p,p} = (H^{p,p} \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

Example 4.3. Tate Hodge structure $\mathbb{Z}(1)$ is defined by

$$H_{\mathbb{Z}} = 2i\pi\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-1,-1}$$

The tensor product of Hodge structures is defined in general. In this case the $n-$tensor product $\mathbb{Z}(1) \otimes \cdots \otimes \mathbb{Z}(1)$ of $\mathbb{Z}(1)$ is defined by

$$\mathbb{Z}(m) := (H_{\mathbb{Z}} = (2i\pi)^m \mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-m,-m})$$

and the Tate twist of a HS $H = (H_{\mathbb{Z}}, \oplus_{p+q=n} H^{p,q})$ is $H(m)$ defined by

$$(H_{\mathbb{Z}} \otimes (2i\pi)^m \mathbb{Z}, \oplus_{p+q=n} H^{p-m,q-m})$$

The DeRham cohomology $H^i(X, \mathbb{R})$ of a compact Kähler manifold is a HS of weight i with lattice defined by the image of integral cohomology $Im : H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{R})$, in particular if we consider the following trace map

$$H^{2n}(X, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}(-n), \quad \omega \to \frac{1}{(2i\pi)^n} \int_X \omega.$$

then Poincaré duality will be compatible with HS

$$H^{n-i}(X, \mathbb{R}) \simeq Hom(H^{n+i}(X, \mathbb{R}), \mathbb{R}(-n)).$$

Theorem 4.4 (Deligne)**.** *Let X be a smooth complex algebraic variety. There exists a decomposition of the complex cohomology spaces into a direct sum of complex subspaces*

$$H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}}(X)$$

defining a HS of weight i on cohomology.

4.1.1. *The Hodge filtration.* To study variations of HS, Griffiths introduced an equivalent structure defined by the Hodge filtration which vary holomorphically. Given a HS: $(H_{\mathbb{Z}}, H^{p,q})$ of weight i, we define a filtration F^* on $H_{\mathbb{C}}$ by subspaces

$$F^p H_{\mathbb{C}} := \oplus_{r \geq p} H^{r,i-r}$$

the filtration F is decreasing such that the following decomposition is satisfied

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{i-p+1} H_{\mathbb{C}}}$$

The Hodge decomposition is then determined by the filtration as

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}},\ p+q=i$$

4.1.2. *The Hodge filtration on the DeRham complex.* The Hodge filtration on cohomology plays an important role since its definition does not use the theory of harmonic forms, although the Hodge result on the filtration is based on such theory. It is defined by a filtration F of the holomorphic DeRham complex by subcomplexes

$$F^p \Omega_X^* := 0 \cdots 0 \to \Omega_X^p \to \cdots \Omega_X^q \to \cdots \Omega_X^n$$

The simple complex, associated to the resolution of each term Ω_X^r by a Dolbeault complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for $r \geq p$

$$F^p \mathcal{E}_X = s(\mathcal{E}_X^{r,*}, \overline{\partial}), \partial)_{r \geq p}$$

is a fine resolution of $(F^p \Omega_X^*, d = \partial)$ and define a Hodge filtration by subcomplexes of $\mathcal{E}_X \otimes_{\mathbb{R}} \mathbb{C}$. We have

$$H^i(F^p \mathcal{E}^*(X), d) \simeq F^p H^i(X, \mathbb{C})$$

Remark 4.5. We will see that the spectral sequence defined by the Hodge filtration degenerates at rank one.

4.2. **Lefschetz decomposition.** Let $[\omega] \in H^2(X, \mathbb{R})$ denotes the Kähler class of X. The cup-product with $[\omega]$ define morphisms

$$L : H^q(X, \mathbb{R}) \to H^{q+2}(X, \mathbb{R}), \quad L : H^q(X, \mathbb{C}) \to H^{q+2}(X, \mathbb{C})$$

Let $n = \dim X$ and define the primitive subspaces

$$H^{n-i}_{prim}(\mathbb{R}) := Ker(L^{i+1} : H^{n-i}(X, \mathbb{R}) \to H^{n+i+2}(X, \mathbb{R}))$$

and similarly for complex coefficients $H^{n-i}_{prim}(\mathbb{C}) \simeq H^{n-i}_{prim}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, then we deduce from the following Lefschetz isomorphism theorem a decomposition of the cohomology in terms of primitive spaces

Theorem 4.6 (Lefschetz)**.** *The iterated linear operator L induces isomorphisms for each i*

$$L^i : H^{n-i}(X, \mathbb{R}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{R}),\ L^i : H^{n-i}(X, \mathbb{C}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{C})$$

It gives rise to a decomposition of the cohomology into direct sum of image of primitive subspaces by L^r, $r \geq 0$

$$H^q(X, \mathbb{R}) = \oplus_{r \geq 0} L^r H^{q-2r}_{prim}(\mathbb{R}), \quad H^q(X, \mathbb{C}) = \oplus_{r \geq 0} L^r H^{q-2r}_{prim}(\mathbb{C})$$

Since the operator L acting on the HS is of bidegree $(1,1)$, the decomposition and the primitive subspaces are compatible with HS.
If X is moreover projective, then the action of L is defined with rational coefficients and the decomposition apply on rational cohomology.

Proof. 1) First we consider the action of L on sheaves, $L = \wedge\omega : \mathcal{E}_X^r \to \mathcal{E}_X^{r+2}$, then we introduce its adjoint operator $\Lambda = L^* : \mathcal{E}_X^r \to \mathcal{E}_X^{r-2}$ which can be defined by $\Lambda = *^{-1} \circ L \circ *$. Note that the operator

$$h = \sum_{p=0}^{2n} (n-p)\Pi^p$$

where Π^p is the projection in degree p on $\mathcal{E}_X^*$ and $n = \dim X$ satisfy the relation

$$[\Lambda, L] = h.$$

from which we can deduce the injectivity of the morphism

$$L^i : \mathcal{E}_X^{n-i} \to \mathcal{E}_X^{n+i}$$

For this we use the formula

$$[L^r, \Lambda] = (r(k-n) + r(r-1))L^{r-1}$$

which is proved by induction on r. Such morphism commutes with the Laplacian and since cohomology classes can be represented by global harmonic sections, it induces an isomorphism on cohomology vector spaces of finite equal dimension

$$L^i : H^{n-i}(X, \mathbb{R}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{R})$$

Moreover the extension of the operator L to complex coefficients is compatible with the bigrading (p,q) since ω is of type $(1,1)$. The decomposition of the cohomology into direct sum of image of primitive subspaces by L^r, $r \geq 0$ follows from the previous isomorphisms.
2) Another proof is based on the representation theory of the Lie algebra sl_2. We represent L by the action on global sections of the operator $L = \wedge\omega : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q+1}(X)$ and its adjoint $\Lambda = L^* : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p-1,q-1}(X)$ is defined by $\Lambda = *^{-1} \circ L \circ *$. We admit the relations

$$[\Lambda, L] = h$$
$$[h, L] = -2L$$
$$[h, \Lambda] = 2\Lambda$$

We deduce from the action of the operators L, h, Λ on the space of harmonic forms $\oplus\mathcal{H}_d^*(X) \simeq \oplus H^*(X)$, a representation of the Lie algebra sl_2 by identifying the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ resp. with } \Lambda,\ L,\ h.$$

Then the theorem follows from the general structure of such representation. □

A geometric interpretation of this theorem in the case of a projective subvariety $X \subset \mathbb{P}^N$ of dimension n is given via Poincaré isomorphism $H^{n-k}(X) \xrightarrow{\sim} H_{n-k}(X)$, so that the operator L corresponds to the intersection with an hyperplane section V of X so that the result is an isomorphism

$$H_{n+k}(X) \xrightarrow{\cap \mathbb{P}^{N-k}} H_{n-k}(X)$$

The primitive cohomology $H^{n-k}_{prim}(X)$ corresponds to a subspace $H^f_{n-k}(X) \subset H_{n-k}(X)$ equal to the image of $H_{n-k}(X - V) \to H_{n-k}(X)$ (f for finite cycles or outside infinity).

4.3. **Polarization of HS.** We define in this subsection a scalar product on cohomology compatible with HS and satisfying relations known as Hodge Riemann relations leading to a polarization of the primitive cohomology which is an additional highly rich structure on cohomology of proper algebraic varieties. Such product is defined via the wedge product on DeRham cohomology, or cup-product on singular cohomology and called sometimes an intersection form (referring implicitly to intersection on homology, with which it is compatible via Poincaré duality).

4.3.1. *Hermitian product on cohomology.* Representing cohomology classes by differential forms, we define a bilinear form

$$Q(\alpha,\beta) = (-1)^{\frac{j(j-1)}{2}} \int_X \alpha \wedge \beta \wedge \omega^{n-j}, \quad \forall [\alpha],[\beta] \in H^j(X,\mathbb{C})$$

where ω is the Kähler class. On projective varieties the Kähler class is in the integral lattice in rational cohomology, the product is defined on rational cohomology and preserves the integral lattice. It is alternating if j is odd, symmetric if j is even and non degenerate by Lefschetz isomorphism and Poincaré duality. By consideration of type, the Hodge and Lefschetz decompositions are orthogonal with respect to Q

$$Q(H^{p,q}, H^{p',q'}) = 0, \quad \text{unless } p = q', q = p'.$$

Proposition 4.7 (Hodge-Riemann bilinear relations)**.** On the primitive component $H^{p,q}_{prim} := H^j(X,\mathbb{C})^{p,q}_{prim}$

$$i^{p-q}Q(\alpha,\overline{\alpha}) > 0, \quad \forall \alpha \neq 0 \in H^{p,q}_{prim}$$

the form $i^{p-q}Q(\alpha,\overline{\alpha})$ is positive definite.

