The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2150-5

Summary School and Conference on Hodge Theory and Related Topics

14 June - 2 July, 2010

Mixed Odge Structures

Fouad El Zein
Institut de Mathematiques de Jussieu
Paris
France

MIXED HODGE STRUCTURES

FOUAD EL ZEIN

Summer School on Hodge Theory and Related Topics
International center for theoretical Physics
14 June 2010 - 2 July 2010
Trieste - ITALY

Abstract. These notes start with a summary of cohomology theory (section 2), necessary to understand the background to Hodge theory (section 3) and the polarization (section 4). The Mixed Hodge structure (MHS) is defined as an object of interest in linear algebra (section 5). After developing the spectral sequence of a filtered complex (section 6), canonical MHS are constructed on cohomology of algebraic varieties (sections 7,8).
The lectures in previous courses cover the first three sections. The course on MHS will be presented in a more elementary language than these notes
Lecture 1:Mixed Hodge structure (section 5).
Lecture 2:Polarization (section 4, 5).
Lecture 3:Logarithmic complex (section 6, 7) .
Lecture 4:Spectral sequence and Hodge theory (section 6, 7).
Lecture 5: MHS on cohomology of complex algebraic varieties (section 8).

1. Introduction

Cohomology theory attach to a manifold a group, invariant under homeomorphisms. Introduced first as a topological invariant, it depends only on the topology of the manifold, but it plays a significant role in the differential study of a manifold as shown in DeRham theory. Its importance in algebraic and analytic geometries is reflected in Hodge theory as well in the introduction by J.P. Serre of the cohomology of coherent sheaves. Inspired by the properties of étale cohomology introduced by A. Grothendieck in the study of varieties over fields with positive characteristic, Deligne established the existence of a linear algebraic structure on cohomology of algebraic varieties over fields of characteristic zero, called mixed Hodge structure (MHS); it reflects geometric and topological properties of algebraic varieties, which means it depends on the geometry and not only the topology of such varieties.
This structure proved to be very useful in the interpretation of various geometric invariants classically attached to algebraic varieties, as Picard group or the Jacobian variety. Deformation of geometric structures result into a linear invariant on the cohomlogy of the family of varieties defined by the deformation, called variation of Hodge structure (VHS) by P. Griffiths. More generally a variation of mixed Hodge structure (VMHS) is attached to an algebraic family of varieties not necessarily proper or smooth.

2. Review of cohomology theories

Many techniques have been used to construct homology and cohomology groups. We mention here :*triangulation, singular chains, Čech coverings* and finally *sheaf theory* via injective or acyclic resolutions. In particular De Rham cohomology defined only on differentiable manifolds, is of special interest in our case. We refer to Dieudonné [3] for an historical account of each contribution.
All these constructions give, when they are defined and under mild assumptions on the topological space, isomorphic cohomology groups. Based on their main common properties on the class of spaces defined as finite euclidean simplicial complexes, the *axiomatization of the theory* occurred in the 1940's. The use of *spectral sequences* , introduced by Leray and developed by Serre, appeared to be a powerful tool in computation, see for example [2]. The most general approach since 1955 is due to Grothendieck, first with the systematic use of *injective resolutions* in *abelian categories* [10], second with the introduction with Verdier [13] of the *derived category* and the use of *hypercohomology*, third with the introduction with M. Artin, of a new (Grothendieck) topology and (étale) cohomology theory developed with many colleagues and students (namely P. Deligne, L. Illusie and J.L. Verdier). However, to introduce MHS we may use only flabby resolutions of Godement [8] to construct the weight filtration W on cohomology with coefficients $\mathbb{Q}$. Dolbeault resolutions are of particular interest in the study of the Hodge filtration F, which is an analytic invariant, on the cohomology with coefficients $\mathbb{C}$.

2.1. **Simplicial objects.** Let Δ_n denote the set of integers $\{0, 1, \dots, n\}$ and $H_{p,q}$ the set of increasing (large sense) mappings from Δ_p to Δ_q for integers $p, q \geq 0$. We define in this way the simplicial category Δ whose objects are $\Delta_n, n \in \mathbf{N}$, with the natural composition of mappings : $H_{pq} \times H_{qr} \to H_{pr}$. The semi-simplicial category $\Delta_>$ is obtained when we consider only the strictly increasing morphisms in Δ.
A simplicial (resp. co-simplicial) object of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from Δ to $\mathcal{C}$. For example a simplicial group consists of a family of groups X_i for each $i \in \mathbf{N}$ and of morphisms $X(f)\colon X_q \to X_p$ for each $f \in H_{p,q}$, compatible with composition. A semi-simplicial (resp. co-semi-simplicial) object of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from $\Delta_>$ to $\mathcal{C}$. We define for $0 \leq i \leq n+1$ the $i-$th face map and denote by

$$\delta_i\colon \Delta_n \to \Delta_{n+1}, \quad i \notin \delta_i(\Delta_n) := Im\, \delta_i$$

the unique increasing mapping such that $i \notin \delta_i(\Delta_n)$. Such suggestive name corresponds to the topological realization of the simplex where δ_i refers to the basic definition of the boundary. Its image by a functor is also denoted abusively by the same symbol $\delta_i\colon X_{n+1} \to X_n$ (resp. $\delta_i\colon X_n \to X_{n+1}$).

2.1.1. *Complexes.* A complex $(Y^\cdot, d)$ (resp. chain complex $(Y_\cdot, d)$) in an abelian category is given by a sequence of objects (Y^n) and morphisms $d_n : Y^n \to Y^{n+1}$ (resp. $d_n : Y_n \to Y_{n-1}$) such that $d_{n+1} \circ d_n = 0$ (resp. $d_{n-1} \circ d_n = 0$) for all integers n. It follows that $Im\, d_n \subset Ker\, d_{n+1}$ (resp. $Im\, d_n \subset Ker\, d_{n-1}$).
Given a complex $(Y^\cdot, d)$, we deduce a chain complex $(X_\cdot, d)$ if we put $X_n = Y^{-n}$, with the same differential $d_n : X_n \to X_{n-1}$. Such construction may be inverted.

Definition 2.1. The cohomology of the complex $(Y^\cdot, d)$ is defined for each n by

$$H^n(Y^\cdot, d) = Ker(d\colon Y^n \to Y^{n+1})/Im(d\colon Y^{n-1} \to Y^n).$$

The n^{th} homology of a chain complex $(X., d)$ is defined as

$$H_n(X., d) = ker(d\colon X_n \to X_{n-1})/Im(d\colon X_{n+1} \to X_n).$$

2.1.2. *Chain complex defined by a simplicial object of an abelian category.* Let $X_n, n \in \mathbb{N}$ be a simplicial object of an abelian category, we define the differential of a chain complex

$$d_{n+1}\colon X_{n+1} \to X_n, \qquad d_{n+1} = \Sigma_{i=0}^{n+1}(-1)^i\delta_i, \qquad \text{then } d_n \circ d_{n+1} = 0.$$

A complex may be defined similarly by a cosimplicial object.

2.2. **Triangulation.** [8;I.3.2], [12 ; 3.1] According to Dieudonné [3, p 26], the word triangulation is due to Weyl but the concept can be found in the paper Analysis situs written by Poincaré in 1895 and which is considered as a foundational article of Homotopy and Homology theory.

Definition 2.2. A combinatorial complex K consists of
i) a set K of elements $\{v\}$ called vertices.
ii) a set S of finite non empty subsets $\{s\}$ of K called simplices, such that
1) any set consisting of exactly one vertex is a simplex,
2) any non empty subset of a simplex is a simplex.
iii) A simplex $\{s\}$ is of dimension n if it contains $n+1$ vertices $|s| = n+1$.

Definition 2.3. 1) A singular simplex of dimension n of K is a mapping $\sigma\colon \Delta_n \to K$ whose image is a simplex of K.
2) Let $\Sigma_n(K)$ be the free abelian group generated by the set $S_n(K)$ of all singular simplices of dim n of K (resp. all singular simplices of dim n of K), then $\Sigma.(K)$ is a simplicial group. We denote also by $\Sigma.(K)$ the chain complex associated to it.
3) The homology of K is defined as the homology of the complex $\Sigma.(K)$.

Remark 2.4 (3;I.3.8). Suppose we are given an order on K such that each simplex is totally ordered, then the set of simplices totally ordered is the same as the set of strictly increasing singular simplices $\sigma\colon \Delta_n \to K$ and denote by $\Sigma_n^+(K)$ the subgroup of $\Sigma_n(K)$ generated by such subset of simplices $s \in K$. Then the subcomplex $\Sigma^+.(K)$ of $\Sigma_.(K)$ has the same homology groups as $\Sigma_.(K)$.

2.2.1. *Realization.* The topological realization of Δ_n is the real simplex Σ_n in $\mathbb{R}^{n+1}$ defined as the set of points $t = (t_o, \dots, t_n)$ such that $t_i \geq 0$ and $\Sigma_{i=0}^n t_i = 1$. As well a topological space $R(K)$, called its realization, is defined by K; it consists of real simplices $R(s)$ embedded in some real vector spaces of dimension big enough, one for each simplex s in K, $R(s)$ and $R(s')$ being glued along $R(s \cap s')$ considered as a face of $R(\sigma)$ and $R(\sigma')$. One way of defining this space is given in [G] as follows

Definition 2.5. The realization $R(K)$ of K is the set of mappings $f\colon K \to \mathbb{R}$ such that
i) The set of elements v of K where $f(v) \neq 0$ is a simplex.
ii) $f(v) \geq 0$ for all $v \in K$ and $\Sigma_{v\in K} f(v) = 1$.

The topology on $R(K)$ is defined as follows. For each simplex σ let $R(s)$ denotes the set of elements in $R(K)$ vanishing on $K - s$. It is isomorphic to a real topological simplex in the finite dimensional real vector subspace generated by $R(v)$ for all v

in s in the vector space of all mappings $f\colon K \to \mathbb{R}$. If v is a vertex of s then $R(v)$ is a point in $R(s)$. A subset U of $R(K)$ is open if it induces an open subset $U \cap R(s)$ in $R(s)$ for each simplex s in K.

Definition 2.6. A triangulation of a topological space E is given by a homeomorphism of E onto a realization $R(K)$ of a combinatorial complex K.

The homology of a triangulated space E is defined as the homology of its triangulation K. This notion becomes interesting when one proves the independence of the choice of the triangulation, which is the case [3]. A natural proof is deduced by comparison of homology with singular homology.

2.3. **Singular homology and cohomology.** It is based on the use of continuous mappings defined on the topological simplex Σ_n and generally adapted for an introduction to homology theory, although triangulations are more intuitive.

Definition 2.7. A singular simplex of dimension n of a topological space E is a continuous mapping $\sigma\colon \Sigma_n \to E$.

Let $\Sigma_n(E)$ denote the set of singular simplices of dimension n, and $S_n(E)$ the free abelian group generated by $\Sigma_n(E)$. Then $\Sigma_n(E)$ (resp. $S_n(E)$) define a simplicial set (resp. group) hence a chain complex; while $Hom_{\mathbb{Z}}(S_n(E), \mathbb{Z})$ define a cosimplicial group, hence a complex.

Definition 2.8. i) The homology groups $H_i(E, \mathbb{Z})$ of the chain complex $S_.(E)$ defined by the simplicial complex are called the singular homology groups of E with coefficients in $\mathbb{Z}$.
ii) The cohomology groups $H^i(E, \mathbb{Z})$ of the cochain complex $Hom_{\mathbb{Z}}(S_.(E), \mathbb{Z})$ defined by the cosimplicial group are called the singular cohomology groups of E with coefficients in $\mathbb{Z}$.

When E is triangulable, and this is the case of differentiable manifolds or algebraic varieties, the singular homology coincides with the homology of any triangulation. Homology and Cohomology groups have in general torsion elements. In Hodge theory we are mainly interested in the image of $H^i(E, \mathbb{Z})$ in $H^i(E, \mathbb{Q})$ which is torsion free.

Definition 2.9. A piecewise smooth singular simplex of dimension n of a topological differentiable manifold X is a map $\sigma\colon \Sigma_n \to X$ which extends to a differentiable map on a neighbourhood of Σ_n.

We form the chain complex $S^{ps}_.(E)$ of piecewise smooth integral chains and define its homology denoted by $H^{ps}_i(X, \mathbb{Z})$

Lemma 2.10. *The natural morphism $H^{ps}_i(X, \mathbb{Z}) \xrightarrow{\sim} H_i(X, \mathbb{Z})$ is an isomorphism.*

2.3.1. *Poincaré duality.* A general intersection theory is defined on triangulable spaces, in particular compact differentiable manifolds, on the level of chains of triangles. It induces an intersection pairing on homology which is used to prove a general Poincaré duality on oriented topological spaces.

Theorem 2.11 (Poincaré duality)**.** *On a compact oriented n-manifold, the intersection pairing*

$$H_i(X, \mathbb{Z}) \times H_{n-i}(X, \mathbb{Z}) \to \mathbb{H}_0(X, \mathbb{R}) \xrightarrow{degree} \mathbb{Z}$$

is unimodular, that is the induced morphism

$$H_i(X,\mathbb{Z}) \to Hom(H_{n-i}(X,\mathbb{Z}),\mathbb{Z})$$

is surjective and its kernel is the torsion subgroup

We need this duality on compact oriented differentiable manifolds where it can be proved using differential forms. For rational coefficients, it defines a perfect duality

$$H_i(X,\mathbb{Q}) \simeq Hom(H_{n-i}(X,\mathbb{Q}),\mathbb{Q}).$$

2.4. **Čech cohomology.** In this case, we can define cohomology groups with coefficients in any sheaf $\mathcal{F}$ of abelian groups.
Let $\mathcal{U} = (U_i)_{i\in I}$ denotes an open covering of a topological space E and S_p the set of sequences $(i_o,\dots,i_p)$ of length $p+1$ in the set of indices I such that $\cap_{j=0}^{p} U_{i_j} \neq \emptyset$ (Remark that this condition defines a structure of combinatorial complex on I as follows : a finite subset $J \in I$ is a simplex if and only if $\cap_{i\in J} U_i \neq \emptyset$). Define

$$U_s = \cap_{i\in s} U_i,\ s\in S_p \quad ; \quad C^p = C^p(\mathcal{U},\mathcal{F}) = \prod_{s\in S_p} \mathcal{F}(U_s)$$

then $C^\cdot$ is a cosimplicial group in the sense that to a map $f\colon \Delta_p \to \Delta_q$ corresponds covariantly with f; if we denote an element β of C^p as $\beta_{i_0,\dots,i_p} \in \mathcal{F}(U_{i_0,\dots,i_p})$, then

$$f^* : C^p \to C^q \ :\ (f^*\alpha)_{i_o\dots i_q} = (\alpha_{i_{f(o)}\dots i_{f(p)}})|_{U_{i_o}\cap\dots\cap U_{i_q}}$$

where we use $|$ for the restriction map of sections on $U_{i_{f(o)}} \cap \dots \cap U_{i_{f(p)}}$ to sections on $U_{i_o} \cap \dots \cap U_{i_q}$. Hence a structure of complex is defined on $C^\cdot$ by the differential

$$d\colon C^p \to C^{p+1}, (d\alpha)_{i_o\dots i_{p+1}} = \Sigma_{j=0}^{p+1}(-1)^j \alpha_{i_o\dots\hat{i}_j\dots i_{p+1}}$$

where we write $\alpha_{i_o\dots\hat{i}_j\dots i_{p+1}}$ for $\delta_j(\alpha)$. The n^{th} cohomology group of $C^\cdot$ is denoted by $\check{H}^n(\mathcal{U},\mathcal{F})$. To obtain a group independent of the covering, one needs to define an inductive limit on the set of all coverings of E using the notion of refinement of coverings, then one put

$$\check{H}^n(X,\mathcal{F}) = \underrightarrow{lim}_{\mathcal{U}} \check{H}^n(\mathcal{U},\mathcal{F})$$

When the space is triangulated, the Čech cohomology groups with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ coincides with simplicial cohomology groups.

2.5. **Hypercohomology.** Modern theory of sheaves lead to a general theory of cohomology $[G, Gr]$. An abelian category $\mathcal{A}$ is said to have enough injective objects if for any object M there exists a monomorphism $M \to I$ into an injective object I of $\mathcal{A}$. In this case, for any complex $X^\cdot$ of objects of $\mathcal{A}$, bounded to the left, and any functor $\Psi : \mathcal{A} \to \mathcal{B}$ to an abelian category $\mathcal{B}$, which is left exact, Grothendieck defines the hypercohomology $\mathbb{H}^*(\Psi, X^\cdot)$ of Ψ at $X^\cdot$ as follows:
i) *Construct first an injective resolution of* $X^\cdot$, that is a complex $I^\cdot$ of injective objects in $\mathcal{A}$ and a morphism $\Phi\colon X^\cdot \to I^\cdot$ which induces an isomorphism on cohomology objects (then Φ is called a quasi-isomorphism and an isomorphism in the derived category $D^+(\mathcal{A})[D]$).
ii) *Define the hypercohomology as an object in* $\mathcal{B}$

$$\mathbb{H}^*(\Psi, X^\cdot) = H^*(\Psi(I^\cdot))$$

In particular, the previous cohomology groups of a topological space X with coefficients in $\mathbb{Z}$ are isomorphic to the hypercohomology of the functor Γ of global sections of sheaves of abelian groups on X applied to the constant sheaf $\underline{\mathbb{Z}}$ defined by the group $\mathbb{Z}$ on X. The proof is based on the use of adequate resolutions. A sheaf is called acyclic for a functor if its hypercohomology vanishes in degree ≥ 1. Using spectral sequence theory, one can show that all resolutions by acyclic sheaves for the functor of global sections, will give isomorphic cohomology groups. Godement [G] uses a particular type of acyclic sheaves called flabby. Any sheaf has natural resolutions by flabby sheaves and injective sheaves are flabby, so we have the same cohomology objects using either resolutions.

2.6. **DeRham cohomology.** Now let X be a differentiable manifold and let $\mathcal{E}_X$ be the sheaf of $\mathcal{C}^\infty$ real-valued functions on X, $\mathcal{E}_X^p$ the $\mathcal{E}_X$-module of $\mathcal{C}^\infty$ differential p-forms on X and consider the complex $\mathcal{E}_X^*$ with its exterior differential.

Definition 2.12. The cohomology groups of the complex of global sections of $\mathcal{E}_X^*$ are called De Rham cohomology

$$H^i_{DR}(X,\mathbb{R}) := H^i(\Gamma(X,\mathcal{E}_X^*)) := H^i(\mathcal{E}^*(X)) := \frac{Ker(d^i : \mathcal{E}^i(X) \to \mathcal{E}^{i+1}(X)}{Im(d^{i-1} : \mathcal{E}^{i-1}(X) \to \mathcal{E}^i(X)}$$

This definition uses the differentiable structure to introduce the above groups called DeRham realization of the cohomology, as they give isomorphic groups with previous definitions of the cohomology.

2.6.1. *DeRham resolution.* By Poincaré lemma this complex is a resolution of the constant sheaf $\mathbb{R}$ and using "partition of unity" one can prove this is a resolution by fine sheaves.

Lemma 2.13. *The complex of sheaves of differential forms $\mathcal{E}_X^*$ is a fine resolution of the constant sheaf $\mathbb{R}$.*

Since fine resolutions are acyclic for the global section functor, we deduce the isomorphisms

$$H^i(X,\mathbb{R}) \simeq H^i_{DR}(X,\mathbb{R})$$

We define also the cohomology of X with complex coefficients using the global sections of complex valued differential forms

$$H^i(X,\mathbb{C}) \xrightarrow{\sim} \mathbb{H}^i(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} H^i(\mathcal{E}^*(X) \otimes_{\mathbb{R}} \mathbb{C})$$

We note that the cohomology spaces are defined naturally over $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$, which provides a natural lattice inside the real vector space and a real structure inside the complex cohomology

$$H^i(X,\mathbb{Z}) \otimes \mathbb{R} \simeq H^i(X,\mathbb{R}), \quad H^i(X,\mathbb{R}) \otimes \mathbb{C} \simeq H^i(X,\mathbb{C})$$

2.6.2. *Comparison with Singular cohomology.* Given a differential form $\omega \in \Gamma(X,\mathcal{E}_X^n)$ and a piecewise smooth singular chain of simplices $\sigma \in S_n^{ps}(X)$ one can define the integral $\int_\sigma \omega$ in $\mathbb{R}$. This integral can be used to construct a map

$$\Gamma(X,\mathcal{E}_X^n) \to S^n_{ps}(X,\mathbb{R}) := Hom_{\mathbb{R}}(S_n^{ps}(X),\mathbb{R})$$

defined by $(\omega,\sigma) = (-1)^{n(n+1)/2}\int_\sigma \omega$.
By Stoke's theorem : $\int_\sigma d\omega = \int_{\partial\sigma}\omega$ where ∂ denotes the boundary map for chains; from which we deduce that the above map extends to a map of complexes.
DeRham's theorem states that the map defined above induces an isomorphism on cohomology groups for each $n \in \mathbb{N}$

Theorem 2.14 (DeRham). *Integration over piecewise smooth singular cycles in a differentiable manifold X, defines an isomorphism of cohomology groups*

$$H^n_{DR}(X,\mathbb{R}) \xrightarrow{\sim} H^n(S^*_{ps}(X,\mathbb{R})) \xrightarrow{\sim} H^n(X,\mathbb{R})$$

compatible with wedge product on DeRham cohomology and cup-product on singular cohomology.

2.6.3. *Poincaré duality.* It can be checked directly on DeRham cohomology

Theorem 2.15 (Poincaré). *The bilinear product*

$$H^{n-i}(X,\mathbb{R}) \otimes H^{n+i}(X,\mathbb{R}) \to H^{2n}(X,\mathbb{R}) \to \mathbb{R} : (\alpha \otimes \beta) \mapsto \alpha \wedge \beta \mapsto \int_X \alpha \wedge \beta$$

defines an isomorphism: $H^{n-i}(X,\mathbb{R}) \simeq Hom(H^{n+i}(X,\mathbb{R}),\mathbb{R})$.

When we identify above the dual space to cohomology with homology, the Poincaré duality is stated as an isomorphism

$$H^{n-i}(X,\mathbb{R}) \simeq H_{n+i}(X,\mathbb{R})$$

Then the cup-product on cohomology corresponds to the topological Intersection theory on the manifold X, such that the duality is stated on homology as follows

$$H_{n-i}(X,\mathbb{R}) \otimes H_{n+i}(X,\mathbb{R}) \xrightarrow{\cap} H_0(X,\mathbb{R}) \xrightarrow{degree} \mathbb{R} : (\alpha \otimes \beta) \mapsto \alpha \cap \beta \mapsto deg\, \alpha \cap \beta$$

3. Harmonic forms

In this section X will denote successively a differentiable , complex analytic, and Kähler compact manifold.

3.1. Harmonic forms on a differentiable manifold.

A Riemannian manifold X is endowed with a *scalar product on its tangent bundle* defining a metric on X. A basic result in analysis states that the *cohomology of a compact smooth Riemannian manifold is represented by real harmonic global differential forms* denoted in degree i by $\mathcal{H}^i(X)$. To define harmonic forms, we need to introduce the *$\star$-operator and the Laplacian.*

3.1.1. *Riemannian metric.* A bilinear form g on X is defined at each point x as a product on the tangent space T_x to X at x

$$g_x(,) : T_x \otimes_{\mathbb{R}} T_x \to \mathbb{R}$$

where g_x vary smoothly with x, that is $h_{ij}(x) := g_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is differentiable in x, then the product is given as $g_x = \sum_{i,j} h_{ij}(x) dx_i \otimes dx_j$. It is called a metric if moreover the matrix of the the product defined by $h_{ij}(x)$ is positive definite, that is: $g_x(u,u) > 0$ for all $u \neq 0 \in T_x$.
An induced metric is defined on $\Omega^1_x := T^*_x := Hom(T_x,\mathbb{R})$ and on Ω^p_x as follows. Let $e_1,\dots,e_n$ be an orthonormal basis, then on Ω^p_x we define the unique metric such that $e^*_{i_1} \wedge \dots, \wedge e^*_{i_p}$ is an orthonormal basis where $(e^*_i)_{i\in I}$ form a dual basis to $(e_i)_{i\in I}$.

3.1.2. *Volume form, L^2 metric.* Suppose now that the manifold is oriented, then an orthonormal positively oriented basis e_i define a nowhere vanishing section of Ω^n_X called volume section vol satisfying $vol_x := e_1^* \wedge \dots, \wedge e_n^*$.

Exercise 3.1. Let $x_1, \dots, x_n$ denote a local ordered set of coordinates on an open subset U of a covering of X, compatible with the orientation. Prove that

$$\sqrt{d}et(h_{ij})dx_1 \wedge \cdots \wedge dx_n$$

defines the global volume form.

