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Hodge theory of maps: Lectures 1-3

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Abstract

These three lectures summarize classical results of Hodge theory concerning algebraic maps, and presumably contain much more material than I'll be able to cover. Lectures 4 and 5, to be delivered by M. A. de Cataldo, will discuss more recent results. I will not try to trace the history of the subject nor attribute the results discussed. Coherently with this policy, the bibliography only contains textbooks and a survey, and no original paper. Furthermore, quite often the results will not be presented in their maximal generality; in particular I'll alway stick to projective maps, even though some results discussed hold more generally.

Contents

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1 Introduction.

Hodge theory gives non-trivial restrictions to the topology of a nonsingular projective variety, or, more generally, of a compact Kähler manifold, such as the parity of odd Betti numbers, the hard Lefschetz theorem, the formality theorem, stating that the real homotopy type of such variety is, if simply connected, determined by the cohomology ring. Similarly, Hodge theory gives non-trivial topological costraints on algebraic maps. This is, broadly speaking, what these lectures are about.

In dealing with maps one is forced to deal with singularities: even assuming that domain and target of an algebraic map are nonsingular, asking that the map is smooth is much too restrictive: there are singular fibres, and this brings into the picture the technical tools to deal with them: stratification theory and topological invariants of singular spaces, such as intersection cohomology. This latter, which turns out to be a good replacement for cohomology when dealing with singular varieties, is better understood as the hypercohomology of a complex of sheaves, and this naturally leads to consider objects in the "constructible" derived category.

The question which we plan to address can also be formulated as follows: *How is the existence of an algebraic map $f : X \to Y$ of complex algebraic varieties reflected in the topological invariants of X?* From this point of view, one is looking for a relative version of Hodge theory, classical theory corresponding to $Y = point$. Hodge theory encodes the algebraic structure of X in linear algebra data on $H^*(X)$. If X is nonsingular and projective this amount to the (p, q) decomposition

$$H^r = \bigoplus_{p+q=r} H^{p,q} \text{ with the simmetry } H^{p,q} = \overline{H^{q,p}},$$

which is possible to enrich with polarization data if a preferred ample line bundle is chosen.

As we will see in lecture 2, for a general X, i.e. maybe singular and non-compact, the (p,q) decomposition is replaced by a more complicated linear algebra object, the mixed Hodge structure, consisting of two filtrations $W_\bullet$ on $H^*(X,\mathbb{Q})$, and $F^\bullet$ on $H^*(X,\mathbb{C})$ with compatibility conditions.

Similarly, given a projective map $f : X \longrightarrow Y$, with X is nonsingular, we look for a linear algebra datum encoding the datum of the map f, with the obvious requirements :

- This datum should be compatible with the Hodge structure of X.
- It should impose strong costraints of linear algebra type.
- It should have a vivid geometric interpretation.

The theorems discussed by de Cataldo in the last lecture of the course, with their Hodge theoretic counterpart give some answers to these questions.

2 The smooth case: E_2-degeneration

We suppose that $f : X \to Y$ is a projective smooth map $f : X \to Y$ of nonsingular (connected) quasi-projective varieties, that is, f factors through some product $Y \times \mathbb{P}^N$ by a closed immersion $X \longrightarrow Y \times \mathbb{P}^N$, and the fibers are nonsingular projective manifolds. The nonsingular hypersurfaces of a fixed degree in some projective space give an interesting example. More generally we have the following:

Example 2.0.1. **(The universal hyperplane section)** Let $X \subseteq \mathbb{P}^n(\mathbb{C})$ be a nonsingular projective variety, denote by $\mathbb{P}^n(\mathbb{C})^\vee$ the dual projective space, whose points are hyperplanes in $\mathbb{P}^n(\mathbb{C})$, and define

$$\mathcal{X} := \{(x, H) \in X \times \mathbb{P}^n(\mathbb{C})^\vee \text{ such that } x \in H \cap X\}.$$

with the second projection $p_2 : \mathcal{X} \longrightarrow \mathbb{P}^n(\mathbb{C})^\vee$. The fibre over the hyperplane H is the hyperplane section $H \cap X \subseteq X$. Since the projection $\mathcal{X} \longrightarrow X$ makes $\mathcal{X}$ into a projective bundle over X, it follows that $\mathcal{X}$ is nonsingular. Let $X^\vee = \{H \in \mathbb{P}^n(\mathbb{C})^\vee$ such that $H \cap X$ is singular$\}$. It is an algebraic subvariety of $\mathbb{P}^n(\mathbb{C})^\vee$, called the dual variety of X.

Set

$$U_{\mathrm{reg}} := \mathbb{P}^n(\mathbb{C})^\vee \setminus X^\vee, \qquad \mathcal{X}_{\mathrm{reg}} := p_2^{-1}(U_{\mathrm{reg}}).$$

Then the restriction

$$p_{2|} : \mathcal{X}_{\mathrm{reg}} \longrightarrow U_{\mathrm{reg}}$$

is a projective smooth map. Let $\mathbb{P}^n(\mathbb{C}) \xrightarrow{v_d} \mathbb{P}^N(\mathbb{C})$ be the $d-$th Veronese imbedding of $\mathbb{P}^n(\mathbb{C})$. Setting $X := v_d(\mathbb{P}^n(\mathbb{C}))$, the construction gives the family of degree d hypersurfaces in $\mathbb{P}^n(\mathbb{C})$.

By a classical result (Ehresmann fibration Lemma), a map as above is a C^∞-fiber bundle, namely, for some manifold F, each point $y \in Y$ has a neighborhood N in the analytic topology, such that there is a diffeomorphism

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{\simeq} & N \times F\ , \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle p_1} & \\ N & & \end{array}$$

where p_1 denotes the projection on the first factor.