Proof. We use

Lemma 4.8. *Let $\omega \in \Omega^{q,p}_{X,x} \subset \mathcal{E}^j_{X,x}$ be a primitive element, then*

$$*\omega = (-1)^{\frac{j(j+1)}{2}} i^{q-p} \frac{L^{n-j}}{(n-j)!}\omega$$

We represent a primitive cohomolgy class by a primitive harmonic form α of degree j, then since $(-1)^j i^{q-p} = (-1)^{q-p} i^{q-p} = i^{p-q}$ we deduce from the lemma $L^{n-j}\overline{\alpha} = i^{q-p}(n-j)!(-1)^{\frac{j(j-1)}{2}}(*\overline{\alpha})$ and the product is written via the L^2-norm, as

$$i^{p-q}Q(\alpha,\overline{\alpha}) = (n-j)! \int_X \alpha \wedge *\overline{\alpha} = (n-j)! ||\alpha||^2_{L^2}$$

□

This result suggest to introduce the Weil operator C on cohomology

$$C(\alpha) = i^{p-q}\alpha, \quad \forall \alpha \in H^{p,q}$$

Notice that C depends on the decomposition, in particular the action of C in a variation of Hodge structure is differentiable in the parameter space.
We deduce from Q a new non degenerate Hermitian product

$$H(\alpha,\beta) = Q(C\alpha,\overline{\beta}), \quad \forall [\alpha],[\beta] \in H^j(X,\mathbb{C})$$

It satisfies for $\alpha,\beta \in H^{p,q}$

$$H(\beta,\alpha) = \overline{H(\alpha,\beta)}$$

since
$\overline{H(\alpha,\beta)} = \overline{Q(i^{p-q}\alpha,\overline{\beta})} = \overline{i}^{p-q}Q(\overline{\alpha},\beta) = (-1)^{2j} i^{p-q} Q(\beta,\overline{\alpha}) = H(\beta,\alpha)$.

Remark 4.9. i)When the class $[\omega] \in H^j(X,\mathbb{Z})$ is integral, which is the case for algebraic varieties, the product Q is integral (with integral value on integral classes).
ii) The integral bilinear form Q extends by linearity to the complex space $H^k_{\mathbb{Z}} \otimes \mathbb{C}$. Its associated form $H(\alpha,\beta) := Q(\alpha,\overline{\beta})$ is not Hermitian if k is odd. One way to overcome this sign problem is to define H as $H(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$, still this form will not be positive definite in general.

Definition 4.10 (Polarization of HS)**.** A Hodge structure ($H_{\mathbb{Z}}, (H_{\mathbb{C}}, F)$) of weight n is polarized if a non degenerate scalar product Q is defined on $H_{\mathbb{Z}}$, alternating if n is odd and symmetric if n is even, such that the Hermitian form on $H_{\mathbb{C}}$ defined as $H(\alpha,\beta) := i^{p-q}Q(\alpha,\overline{\beta})$ for $\alpha,\beta \in H^{p,q}$ is orthogonal to the Hodge decomposition and $i^{p-q}H(\alpha,\alpha)$ is positive definite on the primitive component of type (p,q)(satisfy Hodge-Riemann bilinear relations).

4.3.2. *Polarized subvarieties are algebraic.* A theorem of Kodaira states that a Kähler variety with an integral class $[\omega]$ is projective (which means it can be embedded as a closed analytic subvariety in a projective space, hence it is necessarily an algebraic subvariety.

4.4. **Examples.**

4.4.1. *Projective space.* Let $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ denotes the Chern class of the canonical line bundle; it is dual to the homology class of an hyperplane.

Theorem 4.11. *$\mathbb{H}^i(\mathbb{P}^n,\mathbb{Z}) = 0$ for i odd and $\mathbb{H}^i(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$ for i even with $[H]^i$ the cohomology class of an hyperplane to the power i for the cup product, as generator; hence: $\mathbb{H}^{2r}(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}(-r)$ as HS.*

4.4.2. *Hodge structures of weight 1 and abelian varieties.* Given a HS: $(H_{\mathbb{Z}}, H^{1,0} \oplus H^{0,1})$, the projection on $H^{0,1}$ induces an isomorphism of real vector spaces

$$H_{\mathbb{R}} \to H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} \to H^{0,1}$$

since $\overline{H^{0,1}} = H^{1,0}$. Then we deduce that $H_{\mathbb{Z}}$ is a lattice in the complex space $H^{0,1}$, and the quotient $T := H^{0,1}/H_{\mathbb{Z}}$ is a complex torus . When $H_{\mathbb{Z}}$ is identified with the image of cohomology spaces $Im(H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{R}))$ of a complex manifold X, resp. $H^{0,1}$ with $H^1(X,\mathcal{O})$, the torus T will be identified with the Picard variety $Pic^0(X)$ parameterizing the holomorphic line bundles on X with first Chern class equal to zero as follows. We consider the exact sequence of sheaves defined by $f \mapsto e^{2i\pi f}$

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \to 1$$

where 1 is the neutral element of the sheaf of multiplicative groups $\mathcal{O}^*_X$, and its associated long exact sequence of derived functors of the global sections functor

$$\to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}^*_X) \to H^2(X,\mathbb{Z})$$

where the morphisms can be interpreted geometrically; when the space $H^1(X,\mathcal{O}^*_X)$ is identified with isomorphisms classes of line bundles on X, the last morphism defines the Chern class of the line bundle. Hence the torus T is identified with the isomorphism classes $\mathcal{L}$ with $c_1(\mathcal{L}) = 0$

$$T := \frac{H^1(X,\mathcal{O}_X)}{ImH^1(X,\mathbb{Z})} \simeq Pic^0(X) := Ker(H^1(X,\mathcal{O}^*_X) \xrightarrow{c_1} H^2(X,\mathbb{Z}))$$

It is possible to show that the Picard variety of a smooth projective variety is an abelian variety. (Define a Kähler form with integral class on $Pic^0(X)$).

4.4.3. *Hodge structures of weight 2.*

$$(H_{\mathbb{Z}},\, H^{2,0} \oplus H^{1,1} \oplus H^{0,2};\, H^{0,2} = \overline{H^{2,0}},\, H^{1,1} = \overline{H^{1,1}};\, Q)$$

the intersection form Q is symmetric and $H(\alpha,\beta) = Q(\alpha,\overline{\beta})$ is Hermitian. The decomposition is orthogonal for H with H positive definite on $H^{1,1}$, negative definite on $H^{2,0}$ and $H^{2,1}$.

Lemma 4.12. *The HS is completely determined by the subspace $H^{2,0} \subset H_{\mathbb{C}}$ s.t. $H^{2,0}$ is totally isotropic for Q and the associated Hermitian product H is negative definite on $H^{2,0}$. The signature of Q is $(h^{1,1}, 2h^{2,0})$.*

In the lemma $H^{0,2}$ is determined by conjugation and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$.

4.4.4. *Hodge structures of $K3$ type.*

Lemma 4.13. *Let Q be a symmetric non degenerate form on a lattice $H_{\mathbb{Z}}$ of signature $(h^{1,1}, 2)$. The HS of weight 2 on $H_{\mathbb{C}}$ polarized by Q are parameterized by the complex manifold*

$$\mathcal{D} := \{\omega \in \mathbb{P}(H_{\mathbb{C}}) | Q(\omega,\omega) = 0, Q(\omega,\overline{\omega}) < 0\}$$

Here $H^{2,0}$ must be a line parameterized by $\omega \in \mathbb{P}(H_{\mathbb{C}})$ where the signature is -1 as well on its conjugate and $h^{1,1}$ on $H^{1,1}$.

5. Mixed Hodge Structures (MHS)

Classical Hodge theory apply only for compact Kähler manifolds. In order to describe specific properties of cohomology of algebraic varieties inspired by étale cohomology and the solution of Weil conjectures, Deligne introduced in [6] a far reaching generalization of HS called mixed Hodge structure (MHS) that can be defined on the cohomology of all algebraic varieties.
We are essentially concerned by filtrations of vector spaces. However it is not more difficult to describe this notion in the terminology of abelian categories.
The formal study of MHS, namely the fact that they form an abelian category, will help the reader before he is confronted to their construction.

5.1. **Filtrations.** Let $\mathbb{A}$ denote an abelian category.

Definition 5.1. A decreasing filtration F (resp. increasing) of an object A of $\mathbb{A}$ is a family of sub-objects of $\mathbb{A}$, satisfying

$$\forall n, m, \qquad n \leq m \Longrightarrow F^m(A) \subset F^n(A) (resp. \quad n \leq m \Longrightarrow F_n(A) \subset F_m(A))$$

If F is a decreasing filtration (resp. W an increasing filtration), a shifted filtration $F[n]$ by an integer n is defined as

$$(F[n])^p(A) = F^{n+p}(A).$$

Decreasing filtrations F will be considered for a general study. Statements for increasing filtrations W can be deduced by the change of indices $W_n(A) = F^{-n}(A)$.
A filtration is finite if there exists integers n and m such that $F^n(A) = A$ and $F^m(A) = 0$.
A morphism of filtered objects $(A, F) \xrightarrow{f} (B, F)$ is a morphism $A \xrightarrow{f} B$ satisfying

$f(F^n(A)) \subset F^n(B)$ for all $n \in \mathbb{Z}$. Filtered objects (resp. of finite filtration) form an additive category with existence of *kernel* and *cokernel* of a morphism with natural induced filtrations as well image and coimage, however the image and coimage will not be necessarily filtered-isomorphic, which is the main obstruction to obtain an abelian category.
In the case of the category of modules over a ring, a morphism $f : (A, F) \to (B, F)$ is strict if

$$f(F^n(A) = f(A) \cap F^n(B)$$

so that any element $b \in F^n(B) \cap ImA$ is already in $Im\, F^n(A)$.
More generally a morphism $f : (A, F) \to (B, F)$ is strict if it induces a filtered isomorphism $Coim(f) \to Im(f)$ from the coimage to the image of f.