We deduce from the volume form a global pairing called the L^2 metric

$$\forall \psi,\, \eta \in \mathcal{E}^i(X), (\psi,\, \eta)_{L^2} = \int_X g_x(\psi(x),\, \eta(x))vol(x)$$

3.1.3. *Laplacian.* We prove the existence of an adjoint $d^* : \mathcal{E}_X^{i+1} \to \mathcal{E}_X^i$ to the differential $d : \mathcal{E}_X^i \to \mathcal{E}_X^{i+1}$, satisfying

$$(d\psi, \eta)_{L^2} = (\psi, d^*\eta)_{L^2}, \quad \forall \psi \in \mathcal{E}^i(X), \eta \in \mathcal{E}^{i+1}(X).$$

The adjoint operator is defined by constructing first an operator $*$

$$\mathcal{E}_X^i \xrightarrow{*} \mathcal{E}_X^{n-i}$$

by requiring at each point $x \in X$

$$(\psi,\, \eta)vol_x = \psi_x \wedge *\eta_x, \quad \forall \psi,\, \eta \in \mathcal{E}^i_{X,x}.$$

The section vol defines an isomorphism $\mathbb{R} \simeq \mathcal{E}_X^n$, inducing the isomorphism $Hom(\mathcal{E}_X^i, \mathbb{R}) \simeq Hom(\mathcal{E}_X^i, \mathcal{E}_X^n) \simeq \mathcal{E}_X^{n-i}$. The definition of $*$ is obtained by composition with the isomorphism $\mathcal{E}_X^i \to Hom(\mathcal{E}_X^i, \mathbb{R})$ defined by the scalar product on $\mathcal{E}_X^i$. We have on $\mathcal{E}_X^i$

$$*^2 = (-1)^{i(n-i)} Id, \quad d^* = (-1)^{n+in+1} * \circ d \circ *.$$

Lemma 3.2. *We have for all $\alpha, \beta \in \mathcal{E}^i(X)$*

$$(\alpha,\, \beta)_{L^2} = \int_X \alpha \wedge *\beta.$$

Definition 3.3. The Laplacian Δ is defined as

$$\Delta = d^* \circ d + d \circ d^*$$

Harmonic forms are defined as the solutions of the Laplacian

$$\mathcal{H}^i(X) = \{\psi \in \mathcal{E}^i(X) : \Delta(\psi) = 0\}$$

The importance of harmonic forms in Hodge theory stems from the representation of cohomology. However they have the following interesting property.
Consider $\phi \in \mathcal{E}^i(X)$ satisfying $d\phi = 0$ and its cohomology class $\phi + d\psi \in \mathcal{E}^i(X)$. The Laplace equation defines exactly the form in this class with minimal L^2-norm. First note that the class is an affine subspace of the $\mathcal{E}^i(X)$. To show the existence of the solution we introduce the Hilbert space of L^2-forms; then the closure of the affine subspace contains a unique element of minimal distance to 0. A remarkable result in the theory of elliptic differential equations [Warner, Wells] proves the regularity of the solution that is, such element is in fact in the affine space and not only in its completion. The link to Laplace equation is as follows. An element ϕ has minimal norm if

$$\forall t \in \mathbb{R}, \quad \|\phi\|^2_{L^2} \le \|\phi + d\psi\|^2_{L^2}$$

The expansion of the right term in t, $\|\phi\|^2_{L^2} + 2t(\phi, d\psi)_{L^2} + O(t^2)$ shows that the inequality holds if and only if $(\phi, d\psi)_{L^2} = 0$, and equivalently $(d^*\phi, \psi)_{L^2} = 0$ for all ψ, hence $d^*\phi = 0$. That is $d\phi = 0$ (ϕ is closed), $d^*\phi = 0$ (ϕ is co-closed) and $\Delta(\phi) = 0$. Reciprocally if $\Delta(\phi) = 0$, then $0 = (\Delta(\phi), \phi)_{L^2} = (dd^*\phi, \phi)_{L^2} + (d^*d\phi, \phi)_{L^2} = \|d^*\phi\|^2_{L^2} + \|d\phi\|^2_{L^2}$, hence ϕ is closed and co-closed.
A basic result admitted here, is

Theorem 3.4. *On a compact smooth oriented Riemannian manifold each cohomology class is represented by a unique real harmonic global differential form*

$$\mathcal{H}^i(X) \simeq H^i(X, \mathbb{R}).$$

3.2. **Complex manifolds.** An important feature of a complex manifold is the existence of an almost complex structure J on the real tangent bundle.

3.2.1. *Almost complex structure.* Let V be a real vector space. A almost complex structure on V is defined by a linear map $J : V \to V$ satisfying $J^2 = -1$; then dim.V is even and the eigenvalues are i and $-i$ with associated eigenspaces in $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$

$$V^+ = \{x - iJx : x \in V\} \subset V_{\mathbb{C}},\ V^- = \{x + iJx : x \in V\} \subset V_{\mathbb{C}}.$$

Example 3.5. 1- The space $\mathbb{R}^2$ with the action of J defined on the canonical basis by $J(1,0) := (0,1)$.
The isomorphism $\varphi : \mathbb{C} \xrightarrow{\sim} \mathbb{R}^2, \varphi(1) = (1,0), \varphi(i) = (0,1)$ is compatible with the action of i on $\mathbb{C}$ and the action of J on $\mathbb{R}^2$, hence the action of J amounts to define a complex structure for which J is the multiplication by i.
2- Let V be a complex vector space, then the isomorphism $\varphi : V \to V_{\mathbb{R}}$ with the real underlying space to V transports the action of i to an action J. The image of a complex basis $(e_i)_{i\in[1,n]}$ defines a basis a real basis $\varphi(e_i), J(\varphi(e_i)) = J(i\varphi(e_i), i \in [1,n]$.
Reciprocally, a real vector space W with the action of J corresponds to a structure of complex vector space on V as follows. Let $(e_i)_{i\in[1,n]}$ be a subset of W such that $\{e_i, J(e_i)\}_{i\in[1,n]}$ form a basis of W (first choose a non zero vector e_1 in W and prove that W_1 generated by $e_1, J(e_1)$ is of dimension two, then continue with $e_2 \notin W_1$ and so on). Then there is a complex structure on W for which $e_j, j \in [1,n]$ is a complex basis with the action of $i \in \mathbb{C}$ defined by $i.e_j := J(e_j)$.

3.2.2. *Decomposition into types.* Let V be a complex vector space and $(V_{\mathbb{R}}, J)$ the underlying real vector space with its involution J defined by multiplication by i. Let $W := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ and $W_{\mathbb{C}} := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{C}) \simeq W \otimes \mathbb{C}$, then $Hom_{\mathbb{C}}(V, \mathbb{C})$ embeds into $W_{\mathbb{C}}$ and its image is denoted $W^{1,0}$ while its conjugate with respect to W is the subspace denoted $W^{0,1}$, moreover

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}, \quad \wedge^2 W_{\mathbb{C}} = W^{2,0} \oplus W^{1,1} \oplus W^{0,2}$$

where we define resp. the spaces of forms of type $(2,0)$, $(1,1)$, $(0,2)$ as :

$$W^{2,0} = W^{1,0} \otimes W^{1,0}, W^{1,1} = W^{1,0} \otimes W^{0,1}, W^{0,2} = W^{0,1} \otimes W^{0,1}.$$

3.2.3. *Decomposition into types on the complexified tangent bundle.* The existence of a complex structure on the manifold leads to an almost complex structure on the tangent bundle $T_{X,\mathbb{R}}$ and a decomposition into types of the complexified tangent bundle $T_{X,\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which splits as a direct sum $T^{1,0}_X \oplus T^{0,1}_X$ where $T^{1,0}_{X,z} = \{u - iJu :$

$u \in T_{X,\mathbb{R},z}\}$ and $\overline{T_X^{1,0}} = T_X^{0,1}$. Precisely, if $z_j = x_j + iy_j, j \in [1,n]$ are local complex coordinates, then $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ in $T_{X,\mathbb{R},z}$. The holomorphic tangent space $T_{X,z}$ embeds isomorphically into $T_X^{1,0}$ generated by $\frac{\partial}{\partial z_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ of the form $u_j - iJ(u_j)$. From now on we identify the two bundles

$$T_X \xrightarrow{\sim} T_X^{1,0}$$

The dual of the above decomposition of the tangent space is written as $\mathcal{E}_X^1 \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathcal{E}_X^{1,0} \oplus \mathcal{E}_X^{0,1}$ which induces a decomposition of the sheaves of differential forms into types

$$\mathcal{E}_X^i \otimes_{\mathbb{R}} \mathbb{C} \simeq \oplus_{p+q=i} \mathcal{E}_X^{p,q}$$

In terms of complex local coordinates on an open set U, $\phi \in \mathcal{E}^{p,q}(U)$ is written as a linear combination of

$$dz_I \wedge d\overline{z}_J := dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

where $I = \{i_1, \cdots, i_p\}, J = \{j_1, \ cdots, j_q\}$. The differential decomposes as as well $d = \partial + \overline{\partial}$, where $\overline{\partial} : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q+1}$ and $\partial : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p+1,q}$ are compatible with the decomposition up to a shift on the bidegree, we deduce from $d^2 = \partial^2 + \overline{\partial}^2 + \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$ that the DeRham complex is the simple complex associated to a double complex

$$(\mathcal{E}_X^* \otimes_{\mathbb{R}} \mathbb{C}, d) \simeq s((\mathcal{E}_X^{*,*}, \partial, \overline{\partial})$$

In particular we recover the notion of a subspace of cohomology type (p,q).

$$H^{p,q}(X) := \{\text{Cohomology classes representable by a closed form of type } (p,q)\,\}$$

We have $\overline{H^{p,q}(X)} = H^{q,p}(X)$ but it is not always true that $H^i_{DR}(X) = \oplus_{p+q=i} H^{p,q}(X)$. Hodge theory apply to a complex manifold when exactly such decomposition exists.

Remark 3.6. Given a bundle of real vector spaces with differentiable action of J, the problem to lift this action into a complex structure on the bundle is not straightforward. An important result in this direction is the Newlander-Nirenberg theorem.

Example 3.7. 1 - On a compact complex torus $X = \mathbb{C}^n/\Lambda$, a basis of the cohomology is given by the translation-invariant forms, hence with constant coefficients since the tangent space is isomorphic to a direct sum of the structural sheaf $\mathcal{E}_X^0$. Hence the decomposition reduces to the case of $\mathbb{R}^{2n} = \mathbb{C}^n \oplus \overline{\mathbb{C}^n}$.
2 - The case of a compact Riemann surface , reduces to prove

$$H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X).$$

3.2.4. *Holomorphic Poincaré lemma.* On complex manifolds the sheaf of holomorphic forms Ω_X^1 is an important geometric invariant. The holomorphic version of Poincaré lemma resolution lemma shows that the DeRham complex of holomorphic forms Ω_X^* is a resolution of the constant sheaf $\underline{\mathbb{C}}$. However, since the resolution is not acyclic, cohomology spaces are computed only as hypercohomology of the global sections functor, that is after acyclic resolution.

$$H^i(X,\mathbb{C}) \simeq \mathbb{H}^i(X, \Omega_X^*) := R^i\Gamma(X, \Omega_X^*) \simeq H^i(\mathcal{E}^*(X) \otimes_{\mathbb{R}} \mathbb{C})$$

3.2.5. *Dolbeault cohomology.* The complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for each $r \geq 0$, is called a Dolbeault resolution since Dolbeault's result shows it is a resolution of Ω_X^*.

Lemma 3.8. *The Dolbeault complex $(\mathcal{E}_X^{r,*}, \overline{\partial})$ for $r \geq 0$, is a fine resolution of Ω_X^r, hence*

$$H^i(X, \Omega_X^r) \simeq H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) := H_{\overline{\partial}}^{r,i}(X)$$

The cohomology of the complex of its global sections is called the $\overline{\partial}$ cohomology of X of type (r,i)

$$H_{\overline{\partial}}^{r,i}(X) := H^i(\mathcal{E}_X^{r,*}(X), \overline{\partial}) = \frac{Ker(\overline{\partial}^i : \mathcal{E}^{r,i}(X) \to \mathcal{E}^{r,i+1}(X))}{Im(\overline{\partial}^{i-1} : \mathcal{E}^{r,i-1}(X) \to \mathcal{E}^{r,i}(X))}$$

A cohomology class of X of type (r,i) defines a $\overline{\partial}$ cohomology class of the same type. It will follows that this map defines an isomorphism whenever Hodge theory apply.

3.3. **Hermitian metric.** A Hermitian product h on a complex manifold X is defined continuously on the tangent space.

3.3.1. *Hermitian metric on a complex vector space.* Let V be a complex vector space and $(V_{\mathbb{R}}, J)$ the underlying real vector space with its involution J defined by multiplication by i. A Hermitian product h on V is a product

$$h : V \times V \to \mathbb{C}, \text{ satisfying } h(u,v) = \overline{h(v,u)}$$

which is biadditive, linear in the first term and antilinear in the second term. Considering the restriction to $V_{\mathbb{R}}$

$$h_{|V_{\mathbb{R}}} = \mathcal{R}eh + i\mathcal{I}mh$$

The product defines a Hermitian metric if moreover the real number $h(u, \overline{u}) > 0$ is positive definite for all $u \neq 0 \in V$, then we deduce a Riemannian structure on $V_{\mathbb{R}}$ defined by $g := \mathcal{R}eh$.

Definition 3.9 (The (1,1) form ω)**.** It is associated to the hermitian metric a real $2-$form $\omega := -\mathcal{I}mh$ on $V_{\mathbb{R}}$, satisfying:
$\omega(Ju, v) = -\omega(u, Jv)$, and $\omega(Ju, Jv) = \omega(u, v)$.

Proof. We have a $2-$form since $\omega(v,u) = -\mathcal{I}mh(v,u) = \mathcal{I}mh(u,v) = -\omega(u,v)$. We check
$\omega(u, Jv) = -\mathcal{I}mh(u, iv) = g(u,v), \omega(Ju, v) = -\mathcal{I}mh(iu, v) = -g(u,v)$
$\omega(Ju, Jv) = -\mathcal{I}mh(iu, iv) = -\mathcal{I}mh(u,v) = \omega(u,v)$. □

Remark 3.10. The hermitian form is entirely determined by g or ω since $g(u,v) = \omega(u, Jv)$ and $\omega(u,v) = -g(u, Jv)$, as one can deduce from the identity: $h(u, iv) = -ih(u,v)$.

Let $W := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{R})$ and $W_{\mathbb{C}} := Hom_{\mathbb{R}}(V_{\mathbb{R}}, \mathbb{C}) \simeq W \otimes \mathbb{C}$, recall the decompositions

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}, \quad \wedge^2 W_{\mathbb{C}} = W^{2,0} \oplus W^{1,1} \oplus W^{0,2}$$

Considering $\wedge_{\mathbb{R}}^2 W \subset \wedge^2 W_{\mathbb{C}}$, we introduce the intersection in $\wedge^2 W_{\mathbb{C}}$

$$W_{\mathbb{R}}^{1,1} := W^{1,1} \cap (\wedge_{\mathbb{R}}^2 W) \subset \wedge^2 W_{\mathbb{R}}$$

Lemma 3.11. *The correspondence: $h \to -\mathcal{I}mh$ is a natural isomorphism between the real vector space of Hermitian forms on $V \times V$ and the space $W^{1,1}_{\mathbb{R}}$ of real forms of type* $(1,1)$.

Definition 3.12. A real form of type $(1,1)$ is positive if the corresponding Hermitian form is positive definite.

Let $\{e_i\}$ of V denote a basis of V. The projections $dz_j, j \in [1,n]$, denote a dual basis of the dual space V^*; then the Hermitian product is written as

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\overline{z_j} \in V^* \otimes_{\mathbb{C}} \overline{V}^* \simeq W^{1,0} \otimes W^{0,1}$$

where $h_{ij} = h(e_i, e_j)$ and $\overline{V}$ is the complex conjugate space to V. Let $H = (h_{ij})$ denotes the matrix with coefficients h_{ij} (equal to its conjugate transpose), then

$$h(u,v) = {}^t\overline{V}HU$$

where U, V are the coordinates of u and v written in columns. For a Hermitian metric, the matrix H is positive definite.
We can always suppose the basis orthonormal, that is $H = Id$. If we write $dz_j = dx_j + idy_j$, then

$$h = \sum_j dz_j \otimes d\overline{z_j} = \sum_j (dx_j \otimes dx_j + dy_j \otimes dy_j) - i \sum_j (dx_j \otimes dy_j - dy_j \otimes dx_j)$$

where $\sum_j (dx_j \otimes dx_j + dy_j \otimes dy_j)$ defines the Riemannian metric and
$\omega = \sum_j dx_j \wedge dy_j := \sum_j (dx_j \otimes dy_j - dy_j \otimes dx_j) = \frac{i}{2} \sum_j dz_j \otimes d\overline{z_j}$ (note that we avoid the factor $\frac{1}{2}$ in some definition of $\sum_j dx_j \wedge dy_j$).
The volume form $vol \in \wedge^{2n} V_{\mathbb{R}}$, associated to g, can be computed via ω and is equal to $\frac{1}{n!}\omega^n$. We have

$$\omega = \frac{i}{2} \Sigma_j dz_j \wedge d\overline{z}_j = \Sigma_j dx_j \wedge dy_j, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

3.3.2. *Extensions of h to $W^{p,q}$.* Note that the hermitian product h extends to $h^{1,0}$ on $W^{1,0}$ by the condition that the dual basis $\{dz_j\}_{j \in [1,n]}$ to an orthonormal basis $\{e_j\}_{j \in [1,n]}$ of V, is orthonormal for $h^{1,0}$. On $W^{0,1}$ the basis $\{d\overline{z}_j\}_{j \in [1,n]}$ is orthonormal for the extension $h^{0,1}$.
Independently, g on V extends by the condition of orthonormality to W and by linearity to $W \otimes \mathbb{C}$, then we define h^1 on $W_{\mathbb{C}}$ by $h^1(u,v) := g(u, \overline{v})$. Moreover we define $h^{p,q}$ on $W^{p,q} \wedge^p W^{1,0} \otimes \wedge^q W^{0,1}$ and h^k on $\wedge^k W_{\mathbb{C}}$

Lemma 3.13. *The extension h^k of h is equal to $2^k \sum h^{p,q}$ on $\wedge^k W_{\mathbb{C}} \simeq \sum_{p+q=k} W^{p,q}$.*

In fact, $h^{1,0}(dz_j, dz_j) = 1$ while $h^1(dz_j, dz_j) = g(dx_j + idy_j, dx_j - idy_j)$
$= g(dx_j, dx_j) + g(dy_j, dy_j) + i(g(dy_j, dx_j) - g(dx_j, dy_j)) = 2$. Similarly $h^1(d\overline{z}_j, d\overline{z}_j) = 2$.

3.3.3. *Hermitian metric on X.* The Hermitian product h on the complex manifold X is defined at each point z as a Hermitian product on the holomorphic tangent space T_z to X at z

$$h_z : T_z \times T_z \to \mathbb{C}$$

moreover h_z vary smoothly with z, that is $h_{ij}(z) := h_z(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ is differentiable in z and $h_{ij}(z) = \overline{h_{ji}(z)}$.

The isomorphism $T_z \xrightarrow{\sim} T_z^{1,0}$. The holomorphic tangent space is generated by $\frac{\partial}{\partial z_i}$ and the isomorphism is defined by

$$\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$$

in $T_z^{1,0} \subset T_{X,\mathbb{R},z} \otimes \mathbb{C}$. Hence it is equivalent to define the Hermitian product as

$$h_z : T_z^{1,0} \times T_z^{1,0} \to \mathbb{C},\ h_z = \sum_{i,j} h_{ij}(z) dz_i \otimes d\overline{z_j} \in \mathcal{E}_z^{1,0} \otimes_{\mathbb{C}} \mathcal{E}_z^{0,1}$$

The product is called a Hermitian metric if moreover the real number $h_z(u, \overline{u}) > 0$ is positive definite for all $u \neq 0 \in T_z$; equivalently the matrix of the product defined by $h_{ij}(z)$ (equal to its conjugate transpose) is positive definite.
Considering the $\mathbb{R}-$linear isomorphism

$$T_{X,\mathbb{R},z} \xrightarrow{\sim} T_z^{1,0} : \frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}, \frac{\partial}{\partial y_j} = J(\frac{\partial}{\partial x_j}) \mapsto i\frac{\partial}{\partial z_j}$$

with inverse defined by the real part u of $u + iv$, and compatible with the action of J on $T_{X,\mathbb{R},z}$ and i on $T_z^{1,0}$; the induced product is written as

$$h_{z|T_{X,\mathbb{R},z}} = \mathcal{R}eh_z + i\mathcal{I}mh_z$$

We associate to the metric a Riemannian structure on X defined by $g_z := \mathcal{R}eh_z$. Then X of dimension n is viewed as a Riemann manifold of dimension $2n$ with metric g_z. Note that g is defined over the reals since it can be represented by a real matrix over a basis of the real tangent space.
Since h_z is hermitian, we have $g_z(Ju, Jv) = g_z(u, v)$.

3.3.4. *The (1,1) form* ω. We associate also to the metric a real $2-$form $\omega := -\mathcal{I}mH$ of type $(1,1)$ ($\omega \in \mathcal{E}_X^{1,1} \cap \mathcal{E}_X^2$). The volume form $vol \in \mathcal{E}^{2n}$ defined by g on X, can be defined by ω and is equal to $\frac{1}{n!}\omega^n$. We have locally

$$\omega = \frac{i}{2}\Sigma_i dz_i \wedge d\overline{z}_i, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Reciprocally, the metric can be recovered from ω or from g.

Example 3.14. 1) The Hermitian metric on $\mathbb{C}^n$ is defined by $h := \sum_{i=1}^n dz_i \otimes d\overline{z_j}$ with induced metric the standard metric on $\mathbb{R}^{2n}$

$$H(z, w) = \sum_{i=1}^n z_i \overline{w}_i$$

where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$. By writing $z_k = x_k + iy_k$, it is possible to consider $\mathbb{C}^n = \mathbb{R}^{2n}$, in which case $\mathcal{R}eH$ is the Euclidean inner product and $\mathcal{I}mH$ is a nondegenerate alternating bilinear form, i.e., a symplectic form.
Explicitly, in $\mathbb{C}^2$, the standard Hermitian form is expressed as

$$\begin{aligned} H((z_{1,1}, z_{1,2}), (z_{2,1}, z_{2,2})) =& x_{1,1}x_{2,1} + x_{1,2}x_{2,2} + y_{1,1}y_{2,1} + y_{1,2}y_{2,2} \\ &+ i(x_{2,1}y_{1,1} - x_{1,1}y_{2,1} + x_{2,2}y_{1,2} - x_{1,2}y_{2,2}). \end{aligned}$$

2) If $\Lambda \subset \mathbb{C}^n$ is a full lattice, then the above metric induces a Hermitian metric on the torus $\frac{\mathbb{C}^n}{\Lambda}$.
3) The *Fubini -Study metric on the projective space* $\mathbb{P}^n$. Let $\pi : \mathbb{C}^n - 0 \to \mathbb{P}^n$ be the projection and consider a section $Z : U \to \mathbb{C}^n - 0$ of π on an open subset U, then the form $\omega = \frac{i}{2\pi}\partial\overline{\partial}log||Z||^2$ is a $(1,1)$ form, globally defined on $\mathbb{P}^n$, since we can check

that forms on two open subsets glue together on the intersection. For the section $Z = (1, w_1, \dots, w_n)$, $\omega = \frac{i}{2\pi}\sum_{i=1}^n dw_i \wedge d\overline{w_i} > 0$ is a real form (takes real value on real tangent vectors) and positive that is associated to a positive Hermitian form. Hence the projective space admits a Hermitian metric.

3.4. **Harmonic forms on compact complex manifolds.** In this section, a study of the $\overline{\partial}$ operator (similar to the previous study of d) will lead to the representation of Dolbeault cohomology by $\overline{\partial}$ harmonic forms as stated in Hodge theorem. We choose a Hermitian metric h with associated $(1,1)$ form ω with volume form $vol = \frac{1}{n!}\omega^n$. Then we define the L^2-inner product on $\mathcal{E}_X^{p,q}$ using the underlying Riemannian metric g and the adjoint operator to $\overline{\partial}$

$$\overline{\partial}^* : \mathcal{E}_X^{p,q} \to \mathcal{E}_X^{p,q-1}$$

defined as $\overline{\partial}^* = - * \partial *$ satisfying

$$(\overline{\partial}^* \psi, \eta)_{L^2} = (\psi, \overline{\partial}\eta)_{L^2}$$

so to introduce the $\overline{\partial}-$Laplacian

Definition 3.15. The $\overline{\partial}-$Laplacian is defined as

$$\Delta_{\overline{\partial}} = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}$$

Harmonic forms of type (p,q) are defined as the solutions of the $\overline{\partial}-$Laplacian

$$\mathcal{H}^{p,q}(X) = \{\psi \in \mathcal{E}^{p,q}(X) \otimes \mathbb{C} : \Delta_{\overline{\partial}}(\psi) = 0\}.$$

We define similarly $\partial * = - * \overline{\partial} *$ and a Laplacian

$$\Delta_{\partial} = \partial \circ \partial^* + \partial^* \circ \partial$$

A basic result is

Theorem 3.16 (Hodge theorem)**.** *On a compact complex Hermitian manifold each cohomology class of type (p,q) is represented by a unique $\overline{\partial}-$harmonic global complex differential form of type (p,q)*

$$\mathcal{H}^{p,q}(X) \simeq H^{p,q}_{\overline{\partial}}(X)$$

moreover the space of $\overline{\partial}-$harmonic forms is finite dimensional (hence the space of $\overline{\partial}-$cohomology also).

Corollary 3.17.

$$H^q(X, \Omega^p_X) \simeq \mathcal{H}^{p,q}(X)$$

Corollary 3.18 (Serre duality)**.** The trace map is an isomorphism

$$Tr : H^n(X, \Omega^n_X) \xrightarrow{\sim} H^{2n}(X, \mathbb{C}) \xrightarrow{\sim} \mathbb{C}, \ \omega \mapsto \frac{1}{(2i\pi)^n}\int_X \omega$$

and the pairing is

$$H^p(X, \Omega^n_X) \otimes H^{n-p}(X, \Omega^n_X) \to H^n(X, \Omega^n_X)$$

is nondegenerate.

Notice that we introduce a factor $\frac{1}{(2i\pi)^n}$ of the integral to fit later with the fact that $H^n(X, \Omega^n_X)$ is a Hodge structure of weight $2n$.

3.5. **Kähler manifolds.** In general the exterior product of harmonic forms is not harmonic neither the restriction of harmonic forms to a submanifold is harmonic for the induced metric. The $\overline{\partial}-$Laplacian $\Delta_{\overline{\partial}}$ and the Laplacian Δ_d are not related in general. The theory will be more natural if we add the Kähler condition on the metric, most importantly the components of harmonic forms into type will be harmonic.

Definition 3.19. The Hermitian metric is Kähler if its associated $(1,1)$ form ω is closed: $d\omega = 0$.