In particular the higher direct image sheaves $R^i f_* \mathbb{Q}$, whose stalk at a point $y \in Y$ is

$$(R^i f_* \mathbb{Q})_y \simeq H^i(f^{-1}(y))$$

are *local systems*, i.e. locally constant sheaves of finite dimensional $\mathbb{Q}$-vector spaces. For instance, if Y is simply connected, they are in fact constant sheaves. In general, choosing a base point $y_0 \in Y$, we have associated *monodromy* representations

$$\rho^i : \pi_1(Y, y_0) \to \mathrm{Aut}(H^i(f^{-1}(y_0), \mathbb{Q})). \tag{2.0.1}$$

For general reasons there is the *Leray spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}) \to H^{p+q}(X, \mathbb{Q}).$$

Even if Y is simply connected, the Leray spectral sequence can be non-trivial; for example, in the Hopf fibration $f : S^3 \to S^2$, the differential $d_2 : E_2^{01} \to E_2^{21}$ is non-zero.

Theorem 2.0.2. *The Leray spectral sequence associated to a smooth projective map degenerates at* E_2.

In fact a stronger statement can be proved, namely

Theorem 2.0.3. *There exists an isomorphism in the bounded derived category of sheaves with constructible cohomology (see lecture 3)*

$$Rf_* \mathbb{Q} \simeq \oplus_q R^q f_* \mathbb{Q}[-q].$$

In particular, if Y is simply connected, Theorem 2.0.2 gives an isomorphism of vector spaces

$$H^r(X) \simeq \oplus_{a+b=r} H^a(Y) \otimes H^b(f^{-1}(y_0));$$

the fibration behaves, form the point of view of additive cohomology, as if it were a product. Simple examples ($\mathbb{P}^1$-fibrations, for instance) show that in general the isomorphism cannot be made compatible with the ring structure on cohomology.

Sketch of proof of 2.0.2. Suppose that $\mathcal{L}$ is a relatively ample line bundle on X, and denote by $L \in H^2(X, \mathbb{Q})$ its first Chern class. Let n be the relative dimension of the map: $n := \dim X - \dim Y$. The Hard Lefschetz theorem applied to the fibres of f gives isomorphisms $L^k : H^{n-k}(f^{-1}(y),\mathbb{Q}) \longrightarrow H^{n+k}(f^{-1}(y),\mathbb{Q})$, hence isomorphisms of local systems $L^k : R^{n-q}f_*\mathbb{Q} \longrightarrow R^{n+q}f_*\mathbb{Q}$. On the cohomology of each fibre we have a Lefschetz decomposition

$$H^i(f^{-1}(y),\mathbb{Q}) = \oplus L^a P^{i-2a}(f^{-1}(y),\mathbb{Q}),$$

with $P^i(f^{-1}(y),\mathbb{Q}) := \mathrm{Ker} L^{n-i+1} : H^i(f^{-1}(y),\mathbb{Q}) \longrightarrow H^{2n-i+2}(f^{-1}(y),\mathbb{Q})$ We have a corresponding decomposition of local systems

$$R^i f_*\mathbb{Q} = \oplus L^a \mathcal{P}^{i-2a},$$

where $\mathcal{P}^r$ denotes the local system with stalk $\mathcal{P}^r_y = P^r(f^{-1}(y),\mathbb{Q})$. The differentials of the Leray spectral sequence are compatible with this decomposition. Let us show for example that $d_2 = 0$. It is enough to show this on the summand $H^p(Y,\mathcal{P}^q) \subseteq E_2^{pq}$. We have the following:

$$\begin{array}{ccc} H^p(Y,\mathcal{P}^q) & \xrightarrow{d_2} & H^{p+2}(Y,R^{q-1}f_*\mathbb{Q}) \\ \downarrow{\scriptstyle L^{n-q+1}} & & \downarrow{\scriptstyle L^{n-q+1}} \\ H^p(Y,R^{2n-q+2}f_*\mathbb{Q}) & \xrightarrow{d_2} & H^{p+2}(Y,R^{2n-q+1}f_*\mathbb{Q}). \end{array}$$

The left vertical arrow is the zero map by the definition of the primitive local system, while the right vertical arrow is an isomorphism by Hard Lefschetz applied to the fibres of f, hence $d_2 = 0$.

□

The stronger statement about the splitting in the derived category is obtained in a similar way, considering the spectral sequence associated with the functors, one for each q,

$$\mathrm{Hom}_{\mathcal{D}(Y)}(R^q f_*\mathbb{Q}[-q], Rf_*\mathbb{Q}).$$

The spectral sequence degenerates for the same reason, and in particular the maps

$$\mathrm{Hom}_{\mathcal{D}(Y)}(R^q f_*\mathbb{Q}[-q], Rf_*\mathbb{Q}) \longrightarrow \mathrm{Hom}_{\mathcal{D}(Y)}(R^q f_*\mathbb{Q}, R^q f_*\mathbb{Q}).$$

are surjective. This gives a way of lifting the identity map $R^q f_*\mathbb{Q} \longrightarrow R^q f_*\mathbb{Q}$ to a map $R^q f_*\mathbb{Q}[-q] \longrightarrow Rf_*\mathbb{Q}$ for every q, so as to obtain a map

$$\oplus_q R^q f_*\mathbb{Q}[-q] \longrightarrow Rf_*\mathbb{Q},$$

which induces the identity on all cohomology sheaves, so it is in particular a quasi-isomorphism.

Remark 2.0.4. For singular maps, the Leray spectral sequence is very seldom degenerate. If $f : X \to Y$ is a resolution of the singularities of a projective variety Y whose cohomology has a mixed Hodge structure (see lecture 2) which is not pure, then f^* cannot be injective, and this prohibits degeneration in view of the edge-sequence.

3 Mixed Hodge structures.

3.1 Mixed Hodge structures on the cohomology of algebraic varieties

As the following two elementary examples show, one cannot expect that Hodge theory extends to singular or noncompact varieties.