Definition 5.2. The graded object associated to (A, F) is defined as

$$Gr_F(A) = \oplus_{n\in\mathbb{Z}} Gr^n_F(A) \quad \text{where} \quad Gr^n_F(A) = F^n(A)/F^{n+1}(A)$$

5.1.1. *Induced filtration.* A filtered object (A, F) induces a filtration on a sub-object $i : B \to A$ of A, defined by $F^n(B) = B \cap F^n(A)$. It is the unique filtration s.t. i is strict. Dually, the quotient filtration on A/B is defined by

$$F^n(A/B) = p(F^n(A)) = (B + F^n(A))/B \simeq F^n(A)/(B \cap F^n(A))$$

where $p : A \to A/B$ is the projection.

5.2. **Hodge Structure (HS).** For any sub-ring A of $\mathbb{R}$ equal to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and any A-module H_A, the complex conjugation extends to a conjugation on the space $H_\mathbb{C} = H_A \otimes_A \mathbb{C}$. A filtration F on $H_\mathbb{C}$ has a conjugate filtration $\overline{F}$ s.t. $(\overline{F})^i H_\mathbb{C} = \overline{F^i H_\mathbb{C}}$.

Definition 5.3 (HS1)**.** An A-Hodge structure H of weight n consists of

(1) an A-module of finite type H_A (the lattice).
(2) a finite filtration F on $H_\mathbb{C}$ (the Hodge filtration) s.t. F and its conjuguate $\overline{F}$ satisfy the relation

$$Gr^p_F Gr^q_{\overline{F}}(H_\mathbb{C}) = 0, \quad \text{for } p + q \neq n$$

The HS is real when $A = \mathbb{R}$, rational when $A = \mathbb{Q}$ and integral when $A = \mathbb{Z}$.

5.3. **Opposite filtrations.** Two finite filtrations F and G on an object A of $\mathbb{A}$ are n-opposite if

$$Gr^p_F Gr^q_G(A) = 0 \quad \text{for } p + q \neq n.$$

hence the Hodge filtration F on a HS of weight n is n-opposite to its conjugate $\overline{F}$.
The following constructions will define an equivalence of categories between objects of $\mathbb{A}$ with two n-opposite filtrations and bigraded objects of $\mathbb{A}$ of the following type.

Example 5.4. Let $A^{p,q}$ be a bigraded object of $\mathbb{A}$ s.t. $A^{p,q} = 0$ for all but a finite number of couples (p, q) and $A^{p,q} = 0$ for $p + q \neq n$; then we define two n-opposite filtrations on $A = \sum_{p,q} A^{p,q}$:

$$F^p(A) = \Sigma_{p'\geq p} A^{p',q'}, \quad G^q(A) = \Sigma_{q'\geq q} A^{p',q'}$$

We have $Gr^p_F Gr^q_G(A) = A^{p,q}$.

We admit reciprocally [6]

Proposition 5.5. i) Two finite filtrations F and G on an object A are n-opposite, if and only if

$$\forall p, q, \quad p+q = n+1 \Rightarrow F^p(A) \oplus G^q(A) \simeq A.$$

ii) If F and G are n-opposite, and if we put $A^{p,q} = F^p(A) \cap G^q(A)$ for $p+q = n$, $A^{p,q} = 0$ for $p+q \neq n$, then A is a direct sum of $A^{p,q}$, moreover F and G can be deduced from the bigraded object $A^{p,q}$ of A by the above procedure.

We have an equivalent definition of HS.

Definition 5.6 (HS2)**.** An $A-$HS on H of weight n is a pair of an A-module H_A and a decomposition into a direct sum on $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$

$$H_{\mathbb{C}} = \oplus_{p+q=n} H^{p,q} \quad \text{such that} \quad \overline{H}^{p,q} = H^{q,p}$$

The relation with the previous definition is given by $H^{p,q} = F^p(H_{\mathbb{C}}) \cap \overline{F}^q(H_{\mathbb{C}})$ for $p+q = n$

5.4. **Morphism of HS.** A morphism $f : H \to H'$ of HS, is an homomorphism $f\colon H_A \to H'_A$ such that $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is compatible with the bigrading, or equivalently with Hodge filtration.
If H and H' are of distinct weights, f is necessarily 0.
The HS of weight n form an abelian category. If H is of weight n and H' of weight n', we define a HS, $H \otimes H'$ of weight $n+n'$.

5.4.1. *Hom_{HS}.* The internal Hom to the category of HS, denoted by $Hom_{HS}(H, H')$, is the sub-group of $Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ of elements of type $(0,0)$ in the HS on $Hom(H, H')$.

Example 5.7. Tate HS, $\mathbb{Z}(1)$ is the HS of weight -2, rank 1, purely bigraded of type $(-1,-1)$, and of lattice $2\pi i\mathbb{Z} \subset \mathbb{C}$.
For $n \in \mathbb{Z}$, we define $\mathbb{Z}(n)$ as the n tensor product of $\mathbb{Z}(1)$: $\mathbb{Z}(n)$ is the HS of weight $-2n$, rank 1, purely bigraded of type $(-n,-n)$, with lattice $(2\pi i)^n\,\mathbb{Z} \subset \mathbb{C}$.

Definition 5.8. A polarization is a bilinear morphism $S : H \otimes \overline{H} \to \mathbb{C}$ s.t.

$$S(F^p, c(\overline{F}^q) = S(\overline{F}^p, c(F^q)) = 0 \quad \text{for } p+q > n$$

and moreover $S(C(H)u, c(v))$ is a positive definite Hermitian form on H where $C(H)$ denotes Weil action on H.

Example 5.9. A complex HS of weight 0 on a complex vector space H is given by a decomposition into a direct sum of subspaces

$$H = \oplus_{p\in\mathbb{Z}} H^p$$

A polarization is an Hermitian form on H for which the decomposition is orthogonal and whose restriction to H^p is definite for p even and negative definite for odd p.

5.5. **Mixed Hodge structure (MHS).** This structure has been introduced in [6]. Let A be equal to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, then for an A-module of finite type H_A, $H_A \otimes \mathbb{Q}$ is defined as $H_A \otimes_A \mathbb{Q} := H_{\mathbb{Q}}$, if A is equal to $\mathbb{Z}$, otherwise as $H_A \otimes_{\mathbb{Q}} \mathbb{Q} = H_A$.

Definition 5.10 (Deligne)**.** An A-mixed Hodge structure H consists of
1) an A-module of finite type H_A
2) a finite increasing filtration W of the $A \otimes \mathbb{Q}$ vector space $H_A \otimes \mathbb{Q}$ called the weight filtration

3) a finite decreasing filtration F of the $\mathbb{C}$-vector space $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$, called the Hodge filtration, such that the systems

$$Gr_n^W H = (Gr_n^W (H_{A\otimes\mathbb{Q}}), (Gr_n^W H_{\mathbb{C}}, F))$$

consist of $A \otimes \mathbb{Q} - HS$ of weight n.

The MHS is called real if $A = \mathbb{R}$, rational if $A = \mathbb{Q}$ and integral if $A = \mathbb{Z}$.

Example 5.11. 1) A HS, H of weight n, is a MHS with weight filtration

$$W_i(H_{\mathbb{Q}}) = 0 \quad \text{for} \quad i < n \qquad \text{and} \quad W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} \quad \text{for} \quad i \geq n.$$

2) Let (H^i, F_i) be a finite family of $A-$HS of weight $i \in \mathbb{Z}$; then $H = \oplus_i H^i$ is endowed with the following MHS

$$H_A = \oplus_i H_A^i, \quad W_n = \oplus_{i\leq n} H_A^i \otimes \mathbb{Q}, \quad F = \oplus_i F_i.$$

5.5.1. *Morphism of MHS.* A morphism $f : H \to H'$ of MHS is a morphism $f : H_A \to H'_A$ whose extension to $H_{\mathbb{Q}}$ (resp. $H_{\mathbb{C}}$) is compatible with the filtration W (resp. F, hence $\overline{F}$).

Definition 5.12 (opposite filtrations)**.** Three finite filtrations W (increasing), F and G on an object A of $\mathbb{A}$ are opposite if

$$Gr_F^p Gr_G^q Gr_n^W (A) = 0 \quad \textit{for } p + q \neq n.$$

This condition is symmetric in F and G. It means that F and G induce on $W_n(A)/W_{n-1}(A)$ two n-opposite filtrations, then $Gr_n^W(A)$ is bigraded

$$W_n(A)/W_{n-1}(A) = \oplus_{p+q=n} A^{p,q} \quad \textit{where } A^{p,q} = Gr_F^p Gr_G^q Gr_{p+q}^W (A)$$

Example 5.13. i) A bigraded object $A = \oplus A^{p,q}$ of finite bigrading has the following three opposite filtrations

$$W_n = \oplus_{p+q\leq n} A^{p,q} \quad , F^p = \oplus_{p'\geq p} A^{p'q'} \quad , G^q = \oplus_{q'\geq q} A^{p'q'}$$

ii) In the definition of an $A-$MHS, the filtration $W_{\mathbb{C}}$ on $H_{\mathbb{C}}$ deduced from W by saclar extension, the filtration F and its complex conjugate , form a system $(W_{\mathbb{C}}, F, \overline{F})$ of three opposite filtrations.