Example 3.20. 1) A Hermitian metric on a compact Riemann surface is Kähler since $d\omega$ of degree 3 must vanish.
2) The Enclidean metric on a compact complex torus is Kähler.
3) The projective space with its canonical metric is Kähler.
4) The restriction of a Kähler metric on a submanifold is Kähler with associated $(1,1)$ form induced from the associated $(1,1)$ form on the ambient manifold.
5) The product of two Kähler manifolds is Kähler.

Corollary 3.21. 1) The fundamental class of a subvariety $Y \subset X$ is non zero in $H^*_{DR}(X,\mathbb{C})$. 2) The even Betti numbers $b_{2q}(X)$ are positive. Moreover

$$H^0(X,\Omega^q_X) \hookrightarrow H^q_{DR}(X,\mathbb{C})$$

We have in this case the following important relations of the Laplacians

Lemma 3.22.

$$\Delta_d = 2\Delta_{\overline{\partial}} = 2\Delta_{\partial}$$

Consider the projections

$$\Pi^{p,q} : \mathcal{E}^r(X) \otimes \mathbb{C} \to \mathcal{E}^{p,q}(X)$$

Corollary 3.23. The projection on the (p,q) component of an harmonic form is harmonic and we have a natural decomposition

$$\mathcal{H}^r(X) \otimes \mathbb{C} = \oplus_{p+q=r}\mathcal{H}^{p,q}(X),\ \mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$$

One prove first that the Laplacian Δ_d commutes with the projection $\Pi^{p,q}$, hence the (p,q) components of an harmonic form, are harmonic. Since Δ_d is real, we deduce the conjugation property.

3.5.1. *Hodge decomposition.* Recall the definition of the type of cohomology

$$H^{p,q}(X) = \frac{Z^{p+q}_d(X)}{d\mathcal{E}^*(X) \cap Z^{p+q}_d(X)} \text{ where } Z^{p+q}_d(X) = Ker\, d \cap \mathcal{E}^{p,q}(X)$$

Theorem 3.24 (Hodge decomposition)**.** *Let X be a compact Kähler manifold. There is an isomorphism of cohomology classes of type (p,q) with harmonic forms of the same type*

$$H^{p,q}(X) \simeq \mathcal{H}^{p,q}(X).$$

which results into a decomposition of the complex cohomology spaces into a direct sum of complex subspaces

$$H^i(X,\mathbb{C}) = \oplus_{p+q=i}H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}}(X)$$

called Hodge decomposition.

Corollary 3.25. There exists a Hodge decomposition on the cohomology of a smooth complex projective variety.

3.5.2. *Applications to Hodge theory.* We remark first the isomorphisms

$$H^{p,q}(X) \simeq H^{p,q}_{\overline{\partial}}(X) \simeq H^q(X, \Omega^p_X)$$

which shows in particular for $q = 0$

$$H^{p,0}(X) \simeq H^0(X, \Omega^p_X)$$

1) The holomorphic forms are closed and harmonic for any Kähler metric.
2) There are no non zero global holomorphic forms on $\mathbb{P}^n$ and more generally

$$H^{p,q}_{\overline{\partial}}(\mathbb{P}^n) \simeq H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}) = \begin{cases} 0, & \text{if } p \neq q \\ \mathbb{C} & \text{if } p = q \end{cases}$$

Exercise 3.26 (Holomorphic invariants in cohomology)**.** Any holomorphic $p-$form on a compact Kähler manifold is closed and it is 0 iff it is exact. Equivalently, there is a natural injective morphism: global holomorphic $p-$forms $\to H^p_{DR}$ with image $H^{p,0}$.

Proof. Let ϕ be holomorphic of type $p, 0$, then $\overline{\partial}\phi = 0$ and $\overline{\partial}^* \phi$ of type $p, -1$ must vanish, hence the Laplacian $\Delta_{\overline{\partial}}\phi = 0$, hence on Kähler manifold $\Delta_d \phi = 0$ and $d\phi = 0$.
Since ϕ is harmonic, it is orthogonal to the space of exact forms, hence if it is exact it must be 0. ☐

Remark 3.27. i) The form zdw on $\mathbb{C}^2$ is not closed.
ii) The spaces $H^{p,q}$ in H^{p+q}_{DR} are isomorphic to the holomorphic invariant $H^q(X, \Omega^p)$, but for a family the embedding of $H^{p,q}(X_t)$ into DeRham cohomology is not holomorphic.

Exercise 3.28 (Riemann Surface)**.** Let $\overline{C}$ be a compact Riemann surface, and let $C = \overline{C} - x_1, \dots, x_m$ be the open surface with m points in $\overline{C}$ deleted. Consider the long exact sequence

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \oplus_{i=1}^{i=m} \mathbb{Z} \to H^2(\overline{C}, \mathbb{Z}) \simeq \mathbb{Z} \to H^2(C, \mathbb{Z}) = 0$$

then

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m} \mathbb{Z} \to \mathbb{Z})$$

presents $H^1(C, \mathbb{Z})$ as an extension, with weight $W_1 H^1(C, \mathbb{Z}) = H^1(\overline{C}, \mathbb{Z})$ and $W_2 H^1(C, \mathbb{Z}) = H^1(C, \mathbb{Z})$.
The Hodge filtration is defined by the residue

$$0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(log\{x_1, \dots, x_m\}) \to \mathcal{O}_{\{x_1, \dots, x_m\}} \to 0$$

$F^0 H^1(C, \mathbb{C}) = H^1(C, \mathbb{C})$,
$F^1 H^1(C, \mathbb{C}) = H^0(\overline{C}, \Omega^1_{\overline{C}}(log\{x_1, \dots, x_m\})$.
$F^2 H^1(C, \mathbb{C}) = 0$. That is an extension of pure HS

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathrm{Ker}\,(\oplus_{i=1}^{i=m} \mathbb{Z}(-1) \to H^2(\overline{C}) \to 0$$

.

Exercise 3.29 (Algebraic cycles)**.** Let X be a compact Kähler manifold and $\omega := -\mathcal{I}mH$ its real $2-$form of type $(1,1)$ ($\omega \in \mathcal{E}^{1,1}_X \cap \mathcal{E}^2_X$). The volume form $vol \in \mathcal{E}^{2n}$ defined by g on X, can be defined by ω and is equal to $\frac{1}{n!}\omega^n$. We have locally

$$\omega = \frac{i}{2}\Sigma_i dz_i \wedge d\overline{z}_i, \quad vol = \frac{\omega^n}{n!} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Lemma 3.30. *For a manifold X compact and Kähler $H^{p,p}(X,\mathbb{Z}) \neq 0$ for $0 \leq p \leq dim.M$.*

In fact the integral of the volume form $\int_X \omega^n > 0$. It follows that the cohomology class $\omega^n \neq 0 \in H^{2n}(X,\mathbb{C})$, hence the cohomology class $\omega^p \neq 0 \in H^{p,p}(X,\mathbb{C})$ since its cup product with ω^{n-p} is not zero.

3.6. **Cohomology class of a subvariety.** A compact oriented manifold X of real dimension m is triangulated and the sum of the oriented simplices of dim. m define a cycle class $[X] \in H_m(X,\mathbb{Z})$. The Poincaré dual of $[X]$ is called a cohomlology fundamental $[X] \in H^0(X,\mathbb{R})$. In particular, if X is a complex manifold of dim. n, then it has a class $[X] \in H_{2n}(X,\mathbb{Z})$. Let Z be a complex submanifold of codim. p in X and $i : Z \hookrightarrow X$ the embedding, then $i_*[Z] \in H_{2(n-p)}(X,\mathbb{Z})$ corresponds, by rational Poincaré duality, to a class in $H^{2p}(X,\mathbb{Z})$. Let

$$H^{p,p}(X,\mathbb{Z}) := \{a \in H^{2p}(X,\mathbb{Z}), \mathrm{Im}\,(a) \in H^{p,p}(X,\mathbb{C})\}$$

We define similarly the rational version by taking the image $H^{p,p}(X,\mathbb{Z})$ in the rational cohomology and by Poincaré duality its homology version the image $H_{p,p}(X,\mathbb{Z})$ in the rational homology.

Definition 3.31. For a complex compact manifold X the cohomology class $[\eta_Z] \in H^{p,p}(X,\mathbb{Z})$ of a complex submanifold Z of codim p is defined by the following relation

$$\forall \omega \in \mathcal{E}^{n-p,n-p}(X), \int_X \omega \wedge \eta_Z = \int_{i_*[Z]} \omega.$$

Lemma 3.32. *For a manifold X compact and Kähler the rational cohomology class of a complex submanifold Z of codim p is a non zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{Z})$ for $0 \leq p \leq dim\ X$.*

Proof. For a compact Kähler manifold X, let ω be a Kähler form, then the integral on the homology class $[Z]$ of the restriction $i^*\omega$ is positive since it is a Kähler form on Z

$$\int_{i_*[Z]} \wedge^{n-p}\omega = \int_{[Z]} i^* \wedge^{n-p} \omega > 0.$$

□

Lemma 3.33. *For a projective smooth variety X, the fundamental class of an irreducible subvariety Z of codimension p is well defined and it is a non zero element $[\eta_Z] \neq 0 \in H^{p,p}(X,\mathbb{Z})$ for $0 \leq p \leq dimX$.*

The easiest proof is to use a desingularisation $\pi : Z' \to Z$ and define $[Z] = \pi_*[Z'] \in H_{n-p,n-p}(X,\mathbb{Z})$, then take its Poincaré dual in $H^{p,p}(X,\mathbb{Z})$.

Definition 3.34. An $r-$cycle of an algebraic variety X is a formal finite linear combination $\Sigma_{i\in[1,h]} m_i Z_i$ of irreducible subvarieties Z_i with integer coefficients m_i. The group of $r-$cycles is denoted by $\mathcal{Z}_r(X)$.

The class of irreducible subvarieties extend into a linear morphism

$$cl : \mathcal{Z}_r(X) \to H^{r,r}(X, \mathbb{Z})$$

defined modulo linear equivalence of cycles. The cohomology cycles in the image of the cycle class map are called algebraic. Respectively we define the map

$$cl_{\mathbb{Q}} : \mathcal{Z}_r(X) \otimes \mathbb{Q} \to H^{r,r}(X, \mathbb{Q})$$

3.6.1. *Hodge conjecture.* Is the map $cl_{\mathbb{Q}}$ surjective?

4. Polarized Hodge structure

4.1. **Hodge structure.** Having the above properties of cohomology of Kähler manifolds in mind, it appeared to be rewarding to introduce this notion as a formal structure in linear algebra

Definition 4.1. A Hodge structure (HS) of weight n is defined by the data
i)A free abelian group $H_{\mathbb{Z}}$ (the lattice)
ii) A decomposition by complex subspaces

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=n} H^{p,q} \text{ satisfying } \quad H^{p,q} = \overline{H^{q,p}}$$

Remark 4.2. The conjugation on $H_{\mathbb{C}}$ makes sense with respect to $H_{\mathbb{Z}}$. A subspace $V \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $\overline{V} = V$ has a real structure, that is $V = (V \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$. In particular $H^{p,p} = (H^{p,p} \cap H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

Example 4.3. Tate Hodge structure $\mathbb{Z}(1)$ is defined by

$$H_{\mathbb{Z}} = 2i\pi\mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-1,-1}$$

The tensor product of Hodge structures is defined in general. In this case the $n-$tensor product $\mathbb{Z}(1) \otimes \cdots \otimes \mathbb{Z}(1)$ of $\mathbb{Z}(1)$ is defined by

$$\mathbb{Z}(m) := (H_{\mathbb{Z}} = (2i\pi)^m \mathbb{Z} \subset \mathbb{C}, \quad H_{\mathbb{C}} = H^{-m,-m})$$

and the Tate twist of a HS $H = (H_{\mathbb{Z}}, \oplus_{p+q=n} H^{p,q})$ is $H(m)$ defined by

$$(H_{\mathbb{Z}} \otimes (2i\pi)^m \mathbb{Z}, \oplus_{p+q=n} H^{p-m,q-m})$$

The DeRham cohomology $H^i(X, \mathbb{R})$ of a compact Kähler manifold is a HS of weight i with lattice defined by the image of integral cohomology $Im : H^i(X, \mathbb{Z}) \to H^i(X, \mathbb{R})$, in particular if we consider the following trace map

$$H^{2n}(X, \mathbb{R}) \xrightarrow{\sim} \mathbb{R}(-n), \quad \omega \to \frac{1}{(2i\pi)^n} \int_X \omega.$$

then Poincaré duality will be compatible with HS

$$H^{n-i}(X, \mathbb{R}) \simeq Hom(H^{n+i}(X, \mathbb{R}), \mathbb{R}(-n)).$$

Theorem 4.4 (Deligne)**.** *Let X be a smooth complex algebraic variety. There exists a decomposition of the complex cohomology spaces into a direct sum of complex subspaces*

$$H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X), \quad H^{p,q}(X) = \overline{H^{q,p}}(X)$$

defining a HS of weight i on cohomology.

4.1.1. *The Hodge filtration.* To study variations of HS, Griffiths introduced an equivalent structure defined by the Hodge filtration which vary holomorphically. Given a HS: $(H_{\mathbb{Z}}, H^{p,q})$ of weight i, we define a filtration F^* on $H_{\mathbb{C}}$ by subspaces

$$F^p H_{\mathbb{C}} := \oplus_{r \geq p} H^{r,i-r}$$

the filtration F is decreasing such that the following decomposition is satisfied

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{i-p+1} H_{\mathbb{C}}}$$

The Hodge decomposition is then determined by the filtration as

$$H^{p,q} = F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}},\ p+q=i$$

4.1.2. *The Hodge filtration on the DeRham complex.* The Hodge filtration on cohomology plays an important role since its definition does not use the theory of harmonic forms, although the Hodge result on the filtration is based on such theory. It is defined by a filtration F of the holomorphic DeRham complex by subcomplexes

$$F^p\Omega^*_X := 0 \cdots 0 \to \Omega^p_X \to \cdots \Omega^q_X \to \cdots \Omega^n_X$$

The simple complex, associated to the resolution of each term Ω^r_X by a Dolbeault complex $(\mathcal{E}^{r,*}_X, \overline{\partial})$ for $r \geq p$

$$F^p\mathcal{E}_X = s(\mathcal{E}^{r,*}_X, \overline{\partial}), \partial)_{r \geq p}$$

is a fine resolution of $(F^p\Omega^*_X, d = \partial)$ and define a Hodge filtration by subcomplexes of $\mathcal{E}_X \otimes_{\mathbb{R}} \mathbb{C}$. We have

$$H^i(F^p\mathcal{E}^*(X), d) \simeq F^p H^i(X, \mathbb{C})$$

Remark 4.5. We will see that the spectral sequence defined by the Hodge filtration degenerates at rank one.

4.2. **Lefschetz decomposition.** Let $[\omega] \in H^2(X, \mathbb{R})$ denotes the Kähler class of X. The cup-product with $[\omega]$ define morphisms

$$L : H^q(X, \mathbb{R}) \to H^{q+2}(X, \mathbb{R}), \quad L : H^q(X, \mathbb{C}) \to H^{q+2}(X, \mathbb{C})$$

Let $n = \dim X$ and define the primitive subspaces

$$H^{n-i}_{prim}(\mathbb{R}) := Ker(L^{i+1} : H^{n-i}(X, \mathbb{R}) \to H^{n+i+2}(X, \mathbb{R}))$$

and similarly for complex coefficients $H^{n-i}_{prim}(\mathbb{C}) \simeq H^{n-i}_{prim}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, then we deduce from the following Lefschetz isomorphism theorem a decomposition of the cohomology in terms of primitive spaces

Theorem 4.6 (Lefschetz). *The iterated linear operator L induces isomorphisms for each i*

$$L^i : H^{n-i}(X, \mathbb{R}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{R}),\ L^i : H^{n-i}(X, \mathbb{C}) \xrightarrow{\sim} H^{n+i}(X, \mathbb{C})$$

It gives rise to a decomposition of the cohomology into direct sum of image of primitive subspaces by L^r, $r \geq 0$

$$H^q(X, \mathbb{R}) = \oplus_{r \geq 0} L^r H^{q-2r}_{prim}(\mathbb{R}), \quad H^q(X, \mathbb{C}) = \oplus_{r \geq 0} L^r H^{q-2r}_{prim}(\mathbb{C})$$

Since the operator L acting on the HS is of bidegree $(1,1)$, the decomposition and the primitive subspaces are compatible with HS.
If X is moreover projective, then the action of L is defined with rational coefficients and the decomposition apply on rational cohomology.

Proof. 1) First we consider the action of L on sheaves, $L = \wedge\omega : \mathcal{E}^r_X \to \mathcal{E}^{r+2}_X$, then we introduce its adjoint operator $\Lambda = L^* : \mathcal{E}^r_X \to \mathcal{E}^{r-2}_X$ which can be defined by $\Lambda = *^{-1} \circ L \circ *$. Note that the operator

$$h = \sum_{p=0}^{2n} (n-p)\Pi^p$$

where Π^p is the projection in degree p on $\mathcal{E}^*_X$ and $n = \dim X$ satisfy the relation

$$[\Lambda, L] = h.$$

from which we can deduce the injectivity of the morphism

$$L^i : \mathcal{E}^{n-i}_X \to \mathcal{E}^{n+i}_X$$

For this we use the formula

$$[L^r, \Lambda] = (r(k-n) + r(r-1))L^{r-1}$$

which is proved by induction on r. Such morphism commutes with the Laplacian and since cohomology classes can be represented by global harmonic sections, it induces an isomorphism on cohomology vector spaces of finite equal dimension

$$L^i : H^{n-i}(X,\mathbb{R}) \xrightarrow{\sim} H^{n+i}(X,\mathbb{R})$$

Moreover the extension of the operator L to complex coefficients is compatible with the bigrading (p,q) since ω is of type $(1,1)$. The decomposition of the cohomology into direct sum of image of primitive subspaces by L^r, $r \geq 0$ follows from the previous isomorphisms.
2) Another proof is based on the representation theory of the Lie algebra sl_2. We represent L by the action on global sections of the operator $L = \wedge\omega : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q+1}(X)$ and its adjoint $\Lambda = L^* : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p-1,q-1}(X)$ is defined by $\Lambda = *^{-1} \circ L \circ *$. We admit the relations

$$[\Lambda, L] = h$$
$$[h, L] = -2L$$
$$[h, \Lambda] = 2\Lambda$$

We deduce from the action of the operators L, h, Λ on the space of harmonic forms $\oplus\mathcal{H}^*_d(X) \simeq \oplus H^*(X)$, a representation of the Lie algebra sl_2 by identifying the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ resp. with } \Lambda,\ L,\ h.$$

Then the theorem follows from the general structure of such representation. □

A geometric interpretation of this theorem in the case of a projective subvariety $X \subset \mathbb{P}^N$ of dimension n is given via Poincaré isomorphism $H^{n-k}(X) \xrightarrow{\sim} H_{n-k}(X)$, so that the operator L corresponds to the intersection with an hyperplane section V of X so that the result is an isomorphism

$$H_{n+k}(X) \xrightarrow{\cap \mathbb{P}^{N-k}} H_{n-k}(X)$$

The primitive cohomology $H^{n-k}_{prim}(X)$ corresponds to a subspace $H^f_{n-k}(X) \subset H_{n-k}(X)$ equal to the image of $H_{n-k}(X - V) \to H_{n-k}(X)$ (f for finite cycles or outside infinity).

4.3. **Polarization of HS.** We define in this subsection a scalar product on cohomology compatible with HS and satisfying relations known as Hodge Riemann relations leading to a polarization of the primitive cohomology which is an additional highly rich structure on cohomology of proper algebraic varieties. Such product is defined via the wedge product on DeRham cohomology, or cup-product on singular cohomology and called sometimes an intersection form (referring implicitly to intersection on homology, with which it is compatible via Poincaré duality).

4.3.1. *Hermitian product on cohomology.* Representing cohomology classes by differential forms, we define a bilinear form

$$Q(\alpha,\beta) = (-1)^{\frac{j(j-1)}{2}} \int_X \alpha\wedge\beta\wedge\omega^{n-j}, \quad \forall[\alpha],[\beta]\in H^j(X,\mathbb{C})$$

where ω is the Kähler class. On projective varieties the Kähler class is in the integral lattice in rational cohomology, the product is defined on rational cohomology and preserves the integral lattice. It is alternating if j is odd, symmetric if j is even and non degenerate by Lefschetz isomorphism and Poincaré duality. By consideration of type, the Hodge and Lefschetz decompositions are orthogonal with respect to Q

$$Q(H^{p,q},H^{p',q'}) = 0, \quad \text{unless } p=q', q=p'.$$

Proposition 4.7 (Hodge-Riemann bilinear relations)**.** On the primitive component $H^{p,q}_{prim} := H^j(X,\mathbb{C})^{p,q}_{prim}$

$$i^{p-q}Q(\alpha,\overline{\alpha}) > 0, \quad \forall\alpha\neq 0\in H^{p,q}_{prim}$$

the form $i^{p-q}Q(\alpha,\overline{\alpha})$ is positive definite.

Proof. We use

Lemma 4.8. *Let $\omega\in\Omega^{q,p}_{X,x}\subset\mathcal{E}^j_{X,x}$ be a primitive element, then*

$$*\omega = (-1)^{\frac{j(j+1)}{2}} i^{q-p}\frac{L^{n-j}}{(n-j)!}\omega$$

We represent a primitive cohomolgy class by a primitive harmonic form α of degree j, then since $(-1)^j i^{q-p} = (-1)^{q-p} i^{q-p} = i^{p-q}$ we deduce from the lemma $L^{n-j}\overline{\alpha} = i^{q-p}(n-j)!(-1)^{\frac{j(j-1)}{2}}(*\overline{\alpha})$ and the product is written via the L^2-norm, as

$$i^{p-q}Q(\alpha,\overline{\alpha}) = (n-j)!\int_X \alpha\wedge *\overline{\alpha} = (n-j)!||\alpha||^2_{L^2}$$

□

This result suggest to introduce the Weil operator C on cohomology

$$C(\alpha) = i^{p-q}\alpha, \quad \forall\alpha\in H^{p,q}$$

Notice that C depends on the decomposition, in particular the action of C in a variation of Hodge structure is differentiable in the parameter space.
We deduce from Q a new non degenerate Hermitian product

$$H(\alpha,\beta) = Q(C\alpha,\overline{\beta}), \quad \forall[\alpha],[\beta]\in H^j(X,\mathbb{C})$$

It satisfies for $\alpha,\beta\in H^{p,q}$

$$H(\beta,\alpha) = \overline{H(\alpha,\beta)}$$

since
$\overline{H(\alpha,\beta)} = \overline{Q(i^{p-q}\alpha,\overline{\beta})} = \overline{i}^{p-q}Q(\overline{\alpha},\beta) = (-1)^{2j}i^{p-q}Q(\beta,\overline{\alpha}) = H(\beta,\alpha)$.

Remark 4.9. i)When the class $[\omega] \in H^j(X,\mathbb{Z})$ is integral, which is the case for algebraic varieties, the product Q is integral (with integral value on integral classes).
ii) The integral bilinear form Q extends by linearity to the complex space $H^k_{\mathbb{Z}} \otimes \mathbb{C}$. Its associated form $H(\alpha,\beta) := Q(\alpha,\overline{\beta})$ is not Hermitian if k is odd. One way to overcome this sign problem is to define H as $H(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$, still this form will not be positive definite in general.

Definition 4.10 (Polarization of HS)**.** A Hodge structure ($H_{\mathbb{Z}}, (H_{\mathbb{C}}, F)$) of weight n is polarized if a non degenerate scalar product Q is defined on $H_{\mathbb{Z}}$, alternating if n is odd and symmetric if n is even, such that the Hermitian form on $H_{\mathbb{C}}$ defined as $H(\alpha,\beta) := i^{p-q} Q(\alpha,\overline{\beta})$ for $\alpha,\beta \in H^{p,q}$ is orthogonal to the Hodge decomposition and $i^{p-q}H(\alpha,\alpha)$ is positive definite on the primitive component of type (p,q)(satisfy Hodge-Riemann bilinear relations).

4.3.2. *Polarized subvarieties are algebraic.* A theorem of Kodaira states that a Kähler variety with an integral class $[\omega]$ is projective (which means it can be embedded as a closed analytic subvariety in a projective space, hence it is necessarily an algebraic subvariety.

4.4. **Examples.**

4.4.1. *Projective space.* Let $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ denotes the Chern class of the canonical line bundle; it is dual to the homology class of an hyperplane.

Theorem 4.11. *$\mathbb{H}^i(\mathbb{P}^n,\mathbb{Z}) = 0$ for i odd and $\mathbb{H}^i(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}$ for i even with $[H]^i$ the cohomology class of an hyperplane to the power i for the cup product, as generator; hence: $\mathbb{H}^{2r}(\mathbb{P}^n,\mathbb{Z}) = \mathbb{Z}(-r)$ as HS.*

4.4.2. *Hodge structures of weight 1 and abelian varieties.* Given a HS:
$(H_{\mathbb{Z}}, H^{1,0} \oplus H^{0,1})$, the projection on $H^{0,1}$ induces an isomorphism of real vector spaces

$$H_{\mathbb{R}} \to H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1} \to H^{0,1}$$

since $\overline{H^{0,1}} = H^{1,0}$. Then we deduce that $H_{\mathbb{Z}}$ is a lattice in the complex space $H^{0,1}$, and the quotient $T := H^{0,1}/H_{\mathbb{Z}}$ is a complex torus . When $H_{\mathbb{Z}}$ is identified with the image of cohomology spaces $Im(H^1(X,\mathbb{Z}) \to H^1(X,\mathbb{R}))$ of a complex manifold X, resp. $H^{0,1}$ with $H^1(X,\mathcal{O})$, the torus T will be identified with the Picard variety $Pic^0(X)$ parameterizing the holomorphic line bundles on X with first Chern class equal to zero as follows. We consider the exact sequence of sheaves defined by $f \mapsto e^{2i\pi f}$

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

where 1 is the neutral element of the sheaf of multiplicative groups $\mathcal{O}_X^*$, and its associated long exact sequence of derived functors of the global sections functor

$$\to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$$

where the morphisms can be interpreted geometrically; when the space $H^1(X,\mathcal{O}_X^*)$ is identified with isomorphisms classes of line bundles on X, the last morphism defines the Chern class of the line bundle. Hence the torus T is identified with the isomorphism classes $\mathcal{L}$ with $c_1(\mathcal{L}) = 0$

$$T := \frac{H^1(X,\mathcal{O}_X)}{ImH^1(X,\mathbb{Z})} \simeq Pic^0(X) := Ker(H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}))$$

It is possible to show that the Picard variety of a smooth projective variety is an abelian variety. (Define a Kähler form with integral class on $Pic^0(X)$).