Example 3.1.1. Consider the projective plane curve $\mathcal{C}$ of equation $Y^2Z - X^2(X-1) = 0$. It is immediately seen that $\dim H^1(\mathcal{C}) = 1$, hence there cannot be a (p,q)-decomposition $H^1(\mathcal{C},\mathbb{C}) = H^{1,0} \oplus \overline{H^{1,0}}$ on this vector space

Example 3.1.2. Consider $\mathbb{C}^* \subseteq \mathbb{P}^1(\mathbb{C})$. Clearly $\dim H^1(\mathbb{C}^*) = 1$, and again there cannot be a (p,q)-decomposition $H^1(\mathbb{C}^*,\mathbb{C}) = H^{1,0} \oplus \overline{H^{1,0}}$ on this vector space

Basically, there are two possibilities:

1. allow linear algebra structures which are more complicated than the "simple" (p,q) decomposition.
2. consider different topological invariants.

Both possibilities turn out to have remarkable consequences. In this lecture we will consider the first option. The second possibility, leading to the definition of *intersection cohomology*, will be considered in lecture 4.

Definition 3.1.3. (**Mixed Hodge structure**) A (rational)*mixed Hodge structure* consists of the following datum:

1. A vector space $V_{\mathbb{Q}}$ over $\mathbb{Q}$ with a finite increasing filtration (*the weight filtration*)
$$\{0\} = W_a \subseteq W_{a+1} \subseteq \ldots \subseteq W_b = V_{\mathbb{Q}}.$$
2. a finite decreasing filtration (*the Hodge filtration*) on $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes \mathbb{C}$
$$V_{\mathbb{C}} = F^q \supseteq F^{q+1} \supseteq \ldots \supseteq F^m \supseteq F^{m+1} = \{0\}$$

with the condition that, for every k, the filtration $F^\bullet$ induces on $\mathrm{Gr}_k^W V_{\mathbb{C}} = (W_k/W_{k-1}) \otimes \mathbb{C}$, a pure Hodge structure of weight k, namely:

$$\mathrm{Gr}_k^W V_{\mathbb{C}} = \bigoplus_{p+q=k} \left(\mathrm{Gr}_k^W V_{\mathbb{C}}\right)^{p\,q},$$

where

$$\left(\mathrm{Gr}_k^W V_{\mathbb{C}}\right)^{pq} := F^p \mathrm{Gr}_k^W V_{\mathbb{C}} \cap \overline{F^q \mathrm{Gr}_k^W V_{\mathbb{C}}}.$$

Let us recall the following definition

Definition 3.1.4. Strict filtered maps Let $(V, G^\bullet)$, $(V', G'^\bullet)$ two vector spaces endowed with increasing filtrations, and $f : V \longrightarrow V'$ a filtered map, namely a linear map such that $f(G^a V) \subseteq G'^a V'$. The map f is said to be *strict* if, for every a,

$$f(G^a V) = \mathrm{Im} f \cap G'_a V.$$

An analogous definition holds for decreasing filtrations. Morphisms of mixed Hodge structures are just what one expects them to be:

Definition 3.1.5. morphisms of mixed Hodge structures. A map $f : (V_{\mathbb{Q}}, W_\bullet, F^\bullet) \longrightarrow (V'_{\mathbb{Q}}, W'_\bullet, F'^\bullet)$ of mixed Hodge structures is a linear map $f : V_{\mathbb{Q}} \longrightarrow V'_{\mathbb{Q}}$ filtered with respect to $W_\bullet$, $W'_\bullet$, such that $f_{\mathbb{C}} : V_{\mathbb{C}} \longrightarrow V'_{\mathbb{C}}$ is filtered with respect to $F^\bullet$, $F'^\bullet$.

One can similarly define morphisms of mixed Hodge structures of type (k, k); they become just morphisms of mixed Hodge structures after an appropriate Tate twist.

The remarkable formal properties of mixed Hodge structures will be treated in other courses: we just list here some of them which will be useful to keep in mind:

Theorem 3.1.6. *Mixed Hodge structures with morphisms of mixed Hodge structures form an abelian category. A morphism f of mixed Hodge structures is strict with respect to $W_\bullet$, and $f_{\mathbb{C}}$ is strict with respect to $F^\bullet$.*

We have the following remarkable theorem:

Theorem 3.1.7. *The cohomology groups $H^i(Y, \mathbb{Q})$ of a complex algebraic variety Y have a functorial mixed Hodge structure. Furthermore we have the following restrictions on the weights:*

1. *$W_a H^i(Y, \mathbb{Q}) = \{0\}$ for $a < 0$ and $W_a H^i(Y, \mathbb{Q}) = H^i(Y, \mathbb{Q})$ for $a \geq 2i$.*
2. *If Y is nonsingular, then $W_a H^i(Y, \mathbb{Q}) = \{0\}$ for $a < i$, and*

$$W_i H^i(Y, \mathbb{Q}) = \mathrm{Im} H^i(\overline{Y}, \mathbb{Q}) \longrightarrow H^i(Y, \mathbb{Q}),$$

where $\overline{Y}$ is any compactification of Y.

3. If Y is complete, then $W_a H^i(Y,\mathbb{Q}) = H^i(Y,\mathbb{Q})$ for $a \geq i$, and

$$W_{i-1}H^i(Y,\mathbb{Q}) = \mathrm{Ker} H^i(Y,\mathbb{Q}) \longrightarrow H^i(\widetilde{Y},\mathbb{Q})$$

where $\widetilde{Y}$ is a resolution of singularities of Y.

Functoriality here means that the pullback map $f^* : H^i(Y,\mathbb{Q}) \longrightarrow H^i(X,\mathbb{Q})$ associated with an algebraic map $f : X \longrightarrow Y$ is a morphism of mixed Hodge structures.