5.5.2. *A canonical decomposition of the weight filtration.* Hodge decomposition. The definition of weight is given by a filtration and not a decomposition as a HS. The quotient W_n/W_{n-2} is a non split extension of the two pure Hodge structures Gr_n^W and Gr_{n-1}^W, in the category of MHS that we aim to study. The question here is how far we can lift the bigrading on $Gr^W(A)$ and at what price do we can split the weight filtration W. Deligne constructed a decomposition of the object A with its three opposite filtrations. For each pair of integers (p, q), he introduces the subspaces of A

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \overline{F^{q-2}} \cap W_{p+q-3} + \cdots)$$

By construction they are related for $p + q = n$ to the components $A^{p,q}$ of the Hodge decomposition: $Gr_n^W A \simeq \oplus_{n=p+q} A^{p,q}$; however, in general $I^{p,q} \neq \overline{I^{q,p}}$ but $I^{p,q} \equiv \overline{I^{q,p}}$ modulo W_{p+q-2}.

Lemma 5.14. *The projection $W_{p+q} \to Gr_{p+q}^W A \simeq \oplus_{p'+q'=p+q} A^{p',q'}$ induces an isomorphism $\varphi : I^{p,q} \xrightarrow{\sim} A^{p,q}$. Moreover*

$$W_n = \oplus_{p+q\leq n} I^{p,q}, \quad F^p = \oplus_{p'\geq p} I^{p',q'}$$

Theorem 5.15 (Deligne). *i) The category of MHS is abelian.*
ii) The kernel (resp. cokernel) of a morphism $f : H \to H'$ has as integral lattice the kernel (resp. cokernel) K of $f : H_A \to {H'}_A$, moreover $K \otimes \mathbb{Q}$ and $K \otimes \mathbb{C}$ are endowed with induced filtrations (resp. quotient filtrations) by W and F of $H_{A\otimes\mathbb{Q}}$ and $H_{\mathbb{C}}$ (resp. ${H'}_{A\otimes\mathbb{Q}}$ and ${H'}_{\mathbb{C}}$).
iii) Each morphism $f : H \to H'$ is strictly compatible with the filtrations W on $H_{A\otimes\mathbb{Q}}$ and ${H'}_{A\otimes\mathbb{Q}}$ as well the filtrations F on $H_{\mathbb{C}}$ and ${H'}_{\mathbb{C}}$. It induces morphisms of $A \otimes \mathbb{Q}-HS$, $Gr_n^W(f) : Gr_n^W(H_{A\otimes\mathbb{Q}}) \to Gr_n^W({H'}_{A\otimes\mathbb{Q}})$ and morphisms $Gr_F^p(f) : Gr_F^p(H_{\mathbb{C}}) \to Gr_F^p({H'}_{\mathbb{C}})$ strictly compatible with the induced filtrations $W_{\mathbb{C}}$.
iv) The functor Gr_n^W from the category of MHS to the category $A\otimes\mathbb{Q}-HS$ of weight n is exact.
v) The functor Gr_F^p is exact.

5.5.3. *Hodge numbers.* Let H be a MHS and set $H^{pq} = Gr_F^p Gr_{\overline{F}}^q Gr_{p+q}^W H_{\mathbb{C}} = (Gr_{p+q}^W H_{\mathbb{C}})^{p,q}$. The Hodge numbers of H are the integers $h^{pq} = dim_{\mathbb{C}} H^{pq}$, that is the Hodge numbers h^{pq} of the HS: $Gr_{p+q}^W H$.

6. Hypercohomology of a Filtered Complex

The construction of MHS on cohomology groups of algebraic varieties proceeds first by attaching to the variety a bifiltered complex of sheaves (Ω^*, W, F)(essentially a variant of DeRham complex) satisfying certain conditions to insure that the induced filtrations by W and F on cohomology define a MHS. Such bifiltered complexes, called mixed Hodge complexes (MHC), are defined in [Deligne]. Since the construction will be applied respectively in the case of compact Kähler manifolds, the complement of a normal crossing divisor (NCD) and then a simplicial scheme, it has been necessary to establish in [6] a common language and a set of axioms which lead naturally to the construction of a MHS on the hypercohomology of such complexes. For a deep knowledge of sheaf theory, the book by Godement [8] for example is good for a start.

6.1. **Derived categories.** The language of derived categories is the most appropriate at this point. It is based on the fact that a complex is a more rich object than merely its cohomology. However since we need acyclic resolutions and there is in general many choices of such resolutions, one needs to change the structure of the category by changing only the structure of morphisms. Verdier showed that the following two constructions lead to a new category satisfactory for our purpose and which proved to be useful in various domains of geometry.
1) The objects of the category are the complexes.
2) The morphisms of the category are defined after the following modifications
i) Two morphisms are equivalent if they are homotopic.
ii) A morphism is invertible if it induces an isomorphism on cohomology (then it is called a quasi-isomorphism).
These two operations have been carried by Verdier [13]. Although the objects remain the same, the modifications on morphisms can transform the category significantly (for example non zero morphisms may become isomorphic to zero and the inverse ψ of a morphism φ need to satisfy $\psi \circ \varphi$ and $\varphi \circ \psi$ are only homotopic to the identity [1]).
In practice the reader should apply the following rule : the sign $\mathbf{R}T$ *in front of a*

functor T applied to a complex of sheaves K refers to the complex TK' where K' is a resolution of K by $T-$acyclic sheaves .

6.2. **Spectral sequence of a filtered complex.** A filtration F of a complex K by sub-complexes induces a filtration on its cohomology with associated graded cohomology . The spectral sequence ([6], [8]) $E_r(K,F)$ associated to F leads, for large r and under mild conditions, to such graded cohomology defined by the filtration. This chapter is technical and we recommend to follow the proofs at a first reading on an example, and return later with the study of MHC where the spectral sequence becomes interesting and meaningful since it contains deep geometrical information.
The connecting morphism

$$H^{p+q}(W_{-p+r-1}K/W_{-p-r}K)\xrightarrow{\partial}H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K).$$

send W_{-p} to W_{-p-r}. The injection $W_{-p-r}K \to W_{-p-1}K$ induces a morphism

$$H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K) \xrightarrow{\varphi} E_r^{p+r,q-r+1} = Gr^W_{-p-r}H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K)$$

Then $\varphi \circ \partial$ restricted to W_{-p} induces the differential

$$d_r : E_r^{pq} \to E_r^{p+r,q-r+1}.$$

The injection $W_{-p+r-1} \to W_{-p+r}K$ induces the isomorphism

$$H(E_r^{pq}, d_r) \xrightarrow{\sim} E_{r+1}^{pq} = Gr^W_{-p}H^{p+q}(W_{-p+r}K/W_{-p-r-1}K).$$

The spectral sequence $E_i^{p,q}(K,W)$ degenerates at E_r if the differentials d_i are zero for $i \geq r$. Then we write $E_r = E_\infty$. Moreover, in this case

$$E_r^{pq} = E_i^{pq} = E_\infty^{pq} \quad \textit{pour } i \geq r.$$

Proposition 6.1 (Deligne)**.** Let K be a complex with a biregular filtration F. The following conditions are equivalent

1. The spectral sequence defined by F degenerates at $E_1 (E_1 = E_\infty)$
2. The morphisms $d : K^i \to K^{i+1}$ are strictly compatibles to the filtrations.

Proposition 6.2. A filtered quasi-isomorphism $f : (K,F) \to (K',F')$ induces an isomorphism of spectral sequences $E_r^{pq}(f) : E_r^{pq}(K,F) \xrightarrow{\sim} E_r^{p,q}(K',F')$.

Example 6.3. Let $f : X \to Y$ be a continued map of topological spaces. Let $\mathcal{F}$ be an abelian sheaf on X and $\mathcal{F}^*$ a resolution of $\mathcal{F}$ by f_*-acyclic sheaves, then $R^if_*\mathcal{F} \cong H^i(f_*\mathcal{F}^*)$. The hypercohomology spectral sequence of the complex $\mathbf{R}f_*\mathcal{F}^*$ with its canonical filtration with respect to the global section functor $\Gamma(Y,*)$ is

$$E_1^{pq} = H^{2p+q}(Y, R^{-p}f_*\mathcal{F}^*) \Rightarrow Gr^\tau_{-p}H^{p+q}(X,\mathcal{F})$$

it coincides, up to renumbering $E_r^{pq} \to E_{r+1}^{2p+q,-p}$, with Leray's spectral sequence for f and $\mathcal{F}$

$$E_2^{pq} = H^p(Y, R^qf_*\mathcal{F}).$$

Example 6.4. On DeRham complex, the trivial filtration is called the Hodge filtration since it is defined by the subcomplexes

$$F^p(\Omega^*_X) = 0 \mapsto \cdots 0 \mapsto \Omega^p_X \mapsto \cdots \mapsto \Omega^m_X \mapsto 0$$

It induces on DeRham cohomology a filtration by sub-vector spaces

$$F^p\mathbb{H}^n(X,\Omega^*_X) = Im\, \mathbb{H}^n(X, F^p\Omega^*_X).$$

The spectral sequence with respect to the global section functor and the filtration F (in different terms of the filtered complex $(\mathbf{R}\Gamma(X,\Omega_X^*), \mathbf{R}\Gamma(X, F^p\Omega_X^*))$ degenerates at rank one

$$E_1^{pq} = \cdots = E_r^{pq} = \cdots = Gr_F^p \mathbb{H}^{p+q}(X, \Omega_X^*)$$