4.4.3. *Hodge structures of weight 2.*

$$(H_{\mathbb{Z}},\, H^{2,0} \oplus H^{1,1} \oplus H^{0,2};\, H^{0,2} = \overline{H^{2,0}},\, H^{1,1} = \overline{H^{1,1}};\, Q)$$

the intersection form Q is symmetric and $H(\alpha, \beta) = Q(\alpha, \overline{\beta})$ is Hermitian. The decomposition is orthogonal for H with H positive definite on $H^{1,1}$, negative definite on $H^{2,0}$ and $H^{2,1}$.

Lemma 4.12. *The HS is completely determined by the subspace $H^{2,0} \subset H_{\mathbb{C}}$ s.t. $H^{2,0}$ is totally isotropic for Q and the associated Hermitian product H is negative definite on $H^{2,0}$. The signature of Q is $(h^{1,1}, 2h^{2,0})$.*

In the lemma $H^{0,2}$ is determined by conjugation and $H^{1,1} = (H^{2,0} \oplus H^{0,2})^{\perp}$.

4.4.4. *Hodge structures of $K3$ type.*

Lemma 4.13. *Let Q be a symmetric non degenerate form on a lattice $H_{\mathbb{Z}}$ of signature $(h^{1,1}, 2)$. The HS of weight 2 on $H_{\mathbb{C}}$ polarized by Q are parameterized by the complex manifold*

$$\mathcal{D} := \{\omega \in \mathbb{P}(H_{\mathbb{C}}) | Q(\omega, \omega) = 0, Q(\omega, \overline{\omega}) < 0\}$$

Here $H^{2,0}$ must be a line parameterized by $\omega \in \mathbb{P}(H_{\mathbb{C}})$ where the signature is -1 as well on its conjugate and $h^{1,1}$ on $H^{1,1}$.

5. Mixed Hodge Structures (MHS)

Classical Hodge theory apply only for compact Kähler manifolds. In order to describe specific properties of cohomology of algebraic varieties inspired by étale cohomology and the solution of Weil conjectures, Deligne introduced in [6] a far reaching generalization of HS called mixed Hodge structure (MHS) that can be defined on the cohomology of all algebraic varieties.
We are essentially concerned by filtrations of vector spaces. However it is not more difficult to describe this notion in the terminology of abelian categories.
The formal study of MHS, namely the fact that they form an abelian category, will help the reader before he is confronted to their construction.

5.1. **Filtrations.** Let $\mathbb{A}$ denote an abelian category.

Definition 5.1. A decreasing filtration F (resp. increasing) of an object A of $\mathbb{A}$ is a family of sub-objects of $\mathbb{A}$, satisfying

$$\forall n, m, \qquad n \leq m \Longrightarrow F^m(A) \subset F^n(A) (resp. \quad n \leq m \Longrightarrow F_n(A) \subset F_m(A))$$

If F is a decreasing filtration (resp. W an increasing filtration), a shifted filtration $F[n]$ by an integer n is defined as

$$(F[n])^p(A) = F^{n+p}(A).$$

Decreasing filtrations F will be considered for a general study. Statements for increasing filtrations W can be deduced by the change of indices $W_n(A) = F^{-n}(A)$.
A filtration is finite if there exists integers n and m such that $F^n(A) = A$ and $F^m(A) = 0$.
A morphism of filtered objects $(A, F) \xrightarrow{f} (B, F)$ is a morphism $A \xrightarrow{f} B$ satisfying

$f(F^n(A)) \subset F^n(B)$ for all $n \in \mathbb{Z}$. Filtered objects (resp. of finite filtration) form an additive category with existence of *kernel* and *cokernel* of a morphism with natural induced filtrations as well image and coimage, however the image and coimage will not be necessarily filtered-isomorphic, which is the main obstruction to obtain an abelian category.
In the case of the category of modules over a ring, a morphism $f : (A, F) \to (B, F)$ is strict if

$$f(F^n(A) = f(A) \cap F^n(B)$$

so that any element $b \in F^n(B) \cap ImA$ is already in $Im\, F^n(A)$.
More generally a morphism $f : (A, F) \to (B, F)$ is strict if it induces a filtered isomorphism $Coim(f) \to Im(f)$ from the coimage to the image of f.

Definition 5.2. The graded object associated to (A, F) is defined as

$$Gr_F(A) = \oplus_{n\in\mathbb{Z}} Gr^n_F(A) \quad \text{where} \quad Gr^n_F(A) = F^n(A)/F^{n+1}(A)$$

5.1.1. *Induced filtration.* A filtered object (A, F) induces a filtration on a sub-object $i : B \to A$ of A, defined by $F^n(B) = B \cap F^n(A)$. It is the unique filtration s.t. i is strict. Dually, the quotient filtration on A/B is defined by

$$F^n(A/B) = p(F^n(A)) = (B + F^n(A))/B \simeq F^n(A)/(B \cap F^n(A))$$

where $p : A \to A/B$ is the projection.

5.1.2. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a $0-$sequence if $g \circ f = 0$; if B is *filtered*, the cohomology $H = Ker\, g/Im\, f$ is endowed with the quotient filtration of the induced filtration on $Ker\, g$. It is equal to the induced filtration on H by the quotient filtration on $B/Im\, f$ ($H \subset (B/Im\, f)$).

5.1.3. *Two filtrations.* Let A be an object of $\mathbb{A}$ with two filtrations F and G. By definition, $Gr^n_F(A)$ is a quotient of a sub-object of A, and as such, it is endowed with an induced filtration by G. Its associated graded object defines a bigraded object $Gr^n_G Gr^m_F(A)_{n,m\in\mathbb{Z}}$. We refer to [6] for

Lemma 5.3 (Zassenhaus' lemma)**.** *The objects $Gr^n_G Gr^m_F(A)$ and $Gr^m_F Gr^n_G(A)$ are isomorphic.*

5.1.4. Let H be a third filtration on A. It induces a filtration on $Gr_F(A)$, hence on $Gr_G Gr_F(A)$. It induces also a filtration on $Gr_F Gr_G(A)$. These filtrations do not correspond in general under the above isomorphism. In the formula $Gr_H Gr_G Gr_F(A)$, G et H have symmetric role, but not F and G.

5.1.5. *Hom and tensor functor.* If A and B are two filtered objects of $\mathbb{A}$, we define a filtration on the left exact functor Hom

$$F^k Hom(A, B) = \{f : A \to B : \forall n, f(F^n(A)) \subset F^{n+k}(B)\}$$

Hence

$$Hom((A, F), (B, F)) = F^0(Hom(A, B)).$$

If A and B are modules on some ring, we define

$$F^k(A \otimes B) = \sum_m Im(F^m(A) \otimes F^{k-m}(B) \to A \otimes B)$$

5.1.6. *Multifunctor.* In general if $H : \mathbb{A}_1 \times \ldots \times \mathbb{A}_n \to \mathbb{B}$ is a right exact multi-additive functor, we define

$$F^k(H(A_1, ..., A_n)) = \Sigma_{\Sigma k_i = k} Im(H(F^{k_1}A_1, ..., F^{k_n}A_n) \to H(A_1, ..., A_n))$$

and dually if H is left exact

$$F^k(H(A_1, ..., A_n) = \cap_{\Sigma k_i = k} Ker(H(A_1, ..., A_n) \to H(A_1/F^{k_1}A_1, ..., A_n/F^{k_n}A_n))$$

If H is exact, both definitions are equivalent.

5.2. **Hodge Structure (HS).** For any sub-ring A of $\mathbb{R}$ equal to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and any A-module H_A, the complex conjugation extends to a conjugation on the space $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$. A filtration F on $H_{\mathbb{C}}$ has a conjugate filtration $\overline{F}$ s.t. $(\overline{F})^i H_{\mathbb{C}} = \overline{F^i H_{\mathbb{C}}}$.

Definition 5.4 (HS1)**.** An A-Hodge structure H of weight n consists of

(1) an A-module of finite type H_A (the lattice).
(2) a finite filtration F on $H_{\mathbb{C}}$ (the Hodge filtration) s.t. F and its conjuguate $\overline{F}$ satisfy the relation

$$Gr_F^p Gr_{\overline{F}}^q (H_{\mathbb{C}}) = 0, \quad \text{for } p + q \neq n$$

The HS is real when $A = \mathbb{R}$, rational when $A = \mathbb{Q}$ and integral when $A = \mathbb{Z}$.

5.3. **Opposite filtrations.** Two finite filtrations F and G on an object A of $\mathbb{A}$ are n-opposite if

$$Gr_F^p Gr_G^q (A) = 0 \quad \text{for } p + q \neq n.$$

hence the Hodge filtration F on a HS of weight n is n-opposite to its conjugate $\overline{F}$. The following constructions will define an equivalence of categories between objects of $\mathbb{A}$ with two n-opposite filtrations and bigraded objects of $\mathbb{A}$ of the following type.

Example 5.5. Let $A^{p,q}$ be a bigraded object of $\mathbb{A}$ s.t. $A^{p,q} = 0$ for all but a finite number of couples (p, q) and $A^{p,q} = 0$ for $p + q \neq n$; then we define two n-opposite filtrations on $A = \sum_{p,q} A^{p,q}$:

$$F^p(A) = \Sigma_{p' \geq p} A^{p',q'}, \quad G^q(A) = \Sigma_{q' \geq q} A^{p',q'}$$

We have $Gr_F^p Gr_G^q (A) = A^{p,q}$.

We admit reciprocally [6]

Proposition 5.6. i) Two finite filtrations F and G on an object A are n-opposite, if and only if

$$\forall\, p, q, \quad p + q = n + 1 \Rightarrow F^p(A) \oplus G^q(A) \simeq A.$$

ii) If F and G are n-opposite, and if we put $A^{p,q} = F^p(A) \cap G^q(A)$ for $p + q = n$, $A^{p,q} = 0$ for $p + q \neq n$, then A is a direct sum of $A^{p,q}$, moreover F and G can be deduced from the bigraded object $A^{p,q}$ of A by the above procedure.

We have an equivalent definition of HS.

Definition 5.7 (HS2)**.** An $A-$HS on H of weight n is a pair of an A-module H_A and a decomposition into a direct sum on $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$

$$H_{\mathbb{C}} = \oplus_{p+q=n} H^{p,q} \quad \text{such that} \quad \overline{H}^{p,q} = H^{q,p}$$

The relation with the previous definition is given by $H^{p,q} = F^p(H_{\mathbb{C}}) \cap \overline{F}^q(H_{\mathbb{C}})$ for $p + q = n$

5.4. **Morphism of HS.** A morphism $f : H \to H'$ of HS, is an homomorphism $f\colon H_A \to H'_A$ such that $f_{\mathbb{C}}\colon H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is compatible with the bigrading, or equivalently with Hodge filtration.
If H and H' are of distinct weights, f is necessarily 0.
The HS of weight n form an abelian category. If H is of weight n and H' of weight n', we define a HS, $H \otimes H'$ of weight $n + n'$ as follows
i) $(H \otimes H')_A = H_A \otimes H'_A$
ii) the bigrading (resp. the Hodge filtration) of $(H \otimes H')_{\mathbb{C}} = H_{\mathbb{C}} \otimes H'_{\mathbb{C}}$ is the tensor product of the bigradings (resp. Hodge filtrations of $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$).
We define a HS of weight $n' - n$ on $Hom(H, H')$ by the formula

$$F^r Hom(H_{\mathbb{C}}, H'_{\mathbb{C}}) = \{f : H_{\mathbb{C}} \to H'_{\mathbb{C}} : \forall p,\ f(F^p(H_{\mathbb{C}}) \subset F^{p+r}(H'_{\mathbb{C}})$$

In particular H^* the dual to H is a HS of weight $-n$. An homomorphism $f : H_A \to H'_A$ of type (r, r) is strictly compatible with F that is

$$f(F^p(H_{\mathbb{C}}) = F^{p+r}(H'_{\mathbb{C}}) \cap Imf.$$

Similarly the Hodge structure $\wedge^p H$ is of weight pn.

5.4.1. *Hom_{HS}.* The internal Hom to the category of HS, denoted by $Hom_{HS}(H, H')$, is the sub-group of $Hom_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ of elements of type $(0, 0)$ in the HS on $Hom(H, H')$.

Example 5.8. Tate HS, $\mathbb{Z}(1)$ is the HS of weight -2, rank 1, purely bigraded of type $(-1, -1)$, and of lattice $2\pi i\mathbb{Z} \subset \mathbb{C}$.
For $n \in \mathbb{Z}$, we define $\mathbb{Z}(n)$ as the n tensor product of $\mathbb{Z}(1)$: $\mathbb{Z}(n)$ is the HS of weight $-2n$, rank 1, purely bigraded of type $(-n, -n)$, with lattice $(2\pi i)^n\,\mathbb{Z} \subset \mathbb{C}$.

5.4.2. *Complex HS.*

Definition 5.9. A complex HS of weight n on a complex vector space H is given by a pair of $n-$opposite filtrations F and $\overline{F}$, hence a decomposition into a direct sum of subspaces

$$H = \oplus_{p+q=n} H^{p,q}, \quad \text{where } H^{p,q} = F^p \cap \overline{F}^q$$

The two $n-$opposite filtrations F and $\overline{F}$ on a complex HS of weight n can be recovered from the decomposition by the formula

$$F^p = \oplus_{i \geq p} H^{i,n-i} \quad \overline{F}^q = \oplus_{i \leq n-q} H^{i,n-i}$$

Here we don't assume the existence of conjugation although we keep the notation $\overline{F}$. An $A-$HS of weight n defines a complex HS on $H = H_{\mathbb{C}}$.
Let $\overline{H}$ denotes the complex conjugate space of H and $c : H \to \overline{H}$ the $\mathbb{R}-$linear identity isomorphism s.t. $\lambda \times_{\overline{H}} c(v) := c(\overline{\lambda} \times_H v)$.

Definition 5.10. A polarization is a bilinear morphism $S : H \otimes \overline{H} \to \mathbb{C}$ s.t.

$$S(F^p, c(\overline{F}^q) = S(\overline{F}^p, c(F^q)) = 0 \quad \text{for } p + q > n$$

and moreover $S(C(H)u, c(v))$ is a positive definite Hermitian form on H where $C(H)$ denotes Weil action on H.

Example 5.11. A complex HS of weight 0 on a complex vector space H is given by a decomposition into a direct sum of subspaces

$$H = \oplus_{p \in \mathbb{Z}} H^p$$

A polarization is an Hermitian form on H for which the decomposition is orthogonal and whose restriction to H^p is definite for p even and negative definite for odd p.

5.5. **Mixed Hodge structure (MHS).** This structure has been introduced in [6]. Let A be equal to $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, then for an A-module of finite type H_A, $H_A \otimes \mathbb{Q}$ is defined as $H_A \otimes_A \mathbb{Q} := H_{\mathbb{Q}}$, if A is equal to $\mathbb{Z}$, otherwise as $H_A \otimes_{\mathbb{Q}} \mathbb{Q} = H_A$.

Definition 5.12 (Deligne)**.** An A-mixed Hodge structure H consists of
1) an A-module of finite type H_A
2) a finite increasing filtration W of the $A \otimes \mathbb{Q}$ vector space $H_A \otimes \mathbb{Q}$ called the weight filtration
3) a finite decreasing filtration F of the $\mathbb{C}$-vector space $H_{\mathbb{C}} = H_A \otimes_A \mathbb{C}$, called the Hodge filtration, such that the systems

$$Gr_n^W H = (Gr_n^W (H_{A\otimes\mathbb{Q}}), (Gr_n^W H_{\mathbb{C}}, F))$$

consist of $A \otimes \mathbb{Q} - HS$ of weight n.

The MHS is called real if $A = \mathbb{R}$, rational if $A = \mathbb{Q}$ and integral if $A = \mathbb{Z}$.

Example 5.13. 1) A HS, H of weight n, is a MHS with weight filtration

$$W_i(H_{\mathbb{Q}}) = 0 \quad \text{for} \quad i < n \qquad \text{and} \quad W_i(H_{\mathbb{Q}}) = H_{\mathbb{Q}} \quad \text{for} \quad i \geq n.$$

2) Let (H^i, F_i) be a finite family of $A-$HS of weight $i \in \mathbb{Z}$; then $H = \oplus_i H^i$ is endowed with the following MHS

$$H_A = \oplus_i H_A^i, \quad W_n = \oplus_{i\leq n} H_A^i \otimes \mathbb{Q}, \quad F = \oplus_i F_i.$$

5.5.1. *Morphism of MHS.* A morphism $f : H \to H'$ of MHS is a morphism $f : H_A \to H'_A$ whose extension to $H_{\mathbb{Q}}$ (resp. $H_{\mathbb{C}}$) is compatible with the filtration W (resp. F, hence $\overline{F}$).

Definition 5.14 (opposite filtrations)**.** Three finite filtrations W (increasing), F and G on an object A of $\mathbb{A}$ are opposite if

$$Gr_F^p Gr_G^q Gr_n^W (A) = 0 \quad \textit{for } p + q \neq n.$$

This condition is symmetric in F and G. It means that F and G induce on $W_n(A)/W_{n-1}(A)$ two n-opposite filtrations, then $Gr_n^W(A)$ is bigraded

$$W_n(A)/W_{n-1}(A) = \oplus_{p+q=n} A^{p,q} \quad \textit{where } A^{p,q} = Gr_F^p Gr_G^q Gr_{p+q}^W (A)$$

Example 5.15. i) A bigraded object $A = \oplus A^{p,q}$ of finite bigrading has the following three opposite filtrations

$$W_n = \oplus_{p+q\leq n} A^{p,q} \quad , F^p = \oplus_{p'\geq p} A^{p'q'} \quad , G^q = \oplus_{q'\geq q} A^{p'q'}$$

ii) In the definition of an $A-$MHS, the filtration $W_{\mathbb{C}}$ on $H_{\mathbb{C}}$ deduced from W by saclar extension, the filtration F and its complex conjugate , form a system $(W_{\mathbb{C}}, F, \overline{F})$ of three opposite filtrations.

5.5.2. *A canonical decomposition of the weight filtration.* Hodge decomposition. The definition of weight is given by a filtration and not a decomposition as a HS. The quotient W_n/W_{n-2} is a non split extension of the two pure Hodge structures Gr_n^W and Gr_{n-1}^W, in the category of MHS that we aim to study. The question here is how far we can lift the bigrading on $Gr^W(A)$ and at what price do we can split the weight filtration W. Deligne constructed a decomposition of the object A with its three opposite filtrations. For each pair of integers (p, q), he introduces the subspaces of A

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \overline{F^{q-2}} \cap W_{p+q-3} + \cdots)$$

By construction they are related for $p+q=n$ to the components $A^{p,q}$ of the Hodge decomposition: $Gr_n^W A \simeq \oplus_{n=p+q} A^{p,q}$; however, in general $I^{p,q} \neq \overline{I^{q,p}}$ but $I^{p,q} \equiv \overline{I^{q,p}}$ modulo W_{p+q-2}.

Lemma 5.16. *The projection $W_{p+q} \to Gr^W_{p+q} A \simeq \oplus_{p'+q'=p+q} A^{p',q'}$ induces an isomorphism $\varphi : I^{p,q} \xrightarrow{\sim} A^{p,q}$. Moreover*

$$W_n = \oplus_{p+q\leq n} I^{p,q}, \quad F^p = \oplus_{p'\geq p} I^{p',q'}$$

Proof. Injectivity of φ. Let $m = p+q$; modulo W_{m-1}, the image $\varphi(I^{p,q}) \subset A^{p,q} = (F^p \cap \overline{F^q})(Gr^W_m)$. Let $v \in I^{p,q}$ s.t. $\varphi(v) = 0$, then $v \in F^p \cap W_{m-1}$. The class $cl(v)$ in $(F^p \cap \overline{F^q})(Gr^W_{m-1}) = 0$ must vanish, since $p+q > m-1$; so we deduce $v \in F^p \cap W_{m-2}$. This is a step in an inductive argument based on the formula $F^p \oplus \overline{F^{q-r+1}} \simeq W_{m-r}$. Suppose $v \in F^p \cap \overline{F^{q-r+1}} \cap W_{m-r}$; since $(F^p \cap \overline{F^{q-r+1}}) Gr^W_{m-r} = 0$ for $r > 0$, the class $cl(v) = 0 \in Gr^W_{m-r}$ where $v \in F^p \cap W_{m-r-1}$ hence by definition

$$v \in \overline{F^q} \cap W_{m-r-1} + \sum_{r+1\geq i\geq 2} \overline{F^{q-i+1}} \cap W_{m-r-1} + \sum_{i>r+1\geq 2} \overline{F^{q-i+1}} \cap W_{m-i}$$

hence $cl(v) \in (F^p \cap \overline{F^{q-r}}) Gr^W_{m-r-1} = 0$. Since $W_{m-r-1} = 0$ for large r, we deduce $v = 0$.
Surjectivity. Let $\alpha \in A^{p,q}$; there exists $v_0 \in F^p \cap W_m$ (resp. $u_0 \in F^q \cap W_m$) s.t. $\varphi(v_0) = \alpha = \varphi(\overline{u}_0)$, hence $v_0 = \overline{u}_0 + w_0$ with $w_0 \in W_{m-1}$. In Gr^W_{m-1} the class of w_0 decomposes as $cl(w_0) = cl(v_1) + cl(\overline{u}_1)$ with $v_1 \in F^p \cap W_{m-1}$ and $u_1 \in F^q \cap W_{m-1}$; hence there exists $w_1 \in W_{m-2}$ s.t. $v_0 = \overline{u}_0 + v_1 + \overline{u}_1 + w_1$, then $v_0 - v_1 = \overline{u}_0 + \overline{u}_1 + w_1$ with $v_0 - v_1 \in F^p \cap W_m$ and $u_0 + u_1 \in F^q \cap W_m$.
By an increasing inductive argument on k, we can find vectors

$$v_k \in F^p \cap W_m, w_k \in W_{m-1-k}$$
$$u_k \in F^q \cap W_m + F^{q-1} \cap W_{m-2} + F^{q-2} \cap W_{n-3} + ... + F^{q+1-k} \cap W_{m-k}$$

s.t. v_k represents α and $v_k = \overline{u_k} + w_k$, then we decompose the class of w_k in Gr^W_{m-k-1} as above in the inductive step. For large k, $W_{m-1-k} = 0$ and we represent α in $I^{p,q}$.
Moreover, $W_n = W_{n-1} \oplus (\oplus_{p+q=n} I^{p,q})$, which prove the formula for the weight. Next suppose $v \in F^p$ and consider the least integer n s.t. $v \in W_n$. The image of v in $Gr^W_n A$ decomposes into Hodge components of type $(i, n-i)$ with $i \geq p$ since $v \in F^p \cap W_n$. By subtracting the representatives of the components in $I^{i,n-i}$ for $i \geq p$ we push v in W_{n-1} so a decreasing inductive argument proves the formula for F^p. $\square$

Theorem 5.17 (Deligne)**.** *Let $\mathbb{A}$ be an abelian category and $\mathbb{A}'$ the category of objects of $\mathbb{A}$ endowed with three opposite filtrations W (increasing), F and G. The morphisms of $\mathbb{A}'$ are morphisms in $\mathbb{A}$ compatible with the three filtrations.*
i) $\mathbb{A}'$ is an abelian category.
ii) The kernel (resp. cokernel) of a morphism $f : A \to B$ in $\mathbb{A}'$ is the kernel (resp. cokernel) of f in $\mathbb{A}$, endowed with the three induced filtrations from A (resp. quotient of the filtrations in B).
iii) Any morphism $f : A \to B$ in $\mathbb{A}'$ is strictly compatible with the filtrations W, F and G; the morphism $Gr^W(f)$ is compatible with the bigradings of $Gr^W(A)$ and $Gr^W(B)$; the morphisms $Gr_F(f)$ and $Gr_G(f)$ are strictly compatibles with the induced filtration by W.

iv) The forget functors of the filtrations, as well Gr^W, Gr_F, Gr_G, $Gr^W Gr_F \simeq Gr_F Gr^W$, $Gr_F Gr_G Gr^W$ and $Gr_G Gr^W \simeq Gr^W Gr_G$ of $\mathbb{A}'$ in $\mathbb{A}$ are exact.