Remark 3.1.8. Of course, in the case in which Y is nonsingular and projective, we recover its pure Hodge structure: the weight filtration is trivial, namely $W_{i-1}H^i(Y,\mathbb{Q}) = \{0\}$ and $W_i H^i(Y,\mathbb{Q}) = H^i(Y,\mathbb{Q})$, and $F^\bullet$ is the standard filtration associated with the Hodge decomposition. In the example of the nodal curve $W_{-1}H^1(Y,\mathbb{Q}) = \{0\}$ and $W_0 H^1(Y,\mathbb{Q}) = H^1(Y,\mathbb{Q})$, and every class has "type" $(0,0)$. In the example of the punctured line $W_1 H^1(Y,\mathbb{Q}) = \{0\}$ and $W_2 H^1(Y,\mathbb{Q}) = H^1(Y,\mathbb{Q})$, and every class has "type" $(1,1)$.

3.2 The global invariant cycle theorem

A consequence of the restrictions on the weights of the cohomology groups of an algebraic variety is the following

Theorem 3.2.1. (weight principle) *Let $Z \subseteq U \subseteq X$ be inclusions, where X a nonsingular compact variety, $U \subseteq X$ a Zariski dense open subvariety and $Z \subseteq U$ a closed subvariety of X. Then the images in $H^j(Z,\mathbb{Q})$ of the restriction maps from X and from U coincide.*

Sketch of Proof. In

$$H^l(X,\mathbb{Q}) \xrightarrow{a} H^l(U,\mathbb{Q}) \xrightarrow{b} H^l(Z,\mathbb{Q})$$

the maps a and b are strictly compatible with the weight filtration. Hence, it follows from theorem 3.1.7 that $\mathrm{Im}\, b = \mathrm{Im}\, W_l H^l(U,\mathbb{Q}) = \mathrm{Im}\, a$ □

Remark 3.2.2. Despite its innocent-looking appearance, this is an extremely strong statement, imposing non-trivial costraints on the topology of algebraic maps. For contrast look at the real picture $Z = S^1 \subseteq U = \mathbb{C}^* \subseteq X = \mathbb{P}^1(\mathbb{C})$. The restriction map $H^1(\mathbb{P}^1(\mathbb{C}),\mathbb{Q}) \longrightarrow H^1(S^1,\mathbb{Q})$ is zero, while $H^1(\mathbb{C}^*,\mathbb{Q}) \longrightarrow H^1(S^1,\mathbb{Q})$ is an isomorphism.

The following is the global invariant cycle theorem, which follows quite directly from the weight principle above:

Theorem 3.2.3. *Suppose $f : X \to Y$ is a smooth projective map, with Y connected, and let $\overline{X}$ be a nonsingular compactification of X. Then, for $y_0 \in Y$*

$$H^i(f^{-1}(y_0), \mathbb{Q})^{\pi_1(Y,y_0)} = \mathrm{Im}\,\{H^i(\overline{X}, \mathbb{Q}) \longrightarrow H^i(f^{-1}(y_0), \mathbb{Q})\},$$

Remark 3.2.4. The previous proposition is most often used when we have a projective map, not necessarily smooth, $\overline{f} : \overline{X} \longrightarrow \overline{Y}$, with $\overline{X}$ nonsingular. There is a dense Zariski open subset $Y \subseteq \overline{Y}$ such that $X := \overline{f}^{-1}(Y) \longrightarrow Y$ is a smooth map. Then theorem 3.2.3 states that the monodromy invariants in the cohomology of a generic fibre are precisely the classes obtained by restriction from the total space of the family.

Remark also that, while it is clear that a cohomology class in X restricts to a monodromy invariant class in the cohomology of the fibre at y_0, the other assertion is not at all obvious and is in fact specific of algebraic maps. Again, the Hopf fibration gives an example: identify a fibre with S^1: the generator of $H^1(S^1)$ is clearly monodromy invariant, as the monodromy of the Hopf fibration is trivial, but it is not the restriction of a class in S^3, as $H^1(S^3) = 0$.

Sketch of Proof of the global invariant cycle theorem. In force of the theorem 2.0.2 proved in lecture 1, the Leray spectral sequence for f degenerates at E_2, in particular, for all l, the map

$$H^l(X, \mathbb{Q}) \longrightarrow E_2^{0l} = H^0(Y, R^l f_* \mathbb{Q})$$

is surjective. We have the natural identification

$$H^0(Y, R^l f_* \mathbb{Q}) \simeq H^i(f^{-1}(y_0))^{\pi_1(Y,y_0)} \subseteq H^i(f^{-1}(y_0)),$$

and the composition

$$H^l(X, \mathbb{Q}) \longrightarrow H^0(Y, R^l f_* \mathbb{Q}) \xrightarrow{\simeq} H^l(f^{-1}(y_0), \mathbb{Q})^{\pi_1(Y,y_0)} \longrightarrow H^i(f^{-1}(y_0), \mathbb{Q})$$

is a map of (mixed) Hodge structures. By theorem 3.2.1 we have

$$\mathrm{Im}\, H^l(\overline{X}, \mathbb{Q}) \longrightarrow H^i(f^{-1}(y_0)) = \mathrm{Im}\, H^l(\overline{X}, \mathbb{Q}) \longrightarrow H^i(f^{-1}(y_0), \mathbb{Q}) = H^l(f^{-1}(y_0), \mathbb{Q})^{\pi_1(Y,y_0)}.$$

□

3.3 Semisimplicity of monodromy

A consequence of the global invariant cycle theorem 3.2.3 is that the subspace $H^i(f^{-1}(y_0))^{\pi_1(Y,y_0)} \subseteq H^i(f^{-1}(y_0))$ of monodromy invariants is a (pure) sub-Hodge structure of $H^i(f^{-1}(y_0))$. It is interesting to compare this fact with the local situation:

Example 3.3.1. Consider the family of degenerating elliptic curves C_t of equations

$$\{Y^2Z - X(X-t)(X-1) = 0\} \subseteq \mathbb{P}^2(\mathbb{C}) \times \Delta,$$

where t is a coordinate on the disc Δ. The monodromy operator is a length 2 Jordan block, and the subspace of monodromy invariants is one-dimensional, spanned by the vanishing cycle of the degeneration. Hence the subspace cannot be a sub-Hodge structure of the weight 1 Hodge structure $H^1(C_{t_0}\mathbb{Q})$ for $t \neq 0$. As we will se, in the local situation, the local invariant cycle theorem with the associated Clemens-Schmid exact sequence deals with these kind of set-up, defining a *Mixed Hodge structure* on $H^1(C_{t_0}\mathbb{Q})$ whose weight filtration is related to the monodromy.