The proof uses Hodge theory from which we deduce $Gr_F^p\mathbb{H}^{p+q}(X,\Omega_X^*) \simeq H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$, then it follows that dim $E_1^{pq} = \cdots =$ dim E_r^{pq} and $d_r = 0$ for $r > 0$. In particulae we deduce that the morphism

$$\mathbb{H}^n(X, F^p\Omega_X^*) \xrightarrow{\sim} F^p\mathbb{H}^n(X, \Omega_X^*)$$

is an isomorphism. Such result is algebraic and has been obtained directly by Deligne and Illusie. However Hodge theory tells more. Precisely, the conjugate Hodge filtration $\overline{F}$ of F with respect to $\mathbb{H}^n(X,\mathbb{Q}))$ satisfy

$$H^{p,q}(X) = F^p\mathbb{H}^n(X,\Omega_X^*) \cap \overline{F^q\mathbb{H}^n(X,\Omega_X^*)},\ \overline{H^{p,q}} = H^{q,p}.$$

as well the decomposition

$$H^n(X,\mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X).$$

7. Mixed Hodge Complex(MHC)

We consider complexes of abelian sheaves endowed with new structures needed to deduce a MHS on their cohomology. Deligne called them Mixed Hodge complexes (MHC). The technique of mixed cone associates to a morphism of MHC a new MHC, which leads to a MHS on relative cohomology and long exact sequences of MHS. The first example is constructed on the cohomology of a divisor Y with normal crossings (NCD) in a smooth complex complete variety X and the cohomology with compact support of $X - Y$. The dual case is the logarithmic complex constructed defining the MHS on the cohomology of $X - Y$. The Hodge filtration F may be deduced also from the filtration by order of the pole on the sheaf of meromorphic forms holomorphic on $X - Y$. Other examples are given by a construction of natural mixed cones, including the case of cohomology with compact support on $X - Y$. The most general type of MHS is on the cohomology of open NCD.

7.1. **Hodge Complex (HC).** In this section we consider complexes with additional properties that enables us to obtain a HS on their cohomology, and arrive to the definition of a Hodge complex. The cohomology of a compact complex smooth algebraic variety carry a HS whose Hodge filtration is deduced from the filtration on the algebraic DeRham complex, however the proof of the decomposition is reduced to the case of a projective variety, hence compact Kähler manifold.

7.1.1. Let $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. A Hodge A-complex K of weight n consists of

(1) a complex of $A-$modules $K_A \in ObD^+(A)$, s.t. $H^k(K_A)$ is an A-module of finite type for all k,
(2) a filtered complex $(K_\mathbb{C}, F) \in ObD^+F(\mathbb{C})$
(3) an isomorphism $\alpha : K_\mathbb{C} \simeq K_A \otimes \mathbb{C}$ in $D^+(\mathbb{C})$.

The following axioms must be satisfied
(CH 1) the differential d of $K_\mathbb{C}$ is strictly compatible with the filtration F; equivalently, the spectral sequence defined by $(K_\mathbb{C}, F)$ degenerates at $E_1 (E_1 = E_\infty)$,
(CH 2) for all k, the filtration F on $H^k(K_\mathbb{C}) \simeq H^k(K_A) \otimes \mathbb{C}$ define an $A - HS$ of weight $n + k$ on $H^k(K_A)$.

Equivalently, the filtration F is $(n+k)$-opposed to its complex conjugate (which makes sense since $A \subset \mathbb{R}$).

Theorem 7.1. *Let X be a compact Kähler manifold and consider*

1. *$K_{\mathbb{Z}}$ the complex reduced to a constant sheaf $\mathbb{Z}$ on X in degree zero.*
2. *$K_{\mathbb{C}}$ the De Rham holomorphic complex Ω_X^* with its Hodge filtration $F = \sigma$*

$$F^p\Omega_X^* := 0 \to \cdots \to \Omega_X^p \to \Omega_X^{p+1} \cdots \to \Omega_X^n \to 0$$

3. *The quasi-isomorphism $\alpha\colon K_{\mathbf{Z}} \otimes \mathbb{C} \xrightarrow{\approx} \mathbb{C} \xrightarrow{\approx} \Omega_X^*$ (Poincaré's lemma).*

Then $(K_Z, (K_{\mathbb{C}}, F), \alpha)$ is a CHC of weight.

We need to prove that the complex $\mathbf{R}\Gamma(X, \Omega_X^*)$ defined by the derived global section functor Γ is a HC. In simple terms, the global section of the filtered resolution $(\mathcal{E}_X, F)$ defined by Dolbeault resolutions of $F^p\Omega_X^*$ is a HC. This means exactly that the induced Hodge filtration F on $H^n(X, \mathbb{C})$ defines a HS, hence the result is just a reformulation of classical Hodge theory.

7.1.2. Let $\mathcal{L}$ be a rational local system with a polarization, rationally defined, on the associated local system $\mathcal{L}_{\mathbb{C}} = \mathcal{L} \otimes_{\mathbb{Q}} \mathbb{C}$, then the spectral sequence defined by the Hodge filtration on DeRham complex with coefficients $\Omega_X^* \otimes_{\mathbb{C}} \mathcal{L}$ degenerates at rank 1

$$E_1^{pq} = H^q(X, \Omega_X^p(\mathcal{L})) \Rightarrow H^{p+q}(X, \mathcal{L}_{\mathbb{C}})$$

and the induced filtration by F on cohomology defines a HS. The proof is similar to the constant case.

7.1.3. A CHC is defined by the Hodge filtration on DeRham complex for any complex algebraic manifold X. Using the existence of a projective manifold above X and birational to X (Chow's lemma), and resolution of singularities, Deligne deduces a Hodge decomposition in this way on the cohomology of any compact complex algebraic manifold not necessarily Kähler. The HS defined in this way are functorial for morphisms of algebraic varieties.

7.2. An A- mixed Hodge complex (MHC) K consists of

1. a complex $K_A \in ObD^+(A)$ s.t. $H^k(K_A)$ is an A-module of finite type for all k,
2. a filtered complex $(K_{A\otimes\mathbb{Q}}, W) \in ObD^+F(A \otimes \mathbb{Q})$ with an increasing filtration W,
3. an isomorphism $K_{\mathbb{Q}} \xrightarrow{\sim} K_A \otimes \mathbb{Q}$ in $D^+(A \otimes \mathbb{Q})$,
4. a bifiltered complex $(K_{\mathbb{C}}, W, F) \in ObD^+F_2(\mathbb{C})$ with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_{\mathbb{C}}, W) \xrightarrow{\sim} (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C}$ in $D^+F(\mathbb{C})$.

Moreover, the following axiom is satisfied
(MHC) For all n, the system consisting of the complex $Gr_n^W(K_{A\otimes\mathbb{Q}}) \in ObD^+(A \otimes \mathbb{Q})$, the complex $(Gr_n^W(K_{\mathbb{C}}, F) \in ObD^+F(\mathbb{C})$ with induced F and the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$, is an $A \otimes \mathbb{Q} - HC$ of weight n.

7.2.1. *MHS on the cohomology of a MHC.*

Theorem 7.2 (Deligne). *Let K be an $A-MHC$.*
(i)The filtration $W[n]$ of $H^n(K_{A\otimes\mathbb{Q}}) \simeq H^n(K_A)\otimes\mathbb{Q}$ and the filtration F on $H^n(K_{\mathbb{C}}) \simeq H^n(K_A)\otimes_A \mathbb{C}$ define on $H^n(K)$ an $A-MHS$ i.e.
$(H^n(K_A), (H^n(K_{A\otimes\mathbb{Q}}), W), (H^n(K_{\mathbb{C}}), W, F)$ is an A-MHS.
(ii)On the terms ${}_WE_r^{pq}$ of the spectral sequence of $(K_{\mathbb{C}}, W)$, the recurrent filtration and the two direct filtrations defined by F coincide $F_d = F_{rec} = F_{d^*}$.
(iii) The morphisms $d_1 : {}_WE_1^{p,q} \to {}_WE_1^{p+1,q}$ are strictly compatible with the filtration F.
(iv) The spectral sequence of $(K_{A\otimes\mathbb{Q}}, W)$ degenerates at E_2 (${}_WE_2 = {}_WE_\infty$).
(v) The spectral sequence of $(K_{\mathbb{C}}, F)$ degenerates at $E_1({}_FE_1 = {}_FE_\infty)$.
(vi) The spectral sequence of the complex $Gr_F^p(K_{\mathbb{C}})$, with the induced filtration W, degenerates at E_2.

7.3. MHS of a normal crossing divisor (NCD). Let Y be a normal crossing divisor in a proper complex smooth algebraic variety. We suppose the irreducible components $(Y_i)_{i\in I}$ of Y smooth and ordered.

7.3.1. *Mayer-Vietoris resolution.* Let S_q denotes the set of strictly increasing sequences $\sigma = (\sigma_0, ..., \sigma_q)$ on the ordered set of indices I, $Y_\sigma = Y_{\sigma_0} \cap ... \cap Y_{\sigma_q}, Y_{\underline{q}} = \sum_{\sigma\in S_q} Y_\sigma$ and for all $j \in [0,q], q \geq 1$ let $\lambda_{j,\underline{q}} : Y_{\underline{q}} \to Y_{\underline{q-1}}$ denotes a map inducing for each σ the embedding $\lambda_{j,\sigma} : Y_\sigma \to Y_{\sigma(\widehat{j})}$ where $\sigma(\widehat{j}) = (\sigma_0, ..., \widehat{\sigma_j}, ..., \sigma_q)$ is obtained by deleting σ_j. Let $\Pi_q : Y_{\underline{q}} \to Y$ (or simply Π) denotes the canonical projection and $\lambda^*_{j,\underline{q}} : \Pi_*\mathbf{Z}_{Y_{\underline{q-1}}} \to \Pi_*\mathbf{Z}_{Y_{\underline{q}}}$ the morphism defined by $\lambda_{j,\underline{q}}$ for $j \in [0,q]$.