Proof. A morphism compatible with the filtrations is necessarily compatible with the above decomposition into $\oplus I^{p,q}$. The proof of (i) follows from the preceding statements.
ii) Let $f : A \to B$ in $\mathbb{A}'$ and consider on $K = Ker(f)$ the induced filtrations from A. The morphism $Gr^W K \to Gr^W A$ is injective, since it is injective on the corresponding terms $I^{p,q}$; moreover, the filtration F (resp. G) of K induces on $Gr^W K$ the inverse image of the filtration F (resp. G) on $Gr^W A$. The sub-object $Gr^W K$ of $Gr^W A$ is compatible with the bigrading of $Gr^W A$, that is

$$Gr^W K = \oplus_{p,q}((Gr^W K) \cap A^{p,q}) \text{ where } Gr_F^p Gr_G^q Gr_n^W K \hookrightarrow Gr_F^p Gr_G^q Gr_n^W A$$

Hence the filtrations W, F, G on K are opposite and K is a kernel of f in $\mathbb{A}'$. This and the dual result, prove (ii). If f is a morphism in $\mathbb{A}'$, the canonical morphism $Coim(f) \to Im(f)$ is an isomorphism in $\mathbb{A}$; it follows from (iii), it is an isomorphism in $\mathbb{A}'$, which is therefore abelian. Forget the filtrations is an exact functor after (ii). Hence the exactness of the other functors in (iv) results. $\square$

We can deduce now the following

Theorem 5.18 (Deligne)**.** *i) The category of MHS is abelian.*
ii) The kernel (resp. cokernel) of a morphism $f : H \to H'$ has as integral lattice the kernel (resp. cokernel) K of $f : H_A \to H'_A$, moreover $K \otimes \mathbb{Q}$ and $K \otimes \mathbb{C}$ are endowed with induced filtrations (resp. quotient filtrations) by W and F of $H_{A\otimes\mathbb{Q}}$ and $H_\mathbb{C}$ (resp. $H'_{A\otimes\mathbb{Q}}$ and $H'_\mathbb{C}$).
iii) Each morphism $f : H \to H'$ is strictly compatible with the filtrations W on $H_{A\otimes\mathbb{Q}}$ and $H'_{A\otimes\mathbb{Q}}$ as well the filtrations F on $H_\mathbb{C}$ and $H'_\mathbb{C}$. It induces morphisms of $A \otimes \mathbb{Q}-HS$, $Gr_n^W(f) : Gr_n^W(H_{A\otimes\mathbb{Q}}) \to Gr_n^W(H'_{A\otimes\mathbb{Q}})$ and morphisms $Gr_F^p(f)$: $Gr_F^p(H_\mathbb{C}) \to Gr_F^p(H'_\mathbb{C})$ strictly compatible with the induced filtrations $W_\mathbb{C}$.
iv) The functor Gr_n^W from the category of MHS to the category $A\otimes\mathbb{Q}-HS$ of weight n is exact.
v) The functor Gr_F^p is exact.

5.5.3. *Hodge numbers.* Let H be a MHS and set $H^{pq} = Gr_F^p Gr_{\overline{F}}^q Gr_{p+q}^W H_\mathbb{C} = (Gr_{p+q}^W H_\mathbb{C})^{p,q}$. The Hodge numbers of H are the integers $h^{pq} = dim_\mathbb{C} H^{pq}$, that is the Hodge numbers h^{pq} of the HS: $Gr_{p+q}^W H$.

5.5.4. *Complex MHS.* Although the cohomolgy of algebraic varieties carry MHS defined over $\mathbb{Z}$, we may need to work in analysis without such structure over $\mathbb{Z}$.

Definition 5.19. A complex MHS of weight n on a complex vector space H is given by an increasing filtration W and two decreasing filtrations F and G such that $(Gr_k^W H, F, G)$ with the induced filtrations is a complex HS of weight $n + k$.

For $n = 0$, we say a complex MHS. The definition of complex HS of weight n is obtained in the particular case when $W_n = H$ and $W_{n-1} = 0$.
Let $\overline{H}$ denotes the complex conjugate space of H and $c : H \to \overline{H}$ the $\mathbb{R}-$linear identity isomorphism s.t. $\lambda ._{\overline{H}} c(v) := c(\overline{\lambda} ._H v)$.

Definition 5.20. i) A complex MHS of weight n is graded polarisable if there exists a bilinear morphism $S : H \otimes \overline{H} \to \mathbb{C}$ s.t.

$$S(F^p, c(G^q) = S(G^p, c(F^q) = 0 \quad \text{for } p + q > n$$

and moreover $(Gr_k^W H, F, G)$ is an induced polarized complex HS.

5.6. **Variation of complex MHS.** The structure which appears in deformation theory on cohomology of the fibers of a morphism of algebraic varieties leads to introduce the concept of Variation of MHS.

Definition 5.21. i) A variation (VHS) of complex HS on a complex manifold X of weight n is given by a data $(\mathcal{H}, F, \overline{F})$ where $\mathcal{H}$ is a complex local system, F and $\overline{F}$ are two decreasing filtrations by sub-bundles of the vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{H}$ s.t. for each point $x \in X$, the data $(\mathcal{H}(x), F(x), \overline{F}(x))$ form a HS of weight n. Moreover, the connection ∇ defined by the local system satisfy Griffiths tranversality

$$(\nabla F^p) \subset F^{p-1}, \quad (\nabla \overline{F}^p) \subset \overline{F}^{p-1}$$

ii) A variation (VMHS) of complex MHS of weight n on X is given by the following data

$$(\mathcal{H}, W, F, \overline{F})$$

where $\mathcal{H}$ is a complex local system, W an increasing filtration by sub-local systems, F and $\overline{F}$ two decreasing filtrations by sub-bundles of the vector bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{H}$ satisfying Griffiths tranversality

$$(\nabla F^p) \subset F^{p-1}, \quad (\nabla \overline{F}^p) \subset \overline{F}^{p-1}$$

s.t. $(Gr_k^W \mathcal{H}, F, \overline{F})$ with the induced filtrations is a complex VHS of weight $n + k$.

For $n = 0$ we just say a complex VMHS. Let $\overline{\mathcal{H}}$ be the conjugate local system of $\mathcal{L}$. A linear morphism $S : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}_X$ defines a polarization of a VHS if it defines a polarization at each point $x \in X$. A complex MHS of weight n is graded polarisable if $(Gr_k^W \mathcal{H}, F, \overline{F})$ is a polarized VHS.

6. Hypercohomology of a Filtered Complex

The construction of MHS on cohomology groups of algebraic varieties proceeds first by attaching to the variety a bifiltered complex of sheaves (Ω^*, W, F)(essentially a variant of DeRham complex) satisfying certain conditions to insure that the induced filtrations by W and F on cohomology define a MHS. Such bifiltered complexes, called mixed Hodge complexes (MHC), are defined in [Deligne]. Since the construction will be applied respectively in the case of compact Kähler manifolds, the complement of a normal crossing divisor (NCD) and then a simplicial scheme, it has been necessary to establish in [6] a common language and a set of axioms which lead naturally to the construction of a MHS on the hypercohomology of such complexes. For a deep knowledge of sheaf theory, the book by Godement [8] for example is good for a start.

6.1. **Derived categories.** The language of derived categories is the most appropriate at this point. It is based on the fact that a complex is a more rich object than merely its cohomology. However since we need acyclic resolutions and there is in general many choices of such resolutions, one needs to change the structure of the category by changing only the structure of morphisms. Verdier showed that the following two constructions lead to a new category satisfactory for our purpose and which proved to be useful in various domains of geometry.
1) The objects of the category are the complexes.
2) The morphisms of the category are defined after the following modifications

i) Two morphisms are equivalent if they are homotopic.
ii) A morphism is invertible if it induces an isomorphism on cohomology (then it is called a quasi-isomorphism).
These two operations have been carried by Verdier [13]. Although the objects remain the same, the modifications on morphisms can transform the category significantly (for example non zero morphisms may become isomorphic to zero and the inverse ψ of a morphism φ need to satisfy $\psi \circ \varphi$ and $\varphi \circ \psi$ are only homotopic to the identity [1]).
In practice the reader should apply the following rule : the sign $\mathbf{R}T$ *in front of a functor* T *applied to a complex of sheaves* K *refers to the complex* TK' *where* K' *is a resolution of* K *by* $T-$*acyclic sheaves* .

6.2. **Filtered derived category.** Let K be a complex of objects of an abelian category $\mathcal{A}$, filtered by subcomplexes F^i. The filtration is biregular if it induces a finite filtration on each component of K . A morphism of complexes is called a quasi-isomorphism if it induces an isomorphism on the cohomology. The filtration F of a bifiltered complex (K, F, W) induces by restriction a filtration F on the terms W^iK, which induces on its own a quotient filtration F on $Gr^i_W K$. We define in this way $Gr_F Gr_W K$.

Definition 6.1. (i) A morphism $f : (K, F) \to (K', F')$ of complexes with biregular filtrations is a filtered quasi-isomorphism if $Gr_F(f)$ is a quasi-isomorphism.
(ii) A morphism $f : (K, F, W) \to (K', F', W')$ of complexes with biregular filtrations is a bifiltered quasi-isomorphism if $Gr_F Gr_W(f)$ is a quasi-isomorphism.

6.2.1. For an abelian category $\mathcal{A}$, let $F\mathcal{A}$ (resp. $F_2\mathcal{A}$) denotes the category of filtered objects (resp. bifilered) of $\mathcal{A}$ with finite filtration(s), $C^+\mathcal{A}$ (resp. $C^+F\mathcal{A}, C^+F_2\mathcal{A}$) the category of complexes of $\mathcal{A}$ (resp. $F\mathcal{A}, F_2\mathcal{A}$) bounded on the left (zero in degrees near $-\infty$) with morphisms of complexes (resp. morphisms respecting the filtrations).
An homotopy from a morphism $u : K \to K'$ to u', is a family of morphisms $k^{i+1} : K^{i+1} \to K'^{i}$ for $i \in \mathbb{Z}$ of degree -1, satisfying $u - u' = d_{K'} \circ h + h \circ d_K$; in presence of filtrations, the homotopy should respect the filtrations.

6.2.2. Let $K^+\mathcal{A}$ (resp. $K^+F\mathcal{A}, K^+F_2\mathcal{A}$) be the category whose objects are complexes bounded below of objects of $\mathcal{A}$ (resp. filtered, bifiltered complexes), and whose morphisms are equivalence classes modulo homotopy (resp. homotopy respecting the filtrations).

6.2.3. Let $D^+\mathcal{A}$ (resp. $D^+F\mathcal{A}$, $D^+F_2\mathcal{A}$) be the category deduced from $K^+\mathcal{A}$ (resp. $K^+F\mathcal{A}, K^+F_2\mathcal{A}$) by inverting the quasi-isomorphisms (resp. the filtered, bifiltered quasi-isomorphisms). That is the derived category (resp. derived filtered, bifiltered) as defined by Verdier (resp. [Deligne]). What is important to retain is the following : the objects of $D^+\mathcal{A}$ (resp.$D^+F\mathcal{A}$, $D^+F_2\mathcal{A}$) are complexes of objects of $\mathcal{A}$, hence the same as those of $K^+\mathcal{A}$ (resp.$K^+F\mathcal{A}$, $K^+F_2\mathcal{A}$). An isomorphism $f : K_1 \to K_2$ in $D^+\mathcal{A}$ (resp.$D^+F\mathcal{A}$, $D^+F_2\mathcal{A}$) is represented by one of the diagrams of morphisms

$$K_1 \xleftarrow{g_1 \approx} {K'}_1 \xrightarrow{f_1} K_2 \quad , \quad K_1 \xrightarrow{f_2} {K'}_2 \xleftarrow{g_2 \approx} K_2$$

in $K^+\mathcal{A}$ (resp.$K^+F\mathcal{A}, K^+F_2\mathcal{A}$) where g_1 and g_2 are quasi-isomorphisms (resp. filtered, bifiltered).

6.2.4. *Triangles.* The complex TK or $K[1]$ is the complex (K, d_K) with shifted degrees defined as

$$(TK)^i = K^{i+1}, d_{TK} = -d_K$$

Let $u : K \to K'$ be a morphism of $C^+\mathcal{A}$(resp. $C^+F\mathcal{A}$, $C^+F_2\mathcal{A}$), the cone $C(u)$ is the complex $TK \oplus K'$ with the differential $\left(\begin{smallmatrix} -d_K & 0 \\ u & d_{K'} \end{smallmatrix}\right)$ (resp. endowed with the sum of the filtrations). The exact sequence associated to $C(u)$ is

$$0 \to K' \to C(u) \to TK \to 0$$

Let h denotes an homotopy (resp. filtered, bifiltered) from a morphism $u : K \to K'$ to u', we define an isomorphism (resp. filtered, bifiltered) $I_h : C(u) \xrightarrow{\sim} C(u')$ by the matrix $\left(\begin{smallmatrix} Id & 0 \\ h & Id \end{smallmatrix}\right)$ acting on $TK \oplus K'$, which commute to the injections of K' in $C(u)$ and $C(u')$, and to the projections on TK.
Let h and h' be two homotopies of u to u'. A second homotopy of h to h', that is a family of morphisms $k^{i+2} : K^{i+2} \to K'^i$ for $i \in \mathbb{Z}$, satisfying $h - h' = d_{K'} \circ k - k \circ d_K$, defines an homotopy of I_h to $I_{h'}$.
A distinguished triangle (or exact) in $K^+\mathcal{A}$ (resp. $K^+F\mathcal{A}$, $K^+F_2\mathcal{A}$)) is a sequence of complexes isomorphic to the image of an exact sequence associated to a cone in $C^+\mathcal{A}$ (resp.$C^+F\mathcal{A}$, $C^+F_2\mathcal{A}$).
The derived category (resp. derived filtered, bifiltered) is a triangulated category (that is endowed with a class of distinguished triangles). A distinguished triangle in $D^+\mathcal{A}$ (resp.$D^+F\mathcal{A}$, $D^+F_2\mathcal{A}$) is a sequence of complexes isomorphic to the image of a distinguished triangle in $K^+\mathcal{A}$ (resp.$K^+F\mathcal{A}$, $K^+F_2\mathcal{A}$).

6.3. **Filtered complexes.** We follow in this section Deligne's notations in [6] where all complexes are bounded below. There exist different point of views in the literature.

6.3.1. Consider a left exact functor $T : \mathcal{A} \to \mathcal{B}$, (A, F) a filtered object of finite filtration, and TF the filtration of TA by its sub-objects $TF^p(A)$ (here we use T left exact). If $Gr_F(A)$ is T-acyclic, the $F^p(A)$ are T-acyclic as successive extensions of T-acyclic objects. Hence, the image by T of the sequence $o \to F^{p+1}(A) \to F^p(A) \to Gr_F^p(A) \to o$ is distinguished, and

$$Gr_{TF}TA \simeq TGr_FA.$$

6.3.2. Let A be an object with two finite filtrations F and W such that Gr_FGr_WA is T-acyclic. Then the objects Gr_FA andGr_WA are T-acyclic, as well $F^q(A) \cap W^p(A)$. The sequences

$$o \to T(F^q \cap W^{p+1}) \to T(F^q \cap W^p) \to T((F^q \cap W^p)/(F^q \cap W^{p+1})) \to o$$

are exact, and $T(F^q(Gr_W^pA))$ is the image in $T(Gr_W^p(A)$ of $T(F^p \cap W^q)$. The diagram

$$\begin{array}{ccccc} T(F^q \cap W^p) & \to & T(F^qGr_W^pA) & \to & TGr_W^pA \\ \simeq\downarrow & & & & \simeq\downarrow \\ TF^q \cap TW^p & & \to & & Gr_{TW}^pTA \end{array}$$

shows that the isomorphism relative to W transform the filtration $Gr_{TW}(TF)$ on $Gr_{TW}TA$ into the filtration $T(Gr_W(F))$ on TGr_WA.

6.3.3. Whenever the category $\mathcal{A}$ has enough injectives, the following derived functors of a left exact functor are defined functorially

$$\mathbf{R}T : D^+(\mathcal{A}) \to D^+(\mathcal{B}),\ \mathbf{R}T : D^+F(\mathcal{A}) \to D^+F(\mathcal{B}),\ \mathbf{R}T : D^+F_2(\mathcal{A}) \to D^+F_2(\mathcal{B})$$

as follows
a) Let K be a complex in $D^+(\mathcal{A})$ (resp. $D^+F(\mathcal{A}), D^+F_2(\mathcal{A})$). We choose a quasi-isomorphism $i : K \xrightarrow{\approx} I$ (resp. filtered, bifiltered), where the components of I are injectives in each degree (resp. with injective filtrations). For example, I is the simple complex defined by an injective Cartan-Eilenberg resolution of K.
b) We define $\mathbf{R}TK = T(I)$.
c) A morphism $f : K \to K'$ gives rise to a morphism $\mathbf{R}T(K) \to \mathbf{R}T(K')$ functorially, since f can be extended on the injective resolutions to a morphism $I(f) : T(I) \to T(I')$, defined uniquely up to homotopy.
The hypercohomology R^iTK is equal to the cohomology $H^i(\mathbf{R}TK)$. In particular, for a different choice of an injective resolution J of K, we have an isomorphism $T(I) \simeq T(J)$ in $D^+(\mathcal{B})$.

Remark 6.2. A single sheaf $\mathcal{F}$ is $T-$acyclic when $R^iT(\mathcal{F}) = 0$ for $i > 0$. An important result shows that for any resolution A of a complex K with acyclic sheaves, TA is isomorphic to the derived complex $\mathbf{R}TK$.

6.3.4. Let F be a biregular filtration of K. A filtered T-acyclic resolution of K is given by a filtered quasi-isomorphism $i\colon K \to K'$ to a complex with a biregular filtration s.t. $Gr^p_F(K'^n)$ are acyclic for T. If K' is such a resolution, the $F^p(K'^n)$ are acyclic for T and the derived functor is defined as follows
a) We choose a filtered quasi-isomorphism $i : (K, F) \to (K', F')$ s.t. the components of $Gr^p_{F'}(K')$ are acyclic for T.
b) We define $\mathbf{R}T(K, F) = (TK', TF')$. For a different choice (K'', F'') of (K', F') we have an isomorphism $(TK'', TF'') \simeq (TK', TF')$ in $D^+F(\mathcal{B})$. In fact we have

$$\mathbf{R}T(Gr_F K) \simeq Gr_{TF'}T(K') \simeq Gr_{TF''}T(K'')$$

The complex $T(K', F')$ is functorial in (K', F'), and a filtered quasi-isomorphism $f : (K_1, F_1) \to (K_2, F_2)$ induces an isomorphism $\mathbf{R}T(f) : \mathbf{R}T(K_1, F_1) \simeq \mathbf{R}T(K_2, F_2)$.

6.3.5. Let F and W be two biregular filtrations of K. A bifiltered T-acyclic resolution of K is a bifiltered quasi- isomorphism $i : K \to K'$ of K to a bifiltered complex biregular for each filtration s.t. $Gr^p_F Gr^q_W(K'^n)$ is acyclic for T. If K' is such a resolution, the terms $F^pK'^n \cap W^qK'^n$ are acyclic for T, then the derived functor is defined as follows
a) We chose a bifiltered quasi-isomorphism $i : (K, F, W) \xrightarrow{\approx} (K', F, W)$ s.t. the components of $Gr^p_F Gr^q_W K'$ are acyclic for T (such i always exists). In the particular case where $\mathcal{A}$ is the category of sheaves of A-modules on a topological space X, and where T is the functor Γ of $\mathcal{A}$ to the category of A-modules, an example of bifiltered T-acyclic resolution of K is the simple complex associated to the double complex defined by Godement resolution $\mathcal{G}^*$ of K, filtered by $\mathcal{G}^*(F^pK)$ and by $\mathcal{G}^*(W^nK)$. Since $\mathcal{G}^*$ is exact, we have

$$Gr_F Gr_W(\mathcal{G}^*K) \simeq \mathcal{G}^*(Gr_F Gr_W K)$$

b) We define $\mathbf{R}T(K, F, W) = (TK', TF, TW)$.
If K' is a bifiltered T-acyclic resolution of K, the complex TK' is filtered by TF^pK' and by TW^qK'. Moreover, $Gr^n_W K'$ is a filtered T-acyclic resolution for F of $Gr^n_W K$,

hence $Gr^n_F Gr^m_W K'$ is a T-acyclic resolution of $Gr^n_F Gr^m_W K$, and $TGr_F K' \simeq Gr_F TK'$ (as a W-filtered complex), $TGr_W K' \simeq Gr_W TK'$ (as a F-filtered complex) and $TGr_F Gr_W K' \simeq Gr_F Gr_W TK'$. We deduce

$$\mathbf{R}TGr_F Gr_W K \simeq Gr_F Gr_W TK'$$

For a different choice (K'', F, W) of (K', W, F), we have an isomorphism

$$(TK'', TF, TW) \simeq (TK', TF, TW) \text{ in } D^+F_2(\mathcal{B}).$$

The complex $T(K', F, W)$ is functorial in (K, F, W) and a bifiltered quasi-isomorphism $f : K_1 \to K_2$ induces an isomorphism $\mathbf{R}T(f) : \mathbf{R}TK_1 \xrightarrow{\sim} \mathbf{R}TK_2$.

6.4. **Spectral sequence of a filtered complex.** A filtration F of a complex K by sub-complexes induces a filtration on its cohomology with associated graded cohomology . The spectral sequence ([6], [8]) $E_r(K, F)$ associated to F leads, for large r and under mild conditions, to such graded cohomology defined by the filtration. This chapter is technical and we recommend to follow the proofs at a first reading on an example, and return later with the study of MHC where the spectral sequence becomes interesting and meaningful since it contains deep geometrical information.
Let K be a complex of objects of $\mathcal{A}$, with a biregular filtration F.

6.4.1. To construct the terms $E^{pq}_r(K, F)$, or simply E^{pq}_r of the spectral sequence defined by F, we set for $r \geq 1$

$$Z^{pq}_r = Ker(d : F^p K^{p+q} \to K^{p+q+1}/F^{p+r}K^{p+q+1})$$

and we define dually B^{pq}_r by the formula

$$K^{p+q}/B^{pq}_r = coker(d : F^{p-r+1}K^{p+q-1} \to K^{p+q}/F^{p+1}(K^{p+q}).$$

Such formula still makes sense for $r = \infty$ if we set, for a filtered object (A, F), $F^{-\infty}(A) = A$ and $F^{\infty}(A) = 0$

$$Z^{pq}_\infty = Ker(d : F^p K^{p+q} \to K^{p+q+1})$$

$$K^{p+q}/B^{pq}_\infty = coker(d : K^{p+q+1} \to K^{p+q}/F^{p+1}K^{p+q})$$

The notations here are similar to [6] but different from [Go]. We set by definition

$$\begin{aligned} E^{pq}_r &= Im(Z^{pq}_r \to K^{p+q}/B^{pq}_r) = Z^{pq}_r/(B^{pq}_r \cap Z^{pq}_r) \\ &= Ker(K^{p+q}/B^{pq}_r \to K^{p+q}/(Z^{pq}_r + B^{pq}_r)) \end{aligned}$$

$$E^{pq}_\infty = Z^{pq}_\infty / B^{pq}_\infty \cap Z^{pq}_\infty.$$

The differential $d : Z^{pq}_r \to Z^{p+r,q-r+1}_r$ induces a differential $d_r : E^{pq}_r \to E^{p+r,q-r+1}_r$. For $r < \infty$, the terms E_r form a complex of objects, graded by the degree $p-r(p+q)$ (or a direct sum of complexes with index $p-r(p+q)$); moreover E_{r+1} is isomorphic to the cohomology of this complex:

$$E^{pq}_{r+1} = H(E^{p-r,q+r-1}_r \xrightarrow{d_r} E^{pq}_r \xrightarrow{d_r} E^{p+r,q-r+1}_r).$$

for $r = 0$, we set

$$E^{pq}_0 = Gr^p_F(K^{p+q}).$$

Consider the filtration F on $H^*(K)$ defined by

$$F^i H^*(K) = Im(H^*(F^i K) \to H^*(K)).$$

Then we have

$$E_\infty^{pq} = Gr_F^p(H^{p+q}(K)).$$
$$E_1^{pq} = H^{p+q}(Gr_F^p(K)).$$

The differentials d_1 are the connecting morphisms defined by the exact sequences

$$0 \to Gr_F^{p+1}K \to F^pK/F^{p+2}K \to Gr_F^pK \to 0.$$

Remark 6.3. The precedent formulas are defined for an increasing filtration on K by a change of indices.
Let W be an increasing filtration on K. We set for all j, $n \leq m$ and $n \leq i \leq m$

$$W_iH^j(W_mK/W_nK) = Im(H^j(W_iK/W_nK) \to H^j(W_mK/W_nK)$$

then we define with these notations, for all $r \geq 1, p$ and q

$$E_r^{pq} = Gr_{-p}^W H^{p+q}(W_{-p+r-1}K/W_{-p-r}K).$$

This definition coincides with the previous one, written for W, as we can check

$$E_r^{pq}(K,W) = Z_r^{pq}/B_r^{pq} \cap Z_r^{pq} \quad =$$
$$Ker(d : W_{-p}K^{p+q} \to K^{p+q+1}/W_{-p-r}K^{p+q+1})/(W_{-p-1}K^{p+q}+dW_{-p+r-1}K^{p+q-1}) \cap Z_r^{pq}$$
$$= \frac{Ker(d : W_{-p}K^{p+q}/W_{-p-r}K^{p+q} \to W_{-p+r-1}K^{p+q+1}/W_{-p-r}K^{p+q+1})}{((W_{-p-1}K^{p+q} + dW_{-p+r-1}K^{p+q-1})/W_{-p-r}K^{p+q}) \cap Z_\infty^{pq}} =$$
$$\frac{Z_\infty^{pq}}{B_\infty^{pq} \cap Z_\infty^{pq}} = E_\infty^{pq}(W_{-p+r-1}K/W_{-p-r}K) = Gr_{-p}^W H^{pq}(W_{-p+r-1}K/W_{-p-r}K)$$

Consider the exact sequence

$$0 \to W_{-p-r}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-2r}K \to W_{-p+r-1}K/W_{-p-r}K \to 0.$$

The connecting morphism

$$H^{p+q}(W_{-p+r-1}K/W_{-p-r}K) \xrightarrow{\partial} H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K).$$

send W_{-p} to W_{-p-r}. The injection $W_{-p-r}K \to W_{-p-1}K$ induces a morphism

$$H^{p+q+1}(W_{-p-r}K/W_{-p-2r}K) \xrightarrow{\varphi} E_r^{p+r,q-r+1} = Gr_{-p-r}^W H^{p+q+1}(W_{-p-1}K/W_{-p-2r}K)$$

Then $\varphi \circ \partial$ restricted to W_{-p} induces the differential

$$d_r : E_r^{pq} \to E_r^{p+r,q-r+1}.$$

The injection $W_{-p+r-1} \to W_{-p+r}K$ induces the isomorphism

$$H(E_r^{pq}, d_r) \xrightarrow{\sim} E_{r+1}^{pq} = Gr_{-p}^W H^{p+q}(W_{-p+r}K/W_{-p-r-1}K).$$

The spectral sequence $E_i^{p,q}(K,W)$ degenerates at E_r if the differentials d_i are zero for $i \geq r$. Then we write $E_r = E_\infty$. Moreover, in this case

$$E_r^{pq} = E_i^{pq} = E_\infty^{pq} \quad pour\ i \geq r.$$

Proposition 6.4 (Deligne)**.** Let K be a complex with a biregular filtration F. The following conditions are equivalent

(1) The spectral sequence defined by F degenerates at $E_1 (E_1 = E_\infty)$
(2) The morphisms $d : K^i \to K^{i+1}$ are strictly compatibles to the filtrations.