The fact that the space of monodromy invariants is a sub-Hodge structure can be refined as follows: Recall that a representation is said to be *irreducible* if it has no non-trivial invariant subspace, i.e. if it is a simple object in the category of representations.

Theorem 3.3.2. (Semisimplicity theorem) *Suppose $f : X \to Y$ is a smooth projective map of quasi-projective manifolds. Then the monodromy representations ρ^i defined in (2.0.1) are semisimple, namely they split as a direct sum of irreducible representations.*

Again one can compare with the local set-up of a family over a disc which is smooth outside 0. In this case, by the *monodromy theorem* the monodromy operator T is *quasi-unipotent*, namely $(T^a - I)^b = 0$ (compare with example 3.3.1). Again, the semisimplicity of global monodromy is a specific property of algebraic geometry. For examples there exist Lefschetz pencils on symplectic varieties of dimension 4 with non semisimple monodromy. We will not discuss the proof of 3.3.2, but let us summarize what we have so far.

Theorem 3.3.3. *Suppose $f : X \to Y$ is a smooth projective map of quasi-projective manifolds of relative dimension n, $\mathcal{L}$ a relatively ample line bundle, and L its first Chern class.*

- **Decomposition** *There is an isomorphism in* $\mathcal{D}_Y$

$$Rf_*\mathbb{Q} \simeq \oplus_q R^q f_*\mathbb{Q}[-q]$$

- **Hard Lefschetz along the fibres** *Cupping with* L *defines isomorphisms of local systems on* Y

$$L^k : R^{n-q} f_*\mathbb{Q} \longrightarrow R^{n+q} f_*\mathbb{Q}.$$

- **Semisimplicity** *The local systems* $R^q f_*\mathbb{Q}$ *are, for every* q*, semisimple local systems.*

As will be explained in the next lectures, the three statements above will generalize to any projective map, once the local systems are replaced by *intersection cohomology complexes of semisimple local systems.* More precisely, the category of local systems is replaced by a different category, that of *perverse sheaves*, which has strikingly similar formal properties.

4 Two classical theorems on surfaces. The local invariant cycle theorem

4.1 Homological interpretation of the contraction criterion and Zariski's lemma

Let $f : X \longrightarrow Y$ be a birational proper map with X a nonsingular surface, and Y normal; let $y \in Y$ be a singular points. Set $C := f^{-1}(y)$ and let C_i, for $i = 1, \dots k$, be its irreducible components. Then the intersection numbers (C_i, C_j) give a symmetric matrix, hence a symmetric bilinear form, called the *intersection form*, on the vector space generated by the C_i's.

Example 4.1.1. Let $C \subseteq \mathbb{P}^n$ be a nonsingular projective curve, and let $\mathcal{C} \subseteq \mathbb{C}^{n+1}$ be the affine cone over C, with vertex o. Blowing up the vertex we get a nonsingular surface $\widetilde{\mathcal{C}}$, the total space of the line bundle $\mathcal{O}(-1)_{|C}$, with the blow-down map $\beta : \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$. We have $\beta^{-1}(o) = C$, where C is imbedded in $\widetilde{\mathcal{C}}$ as the zero section. In this case the intersection form amount to the self-intersection number $C^2 = -\deg C$.

We have the classical theorem:

Theorem 4.1.2. *The* intersection form (C_i, C_j) *associated to an exceptional curve* $C = \bigcup C_i$ *is negative definite.*

There is a similar statement for surfaces mapping to a curve: the set-up is as follows: $f : X \longrightarrow Y$ is a projective map, with X a nonsingular surface, and Y a nonsingular curve. Let $y \in Y$, and $C := f^{-1}(y)$ and let C_i, for $i = 1, \dots, k$, be its irreducible components. As before, the intersection numbers (C_i, C_j) define a symmetric bilinear form, the *intersection form*, on the vector space $\mathcal{V}$ generated by the C_i's. If $f^{-1}(y) = \sum n_i C_i$, clearly its intersection with every element in $\mathcal{V}$ vanishes, being algebraically equivalent to every other fibre. The following is known as *Zariski's lemma*

Theorem 4.1.3. *The intersection form is negative-semidefinite. Its radical is spanned by the class* $\sum n_i C_i$ *of the fibre of* y.

Clearly the statement aboove is empty whenever the fibre over y is irreducible.

We are going to give an interpretation of these two classical results in terms of splittings of the derived direct image sheaf $Rf_*\mathbb{Q}$. To do this we need to introduce the constructible derived category.

Recall that a *stratification* Σ of an algebraic variety Y is a decomposition $Y = \coprod_{l \geq 0} S_l$ where the $S_l \subseteq Y$ are locally closed and nonsingular subvarieties of pure complex dimension l. A sheaf on Y is said to be *constructible* if there exists a stratification of the variety such that the restriction of the sheaf to each stratum is a locally constant sheaf of finite rank. The category $\mathcal{D}_Y$ that we are interested in has

- *Objects*: bounded complexes of sheaves

  $$K^{\bullet} : \ldots \longrightarrow K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \ldots$$

  of $\mathbb{Q}$-vector spaces, such that their cohomology sheaves

  $$\mathcal{H}^i(K^{\bullet}) := \mathrm{Ker} d^i / \mathrm{Im} d^{i-1}$$

  are constructible.