Definition 7.3 (Mayer-Vietoris resolution of $\mathbb{Z}_Y$). It is defined by the following complex of sheaves $\Pi_*\mathbb{Z}_{Y_{\underline{\cdot}}}$

$$0 \to \Pi_*\mathbb{Z}_{Y_{\underline{0}}} \to \Pi_*\mathbb{Z}_{Y_{\underline{1}}} \to \cdots \to \Pi_*\mathbb{Z}_{Y_{\underline{q-1}}} \xrightarrow{\delta_{q-1}} \Pi_*\mathbb{Z}_{Y_{\underline{q}}} \to \cdots$$

where $\delta_{q-1} = \sum_{j\in[0,q]} (-1)^j \lambda^*_{j,\underline{q}}$.

This resolution is associated to a resolution of Y by topological spaces in the following sense. Consider the diagram of spaces over Y

$$Y_{\underline{\cdot}} = \left(\begin{array}{ccccccc} & & & \leftarrow & & \vdots & \\ Y_{\underline{0}} & \overset{\leftarrow}{\leftarrow} & Y_{\underline{1}} & \leftarrow & \cdots & Y_{\underline{q-1}} \overset{\lambda_{j,\underline{q}}}{\longleftarrow} & Y_{\underline{q}} \cdots \\ & & & \leftarrow & & \vdots & \end{array} \right) \xrightarrow{\Pi} Y$$

This diagram is the strict simplicial scheme associated in [7] to the normal crossing divisor Y, called here after Mayer-Vietoris. The Mayer-Vietoris complex is canonically associated as direct image by Π of the sheaf $\mathbb{Z}_{Y_{\underline{\cdot}}}$ equal to $\mathbb{Z}_{Y_{\underline{i}}}$ on $Y_{\underline{i}}$.

7.3.2. *The cohomological mixed Hodge complex of a NCD.* The weight filtration W on $\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}$ is defined by

$$W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}) = \sigma_{\cdot\geq q}\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}} = \Pi_*\sigma_{\cdot\geq q}\mathbb{Q}_{Y_{\underline{\cdot}}}, \quad Gr^W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}) \simeq \Pi_*\mathbb{Q}_{Y_{\underline{q}}}[-q]$$

We introduce the complexes $\Omega^*_{Y_{\underline{i}}}$ of differential forms on $Y_{\underline{i}}$. The simple complex $s(\Omega^*_{Y_{\underline{\cdot}}})$ is associated to the double complex$\Pi_*\Omega^*_{Y_{\underline{\cdot}}}$ with the exterior differential d of forms and the differential $\delta.$ defined by $\delta_{q-1} = \sum_{j\in[0,q]}(-1)^j \lambda^*_{j,\underline{q}}$ on $\Pi_*\Omega^*_{Y_{\underline{q-1}}}$. The weight W, and Hodge F filtrations are defined as

$$W_{-q} = s(\sigma_{\cdot \geq q}\Omega^*_{Y_{\underline{\cdot}}}) = s(0 \to \cdots 0 \to \Pi_*\Omega^*_{Y_{\underline{q}}} \to \Pi_*\Omega^*_{Y_{\underline{q+1}}} \to \cdots)$$

$$F^p = s(\sigma_{*\geq p}\Omega^*_{Y_{\underline{\cdot}}}) = s(0 \to \cdots 0 \to \Pi_*\Omega^p_{Y_{\underline{\cdot}}} \to \Pi_*\Omega^{p+1}_{Y_{\underline{\cdot}}} \to \cdots)$$

We have a filtered isomorphism in $D^+F(Y,\mathbb{C})$

$$(Gr^W_{-q}s(\Omega^*_{Y_{\underline{\cdot}}}), F) \simeq (\Pi_*\Omega^*_{Y_{\underline{q}}}[-q], F) \quad \text{in} \quad D^+F(Y,\mathbb{C}).$$

inducing isomorphisms in $D^+(Y,\mathbb{C})$

$$(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W) \otimes \mathbb{C} = (\mathbb{C}_{Y_{\underline{\cdot}}}, W) \xrightarrow{\alpha \sim} (s(\Omega^*_{Y_{\underline{\cdot}}}), W)$$

$$Gr^W_{-q}(\Pi_*\mathbb{C}_{Y_{\underline{\cdot}}}) \simeq \Pi_*\mathbb{C}_{Y_{\underline{q}}}[-q] \xrightarrow{\sim} \Pi_*\Omega^*_{Y_{\underline{q}}}[-q] \simeq Gr^W_{-q}s(\Omega^*_{Y_{\underline{\cdot}}})$$

Let **K** be the system consisting of

$$(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W), \mathbb{Q}_Y \simeq \Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, (s(\Omega^*_{Y_{\underline{\cdot}}}), W, F), \quad (\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W) \otimes \mathbb{C} \simeq (s(\Omega^*_{Y_{\underline{\cdot}}}), W)$$

Proposition 7.4. The system **K** associated to a normal crossing divisor Y with smooth proper irreducible components, is a CMHC on Y. It defines a functorial MHS on the cohomology $H^i(Y,\mathbb{Q})$, with weights varying between 0 and i.

In terms of Dolbeault resolutions : $(s(\mathcal{C})^{*,*}_{Y_{\underline{\cdot}}}, W, F)$, the statement means that the complex of global sections $\Gamma(Y, s(\mathcal{C})^{*,*}_{Y_{\underline{\cdot}}}, W, F) := (\mathbb{R}\Gamma(Y,\mathbb{C}), W, F)$ is a MHC in the following sense

$$\begin{aligned} &Gr^W_{-i}(\mathbb{R}\Gamma(Y,\mathbb{C}) := (\Gamma(Y, W_{-i}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}})/\Gamma(Y, W_{-i-1}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}}), F) \\ &\simeq (\Gamma(Y, Gr^W_{-i}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}}), F) \simeq (\mathbb{R}\Gamma(Y_i, \Omega^*_{Y_{\underline{i}}}[-i]), F)\end{aligned}$$

is a HC of weight $-i$ in the sense that

$$(H^n(Gr^W_{-i}\mathbb{R}\Gamma(Y,\mathbb{C})), F) \simeq (H^{n-i}(Y_{\underline{i}},\mathbb{C}), F)$$

is a HS of weight $n-i$.
The terms of the spectral sequence $E_1(K,W)$ of (K,W) are written as

$${}_WE_1^{pq} = \mathbb{H}^{p+q}(Y, Gr^W_{-p}(s\Omega^*_{Y_{\underline{\cdot}}})) \simeq \mathbb{H}^{p+q}(Y, \Pi_*\Omega_{Y_{\underline{p}}}[-p]) \simeq H^q(Y_{\underline{p}},\mathbb{C})$$

They carry the HS of the space $Y_{\underline{p}}$. The differential is a combinatorial restriction map
$d_1 = \sum_{j\leq p+1}(-1)^j\lambda^*_{j,p+1} : H^q(Y_{\underline{p}},\mathbb{C}) \to H^q(Y_{\underline{p+1}},\mathbb{C})$
is a morphism of HS. The spectral sequence degenerates at E_2 ($E_2 = E_\infty$).

Corollary 7.5. The HS on $Gr^W_q H^{p+q}(Y,\mathbb{C})$ is the cohomology of the complex of HS defined by $(H^q(Y_{\underline{p}},\mathbb{C}), d_1)$:

$$(Gr^W_q H^{p+q}(Y,\mathbb{C}), F) \simeq ((H^p(H^q(Y_{\underline{p}},\mathbb{C}), d_1), F)$$

7.4. Logarithmic complex. Now we construct the MHS on the cohomology of a smooth complex algebraic variety V. By a result of Nagata, V can be embedded as an open Zariski subset of a complete variety X. After Hironaka's desingularisation theorem in characteristic zero, we can suppose X smooth and $Y = X - V$ a NCD with smooth components. Hence we reduce the construction of the MHS to the complement $X - Y$ of a NCD. we show that this construction is independent of the choice of X and Y s.t. $V = X - Y$.
With the previous notations this case is the Poincaré's dual of the cohomology with compact support on $X - Y$ which can be constructed by the double DeRham complex $\Omega^*_X \to \Omega^*_{Y_{\cdot}}$.
However we introduce here the logarithmic complex, important for its own properties as well to obtain functorial constructions later. Let X be a complex manifold and Y be a NCD in X. By definition, at each point $y \in Y$, there exist local coordinates $(z_i)_{i\in[1,n]}$ of X s.t. Y is defined at y by the equation $\Pi_{i\in I\subset[1,n]} z_i = 0$. Let $X^* = X - Y$ and $j : X^* \to X$ denotes the embedding of X^* into X.