Proposition 6.5. A filtered quasi-isomorphism $f : (K,F) \to (K',F')$ induces an isomorphism of spectral sequences $E_r^{pq}(f) : E_r^{pq}(K,F) \xrightarrow{\sim} E_r^{p,q}(K',F')$.

Corollary 6.6. Let $f : (K,F) \to (K',F')$ be a filtered morphism. The following assertions are equivalent

(1) f is a filtered quasi-isomorphism.
(2) $E_1^{pq}(f) : E_1^{pq}(K,F) \to E_1^{pq}(K',F')$ is an isomorphism
(3) $E_r^{pq}(f) : E_r^{pq}(K,F) \to E_r^{pq}(K',F')$ is an isomorphism for all $r \geq 1$.

6.4.2. Let $T : \mathcal{A} \to \mathcal{B}$ be a functor, and (K,F) an object of $D^+F\mathcal{A}$. The complex $\mathbf{R}T(K,F)$ defines a spectral sequence

$$E_1^{p,q} = R^{p+q}T(Gr_F^p) \Rightarrow Gr_F^p R^{p+q}T(K)$$

This is the hypercohomology spectral sequence of the filtered complex K. It is written for an increasing filtration W on K as

$$E_1^{p,q} = R^{p+q}T(Gr_{-p}^W) \Rightarrow Gr_{-p}^W R^{p+q}T(K)$$

It depends functorially on K and a filtered quasi-isomorphism induces an isomorphism of spectral sequences. The differentials d_1 of this spectral sequence are the connecting morphisms defined by the short exact sequences

$$0 \to Gr^{p+1}K \to F^pK/F^{p+2}K \to Gr^pK \to 0.$$

6.4.3. Let K be a complex. We denote by $\tau_{\leq p}(K)$ the following sub-complex

$$\tau_{\leq p}\ (K)^n = \begin{cases} K^n & for\ n < p \\ Ker(d) & for\ n = p \\ 0 & for\ n > p. \end{cases}$$

The filtration of K by the $\tau_{\leq p}(K)$ is called canonical, then

$$H^i(\tau_{\leq p}(K)) = H^i(K) \text{ if } i = p, \text{ and } 0 \text{ if } i > p.$$

A quasi-isomorphism $f : K \to K'$ is necessarily a filtered quasi-isomorphism for the canonical filtrations.

6.4.4. The sub-complexes $\sigma_{\geq p}(K)$ of K

$$\sigma_{\geq p}\ (K)^n = \begin{cases} 0 & if\ n < p \\ K^n & if\ n \geq p. \end{cases}$$

define a biregular filtration, the trivial filtration of K (called bête by Deligne). The hypercohomology spectral sequences of a left exact functor attached to the trivial and canonical filtrations of K are the two natural hypercohomology spectral sequences of K.

Example 6.7. Let $f : X \to Y$ be a continued map of topological spaces. Let $\mathcal{F}$ be an abelian sheaf on X and $\mathcal{F}^*$ a resolution of $\mathcal{F}$ by f_*-acyclic sheaves, then $R^if_*\mathcal{F} \cong H^i(f_*\mathcal{F}^*)$. The hypercohomology spectral sequence of the complex $\mathbf{R}f_*\mathcal{F}^*$ with its canonical filtration with respect to the global section functor $\Gamma(Y,*)$ is

$$E_1^{pq} = H^{2p+q}(Y, R^{-p}f_*\mathcal{F}^*) \Rightarrow Gr_{-p}^{\tau}H^{p+q}(X,\mathcal{F})$$

it coincides, up to renumbering $E_r^{pq} \to E_{r+1}^{2p+q,-p}$, with Leray's spectral sequence for f and $\mathcal{F}$

$$E_2^{pq} = H^p(Y, R^qf_*\mathcal{F}).$$

Example 6.8. On DeRham complex, the trivial filtration is called the Hodge filtration since it is defined by the subcomplexes

$$F^p(\Omega^*_X) = 0 \mapsto \cdots 0 \mapsto \Omega^p_X \mapsto \cdots \mapsto \Omega^m_X \mapsto 0$$

It induces on DeRham cohomology a filtration by sub-vector spaces

$$F^p\mathbb{H}^n(X, \Omega^*_X) = Im\, \mathbb{H}^n(X, F^p\Omega^*_X).$$

The spectral sequence with respect to the global section functor and the filtration F (in different terms of the filtered complex $(\mathbf{R}\Gamma(X, \Omega^*_X), \mathbf{R}\Gamma(X, F^p\Omega^*_X))$ degenerates at rank one

$$E_1^{pq} = \cdots = E_r^{pq} = \cdots = Gr^p_F\mathbb{H}^{p+q}(X, \Omega^*_X)$$

The proof uses Hodge theory from which we deduce $Gr^p_F\mathbb{H}^{p+q}(X, \Omega^*_X) \simeq H^{p,q}(X) \simeq H^q(X, \Omega^p_X)$, then it follows that dim $E_1^{pq} = \cdots =$ dim E_r^{pq} and $d_r = 0$ for $r > 0$. In particulae we deduce that the morphism

$$\mathbb{H}^n(X, F^p\Omega^*_X) \xrightarrow{\sim} F^p\mathbb{H}^n(X, \Omega^*_X)$$

is an isomorphism. Such result is algebraic and has been obtained directly by Deligne and Illusie. However Hodge theory tells more. Precisely, the conjugate Hodge filtration $\overline{F}$ of F with respect to $\mathbb{H}^n(X, \mathbb{Q}))$ satisfy

$$H^{p,q}(X) = F^p\mathbb{H}^n(X, \Omega^*_X) \cap \overline{F^q\mathbb{H}^n(X, \Omega^*_X)},\ \overline{H^{p,q}} = H^{q,p}.$$

as well the decomposition

$$H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X).$$

6.5. **Two filtrations.** This section relates results on various induced filtrations on terms of a spectral sequence, contained in [6](lemma on two filtrations). A filtration F of a complex by sub-complexes define a spectral sequence $E_r(K, F)$. A second filtration W induces in various ways filtrations on $E_r(K, F)$, different in general. We discuss here axioms on W and F, at the level of complexes, in order to get compatibility of the various induced filtrations by W on the spectral sequence of (K, F). What we have in mind is to find axioms leading to define the MHS with induced filtrations W and F on cohomology. The proofs are technical and may be assumed for a first reading.

6.5.1. Let (K, F, W) be a bifiltered complex of objects of an abelian category, bounded below. The filtration F, assumed to be biregular, induces on the terms E_r^{pq} of the spectral sequence $E(K, W)$ various filtrations as follows

Definition 6.9. The first direct filtration on $E_r(K, W)$ is the filtration F_d defined by images

$$F^p_d(E_r(K, W)) = Im(E_r(F^pK, W) \to E_r(K, W)).$$

Dually, the second direct filtration F_{d^*} on $E_r(K, W)$ is defined by kernels

$$F^p_{d^*}(E_r(K, W)) = Ker(E_r(K, W) \to E_r(K/F^pK, W)).$$

Lemma 6.10. *On E_0 and E_1, we have $F_d = F_{d^*}$.*

For r=0 or 1, we have $B_r^{pq} \subset Z_r^{pq}$.

Definition 6.11. The recurrent filtration F_{rec} on E_r^{pq} is defined as follows
i) On $E_0^{pq}, F_{rec} = F_d = F_{d^*}$.
ii) On E_{r+1}^{pq}, as a quotient of a sub-object of E_r^{pq}, the recurrent filtration F_{rec} is induced by F_{rec} on E_r^{pq}.

6.5.2. The precedent definitions of direct filtrations apply to E_∞^{pq} as well and they are compatible with the isomorphism $E_r^{pq} \simeq E_\infty^{pq}$ for large r. We deduce, via this isomorphism a recurrent filtration F_{rec} on E_∞^{pq}. The filtrations F and W induce each a filtration on $H^*(K)$. The isomorphism $E_\infty^{**} \simeq Gr_W^* H^*(K)$ leads to a new filtration F on E_∞^{pq}.

Proposition 6.12. (i) The morphisms d_r are compatible with the first direct filtration. If E_{r+1}^{pq} is considered as a quotient of a sub-object of E_r^{pq}, the first direct filtration on E_{r+1}^{pq} is finer then the filtration F' induced by the first direct filtration on E_r^{pq} : $F_d(E_{r+1}^{pq}) \subset F'(E_{r+1}^{pq})$.
(ii) Dually, the morphisms d_r are compatible with the second direct filtration, and the second direct filtration on E_{r+1}^{pq} is less fine then the filtration F'' induced from E_r^{pq}: $F''(E_{r+1}^{pq}) \subset F_{d^*}(E_{r+1}^{pq})$.
(iii) $F_d(E_r^{pq}) \subset F_{rec}(E_r^{pq}) \subset F_{d^*}(E_r^{pq})$.
(iv) On E_∞^{pq}, the filtration induced by the filtration F on $H^*(K)$ satisfy $F_d(E_\infty^{pq}) \subset F(E_\infty^{pq}) \subset F_{d^*}(E_\infty^{pq})$.

6.5.3. The bifiltered complex (K, F, W) is said to be $F-$split if the filtered complex (K, W) is isomorphic to a sum of filtered complexes $(K_n, W_n)_{n\in\mathbb{Z}}$

$$(K, W) = \oplus_n (K_n, W_n) \quad \text{with} \quad F^n K = \oplus_{n' \geq n} K_{n'}$$

6.6. Let r_0 be an integer ≥ 0, or $+\infty$ and consider the following condition
(2filt, r_0): For each non negative integer $r < r_0$, the differentials d_r of the graded complex $E_r(K, W)$ are strictly compatible with the recurrent filtration defined by F.

6.6.1. It is clear that the condition (2filt, ∞) is satisfied by $F-$split (K, F, W). Reciprocally, if the condition (2filt, ∞) is satisfied, the exact sequence

$$0 \to (F^{i+1}K, W) \to (F^i K, W) \to (Gr_F^i K, W) \to 0$$

behaves like it was split. For example we have exact sequences for all i and j

$$0 \to Gr_W H^j(F^{i+1}K) \to Gr_W H^j(F^i K) \to Gr_W H^j(Gr_F^i K) \to 0,$$

the term Gr_F of the spectral sequence $E(K, W)$ is isomorphic to the spectral sequence $E(Gr_F K, W)$, and the spectral sequence $E(K, F)$ degenerates at rank 1 ($E_1 = E_\infty$).

Theorem 6.13 (Deligne)**.** *Let K be a complex with two filtrations W and F, where F is biregular.*
(i) If (K, F, W) satisfy (2 filt, r_0), then for $r \leq r_0$ the sequence

$$0 \to E_r(F^p K, W) \to E_r(K, W) \to E_r(K/F^p K, W) \to 0$$

is exact, and for $r = r_0 + 1$, the sequence

$$E_r(F^p K, W) \to E_r(K, W) \to E_r(K/F^p K, W)$$

is exact. In particular for $r \leq r_0 + 1$, the two direct and the recurrent filtration on $E_r(K, W)$ coincide $F_d = F_{rec} = F_{d^*}$.
(ii) If (K, F, W) satisfy (2 filt, ∞), then the spectral sequence $E(K, F)$ degenerates at E_1 ($E_1 = E_\infty$) and the filtrations F_d, F_{rec}, F_{d^*} coincide with the filtration F on E_∞ induced by $(H^*(K), F)$.

Corollary 6.14. (i) In the conditions of case (i) of the theorem the condition (2 filt, r_0) is satisfied by the complex $(F^aK/F^bK, W, F)$ if it is already satisfied by (K, W, F).
(ii) In the conditions of case (ii) of the theorem, we have a strict filtered exact sequence

$$0 \to (H(F^pK), W) \to (H(K), W) \to (H(K/F^pK), W) \to 0$$

and an isomorphism of spectral sequences

$$Gr_F^p(E.(K, W)) \simeq E.(Gr_F^p K, W).$$

7. Mixed Hodge Complex(MHC)

We consider complexes of abelian sheaves endowed with new structures needed to deduce a MHS on their cohomology. Deligne called them Mixed Hodge complexes (MHC). The technique of mixed cone associates to a morphism of MHC a new MHC, which leads to a MHS on relative cohomology and long exact sequences of MHS. The first example is constructed on the cohomology of a divisor Y with normal crossings (NCD) in a smooth complex complete variety X and the cohomology with compact support of $X - Y$. The dual case is the logarithmic complex constructed defining the MHS on the cohomology of $X - Y$. The Hodge filtration F may be deduced also from the filtration by order of the pole on the sheaf of meromorphic forms holomorphic on $X - Y$. Other examples are given by a construction of natural mixed cones, including the case of cohomology with compact support on $X - Y$. The most general type of MHS is on the cohomology of open NCD.

7.1. Hodge Complex (HC). In this section we consider complexes with additional properties that enables us to obtain a HS on their cohomology, and arrive to the definition of a Hodge complex. The cohomology of a compact complex smooth algebraic variety carry a HS whose Hodge filtration is deduced from the filtration on the algebraic DeRham complex, however the proof of the decomposition is reduced to the case of a projective variety, hence compact Kähler manifold.

7.1.1. Let $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. A Hodge A-complex K of weight n consists of

(1) a complex of $A-$modules $K_A \in ObD^+(A)$, s.t. $H^k(K_A)$ is an A-module of finite type for all k,
(2) a filtered complex $(K_{\mathbb{C}}, F) \in ObD^+F(\mathbb{C})$
(3) an isomorphism $\alpha : K_{\mathbb{C}} \simeq K_A \otimes \mathbb{C}$ in $D^+(\mathbb{C})$.

The following axioms must be satisfied
(CH 1) the differential d of $K_{\mathbb{C}}$ is strictly compatible with the filtration F; equivalently, the spectral sequence defined by $(K_{\mathbb{C}}, F)$ degenerates at $E_1 (E_1 = E_\infty)$,
(CH 2) for all k, the filtration F on $H^k(K_{\mathbb{C}}) \simeq H^k(K_A) \otimes \mathbb{C}$ define an $A - HS$ of weight $n + k$ on $H^k(K_A)$.
Equivalently, the filtration F is $(n + k)$-opposed to its complex conjugate (which makes sense since $A \subset \mathbb{R}$).

7.1.2. Let X be a topological space. An A-cohomological Hodge complex $(CHC)K$ of weight n on X, consists of

(1) a complex $K_A \in ObD^+(X, A)$
(2) a filtered complex $(K_{\mathbb{C}}, F) \in ObD^+F(X, \mathbb{C})$
(3) an isomorphism $\alpha : K_{\mathbb{C}} \xrightarrow{\sim} K_A \otimes \mathbb{C}$ in $D^+(X, \mathbb{C})$.

Moreover, the following axiom must be satisfied
(CHC) The triple $(\mathbf{R}\Gamma(K_A), \mathbf{R}\Gamma(K_{\mathbb{C}}, F), \mathbf{R}\Gamma(\alpha))$ is a HC of weight n.
When $A = \mathbb{Z}$, we say simply HC or CHC.

Remark 7.1. If (K, F) is a HC (resp. CHC) of weight n, then $(K[m], F[p])$ is a HC (resp. CHC) of weight $n + m - 2p$.

Theorem 7.2. *Let X be a compact Kähler manifold and consider*

(1) *$K_{\mathbb{Z}}$ the complex reduced to a constant sheaf $\mathbb{Z}$ on X in degree zero.*
(2) *$K_{\mathbb{C}}$ the De Rham holomorphic complex Ω_X^* with its Hodge filtration $F = \sigma$*

$$F^p\Omega_X^* := 0 \to \cdots \to \Omega_X^p \to \Omega_X^{p+1} \cdots \to \Omega_X^n \to 0$$

(3) *The quasi-isomorphism $\alpha\colon K_{\mathbf{Z}} \otimes \mathbb{C} \xrightarrow{\approx} \mathbb{C} \xrightarrow{\approx} \Omega_X^*$ (Poincaré's lemma).*

Then $(K_Z, (K_{\mathbb{C}}, F), \alpha)$ is a CHC of weight.

We need to prove that the complex $\mathbf{R}\Gamma(X, \Omega_X^*)$ defined by the derived global section functor Γ is a HC. In simple terms, the global section of the filtered resolution $(\mathcal{E}_X, F)$ defined by Dolbeault resolutions of $F^p\Omega_X^*$ is a HC. This means exactly that the induced Hodge filtration F on $H^n(X, \mathbb{C})$ defines a HS, hence the result is just a reformulation of classical Hodge theory.

7.1.3. Let $\mathcal{L}$ be a rational local system with a polarization, rationally defined, on the associated local system $\mathcal{L}_{\mathbb{C}} = \mathcal{L} \otimes_{\mathbb{Q}} \mathbb{C}$, then the spectral sequence defined by the Hodge filtration on DeRham complex with coefficients $\Omega_X^* \otimes_{\mathbb{C}} \mathcal{L}$ degenerates at rank 1

$$E_1^{pq} = H^q(X, \Omega_X^p(\mathcal{L})) \Rightarrow H^{p+q}(X, \mathcal{L}_{\mathbb{C}})$$

and the induced filtration by F on cohomology defines a HS. The proof is similar to the constant case.

7.1.4. A CHC is defined by the Hodge filtration on DeRham complex for any complex algebraic manifold X. Using the existence of a projective manifold above X and birational to X (Chow's lemma), and resolution of singularities, Deligne deduces a Hodge decomposition in this way on the cohomology of any compact complex algebraic manifold not necessarily Kähler. The HS defined in this way are functorial for morphisms of algebraic varieties.

7.2. An A- mixed Hodge complex (MHC) K consists of

(1) a complex $K_A \in ObD^+(A)$ s.t. $H^k(K_A)$ is an A-module of finite type for all k,
(2) a filtered complex $(K_{A\otimes\mathbb{Q}}, W) \in ObD^+F(A \otimes \mathbb{Q})$ with an increasing filtration W,
(3) an isomorphism $K_{\mathbb{Q}} \xrightarrow{\sim} K_A \otimes \mathbb{Q}$ in $D^+(A \otimes \mathbb{Q})$,
(4) a bifiltered complex $(K_{\mathbb{C}}, W, F) \in ObD^+F_2(\mathbb{C})$ with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_{\mathbb{C}}, W) \xrightarrow{\sim} (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C}$ in $D^+F(\mathbb{C})$.

Moreover, the following axiom is satisfied
(MHC) For all n, the system consisting of the complex $Gr_n^W(K_{A\otimes\mathbb{Q}}) \in ObD^+(A \otimes \mathbb{Q})$, the complex $(Gr_n^W(K_{\mathbb{C}}, F) \in ObD^+F(\mathbb{C})$ with induced F and the isomorphism $Gr_n^W(\alpha) : Gr_n^W(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} Gr_n^W(K_{\mathbb{C}})$, is an $A \otimes \mathbb{Q} - HC$ of weight n.

7.2.1. An A-cohomological mixed Hodge complexe K (CMHC) on a topological space X consists of

1. a complex $K_A \in ObD^+(X, A)$ s.t. $H^k(X, K_A)$ are A-modules of finite type, and $\mathbb{H}^*(X, K_A) \otimes \mathbb{Q} \xrightarrow{\sim} \mathbb{H}^*(X, K_A \otimes \mathbb{Q})$,
2. a filtered complex $(K_{A\otimes\mathbb{Q}}, W) \in ObD^+F(X, A \otimes \mathbb{Q})$ with an increasing filtration W and an isomorphism $K_{A\otimes\mathbb{Q}} \simeq K_A \otimes \mathbb{Q}$ in $D^+(X, A \otimes \mathbb{Q})$,
3. a bifilitered complex $(K_\mathbb{C}, W, F) \in ObD^+F_2(X, \mathbb{C})$ with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_\mathbb{C}, W) \xrightarrow{\sim} (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C}$ in $D^+F(X, \mathbb{C})$.

Moreover, the following axiom is satisfied
(CMHC) For all n, the system consisting of the complex $Gr^W_n(K_{A\otimes\mathbb{Q}}) \in ObD^+(X, A\otimes\mathbb{Q})$, the complex $(Gr^W_n(K_\mathbb{C}, F) \in ObD^+F(X, \mathbb{C})$ with induced F and the isomorphism $Gr^W_n(\alpha) : Gr^W_n(K_{A\otimes\mathbb{Q}}) \otimes \mathbb{C} \xrightarrow{\sim} Gr^W_n(K_\mathbb{C})$, is an $A \otimes \mathbb{Q} - CHC$ of weight n.

Proposition 7.3. If $K = (K_A, K_{A\otimes\mathbb{Q}}, W), (K_\mathbb{C}, W, F))$ is an $A-$CMHC then $\mathbf{R}\Gamma K = (\mathbf{R}\Gamma K_A, \mathbf{R}\Gamma(K_{A\otimes\mathbb{Q}}, W), \mathbf{R}\Gamma(K_\mathbb{C}, W, F))$ is an A-MHC.

Remark 7.4. If (K, W, F) is a MHC (resp. CMHC), then for all m and $n \in \mathbb{Z}, (K[m], W[m-2n], F[n])$ is a MHC (resp. CMHC).

7.2.2. *MHS on the cohomology of a MHC.*

Theorem 7.5 (Deligne)**.** *Let K be an $A-MHC$.*
(i)The filtration $W[n]$ of $H^n(K_{A\otimes\mathbb{Q}}) \simeq H^n(K_A) \otimes \mathbb{Q}$ and the filtration F on $H^n(K_\mathbb{C}) \simeq H^n(K_A) \otimes_A \mathbb{C}$ define on $H^n(K)$ an $A-MHS$ i.e.
$(H^n(K_A), (H^n(K_{A\otimes\mathbb{Q}}), W), (H^n(K_\mathbb{C}), W, F)$ is an A-MHS.
(ii)On the terms ${}_WE^{pq}_r$ of the spectral sequence of $(K_\mathbb{C}, W)$, the recurrent filtration and the two direct filtrations defined by F coincide $F_d = F_{rec} = F_{d^*}$.
(iii) The morphisms $d_1 : {}_WE^{p,q}_1 \to {}_WE^{p+1,q}_1$ are strictly compatible with the filtration F.
(iv) The spectral sequence of $(K_{A\otimes\mathbb{Q}},\ W)$ degenerates at E_2 (${}_WE_2 = {}_WE_\infty$).
(v) The spectral sequence of $(K_\mathbb{C}, F)$ degenerates at $E_1({}_FE_1 = {}_FE_\infty)$.
(vi) The spectral sequence of the complex $Gr^p_F(K_\mathbb{C})$, with the induced filtration W, degenerates at E_2.

Proof. The two direct filtration and the recurrent filtration F_{rec} defined by F coincide on ${}_WE^{pq}_1 = H^{p+q}(Gr^W_{-p}K)$. The axiom on (MHC) shows that the complex $Gr^W_{-p}K$ is a HC of weight $-p$, and the axiom (HC2)) shows that the filtration F_{rec}, induced by F on $Gr^W_{-p}K$, is q-opposed on ${}_WE^{pq}_1$ to its complex conjugate $\overline{F}_{rec}$.
The differential d_1 is compatible with the direct filtrations, hence with F_{rec} and commutes with complex conjugation since it is defined on $A \otimes \mathbb{Q}$, hence it is compatible with $\overline{F}_{rec}$. Then it is strictly compatible with Hodge filtration $F = F_{rec}$, which proves (iii).
We deduce that the two direct filtrations and the recurrent filtration induced by F on ${}_WE^{pq}_2$ coincide. The filtration F_{rec} defined in this way is q-opposed to its complex conjugate and defines a HS of weight q on ${}_WE^{pq}_2$.
The proof of the following lemma is by induction on r

Lemma 7.6. *For $r \geq 0$, the differentials d_r of the spectral sequence ${}_WE$ are strictly compatibles to the recurrent filtration $F = F_{rec}$. For $r \geq 2$, they vanish.*

For $r = 0$ we apply (HC1). We just proved it for $r = 1$ and for $r \geq 2$, it is enough to prove $d_r = 0$. We suppose by induction that the two direct filtrations and the recurrent filtration coincide on ${}_WE_s(s \geq r+1) : F_d = F_{rec} = F_{d^*}$ and ${}_WE_r = \ {}_WE_2$. Then d_r is compatible with the filtration $F_{rec} := F$. On ${}_WE_2^{p,q} = {}_WE_r^{p,q}$, la filtration F is q-opposed to its complex conjugate. Hence the morphism $d_r : {}_WE_r^{p,q} \to {}_WE_r^{p+r,q-r+1}$ satisfy, for $r-1 > 0$

$$d_r({}_WE_r^{p,q}) = d_r(\sum_{a+b=q} F^a({}_WE_r^{p,q}) \cap \overline{F}^b({}_WE_r^{p,q})) \subset$$
$$\sum_{a+b=q} F^a({}_WE_r^{p+r,q-r+1}) \cap \overline{F}^b({}_WE_r^{p+r,q-r+1})) = 0$$

which proves (ii) et (iv).
After the results on two filtrations , the filtration on ${}_WE_\infty^{p,q}$ induced by the filtration F on $H^{p+q}(K)$ coincide with the filtration F_{rec} on ${}_WE_2^{p,q}$. Hence, it is q-opposite to its complex conjugate, which proves (i). $\square$

7.3. MHS of a normal crossing divisor (NCD). Let Y be a normal crossing divisor in a proper complex smooth algebraic variety. We suppose the irreducible components $(Y_i)_{i\in I}$ of Y smooth and ordered.