- *Morphisms*: a map $\phi : K^{\bullet} \longrightarrow L^{\bullet}$ between two objects is defined by a diagram

  $$K^{\bullet} \xrightarrow{\phi'} \widetilde{L}^{\bullet} \xleftarrow{u} L^{\bullet},$$

  where u is a quasi isomorphism, namely a map of complexes of sheaves which induces isomorphisms

  $$\mathcal{H}^i(u) : \mathcal{H}^i(L^{\bullet}) \longrightarrow \mathcal{H}^i(\widetilde{L}^{\bullet})$$

  for every i. There is a natural equivalence relation on such diagrams which we will not discuss. A map in $\mathcal{D}_Y$ is an equivalence class of such diagrams.

The category $\mathcal{D}_Y$ is remarkably stable with respect to many operations, most notably Verdier's duality, to be explained in a future lecture by M.A. de Cataldo. Furthermore, algebraic maps $f : X \longrightarrow Y$ define several functors between $\mathcal{D}_Y$ and $\mathcal{D}_Y$, such as the derived direct image Rf_*, the proper direct image $Rf_!$, the pull-back functors f^*, and $f^!$, related by a rich formalism. For details see the texts in the bibliography. We will use freely this formalism. Recall also that the *(standard) truncation functors* $\tau_{\leq}, \tau_{\geq}$ defined on complexes send quasi-isomorphisms to quasi-isomorphisms and obviously preserve constructibility of cohomology sheaves, hence they define functors in $\mathcal{D}_Y$

Let $f : X \to Y$ be the resolution of singularities of the normal surface Y. For simplicity, let us suppose that Y has a unique singular point y, and as before, let us set $C := f^{-1}(y)$ and $C = \bigcup C_i$.

Let us study the complex $Rf_*\mathbb{Q}_X$.

There is the commutative diagram with Cartesian squares

$$\begin{array}{ccccc} C & \xrightarrow{I} & X & \xleftarrow{J} & U \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} \\ y & \xrightarrow{i} & Y & \xleftarrow{j} & U \end{array}$$

where $U := Y \setminus y = X \setminus Z$.

The distinguished attaching triangle associated to the restriction from Y to U gives

There is the distinguished attaching triangle for $Rf_*\mathbb{Q}_X$:

$$i_!i^!Rf_*\mathbb{Q}_X \longrightarrow Rf_*\mathbb{Q}_X \longrightarrow Rj_*j^*Rf_*\mathbb{Q}_X \simeq Rj_*\mathbb{Q}_{Y\setminus\{y\}} \xrightarrow{[1]} .$$

Let us consider the long exact sequence of the cohomology sheaves at y. Note that $R^if_*\mathbb{Q} = 0$ for $i > 2$, $R^1f_*\mathbb{Q}$, $R^2f_*\mathbb{Q}$ are concentrated at y, while $R^0f_*\mathbb{Q} = \mathbb{Q}$ in force of our hypotheses.

There is a fundamental system of Stein neighborhoods of y such that if N is any neighborhood in the family and $N' := N \setminus \{y\}$ we have

$$\mathcal{H}^a(i_!i^!Rf_*\mathbb{Q}_X)_y = H^a(f^{-1}(N), f^{-1}(N')) \simeq H_{4-a}(C),$$

which vanishes for $a \neq 2, 3, 4$,

$$\mathcal{H}^a(Rf_*\mathbb{Q}_X)_y = H^a(f^{-1}(N)) \simeq H^a(C),$$

which vanishes for $a \neq 0, 1, 2$, and

$$\mathcal{H}^a(Rj_*j^*Rf_*\mathbb{Q}_X)_y = \mathcal{H}^a(Rj_*\mathbb{Q}_{N'})_y \simeq H^a(N').$$

There is the adjunction map $\mathbb{Q}_Y \to Rf_*f^*\mathbb{Q}_Y = Rf_*\mathbb{Q}_X$. This map does not split. We study the obstruction to this failure. As already noted, since $\dim_{\mathbb{C}} f^{-1}(y) = 1$, we have $\tau_{\leq 2}Rf_*\mathbb{Q}_X \simeq Rf_*\mathbb{Q}_X$. The truncation functors yield a map

$$\widetilde{u} \ : \ Rf_*\mathbb{Q}_X \longrightarrow \tau_{\leq 2}Rj_*\mathbb{Q}_{N'}.$$

Consider the truncation distinguished triangle

$$\tau_{\leq 1} Rj_* \mathbb{Q}_{N'} \longrightarrow \tau_{\leq 2} Rj_* \mathbb{Q}_{N'} \longrightarrow \mathcal{H}^2(Rj_* \mathbb{Q}_{N'})[-2] \xrightarrow{[1]} .$$

Note that the last complex is in fact reduced to the skyscraper complex $H^2(N')_y[-2]$. Apply the cohomological functor $\mathrm{Hom}_{\mathcal{D}_Y}(Rf_* \mathbb{Q}_X, -)$ to the triangle and take the associated long exact sequence

$$0 \to \mathrm{Hom}(Rf_* \mathbb{Q}_X, \tau_{\leq 1} Rj_* \mathbb{Q}_{N'}) \to \mathrm{Hom}(Rf_* \mathbb{Q}_X, \tau_{\leq 2} Rj_* \mathbb{Q}_{N'}) \to \mathrm{Hom}(Rf_* \mathbb{Q}_X, H^2(N')[-2]).$$

The map $\widetilde{u}$ maps to a map in $\mathrm{Hom}(Rf_* \mathbb{Q}_X, H^2(N')[-2]) = \mathrm{Hom}(R^2 f_* \mathbb{Q}_X, H^2(N')) = \mathrm{Hom}(H^2(f^{-1}(N)), H^2(f^{-1}(N'))$. This map is just the restriction map, which fits in the long exact sequence

$$\ldots \longrightarrow H_2(C) \xrightarrow{I} H^2(C) \simeq H^2(f^{-1}(N)) \longrightarrow H^2(f^{-1}(N')) \longrightarrow \ldots$$

where $I \in \mathrm{Hom}(H_2(C), H^2(C)) \simeq H^2(C) \otimes H^2(C)$ is just the intersection matrix discussed above. This being nondegenerate, we have that I is an isomorphism, and the restriction map $H^2(f^{-1}(N)) \longrightarrow H^2(f^{-1}(N')$ vanishes. This means that there exists a (unique) lift $\widetilde{v} : Rf_* \mathbb{Q}_X \to \tau_{\leq 1} Rj_* \mathbb{Q}_U$.