Definition 7.6. The logarithmic DeRahm complex of X along Y is a sub-complex $\Omega^*_X(LogY)$ of the complex $\Omega_X(*Y) \subset j_*\Omega^*_{X^*}$ of meromorphic forms along Y, holomorphic on X^*. A section of $\omega \in \Omega^p_X(logY)$ is called a differential form with logarithmic pole along Y and may be written locally as

$$\omega = \Sigma_{i_1,\cdots,i_r}\omega_{i_1,\cdots,i_r}\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_r}}{z_{i_r}};\ \ \omega_{i_1,\cdots,i_r} \text{ holomorphic}$$

The definition is independent of the choice of coordinates; that is ω is written in this form with respect to any set of local coordinates at y. The $\mathcal{O}_X$-module $\Omega^1_X(LogY)$ is locally free with basis $(dz_i/z_i)_{i\in I}$ and $(dz_j)_{j\notin I}$ and $\Omega^p_X(LogY) = \wedge^p\Omega^1_X(LogY)$.
An intrinsic property of the sections ω of the logarithmic complex that can be taken as a definition, is that ω and $d\omega$ have both a pole of order at most 1 at y.
In particular, given a local equation $\prod_{i=1}^k z_i = 0$ of Y at y, a meromorphic section of $j_*\mathcal{O}^*_{X^*}$ along Y is written locally as $f = g\prod_{i=1}^k z_i^{k_i}$ with g invertible, then

$$df/f = dg/g + \sum_1^k k_i dz_i/z_i$$

is the sum of a regular form and a linear combination of the independent vectors dz_i/z_i.
Let $f : X_1 \to X_2$ be a morphism of complex manifolds, with normal crossing divisors Y_i in X_i for $i = 1, 2$, s.t. $f^{-1}(Y_2) = Y_1$. Then, the reciprocal morphism $f^* : f^*(j_{2*}\Omega^*_{X_2^*}) \to j_{1*}\Omega^*_{X_1^*}$ induces a morphism on logarithmic complexes

$$f^* : f^*\Omega^*_{X_2}(LogY_2) \to \Omega^*_{X_1}(LogY_1).$$

7.4.1. *Weight filtration W.* Let $Y = \cup_{i\in I} Y_i$ be the union of smooth irreducible divisors. Let S^q denotes the set of strictly increasing sequences $\sigma = (\sigma_1, ..., \sigma_q)$ in the set of indices I, $Y_\sigma = Y_{\sigma_1 ... \sigma_q} = Y_{\sigma_1} \cap ... \cap Y_{\sigma_q}$, $Y^q = \coprod_{\sigma\in S^q} Y_\sigma$ the disjoint union of Y_σ. Set $Y^0 = X$, $\Pi : Y^q \to Y$ the canonical projection. An increasing filtration W, called the weight, is defined as follows

$$W_m(\Omega^p_X(LogY)) = \sum_{\sigma\in S^m} \Omega^{p-m}_X \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

The sub-$\mathcal{O}_X$-module $W_m(\Omega^p_X(LogY)) \subset \Omega^p_X(LogY)$ is the smallest sub-module stable by exterior multiplication with local sections of Ω^*_X and containing the products $df_1/f_1 \wedge ... \wedge df_k/f_k$ for $k \leq m$ for local sections f_i of $j_*\mathcal{O}^*_{X^*}$ meromorphic along Y.

7.4.2. *The Residue isomorphism.* The Poincaré residue defines an isomorphism

$Res : Gr^W_m(\Omega^p_X(LogY)) \to \Pi_*\Omega^p_{Y^m}[-m] : Res(\alpha \wedge (dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge (dz_{\sigma_m}/z_{\sigma_m})) = \alpha/Y_\sigma$

In the case $p = 1$, it defines

$$Res : \Omega^1_X(LogY) \to \Pi_*\mathcal{O}_{Y^1}$$

We construct its inverse. Consider for $\sigma \in S^m$ the morphism
$\rho_\sigma : \Omega^p_X \to Gr^W_m(\Omega^{p+m}_X(LogY))$ defined locally as

$$\rho_\sigma(\alpha) = \alpha \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

It does not depends on the choice of z_i since for another choice of coordinates z'_i, z_i/z'_i are holomorphic and the difference $(dz_i/z_i) - (dz'_i/z'_i) = d(z_i/z'_i)/(z_i/z'_i)$ is holomorphic too.
Then $\rho_\sigma(\alpha) - \alpha \wedge dz'_{\sigma_1}/z'_{\sigma_1} \wedge ... \wedge dz_{\sigma m}/z_{\sigma_m} \in W_{m-1}\Omega^{p+m}_X(LogY)$, and successively $\rho_\sigma(\alpha) - \rho'_\sigma(\alpha) \in W_{m-1}\Omega^{p+m}_X(LogY)$. We have $\rho_\sigma(z_{\sigma_i} \cdot \beta) = 0$ and $\rho_\sigma(dz_{\sigma_i} \wedge \beta') = 0$ for sections β of Ω^p_X and β' of Ω^{p-1}_X; hence ρ_σ factors by $\overline{\rho}_\sigma : \Pi_*\Omega^p_{Y_\sigma} \to Gr^W_m(\Omega^{p+m}_X(LogY))$ defined locally and glue globally into a morphism of complexes on X

$$\rho : \Pi_*\Omega^*_{Y^m}[-m] \to Gr^W_m(\Omega^*_X(LogY)).$$

Lemma 7.7. *We have*
i) $\underline{H}^i(Gr^W_m\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^m}$ *for* $i = m$ *and vanishes for* $i \neq m$.
ii) $\underline{H}^i(\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^i}$.

Proof. (ii) We deduce from the long exact sequence associated to the short exact sequence

$$0 \to W_r \to W_{r+1} \to Gr^W_{r+1} \to 0$$

by induction on r, the statement:
$\underline{H}^i(W_r\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^i}$ for $i \leq r$ and vanishes for $i > r$. □

7.4.3. *Hodge filtration F.* It is defined on $\Omega^*_X(LogY)$ by the formula $F^p = \sigma_{\geq p}$ including all forms of type (p', q') with $p' \geq p$. We have

$$Res : F^p(Gr^W_m\Omega^*_X(LogY)) \simeq \Pi_*F^{p-m}\Omega^*_{Y^m}[-m]$$

hence a filtered isomorphism

$$Res : (Gr^W_m\Omega^*_X(LogY), F) \simeq (\Pi_*\Omega^*_{Y^m}[-m], F[-m]).$$

Proposition 7.8 (Weight filtration W)**.** The morphisms of filtered complexes

$$(\Omega^*_X(LogY), W) \xleftarrow{\alpha} (\Omega^*_X(LogY), \tau) \xrightarrow{\beta} (j_*\Omega^*_{X^*}, \tau)$$

are filtered quasi-isomorphisms. They define an isomorphism of the spectral sequence of hypercohomology of the filtered complex $(\Omega^*_X(LogY), W)$ on X with Leray spectral sequence for j with complex coefficients. The main point here is that the τ filtration is defined with rational coefficients as $(\mathbf{R}j_*\mathbb{Q}_{X^*}, \tau) \otimes \mathbb{C}$.

7.4.4. *MHS on the hypercohomology of $\Omega^*_X(LogY)$.* Let $j : X^* = X - Y \to X$ denotes the open embedding. The system $\mathbf{K}$

(1) $(\mathbf{K}^{\mathbb{Q}}, W) = (\mathbf{R}j_*\mathbb{Q}_{X^*}, \tau) \in ObD^+F(X,\mathbb{Q})$
(2) $(\mathbf{K}^{\mathbb{C}}, W, F) = (\Omega^*_X(LogY), W, F) \in ObD^+F_2(X,\mathbb{C})$
(3) The isomorphism $(\mathbf{K}^{\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (\mathbf{K}^{\mathbb{C}}, W)$ in $D^+F(X,\mathbb{C})$

form a CMHC on X.

Theorem 7.9 (6)**.** *The system $K = \mathbf{R}\Gamma(X, \mathbf{K})$ is a MHC. It endows the cohomology of V with a canonical MHS.*

Proof. The weight spectral sequence at rank 1 is written as

$$_WE_1^{pq}(\mathbf{R}\Gamma(X, \Omega^*_X(LogY)) = \mathbb{H}^{p+q}(X, Gr^W_{-p}\Omega^*_X(LogY)) \simeq \mathbb{H}^{p+q}(X, \Pi_*\Omega^*_{Y^{-p}}[p])$$
$$\simeq H^{2p+q}(Y^{-p},\mathbb{C}) \Rightarrow Gr^W_q H^{p+q}(V,\mathbb{C}).$$

where the double arrow means that the spectral sequence degenerates to the cohomology graded with respect to W induced by the weight on the complex level. In fact we show that it degenerates at rank 2. The differential d_1

$$d_1 = \sum_{j=1}^{-p}(-1)^{j+1}G(\lambda_{j,-p}) = G : H^{2p+q}(Y^{-p},\mathbb{C}) \longrightarrow H^{2p+q+2}(Y^{-p-1},\mathbb{C})$$

is equal to an alternate Gysin morphism, Poincaré's dual to the alternate restriction morphism

$$\rho = \sum_{j=1}^{-p}(-1)^{j+1}\lambda^*_{j,-p} : H^{2n-q}(Y^{-p-1},\mathbb{C}) \to H^{2n-q}(Y^{-p},\mathbb{C})$$

hence the first term

$$(_WE_1^{pq}, d_1)_{p\in\mathbb{Z}} = (H^{2p+q}(Y^{-p},\mathbb{C}), d_1)_{p\in\mathbb{Z}}$$

is viewed as a complex in the category of HS of weight q. It follows that the terms $_WE_2^{pq} = H^p(_WE_1^{*,q}, d_1)$ are endowed with a HS of weight q. The differential d_2 which is compatible with the induced HS, being a morphism from E_2^{pq} a HS of weight q to $E_2^{p+2,q-1}$ a HS of weight $q-1$, is necessarily zero. The proof consists of a recurrent argument to show in this way that the differentials d_i for $i \geq 2$ are zero. □

Exercise 7.10 (Riemann Surface)**.** Let $\overline{C}$ be a connected compact Riemann surface, $Y = \{x_1, \ldots, x_m\}$ a subset of m points, and $C = \overline{C} - Y$ the open surface with m points in $\overline{C}$ deleted. Consider the long exact sequence

$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to H^2_Y(\overline{C},\mathbb{Z}) = \oplus_{i=1}^{i=m}\mathbb{Z} \to H^2(\overline{C},\mathbb{Z}) \simeq \mathbb{Z} \to H^2(C,\mathbb{Z}) = 0$$

then

$$0 \to H^1(\overline{C},\mathbb{Z}) \to H^1(C,\mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m}\mathbb{Z} \to \mathbb{Z})$$

represents $H^1(C,\mathbb{Z})$ as an extension, with weight $W_1H^1(C,\mathbb{Z}) = H^1(\overline{C},\mathbb{Z})$ and $W_2H^1(C,\mathbb{Z}) = H^1(C,\mathbb{Z})$.
The Hodge filtration is defined by the residue

$$0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(log\{x_1,\ldots,x_m\}) \to \mathcal{O}_{\{x_1,\ldots,x_m\}} \to 0$$

$F^0H^1(C,\mathbb{C}) = H^1(C,\mathbb{C})$,
$F^1H^1(C,\mathbb{C}) = H^0(\overline{C}, \Omega^1_{\overline{C}}(log\{x_1,\dots,x_m\})$.
$F^2H^1(C,\mathbb{C}) = 0$. That is an extension of two different weights

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathrm{Ker}\,(\oplus_{i=1}^{i=m}\mathbb{Z}(-1) \to H^2(\overline{C})) \to 0$$

.