7.3.1. *Mayer-Vietoris resolution.* Let S_q denotes the set of strictly increasing sequences $\sigma = (\sigma_0, ..., \sigma_q)$ on the ordered set of indices I, $Y_\sigma = Y_{\sigma_0} \cap ... \cap Y_{\sigma_q}, Y_{\underline{q}} = \sum_{\sigma \in S_q} Y_\sigma$ and for all $j \in [0,q], q \geq 1$ let $\lambda_{j,\underline{q}} : Y_{\underline{q}} \to Y_{\underline{q-1}}$ denotes a map inducing for each σ the embedding $\lambda_{j,\sigma} : Y_\sigma \to Y_{\sigma(\widehat{j})}$ where $\sigma(\widehat{j}) = (\sigma_0, ..., \widehat{\sigma_j}, ..., \sigma_q)$ is obtained by deleting σ_j. Let $\Pi_q : Y_{\underline{q}} \to Y$ (or simply Π) denotes the canonical projection and $\lambda^*_{j,\underline{q}} : \Pi_*\mathbf{Z}_{Y_{\underline{q-1}}} \to \Pi_*\mathbf{Z}_{Y_{\underline{q}}}$ the morphism defined by $\lambda_{j,\underline{q}}$ for $j \in [0,q]$.

Definition 7.7 (Mayer-Vietoris resolution of $\mathbb{Z}_Y$)**.** It is defined by the following complex of sheaves $\Pi_*\mathbb{Z}_{Y_{\underline{\cdot}}}$

$$0 \to \Pi_*\mathbb{Z}_{Y_{\underline{0}}} \to \Pi_*\mathbb{Z}_{Y_{\underline{1}}} \to \cdots \to \Pi_*\mathbb{Z}_{Y_{\underline{q-1}}} \stackrel{\delta_{q-1}}{\to} \Pi_*\mathbb{Z}_{Y_{\underline{q}}} \to \cdots$$

where $\delta_{q-1} = \sum_{j\in[0,q]}(-1)^j\lambda^*_{j,\underline{q}}$.

This resolution is associated to a resolution of Y by topological spaces in the following sense. Consider the diagram of spaces over Y

$$Y_{\underline{\cdot}} = \left(\begin{array}{ccccccc} & & & & & \vdots & \\ & & & \leftarrow & & & \\ Y_{\underline{0}} & \begin{smallmatrix}\leftarrow\\ \leftarrow\end{smallmatrix} & Y_{\underline{1}} & \leftarrow & \cdots \quad Y_{\underline{q-1}} & \xleftarrow{\lambda_{j,q}} & Y_{\underline{q}} \cdots \\ & & & \leftarrow & & & \\ & & & & & \vdots & \end{array} \right) \xrightarrow{\Pi} Y$$

This diagram is the strict simplicial scheme associated in [7] to the normal crossing divisor Y, called here after Mayer-Vietoris. The Mayer-Vietoris complex is canonically associated as direct image by Π of the sheaf $\mathbb{Z}_{Y_{\underline{\cdot}}}$ equal to $\mathbb{Z}_{Y_{\underline{i}}}$ on $Y_{\underline{i}}$.

7.3.2. *The cohomological mixed Hodge complex of a NCD.* The weight filtration W on $\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}$ is defined by

$$W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}) = \sigma_{\cdot \geq q}\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}} = \Pi_*\sigma_{\cdot \geq q}\mathbb{Q}_{Y_{\underline{\cdot}}}, \quad Gr^W_{-q}(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}) \simeq \Pi_*\mathbb{Q}_{Y_{\underline{q}}}[-q]$$

We introduce the complexes $\Omega^*_{Y_{\underline{i}}}$ of differential forms on $Y_{\underline{i}}$. The simple complex $s(\Omega^*_{Y_{\underline{\cdot}}})$ is associated to the double complex$\Pi_*\Omega^*_{Y_{\underline{\cdot}}}$ with the exterior differential d of forms and the differential δ. defined by $\delta_{q-1} = \sum_{j\in[0,q]}(-1)^j\lambda^*_{j,\underline{q}}$ on $\Pi_*\Omega^*_{Y_{\underline{q-1}}}$. The weight W, and Hodge F filtrations are defined as

$$W_{-q} = s(\sigma_{\cdot\geq q}\Omega^*_{Y_{\underline{\cdot}}}) = s(0 \to \cdots 0 \to \Pi_*\Omega^*_{Y_{\underline{q}}} \to \Pi_*\Omega^*_{Y_{\underline{q+1}}} \to \cdots)$$

$$F^p = s(\sigma_{*\geq p}\Omega^*_{Y_{\underline{\cdot}}}) = s(0 \to \cdots 0 \to \Pi_*\Omega^p_{Y_{\underline{\cdot}}} \to \Pi_*\Omega^{p+1}_{Y_{\underline{\cdot}}} \to \cdots)$$

We have a filtered isomorphism in $D^+F(Y,\mathbb{C})$

$$(Gr^W_{-q}s(\Omega^*_{Y_{\underline{\cdot}}}), F) \simeq (\Pi_*\Omega^*_{Y_{\underline{q}}}[-q], F) \quad \text{in} \quad D^+F(Y,\mathbb{C}).$$

inducing isomorphisms in $D^+(Y,\mathbb{C})$

$$(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W) \otimes \mathbb{C} = (\mathbb{C}_{Y_{\underline{\cdot}}}, W) \xrightarrow{\alpha\,\sim} (s(\Omega^*_{Y_{\underline{\cdot}}}), W)$$

$$Gr^W_{-q}(\Pi_*\mathbb{C}_{Y_{\underline{\cdot}}}) \simeq \Pi_*\mathbb{C}_{Y_{\underline{q}}}[-q] \xrightarrow{\sim} \Pi_*\Omega^*_{Y_{\underline{q}}}[-q] \simeq Gr^W_{-q}s(\Omega^*_{Y_{\underline{\cdot}}})$$

Let $\mathbf{K}$ be the system consisting of

$$(\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W), \mathbb{Q}_Y \simeq \Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, (s(\Omega^*_{Y_{\underline{\cdot}}}), W, F), \quad (\Pi_*\mathbb{Q}_{Y_{\underline{\cdot}}}, W) \otimes \mathbb{C} \simeq (s(\Omega^*_{Y_{\underline{\cdot}}}), W)$$

Proposition 7.8. The system $\mathbf{K}$ associated to a normal crossing divisor Y with smooth proper irreducible components, is a CMHC on Y. It defines a functorial MHS on the cohomology $H^i(Y,\mathbb{Q})$, with weights varying between 0 and i.

In terms of Dolbeault resolutions : $(s(\mathcal{C})^{*,*}_{Y_{\underline{\cdot}}}, W, F)$, the statement means that the complex of global sections $\Gamma(Y, s(\mathcal{C})^{*,*}_{Y_{\underline{\cdot}}}, W, F) := (\mathbb{R}\Gamma(Y,\mathbb{C}), W, F)$ is a MHC in the following sense

$$\begin{aligned} Gr^W_{-i}(\mathbb{R}\Gamma(Y,\mathbb{C}) &:= (\Gamma(Y, W_{-i}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}})/\Gamma(Y, W_{-i-1}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}}), F) \\ &\simeq (\Gamma(Y, Gr^W_{-i}s(\mathcal{C}^{*,*}_{Y_{\underline{\cdot}}}), F) \simeq (\mathbb{R}\Gamma(Y_i, \Omega^*_{Y_{\underline{i}}}[-i]), F) \end{aligned}$$

is a HC of weight $-i$ in the sense that

$$(H^n(Gr^W_{-i}\mathbb{R}\Gamma(Y,\mathbb{C})), F) \simeq (H^{n-i}(Y_{\underline{i}},\mathbb{C}), F)$$

is a HS of weight $n-i$.
The terms of the spectral sequence $E_1(K,W)$ of (K,W) are written as

$$_WE_1^{pq} = \mathbb{H}^{p+q}(Y, Gr^W_{-p}(s\Omega^*_{Y_{\underline{\cdot}}})) \simeq \mathbb{H}^{p+q}(Y, \Pi_*\Omega_{Y_{\underline{p}}}[-p]) \simeq H^q(Y_{\underline{p}},\mathbb{C})$$

They carry the HS of the space $Y_{\underline{p}}$. The differential is a combinatorial restriction map
$d_1 = \sum_{j\leq p+1}(-1)^j\lambda^*_{j,p+1} : H^q(Y_{\underline{p}},\mathbb{C}) \to H^q(Y_{\underline{p+1}},\mathbb{C})$
is a morphism of HS. The spectral sequence degenerates at E_2 ($E_2 = E_\infty$).

Corollary 7.9. The HS on $Gr^W_q H^{p+q}(Y,\mathbb{C})$ is the cohomology of the complex of HS defined by $(H^q(Y_{\underline{p}},\mathbb{C}), d_1)$:

$$(Gr^W_q H^{p+q}(Y,\mathbb{C}), F) \simeq ((H^p(H^q(Y_{\underline{p}},\mathbb{C}), d_1), F)$$

7.4. Logarithmic complex. Now we construct the MHS on the cohomology of a smooth complex algebraic variety V. By a result of Nagata, V can be embedded as an open Zariski subset of a complete variety X. After Hironaka's desingularisation theorem in characteristic zero, we can suppose X smooth and $Y = X - V$ a NCD with smooth components. Hence we reduce the construction of the MHS to the complement $X - Y$ of a NCD. we show that this construction is independent of the choice of X and Y s.t. $V = X - Y$.
With the previous notations this case is the Poincaré's dual of the cohomology with compact support on $X - Y$ which can be constructed by the double DeRham complex $\Omega^*_X \to \Omega^*_{Y_{\cdot}}$.
However we introduce here the logarithmic complex, important for its own properties as well to obtain functorial constructions later. Let X be a complex manifold and Y be a NCD in X. By definition, at each point $y \in Y$, there exist local coordinates $(z_i)_{i\in[1,n]}$ of X s.t. Y is defined at y by the equation $\Pi_{i\in I\subset[1,n]} z_i = 0$. Let $X^* = X - Y$ and $j : X^* \to X$ denotes the embedding of X^* into X.

Definition 7.10. The logarithmic DeRahm complex of X along Y is a sub-complex $\Omega^*_X(LogY)$ of the complex $\Omega_X(*Y) \subset j_*\Omega^*_{X^*}$ of meromorphic forms along Y, holomorphic on X^*. A section of $\omega \in \Omega^p_X(logY)$ is called a differential form with logarithmic pole along Y and may be written locally as

$$\omega = \Sigma_{i_1,\cdots,i_r}\omega_{i_1,\cdots,i_r}\frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_r}}{z_{i_r}}; \ \ \omega_{i_1,\cdots,i_r} \text{ holomorphic}$$

The definition is independent of the choice of coordinates; that is ω is written in this form with respect to any set of local coordinates at y. The $\mathcal{O}_X$-module $\Omega^1_X(LogY)$ is locally free with basis $(dz_i/z_i)_{i\in I}$ and $(dz_j)_{j\notin I}$ and $\Omega^p_X(LogY) = \wedge^p\Omega^1_X(LogY)$.
An intrinsic property of the sections ω of the logarithmic complex that can be taken as a definition, is that ω and $d\omega$ have both a pole of order at most 1 at y.
In particular, given a local equation $\prod_{i=1}^k z_i = 0$ of Y at y, a meromorphic section of $j_*\mathcal{O}^*_{X^*}$ along Y is written locally as $f = g\prod_{i=1}^k z_i^{k_i}$ with g invertible, then

$$df/f = dg/g + \sum_1^k k_i dz_i/z_i$$

is the sum of a regular form and a linear combination of the independent vectors dz_i/z_i.
Let $f : X_1 \to X_2$ be a morphism of complex manifolds, with normal crossing divisors Y_i in X_i for $i = 1, 2$, s.t. $f^{-1}(Y_2) = Y_1$. Then, the reciprocal morphism $f^* : f^*(j_{2*}\Omega^*_{X_2^*}) \to j_{1*}\Omega^*_{X_1^*}$ induces a morphism on logarithmic complexes

$$f^* : f^*\Omega^*_{X_2}(LogY_2) \to \Omega^*_{X_1}(LogY_1).$$

7.4.1. *Weight filtration* W. Let $Y = \cup_{i\in I} Y_i$ be the union of smooth irreducible divisors. Let S^q denotes the set of strictly increasing sequences $\sigma = (\sigma_1, ..., \sigma_q)$ in the set of indices I, $Y_\sigma = Y_{\sigma_1...\sigma_q} = Y_{\sigma_1} \cap ... \cap Y_{\sigma_q}, Y^q = \coprod_{\sigma\in S^q} Y_\sigma$ the disjoint union of Y_σ. Set $Y^0 = X$, $\Pi : Y^q \to Y$ the canonical projection. An increasing filtration W, called the weight, is defined as follows

$$W_m(\Omega^p_X(LogY)) = \sum_{\sigma\in S^m} \Omega^{p-m}_X \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

The sub-$\mathcal{O}_X$-module $W_m(\Omega^p_X(LogY)) \subset \Omega^p_X(LogY)$ is the smallest sub-module stable by exterior multiplication with local sections of Ω^*_X and containing the products $df_1/f_1 \wedge ... \wedge df_k/f_k$ for $k \leq m$ for local sections f_i of $j_*\mathcal{O}^*_{X^*}$ meromorphic along Y.

7.4.2. *The Residue isomorphism.* The Poincaré residue defines an isomorphism

$$Res : Gr^W_m(\Omega^p_X(LogY)) \to \Pi_*\Omega^p_{Y^m}[-m] : Res(\alpha\wedge(dz_{\sigma_1}/z_{\sigma_1}\wedge ...\wedge(dz_{\sigma_m}/z_{\sigma_m})) = \alpha/Y_\sigma$$

In the case $p = 1$, it defines

$$Res : \Omega^1_X(LogY) \to \Pi_*\mathcal{O}_{Y^1}$$

We construct its inverse. Consider for $\sigma \in S^m$ the morphism
$\rho_\sigma : \Omega^p_X \to Gr^W_m(\Omega^{p+m}_X(LogY))$ defined locally as

$$\rho_\sigma(\alpha) = \alpha \wedge dz_{\sigma_1}/z_{\sigma_1} \wedge ... \wedge dz_{\sigma_m}/z_{\sigma_m}$$

It does not depends on the choice of z_i since for another choice of coordinates z'_i, z_i/z'_i are holomorphic and the difference $(dz_i/z_i) - (dz'_i/z'_i) = d(z_i/z'_i)/(z_i/z'_i)$ is holomorphic too.
Then $\rho_\sigma(\alpha) - \alpha \wedge dz'_{\sigma_1}/z'_{\sigma_1} \wedge ... \wedge dz_{\sigma m}/z_{\sigma_m} \in W_{m-1}\Omega^{p+m}_X(LogY)$, and successively $\rho_\sigma(\alpha) - \rho'_\sigma(\alpha) \in W_{m-1}\Omega^{p+m}_X(LogY)$. We have $\rho_\sigma(z_{\sigma_i} \cdot \beta) = 0$ and $\rho_\sigma(dz_{\sigma_i} \wedge \beta') = 0$ for sections β of Ω^p_X and β' of Ω^{p-1}_X; hence ρ_σ factors by $\overline{\rho}_\sigma : \Pi_*\Omega^p_{Y_\sigma} \to Gr^W_m(\Omega^{p+m}_X(LogY))$ defined locally and glue globally into a morphism of complexes on X

$$\rho : \Pi_*\Omega^*_{Y^m}[-m] \to Gr^W_m(\Omega^*_X(LogY)).$$

Lemma 7.11. *We have*
i) $\underline{H}^i(Gr^W_m\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^m}$ *for* $i = m$ *and vanishes for* $i \neq m$.
ii) $\underline{H}^i(\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^i}$.

Proof. (ii) We deduce from the long exact sequence associated to the short exact sequence

$$0 \to W_r \to W_{r+1} \to Gr^W_{r+1} \to 0$$

by induction on r, the statement:
$\underline{H}^i(W_r\Omega^*_X(LogY)) \simeq \Pi_*\mathbf{C}_{Y^i}$ for $i \leq r$ and vanishes for $i > r$. □

7.4.3. *Hodge filtration F.* It is defined on $\Omega^*_X(LogY)$ by the formula $F^p = \sigma_{\geq p}$ including all forms of type (p', q') with $p' \geq p$. We have

$$Res : F^p(Gr^W_m\Omega^*_X(LogY)) \simeq \Pi_*F^{p-m}\Omega^*_{Y^m}[-m]$$

hence a filtered isomorphism

$$Res : (Gr^W_m\Omega^*_X(LogY), F) \simeq (\Pi_*\Omega^*_{Y^m}[-m], F[-m]).$$

Proposition 7.12 (Weight filtration W)**.** The morphisms of filtered complexes

$$(\Omega^*_X(LogY), W) \xleftarrow{\alpha} (\Omega^*_X(LogY), \tau) \xrightarrow{\beta} (j_*\Omega^*_{X^*}, \tau)$$

are filtered quasi-isomorphisms. They define an isomorphism of the spectral sequence of hypercohomology of the filtered complex $(\Omega^*_X(LogY), W)$ on X with Leray spectral sequence for j with complex coefficients. The main point here is that the τ filtration is defined with rational coefficients as $(\mathbf{R}j_*\mathbb{Q}_{X^*}, \tau) \otimes \mathbb{C}$.

7.4.4. *MHS on the hypercohomology of $\Omega^*_X(LogY)$.* Let $j : X^* = X - Y \to X$ denotes the open embedding. The system $\mathbf{K}$

(1) $(\mathbf{K}^{\mathbb{Q}}, W) = (\mathbf{R}j_*\mathbb{Q}_{X^*}, \tau) \in ObD^+F(X, \mathbb{Q})$
(2) $(\mathbf{K}^{\mathbb{C}}, W, F) = (\Omega^*_X(LogY), W, F) \in ObD^+F_2(X, \mathbb{C})$
(3) The isomorphism $(\mathbf{K}^{\mathbb{Q}}, W) \otimes \mathbb{C} \simeq (\mathbf{K}^{\mathbb{C}}, W)$ in $D^+F(X, \mathbb{C})$

form a CMHC on X.

Theorem 7.13 (6)**.** *The system $K = \mathbf{R}\Gamma(X, \mathbf{K})$ is a MHC. It endows the cohomology of V with a canonical MHS.*

Proof. The weight spectral sequence at rank 1 is written as

$$\begin{aligned} &{}_WE_1^{pq}(\mathbf{R}\Gamma(X, \Omega^*_X(LogY)) = \mathbb{H}^{p+q}(X, Gr^W_{-p}\Omega^*_X(LogY)) \simeq \mathbb{H}^{p+q}(X, \Pi_*\Omega^*_{Y^{-p}}[p]) \\ &\simeq H^{2p+q}(Y^{-p}, \mathbb{C}) \Rightarrow Gr^W_q H^{p+q}(V, \mathbb{C}). \end{aligned}$$

where the double arrow means that the spectral sequence degenerates to the cohomology graded with respect to W induced by the weight on the complex level. In fact we show that it degenerates at rank 2. The differential d_1

$$d_1 = \sum_{j=1}^{-p} (-1)^{j+1} G(\lambda_{j,-p}) = G : H^{2p+q}(Y^{-p}, \mathbb{C}) \longrightarrow H^{2p+q+2}(Y^{-p-1}, \mathbb{C})$$

is equal to an alternate Gysin morphism, Poincaré's dual to the alternate restriction morphism

$$\rho = \sum_{j=1}^{-p} (-1)^{j+1} \lambda^*_{j,-p} : H^{2n-q}(Y^{-p-1}, \mathbb{C}) \to H^{2n-q}(Y^{-p}, \mathbb{C})$$

hence the first term

$$({}_WE_1^{pq}, d_1)_{p\in\mathbb{Z}} = (H^{2p+q}(Y^{-p}, \mathbb{C}), d_1)_{p\in\mathbb{Z}}$$

is viewed as a complex in the category of HS of weight q. It follows that the terms ${}_WE_2^{pq} = H^p({}_WE_1^{*,q}, d_1)$ are endowed with a HS of weight q. The differential d_2 which is compatible with the induced HS, being a morphism from E_2^{pq} a HS of weight q to $E_2^{p+2,q-1}$ a HS of weight $q-1$, is necessarily zero. The proof consists of a recurrent argument to show in this way that the differentials d_i for $i \geq 2$ are zero. □

Exercise 7.14 (Riemann Surface)**.** Let $\overline{C}$ be a connected compact Riemann surface, $Y = \{x_1, \ldots, x_m\}$ a subset of m points, and $C = \overline{C} - Y$ the open surface with m points in $\overline{C}$ deleted. Consider the long exact sequence

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to H^2_Y(\overline{C}, \mathbb{Z}) = \oplus_{i=1}^{i=m}\mathbb{Z} \to H^2(\overline{C}, \mathbb{Z}) \simeq \mathbb{Z} \to H^2(C, \mathbb{Z}) = 0$$

then

$$0 \to H^1(\overline{C}, \mathbb{Z}) \to H^1(C, \mathbb{Z}) \to \mathbb{Z}^{m-1} \simeq Ker(\oplus_{i=1}^{i=m}\mathbb{Z} \to \mathbb{Z})$$

represents $H^1(C, \mathbb{Z})$ as an extension, with weight $W_1H^1(C, \mathbb{Z}) = H^1(\overline{C}, \mathbb{Z})$ and $W_2H^1(C, \mathbb{Z}) = H^1(C, \mathbb{Z})$.
The Hodge filtration is defined by the residue

$$0 \to \Omega^1_{\overline{C}} \to \Omega^1_{\overline{C}}(log\{x_1, \ldots, x_m\}) \to \mathcal{O}_{\{x_1,\ldots,x_m\}} \to 0$$

$F^0H^1(C,\mathbb{C}) = H^1(C,\mathbb{C})$,
$F^1H^1(C,\mathbb{C}) = H^0(\overline{C},\Omega^1_{\overline{C}}(log\{x_1,\dots,x_m\})$.
$F^2H^1(C,\mathbb{C}) = 0$. That is an extension of two different weights

$$0 \to H^1(\overline{C}) \to H^1(C) \to \mathrm{Ker}\,(\oplus_{i=1}^{i=m}\mathbb{Z}(-1) \to H^2(\overline{C})) \to 0$$

.

Exercise 7.15 (Hypersurfaces)**.** Let $Y \subset P$ be a smooth hypersurface in a projective space P. To describe Hodge theory on $U = P - Y$ we may use by a result of Grothendieck on algebraic DeRham cohomology, forms on P meromorphic along Y and holomorphic on U, denoted as a sheaf by $\Omega_P(*Y) \subset j_*\Omega_U$ where $j : U \to P$, or the logarithmic complex by Deligne's result.
1) In the case of a curve Y in a plane, holomorphic one forms are residues of rational $2-$forms on P with simple pole along the curve.
For example, if the homogeneous equation is $F = 0$, $\Omega^2_P(LogY) = \Omega^2_P(Y)$ and we have an exact sequence

$$0 \to \Omega^2_P \to \Omega^2_P(Y) \to \Omega^1_Y \to 0$$

Since $h^{2,0} = h^{2,1} = 0$, $H^0(P,\Omega^2_P) = H^1(P,\Omega^2_P) = 0$, so we deduce the isomorphism

$$H^0(P,\Omega^2_P(Y)) \xrightarrow{Res} H^0(Y,\Omega^1_Y)$$

In homogeneous coordinates, we take the residue along Y of the rational form

$$\frac{A(z_0dz_1 \wedge dz_2 - z_1dz_0 \wedge dz_2 + z_2dz_0 \wedge dz_1)}{F}$$

where A is homogeneous of degree $d-3$ if F has degree d.
2) In general, we consider the exact sequence for relative cohomology (or cohomology with support)

$$H^{k-1}(U) \xrightarrow{\partial} H^k_Y(P) \to H^k(P) \xrightarrow{j^*} H^k(U)$$

which reduces via Thom's isomorphism, to

$$H^{k-1}(U) \xrightarrow{r} H^{k-2}(Y) \xrightarrow{i_*} H^k(P) \xrightarrow{j^*} H^k(U)$$

where r is the topological Leray's residue map dual to the tube over cycle map $\tau : H_{k-2}(Y) \to H_{k-1}(U)$ associating to a cycle c the boundary in U of a tube over c, and i_* is Gysin map, Poincaré dual to the map i^* in cohomology.
For $P = \mathbb{P}^{n+1}$ and n odd, the map r is an isomorphism

$$H^{n-1}(Y) \simeq H^{n+1}(P) \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = 0 \xrightarrow{j^*} H^{n+2}(U)$$

and for n even the map r is injective

$$H^{n+1}(P) = 0 \to H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) = \mathbb{Q} \xrightarrow{j^*} H^{n+2}(U)$$

then

$$r : H^{n+1}(U) \xrightarrow{\sim} H^n_{\mathrm{prim}}(X)$$

If Y is ample in P such $U = P - Y$ is affine

$$H^{n-1}(Y) \xrightarrow{i_*} H^{n+1}(P) \xrightarrow{j^*} H^{n+1}(U) \xrightarrow{r} H^n(Y) \xrightarrow{i_*} H^{n+2}(P) \to H^{n+2}(U) = 0$$

then we define the variable cohomology

$$H^n_{var}(Y) = Ker\,(\,H^n(Y) \xrightarrow{i_*} H^{n+2}(P))$$

equal also to $= Im\ r$. The fixed cohomology is

$$H^n_{fix}(Y) = i_* H^n(P)$$

In particular, if $H^{n+1}_{\mathrm{prim}}(P) = 0$

$$H^{n+1}(U, \mathbb{Q}) \xrightarrow{r\sim} H^n_{var}(Y, \mathbb{Q})$$

Moreover

$$H^n(Y, \mathbb{Q}) = H^n_{var}(Y, \mathbb{Q}) \oplus H^n_{fix}(Y, \mathbb{Q})$$

8. MHS on the cohomology of a complex algebraic variety

The aim of this section is to prove

Theorem 8.1 (6). *The cohomology of a complex algebraic variety carry a natural MHS.*

8.1. Simplicial varieties. To construct a natural MHS on the cohomology of any algebraic variety X, not necessarily smooth or compact, Deligne consider a semi-simplicial variety which is a cohomological resolution in some sense of the original variety (descent theorem). On each term of the simplicial resolution, which consists of the complement of a NCD in a smooth compact complex variety, the various logarithmic complexes are connected by functorial relations and referred to as a simplicial CMHC giving rise to the CMHC defining the MHS we are looking for on the cohomology of X. Although such construction is technically elaborate, the above abstract development of MHC structure leads easily to the result without further difficulty.