Taking the cone of this map, one obtains a distinguished triangle

$$C \to f_* \mathbb{Q}_X \to \tau_{\leq 1} j_* \mathbb{Q}_U \xrightarrow{[1]} .$$

An argument similar to the previous one, shows that $\widetilde{v}$ admits a canonical splitting so that there is a canonical isomorphism in $\mathcal{D}_Y$:

$$Rf_* \mathbb{Q}_X \; \simeq \; \tau_{\leq 1} Rj_* \mathbb{Q}_U \; \oplus \; H^2(C)_y[-2].$$

The upshot is that $\mathbb{Q}_Y \to Rf_* \mathbb{Q}_X$ does not split, however, in looking for the non-existing map $Rf_* \mathbb{Q}_X \to \mathbb{Q}_Y$ we find the remarkable and more interesting splitting map $\widetilde{v}$.

Why is this interesting? Because, up to shift, the complex $\tau_{\leq 1} Rj_* \mathbb{Q}_{N'}$ is, by definition, the *intersection cohomology complex* (see next lecture) IC_Y of Y. The complex $H^2(C)_y[-2]$ is, up to shift, the intersection cohomology complex of y with multiplicity $b_2(Z)$. We may re-write the splitting as

$$Rf_* \mathbb{Q}_X[2] \; \simeq \; IC_Y \oplus IC_y^{b_2(Z)}$$

thus obtaining a first non trivial example of the decomposition Theorem.

We can do something similar for a projective map $f : X \longrightarrow Y$ with X a nonsingular surface, and Y a nonsingular curve. For simplicity, we assume that the generic fibre is connected. Set $j : Y' \longrightarrow Y$ the open imbedding of the regular value locus of f. The restriction $f' : X' := f^{-1}(Y') \longrightarrow Y'$ is a smooth family of curves, and by theorem 2.0.2 we have a splitting in $\mathcal{D}_{Y'}$

$$Rf'_*\mathbb{Q}_{X'} \simeq R^0f'_*\mathbb{Q}_{X'} \oplus R^1f'_*\mathbb{Q}_{X'}[-1] \oplus R^2f'_*\mathbb{Q}_{X'}[-2], \tag{4.1.1}$$

which we may re-write as

$$Rf'_*\mathbb{Q}_{X'} \simeq \mathbb{Q}_{Y'} \oplus R^1f'_*\mathbb{Q}_{X'}[-1] \oplus \mathbb{Q}_{Y'}[-2], \tag{4.1.2}$$

as we have $R^0f'_*\mathbb{Q}_{X'} = R^2f'_*\mathbb{Q}_{X'} = \mathbb{Q}_{Y'}$, since the fibres over Y' are nonsingular and connected.

If we try to investigate whether we can extend the splitting (4.1.2) to Y, we see that this time, at the crucial step, we may use Zariski's lemma, to conclude that there is an isomorphism

$$Rf_*\mathbb{Q}_X \simeq \mathbb{Q}_Y \oplus j_*R^1f'_*\mathbb{Q}_{X'}[-1] \oplus \mathbb{Q}[-2] \oplus (\oplus V_{y_i}) \tag{4.1.3}$$

where V_{y_i} is a skyscraper sheaf concentrated at the points y_i where the fibre is reducible; the dimension of V_{y_i} equals the number of irreducible components of the fibre over y_i minus one.

Again, it turns out that the sheaf $j_*R^1f'_*\mathbb{Q}_{Y'}$ (j_* is the non-derived direct image sheaf: $j_*R^1f'_*\mathbb{Q}_{Y'} = \tau_{\geq 0}Rj_*R^1f'_*\mathbb{Q}_{Y'}$) is the *intersection cohomology complex* of the local system $R^1f'_*\mathbb{Q}_{Y'}$. The two examples discussed above turn out to be special cases of the general theorems, to be discussed in the last lecture of this course.

4.2 the local invariant cycle theorem, the limit mixed Hodge structure and the Clemens-Schmid exact sequence.

We consider again the family of curves $f : X \longrightarrow Y$ and, specifically, the sheaf theoretic decomposition 4.1.3. Let $y \in Y \setminus Y'$, and take the stalk at y of the first cohomology sheaf $\mathcal{H}^1$. Pick a disc N around y, and let $y_0 \in N \setminus \{y\}$. We have a monodromy operator $T : H^1(f^{-1}(y_0), \mathbb{Q}) \longrightarrow H^1(f^{-1}(y_0), \mathbb{Q})$. Recall that if $\mathcal{V}$ is a local system on the punctured disc, with monodromy T, then $(j_*\mathcal{V})_0 = H^0(\Delta^*, \mathcal{V}) \simeq \mathrm{Ker}(T - I)$. We find

$$H^1(f^{-1}(y), \mathbb{Q}) = (R^1f_*\mathbb{Q})_y = (j_*R^1f'_*\mathbb{Q}_{Y'})_y = \mathrm{Ker}(T - I) \subseteq H^1(f^{-1}(y_0), \mathbb{Q}).$$

This is, in this particular case, the content of the *local invariant cycle theorem*, whose general statement is:

Theorem 4.2.1. Local invariant cycle theorem *Let $f : X \to \Delta$ a projective flat map, smooth over the puntured disc Δ^*, and assume X nonsingular. Denote by $X_0 := f^{-1}(0)$ the central fibre, let $t_0 \in \Delta^*$ be a fixed base-point, and $X_{t_0} := f^{-1}(t_0)$. Let $T : H^k(X_{t_0}) \longrightarrow H^k(X_{t_0})$ be the monodromy around 0. Then, for every k, the sequence*

$$H^k(X_0, \mathbb{Q}) \longrightarrow H^k(X_{t_0}, \mathbb{Q}) \xrightarrow{T-I} H^k(X_{t_0}, \mathbb{Q})$$

is exact.