Exercise 7.11 (Hypersurfaces)**.** Let $Y \subset P$ be a smooth hypersurface in a projective space P. To describe Hodge theory on $U = P - Y$ we may use by a result of Grothendieck on algebraic DeRham cohomology, forms on P meromorphic along Y and holomorphic on U, denoted as a sheaf by $\Omega_P(*Y) \subset j_*\Omega_U$ where $j : U \to P$, or the logarithmic complex by Deligne's result.
1) In the case of a curve Y in a plane, holomorphic one forms are residues of rational $2-$forms on P with simple pole along the curve.
For example, if the homogeneous equation is $F = 0$, $\Omega^2_P(LogY) = \Omega^2_P(Y)$ and we have an exact sequence

$$0 \to \Omega^2_P \to \Omega^2_P(Y) \to \Omega^1_Y \to 0$$

Since $h^{2,0} = h^{2,1} = 0$, $H^0(P,\Omega^2_P) = H^1(P,\Omega^2_P) = 0$, so we deduce the isomorphism

$$H^0(P,\Omega^2_P(Y)) \xrightarrow{Res} H^0(Y,\Omega^1_Y)$$

In homogeneous coordinates, we take the residue along Y of the rational form

$$\frac{A(z_0dz_1\wedge dz_2 - z_1dz_0\wedge dz_2 + z_2dz_0\wedge dz_1)}{F}$$

where A is homogeneous of degree $d-3$ if F has degree d.
2) In general, we consider the exact sequence for relative cohomology (or cohomology with support)

$$H^{k-1}(U) \xrightarrow{\partial} H^k_Y(P) \to H^k(P) \xrightarrow{j^*} H^k(U)$$

which reduces via Thom's isomorphism, to

$$H^{k-1}(U) \xrightarrow{r} H^{k-2}(Y) \xrightarrow{i_*} H^k(P) \xrightarrow{j^*} H^k(U)$$

where r is the topological Leray's residue map dual to the tube over cycle map $\tau : H_{k-2}(Y) \to H_{k-1}(U)$ associating to a cycle c the boundary in U of a tube over c, and i_* is Gysin map, Poincaré dual to the map i^* in cohomology.
For $P = \mathbb{P}^{n+1}$ and n odd, the map r is an isomorphism

$$H^{n-1}(Y) \simeq H^{n+1}(P) \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = 0 \xrightarrow{j^*} H^{n+2}(U)$$

and for n even the map r is injective

$$H^{n+1}(P) = 0 \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = \mathbb{Q} \xrightarrow{j^*} H^{n+2}(U)$$

then

$$r : H^{n+1}(U) \xrightarrow{\sim} H^n_{\mathrm{prim}}(X)$$

If Y is ample in P such $U = P - Y$ is affine

$$H^{n-1}(Y) \xrightarrow{i_*} H^{n+1}(P) \xrightarrow{j^*} H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) \to H^{n+2}(U) = 0$$

then we define the variable cohomology

$$H^n_{var}(Y) = Ker\,(\,H^n(Y) \xrightarrow{i_*} H^{n+2}(P))$$

equal also to $= Im\ r$. The fixed cohomology is

$$H^n_{fix}(Y) = i_* H^n(P)$$

In particular, if $H^{n+1}_{\mathrm{prim}}(P) = 0$

$$H^{n+1}(U, \mathbb{Q}) \xrightarrow{r\sim} H^n_{var}(Y, \mathbb{Q})$$

Moreover

$$H^n(Y, \mathbb{Q}) = H^n_{var}(Y, \mathbb{Q}) \oplus H^n_{fix}(Y, \mathbb{Q})$$

8. MHS on the cohomology of a complex algebraic variety

The aim of this section is to prove

Theorem 8.1 (6)**.** *The cohomology of a complex algebraic variety carry a natural MHS.*

8.1. MHS on the cohomology of a complete embedded algebraic variety. For embedded varieties into smooth varieties , the MHS on cohomology can be deduced by a simple method, using exact sequences from MHS already constructed for NCD, which should easily convince of the natural aspect of this theory. The technical ingredients consist of Poincaré duality and the trace (or Gysin) morphism.

Let $p : X' \to X$ be a proper morphism of complex smooth varieties of same dimension, Y a closed subvariety of X and $Y' = p^{-1}(Y)$. We suppose that Y' is a NCD in X' and the restriction of p induces an isomorphism $p_{/X'-Y'} : X' - Y' \xrightarrow{\sim} X - Y$.

$$\begin{array}{ccccc} Y' & \xrightarrow{i'} & X' & \xleftarrow{j'} & X' - Y' \\ \downarrow p_Y & & \downarrow p & & \downarrow p_{X-Y} \\ Y & \xrightarrow{i} & X & \xleftarrow{j} & X - Y \end{array}$$

The trace morphism Trp is defined as Poincaré dual to the inverse image p^* on cohomology. It can be defined at the level of sheaf resolutions of $\mathbb{Z}_{X'}$ and $\mathbb{Z}_X$ constructed by Verdier, that is in derived category $Tr\,p$: $\mathbf{R}p_* \mathbb{Z}_{X'} \to \mathbb{Z}_X$ hence we deduce by restriction morphisms depending on the embeddings of Y and Y' into X'. $(Tr\,p)/_Y : \mathbf{R}p_* \mathbb{Z}_{Y'} \to \mathbb{Z}_Y$, $(Tr\,p)/_Y : H^i(Y', \mathbb{Z}) \to H^i(Y, \mathbb{Z})$, $H^i_c(Y', \mathbb{Z}) \to H^i_c(Y, \mathbb{Z})$.

Remark 8.2. Let U be a neighbourhood of Y in X, retract by deformation onto Y s.t. $U' = p^{-1}(U)$ is a retract by deformation onto Y'; this is the case if Y is a sub-variety of X. Then the morphism $(Tr\,p)/_Y$ is deduced from $Tr(p/_U)$ in the diagram

$$\begin{array}{ccc} H^i(Y', \mathbb{Z}) & \xleftarrow{\sim} & H^i(U', \mathbb{Z}) \\ \downarrow (Tr\,p)/_Y & & \downarrow Tr(p/_U) \\ H^i(Y, \mathbb{Z}) & \xleftarrow{\sim} & H^i(U, \mathbb{Z}) \end{array}$$

Consider now the diagram

$$\begin{array}{ccccc} \mathbf{R}\Gamma_c(X' - Y', \mathbb{Z}) & \xrightarrow{j'_*} & \mathbf{R}\Gamma(X', \mathbb{Z}) & \xrightarrow{i'^*} & \mathbf{R}\Gamma(Y', \mathbb{Z}) \\ Trp \downarrow & & \downarrow Trp & & \downarrow (Trp)_{|Y} \\ \mathbf{R}\Gamma_c(X - Y, \mathbb{Z}) & \xrightarrow{j_*} & \mathbf{R}\Gamma(X, \mathbb{Z}) & \xrightarrow{i^*} & \mathbf{R}\Gamma(Y, \mathbb{Z}) \end{array}$$

Proposition 8.3. i) The morphism $(p_Y)^*\colon H^i(Y,\mathbb{Z}) \to H^i(Y',\mathbb{Z})$ is injective with retraction $(Tr\,p)_{/Y}$.
ii) We have a quasi-isomorphism of $i_*\mathbb{Z}_Y$ with the cone $C(i'^* - Tr\,p)$ of the morphism $i'^* - Tr\,p$. The long exact sequence associated to the cone splits into short exact sequences
$0 \to H^i(X',\mathbb{Z}) \xrightarrow{i'^* - Tr\,p} H^i(Y',\mathbb{Z}) \oplus H^i(X,\mathbb{Z}) \xrightarrow{(Tr\,p)_{/Y} + i^*} H^i(Y,\mathbb{Z}) \to 0$
Moreover $i'^* - Tr\,p$ is a morphism of MHS.

Definition 8.4. The MHS of Y is defined as cokernel of $i'^* - Tr\,p$ via its isomorphism with $H^i(Y,\mathbb{Z})$, induced by $(Tr\,p)_{/Y} + i^*$. It coincides with Deligne's MHS.

This result shows the uniqueness of the theory of MHS, once the MHS of the NCD Y' has been constructed.
The above technique consists in the realization of the MHS on the cohomology of Y as relative cohomology with MHS structures.

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