8.1.1. *Cohomology of topological simplicial spaces.* In what follows we could restrict the constructions to semi-simplicial spaces which underly the simplicial spaces and work only with semi-simplicial spaces since we use only the face maps.
Recall the definition of the simplicial category Δ whose objects are $\Delta_n, n \in \mathbf{N}$, the set of integers $\{0, 1, \ldots, n\}$ and morphisms $H_{p,q}$, the set of increasing (large sense) mappings from Δ_p to Δ_q for integers $p, q \geq 0$, with the natural composition of mappings : $H_{pq} \times H_{qr} \to H_{pr}$. The semi-simplicial category $\Delta_>$ is obtained when we consider only the strictly increasing morphisms in Δ.
A simplicial (resp. co-simplicial) object of a category $\mathcal{C}$ is a contravariant (resp. covariant) functor from Δ to $\mathcal{C}$.

8.1.2. *Sheaves.* A sheaf $F^\cdot$ on a simplicial space $X.$ is defined by
1) A family of sheaves F^n on X_n,
2) for each $f : \Delta_n \to \Delta_m$ with $X.(f) : X_m \to X_n$, an $X.(f)-$morphism $F.(f)$ from F^n to F^m, that is $X.(f)^* F^n \to F^m$ on X_m satisfying for all $g : \Delta_r \to \Delta_n$, $F.(f \circ g) = F.(f) \circ F.(g)$.
A morphism $u : F^\cdot \to G^\cdot$ is a family of morphisms $u^n : F^n \to G^n$ s.t. for all $f :$ $\Delta_n \to \Delta_m$, $u^m F^\cdot(f) = G^\cdot(f) u^n$ where the left term is: $X.(f)^* F^n \to F^m \xrightarrow{u^m} G^m$ and the right term is $X.(f)^* F^n \xrightarrow{X.(f)^*(u_n)} X.(f)^* G^n \to G^m$.
We may consider the derived category of cosimplicial $A-$modules.

8.1.3. A topological space S defines a simplicial constant space $S.$ s.t. $S_n = S$ for all n and $S.(f) = Id$ for all f. An augmented simplicial space $\pi : X. \to S$ is defined by a morphism of simplicial spaces $\pi^\cdot : (X.) \to (S.)$.

8.1.4. The structural sheaves $\mathcal{O}_{X_n}$ of a simplicial analytic space form a simplicial sheaf of rings. Let $\pi : X. \to S$ be an augmentation to an analytic space S, the DeRham complex of sheaves $\Omega^*_{X_n/S}$ for various n form a complex of sheaves on X. denoted $\Omega^*_{X./S}$.
A simplicial sheaf $F^\cdot$ on a constant simplicial space $S.$ defined by S is a a co-simplicial sheaf on S; hence if $F^\cdot$ is abelian, it defines a complex with
$d = \sum_i (-1)^i \delta_i : F^n \to F^{n+1}$.
A complex of abelian sheaves K on $S.$, denoted by $K^{n,m}$ with m the cosimplicial degree, defines a simple complex sK:

$$(sK)^n := \oplus_{p+q=n} K^{pq}; \quad d(x^{pq} = d_K(x^{pq}) + (-1)^i \delta_i x^{pq}.$$

The filtration L with respect to the second degree will be useful

$$L^r(sK) = s(K^{pq})_{q \geq r}.$$

8.1.5. *Direct image on derived category of abelian sheaves.* Consider an augmented simplicial space $a : X. \to S$, we define a functor denoted $\mathbf{R}a_*$ for complexes K of abelian sheaves on $(X.)$. We may view S as a constant simplicial scheme $S.$ and a as a morphism $a^\cdot : (X.) \to (S.)$. In the first step we construct an injective or flabby complex I quasi-isomorphic to K ($K \simeq I$); we can always take Godement resolutions for example, then in each degree p, $a^q_* I^p$ on $S_q = S$ defines for varying q a cosimplicial sheaf on S denoted $a^\cdot_* I^p$, and a differential graded complex for varying p, which is a double complex whose associated simple complex is denoted $sa_* I := \mathbf{R}a_* K$

$$(\mathbf{R}a_*K)^n := \oplus_{p+q=n} a^q_* I^{p,q}; \quad dx^{pq} = d_I(x^{pq}) + (-1)^p \Sigma_{i=0}^{q+1} (-1)^i \delta_i x^{pq} \in (\mathbf{R}a_*K)^{n+1}$$

where q is the simplicial index ($\delta_i(x^{pq}) \in I^{p,q+1}$ and p is the degree. In particular for S a point we define the hypercohomology of K

$$\mathbf{R}\Gamma(X., K) := s\mathbf{R}\Gamma^\cdot(X., K); \quad \mathbb{H}^i(X., K) := H^i(\mathbf{R}\Gamma(X., K)).$$

The filtration L on $sa_*I := \mathbf{R}a_*K$ defines a spectral sequence

$$E_1^{pq} = R^q a_{p*}(K_{|X_p}) := H^q(a_{p*}(K_{|X_p}) \Rightarrow H^{p+q}(\mathbf{R}a_*K) := R^{p+q} a_* K$$

8.1.6. *Topological realization.* Recall that a morphism of simplices $f : \Delta_n \to \Delta_m$ has a geometric realization $|f| : |\Delta_n| \to |\Delta_m|$ as the affine map defined when we identify a simplex Δ_n with the vertices of its affine realization in $\mathbb{R}^{\Delta_n}$. We construct the topological realization of a topological semi-simplicial space $X.$ as the quotient of the topological space $Y = \coprod_{n \geq 0} X_n \times |\Delta_n|$ by the equivalence relation $\mathcal{R}$ generated by the identifications

$$\forall f : \Delta_n \to \Delta_m, x \in X_m, a \in |\Delta_n|, \quad (x, |f|(a)) \equiv (X.(f)(x), a)$$

The topological realization $|X.|$ is the quotient space of Y, modulo the relation $\mathcal{R}$, with its quotient topology. The construction above of the cohomology amounts to the computation of the cohomology of the topological space $|X.|$ with coefficient in an abelian group A

$$H^i(X., A) \simeq H^i(|X.|, A).$$

8.1.7. *Cohomological descent.* Let $a : X. \to S$ be an augmented simplicial scheme. Any abelian sheaf F on S, lifts to a sheaf a^*F on $X.$ and we have a natural morphism

$$\varphi(a) : F \to \mathbf{R}a_* a^* F \text{ in } D^+(S)$$

Definition 8.2 (cohomological descent)**.** The morphism $a : X. \to S$ is of cohomological descent if the natural morphism $\varphi(a)$ is an isomorphism in $D^+(S)$ for all abelian sheaves F on S.

The definition amounts to the following conditions

$$F \xrightarrow{\sim} Ker(a_{0*}a_0^*F \xrightarrow{\delta_1-\delta_0} a_1^*F); \quad R^i a_* a^* F = 0 \text{ for } i > 0.$$

In this case for all complexes K in $D^+(S)$

$$\mathbf{R}\Gamma(S, K) \simeq \mathbf{R}\Gamma(X., a^*K)$$

and we have a spectral sequence

$$E_1^{pq} = \mathbb{H}^q(X_p, a_p^*K) \Rightarrow \mathbb{H}^{p+q}(S, K), \quad d_1 = \sum_i (-1)^i \delta_i : E_1^{p,q} \to E_1^{p+1,q}.$$

8.1.8. *Simplicial varieties.* A complex simplicial variety $X.$ is smooth (resp. proper) if every X_n is smooth (resp. compact).

Definition 8.3 (NCD)**.** A normal crossing divisor is a family $Y_n \subset X_n$ of normal crossing divisors such that the family of open subsets $U_n := X_n - Y_n$ form a simplicial subvariety $U.$ of $X.$ s.t. the family of filtered logarithmic complexes $(\Omega^*_{X_n}(LogY_n))_{n\geq 0}, W)$ form a filtered complex on $(X.)$.

Theorem 8.4. *For each separated complex variety S, there exist*
i) A simplicial variety proper and smooth $X.$ over $\mathbb{C}$
ii) A normal crossing divisor $Y.$ in $X.$ and an augmentation $a : U. = (X. - Y.) \to S$ satisfying the cohomological descent property. Hence for all abelian sheaves F on S, $F \xrightarrow{\sim} \mathbf{R}a_* a^* F$.
Moreover, for each morphism $f : S \to S'$, there exists a morphism $f. : X. \to X'.$ of simplicial varieties proper and smooth with normal crossing divisors $Y.$ and $Y'.$ and augmented complements $a : U. \to S$ and $a' : U'. \to S'$ satisfying the cohomological descent property, such that the restriction $u : U. \to U'.$ of f commutes with the augmentations.

The proof is based on Hironaka's desingularisation theorem and on a general contruction of hypercoverings described briefly by Deligne after the general theory of hypercoverings. The desingularisation is carried at each step of the contruction.

Remark 8.5. We can (and we will) assume the NCD with smooth irreducible components.

8.1.9. An A-cohomological mixed Hodge complexe K (CMHC) on a topological simplicial space $X.$ consists of

(1) a complex $K_A \in ObD^+(X., A)$ s.t. $H^k(X., K_A)$ are A-modules of finite type, and $H^*(X., K_A) \otimes \mathbb{Q} \xrightarrow{\sim} H^*(X., K_A \otimes \mathbb{Q})$,
(2) a filtered complex $(K_{A\otimes\mathbb{Q}}, W) \in ObD^+F(X., A \otimes \mathbb{Q})$ with an increasing filtration W and an isomorphism $K_{A\otimes\mathbb{Q}} \simeq K_A \otimes \mathbb{Q}$ in $D^+(X., A \otimes \mathbb{Q})$,
(3) a bifilitered complex $(K_\mathbb{C}, W, F) \in ObD^+F_2(X., \mathbb{C})$ with an increasing (resp. decreasing) filtration W (resp. F) and an isomorphism $\alpha : (K_\mathbb{C}, W) \xrightarrow{\sim} (K_{A\otimes\mathbb{Q}}, W) \otimes \mathbb{C}$ in $D^+F(X., \mathbb{C})$.

Moreover, the following axiom is satisfied
(CMH) The restriction of K to each X_n is an $A-$CMHC.

8.1.10. Let $X.$ be a complex compact smooth simplicial algebraic variety with $Y.$ a NCD in $X.$ s.t. $j : U. = (X. - Y.) \to X.$ is an open simplicial subvariety, then

$$(\mathbf{R}j_*\mathbb{Z}, (\mathbf{R}j_*\mathbb{Q}, \tau_{\leq}), (\Omega^*_{X.}(LogY.), W, F))$$

is a CMHC on $(X.)$.

8.1.11. If we apply the global section functor to an $A-$CMHC K on $X.$, we get an $A-$cosimplicial mixed Hodge complex defined as follows
1) A cosimplicial complex $\mathbf{R}\Gamma^\cdot K_A$ in the derived category of cosimplicial $A-$modules,
2) A filtered cosimplicial complex $\mathbf{R}\Gamma^\cdot(K_{A\otimes\mathbb{Q}}, W)$ in the derived category of filtered cosimplicial vector spaces,
3) A bifiltered cosimplicial complex $\mathbf{R}\Gamma^\cdot(K_\mathbb{C}, W, F)$ in the derived category of bifiltered cosimplicial vector spaces,
4) An isomorphism $\mathbf{R}\Gamma^\cdot(K_{A\otimes\mathbb{Q}} \otimes \mathbb{C}, W \otimes \mathbb{C}) \simeq \mathbf{R}\Gamma^\cdot(K_\mathbb{C}, W, F)$ in the derived category of filtered cosimplicial vector spaces.
The hypercohomology of a cosimplicial A-CMHC on $X.$ is such a complex.

8.1.12. *Diagonal filtration.* A cosimplicial mixed Hodge complex K defines a differential graded complex which is viewed as a double complex whose associated simple complex is sK. We deduce on sK a weight filtration by a diagonal process.

Definition 8.6 (Differential graded $A-$MHC)**.** A differential graded (DG) (resp. DG^+) complex (or a complex of graded objects) has two degrees (resp bounded below), the first defined by the complex and the second by the gradings. It can be viewed as a double complex.
A graded differential $A-$MHC is defined by a system of a DG^+-complex (resp. filtered, bifiltered)

$$K, (K, W), (K, W, F)$$

s.t. for each degree n of the grading, the component $(K^{\cdot,n}, W, F)$ is an A-MHC.

A graded differential $A-$MHC is not a MHC. A cosimplicial mixed Hodge complex (K, W, F) defines a $DG^+ - A-$MHC

$$sK, (sK, W), (sK, W, F)$$

the degree of the grading is the cosimplicial degree.

Definition 8.7 (Diagonal filtration)**.** The diagonal filtration $\delta(W, L)$ of W and L on sK is defined by

$$\delta(W, L)_n(sK) := \oplus_{p,q} W_{n+q} K^{p,q} = \sum_r s(W_{n+r}K) \cap L^r(sK).$$

where $L^r(sK) = s(K^{p,q})_{q\geq r}$.
For a bifiltered complex (K, W, F) with a decreasing F, the sum over F is natural.

8.2. Properties. We have

$$Gr_n^{\delta(W,L)}(sK) \simeq \oplus_p Gr^W_{n+p} K^{\cdot,p}[-p]$$

In the case of a DG^+-complex defined as the hypercohomology of a complex (K,W) on a simplicial space $X.$, we have

$$Gr_n^{\delta(W,L)}\mathbf{R}\Gamma K \simeq \oplus_p \mathbf{R}\Gamma(X_p, Gr^W_{n+p} K^{\cdot,p})[-p].$$

and for a bifiltered complex with a decreasing F

$$Gr_n^{\delta(W,L)}(\mathbf{R}\Gamma K, F) \simeq \oplus_p \mathbf{R}\Gamma(X_p, (Gr^W_{n+p} K, F))[-p].$$

Next we need the following

Lemma 8.8. *If $H = (H_A, W, F)$ is an $A-$MHS, a filtration L of H_A is a filtration of in the category of MHS, if and only if for all n, $(Gr^n_L H_A, Gr^n_L(W), Gr^n_L(F)$ is an $A-$MHS.*

Theorem 8.9. *Let K be a graded differential $A-$MHC (for example defined by a cosimplicial $A-$MHC).*
i) Then $(sK, \delta(W,L), F)$ is an $A-$MHC.
The first terms of the weight spectral sequence

$$_{\delta(W,L)}E_1^{pq}(sK\otimes\mathbb{Q}) = \sum_{p=m-n, q=r+n} H^a(Gr^W_n K^{\cdot,m})$$

form the simple complex $({}_{\delta(W,L)}E_1^{pq}, d_1)$ of $A\otimes\mathbb{Q}-$Hodge structures of weight q associated to the double complex

$$\begin{array}{ccccc}
H^{q-(n+1)}(Gr^W_{n+1}K^{\cdot,m+1}) & \xrightarrow{\delta} & H^{q-n}(Gr^W_n K^{\cdot,m+1}) & \xrightarrow{\delta} & H^{q-(n-1)}(Gr^W_{n-1}K^{\cdot,m+1}) \\
 & & d''\uparrow & & d''\uparrow \\
 & & H^{q-n}(Gr^W_n K^{\cdot,m}) & \xrightarrow{\delta} & H^{q-(n-1)}(Gr^W_{n-1}K^{\cdot,m})
\end{array}$$

ii) The terms ${}_LE_r$ for $r>0$, of the spectral sequence defined by $(sK_{A\otimes\mathbb{Q}}, L)$ are endowed with a natural $A-$MHS, with differentials d_r compatible with such structures.
iii) The filtration L on $H^*(sK)$ is a filtration in the category of MHS and

$$Gr^p_L(H^{p+q}((sK), \delta(W,L)[p+q], F) = (E^{pq}_\infty, W, F).$$

8.2.1. In the case of a logarithmic simplicial variety, the cohomology groups $H^n(U., \mathbb{Z})$ are endowed with MHS defined by the following MHC
$K := \mathbf{R}\Gamma(U., \mathbb{Z})$, $\quad (K\otimes\mathbb{Q} := \mathbf{R}\Gamma(U., \mathbb{Q}), \delta(W,L))$,
$(K\otimes\mathbb{C} := \mathbf{R}\Gamma(U., \Omega^*_{X.}(Log Y.)), \delta(W,L)), F)$
satisfying

$$Gr_n^{\delta(W,L)}\mathbf{R}\Gamma(U., \mathbb{Q}) \simeq \oplus_m (Gr^W_{n+m}\mathbf{R}\Gamma(U_m,, \mathbb{Q}))[-p])[-m] \simeq \oplus_m \mathbf{R}\Gamma(Y^{n+m}_m, \mathbb{Q})[-n-2m].$$

with

$$_{\delta(W,L)}E_1^{-a,b} = \oplus_{p+2r=b, q-r=-a} H^p(Y^r_q, Q) \Rightarrow H^{-a+b}(U., \mathbb{Q}).$$

the filtration F induces on ${}_{\delta(W,L)}E_1^{-a,b}$ a HS of weight b and the differentials d_1 are compatible with the HS. The term E_1 is the simple complex associated to the double complex of HS of weight b where G is an alternating Gysin map

$$\begin{array}{ccccc}
H^{b-(2r+2)}(Y^{r+1}_{q+1}, \mathbb{Q})(-r-1) & \xrightarrow{G} & H^{b-2r}(Y^r_{q+1}, \mathbb{Q})(-r) & \xrightarrow{G} & H^{b-2(r-2)}(Y^{r-1}_{q+1}, \mathbb{Q})(-r+1) \\
 & & \sum_i(-1)^i\delta_i\uparrow & & \sum_i(-1)^i\delta_i\uparrow \\
 & & H^{b-2r}(Y^r_q, \mathbb{Q})(-r) & \xrightarrow{G} & H^{b-2(r-2)}(Y^{r-1}_q, \mathbb{Q})(-r+1)
\end{array}$$

where the indices of E_1^{ab} satisfy $-a = q - r$.

Proposition 8.10. i) The MHS on $H^n(U., \mathbb{Z})$ is defined by the graded differential MHC associated to the simplicial MHC defined by the logarithmic complex on each term of $X.$ and it is functorial in the couple $(U., X.)$,
ii) The rational weight spectral sequence degenerates at rank 2 and the HS on E_2 induced by E_1 is isomorphic to the HS on $Gr_W H^n(U., \mathbb{Q})$.
iii)The Hodge numbers h^{pq} of $H^n(U., \mathbb{Q})$ vanish for $p \notin [0, n]$ or $q \notin [0, n]$.
iv) For $Y. = \emptyset$, the Hodge numbers h^{pq} of $H^n(X., \mathbb{Q})$ vanish for $p \notin [0, n]$ or $q \notin [0, n]$ or $p + q > n$.

Definition 8.11. The MHS on the cohomology of a complex algebraic variety $H^n(X, \mathbb{Z})$ is defined by any logarithmic simplicial resolution of X via the isomorphism with $H^n(U., \mathbb{Z})$ defined by the augmentation $a : U. \to X$. It doesn't depends on the resolution and it is functorial in X.

8.3. MHS on the cohomology of a complete embedded algebraic variety. For embedded varieties into smooth varieties , the MHS on cohomology can be deduced by a simple method, using exact sequences from MHS already constructed for NCD, which should easily convince of the natural aspect of this theory. The technical ingredients consist of Poincaré duality and the trace (or Gysin) morphism.
Let $p : X' \to X$ be a proper morphism of complex smooth varieties of same dimension, Y a closed subvariety of X and $Y' = p^{-1}(Y)$. We suppose that Y' is a NCD in X' and the restriction of p induces an isomorphism $p_{/X'-Y'} : X' - Y' \xrightarrow{\sim} X - Y$.

$$\begin{array}{ccccc}
Y' & \xrightarrow{i'} & X' & \xleftarrow{j'} & X' - Y' \\
\downarrow p_Y & & \downarrow p & & \downarrow p_{X-Y} \\
Y & \xrightarrow{i} & X & \xleftarrow{j} & X - Y
\end{array}$$

The trace morphism Trp is defined as Poincaré dual to the inverse image p^* on cohomology. It can be defined at the level of sheaf resolutions of $\mathbb{Z}_{X'}$ and $\mathbb{Z}_X$ constructed by Verdier, that is in derived category $Tr\, p$: $\mathbf{R}p_*\mathbb{Z}_{X'} \to \mathbb{Z}_X$ hence we deduce by restriction morphisms depending on the embeddings of Y and Y' into X'.
$(Tr\, p)/_Y : \mathbf{R}p_*\mathbb{Z}_{Y'} \to \mathbb{Z}_Y$, $(Tr\, p)/_Y : H^i(Y', \mathbb{Z}) \to H^i(Y, \mathbb{Z})$, $H^i_c(Y', \mathbb{Z}) \to H^i_c(Y, \mathbb{Z})$.

Remark 8.12. Let U be a neighbourhood of Y in X, retract by deformation onto Y s.t. $U' = p^{-1}(U)$ is a retract by deformation onto Y'; this is the case if Y is a sub-variety of X. Then the morphism $(Tr\, p)/_Y$ is deduced from $Tr(p/_U)$ in the diagram

$$\begin{array}{ccc}
H^i(Y', \mathbb{Z}) & \xleftarrow{\sim} & H^i(U', \mathbb{Z}) \\
\downarrow (Tr\, p)/_Y & & \downarrow Tr(p/_U) \\
H^i(Y, \mathbb{Z}) & \xleftarrow{\sim} & H^i(U, \mathbb{Z})
\end{array}$$

Consider now the diagram

$$\begin{array}{ccccc}
\mathbf{R}\Gamma_c(X' - Y', \mathbb{Z}) & \xrightarrow{j'_*} & \mathbf{R}\Gamma(X', \mathbb{Z}) & \xrightarrow{i'^*} & \mathbf{R}\Gamma(Y', \mathbb{Z}) \\
Trp \downarrow & & \downarrow Trp & & \downarrow (Trp)_{|Y} \\
\mathbf{R}\Gamma_c(X - Y, \mathbb{Z}) & \xrightarrow{j_*} & \mathbf{R}\Gamma(X, \mathbb{Z}) & \xrightarrow{i^*} & \mathbf{R}\Gamma(Y, \mathbb{Z})
\end{array}$$

Proposition 8.13. i) The morphism $(p_Y)^*$: $H^i(Y, \mathbb{Z}) \to H^i(Y', \mathbb{Z})$ is injective with retraction $(Tr\, p)_{/Y}$.

ii) We have a quasi-isomorphism of $i_*\mathbb{Z}_Y$ with the cone $C(i'^* - Tr\,p)$ of the morphism $i'^* - Tr\,p$. The long exact sequence associated to the cone splits into short exact sequences

$0 \to H^i(X', \mathbb{Z}) \xrightarrow{i'^* - Tr\,p} H^i(Y', \mathbb{Z}) \oplus H^i(X, \mathbb{Z}) \xrightarrow{(Tr\,p)_{/Y} + i^*} H^i(Y, \mathbb{Z}) \to 0$

Moreover $i'^* - Tr\,p$ is a morphism of MHS.

Definition 8.14. The MHS of Y is defined as cokernel of $i'^* - Tr\,p$ via its isomorphism with $H^i(Y, \mathbb{Z})$, induced by $(Tr\,p)_{/Y} + i^*$. It coincides with Deligne's MHS.

This result shows the uniqueness of the theory of MHS, once the MHS of the NCD Y' has been constructed.

The above technique consists in the realization of the MHS on the cohomology of Y as relative cohomology with MHS structures.

Books

Carlson J., Muller-Stach S., Peters C.: Period mappings and Period Domains.
Griffiths P., Harris J.: Principles of algebraic geometry.
Voisin C.: Hodge theory and complex algebraic geometry
Tu Loring: An Introduction to Manifolds, Universitext, Springer.

References

[1] BEILINSON A.A., BERNSTEIN J., DELIGNE P.: Faisceaux Pervers, in *Analyse et Topologie sur les espaces singuliers Vol.I,* Astérisque **100**(1982).
[2] CARTAN H., EILENBERG S.: Homological Algebra, Princeton University Press (1956).
[3] DIEUDONNE J.: A history of algebraic and differential topology, 1900-1960. Birkhauser Boston 1989.
[4] DELIGNE P.: Théorème de Lefschetz et critères de dégénérescence de suites spectrales.Publ.Math.I.H.E.S., 35 (1968), 107-126.
[5] DELIGNE P.: Théorie de Hodge I. Actes du congrès international. Nice (1970) Gauthier-Villars (1971) I, 425-430.
[6] DELIGNE P.: Théorie de Hodge II Publ. Math. IHES 40 (1972) 5-57.
[7] DELIGNE P.: Théorie de Hodge III Publ. Math. IHES 44 (1975) 6-77.
[8] GODEMENT R.: Topologie algébrique et théorie des faisceaux.Publ.Inst. Math. Univ. Strasbourg XIII, Paris Hermann (1958).
[9] GRIFFITHS P., SCHMID W.: Recent developments in Hodge theory, in Discrete subgroups of Lie groups Oxford university press, (1973).
[10] GROTHENDIECK A.: Surquelques points d'algèbre homologique, Tôhoku Math. J. 9 ((1957) 119-221.
[11] GROTHENDIECK A.: On the De Rham cohomology of Algebraic Varieties, Publ. Math. IHES 29 (1966).
[12] SPANIER E.: Algebraic topology. New York, Mac Graw Hill (1966).
[13] VERDIER J.L.:Des catégories dérives des catégories abéliennes, Astrisque 239 (1996).