Note that the degeneration at E_2 theorem 2.0.2 implies that every monodromy invariant cohomology class on a generic fibre X_{t_0} is the restriction of a class on $X \setminus X_0$. The local invariant cycle theorem however says much more: if Δ is small enough, there is a homotopy equivalence $X \simeq X_0$ (the retraction on the central fibre). Theorem 4.2.1 states that every monodromy invariant cohomology class on X_{t_0} comes in fact from a cohomology class on the total space X.

The statement of theorem 4.2.1 can be refined by introducing a rather non-intuitive mixed Hodge structure on $H^k(X_{t_0}, \mathbb{Q})$, the *limit mixed Hodge structure.*

Let $f : X^* \to \Delta^*$ be a projective smooth map over the puntured disc Δ^*. The monodromy operators $T : H^k(X_{t_0}, \mathbb{Q}) \longrightarrow H^k(X_{t_0}\mathbb{Q})$ are quasi-unipotent, namely $(T^a - I)^b = 0$. Taking base change by $\zeta^a : \Delta^* \longrightarrow \Delta^*$ has the effect of replacing the monodromy T with its power T^a, hence we can suppose T unipotent. We define

$$N := \log T = \sum \frac{1}{k}(T - I)^k,$$

noting that the sum is finite. N is nilpotent: Suppose $N^b = 0$ (it actually follows from the monodromy theorem that we can take $b = k + 1$ if we are considering the monodromy of $H^k(X_{t_0})$). We have the following linear algebra result:

Theorem 4.2.2. *(***Monodromy weight filtration.***)There is an increasing filtration of $\mathbb{Q}$-subspaces of $H^k(X_{t_0}, \mathbb{Q})$:*

$$\{0\} \subseteq W_0 \subseteq W_1 \subseteq \ldots W_{2k-1} \subseteq W_{2k} = H^k(X_{t_0})$$

such that

- *$N(W_l) \subseteq W_{l-2}$ for every l*
- *$N^l : \mathrm{Gr}_W^{k+l} \longrightarrow \mathrm{Gr}_W^{k-l}$ is an isomorphism for every l.*

We have the following surprising

Theorem 4.2.3. *(***The limit mixed Hodge structure***) For every k, there exists a decreasing filtration $F^\bullet_{\lim}$ on $H^k(X_{t_0}, \mathbb{C})$, such that $(H^k(X_{t_0}, \mathbb{C}), W_\bullet, F^\bullet_{\lim})$, where $W_\bullet$ is the filtration associated with the endomorphism $\log T$ as described in theorem 4.2.2, is a mixed Hodge structure. Furthermore, the map $\log T : H^k(X_{t_0}, \mathbb{Q}) \longrightarrow H^k(X_{t_0}, \mathbb{Q})$ becomes a map of type $(-1, -1)$ with respect to this limit mixed Hodge structure.*

Even more remarkably, this limit mixed Hodge structure, which, we emphasize, is constructed just from the family over the punctured disc, is related by the mixed Hodge structure of the central fibre, thus giving a delicate interplay between the monodromy properties of the smooth family and the geometry of the central fibre. This is the content of the Clemens-Schmid exact sequence theorem.

Suppose $f : \overline{X} \longrightarrow \Delta$ is a projective map, smooth outside 0, such that the central fibre is a reduced divisor with global normal crossing, namely, in the irreducible components decomposition

$$f^{-1}(0) = \bigcup X_\alpha,$$

the X_α are nonsingular and meet transversely. The *semistable reduction theorem* states that any degeneration over the disc may be brought in semistable form after a finite base change ramified at 0 and a birational modification. The mixed Hodge structure of the cohomology of a normal crossing is easily expressed by a spectral sequence in the category of mixed Hodge structures involving the cohomology groups of the X_α's and their intersections. We have the (co)specialization map, defined as the composition $H^k(X_0, \mathbb{Q}) \xrightarrow{\simeq} H^k(X, \mathbb{Q}) \longrightarrow H^k(X_{t_0}, \mathbb{Q})$.

Theorem 4.2.4. Clemens-Schmid exact sequence*. The specialization map is a map of mixed Hodge structures if we consider on $H^k(X_{t_0}, \mathbb{Q})$ the limit mixed Hodge structure. There is an exact sequence of mixed Hodge structures (with appropriate Tate twists)*

$$\ldots \longrightarrow H_{2\dim X - k}(X_0, \mathbb{Q}) \longrightarrow H^k(X_0, \mathbb{Q}) \longrightarrow H^k(X_{t_0}, \mathbb{Q}) \xrightarrow{N} H^k(X_{t_0}, \mathbb{Q}) \longrightarrow$$
$$\longrightarrow H_{2\dim X - k - 2}(X_0, \mathbb{Q}) \longrightarrow H^{k+2}(X_0, \mathbb{Q}) \longrightarrow H^{k+2}(X_{t_0}, \mathbb{Q}) \xrightarrow{N} \ldots$$

References

[1] M.A. de Cataldo and L. Migliorini: "The Decomposition Theorem and the topology of algebraic maps," *Bull. of the Amer. Math. Soc.*, Vol. 46, n.4, (2009), 535-633.

[2] J. Carlson, S. Müller-Stach, C. Peters: *Period mappings and period domains,* Cambridge Studies in Advanced Mathematics, **85**. Cambridge University Press, Cambridge, 2003.

[3] A. Dimca: *Sheaves in topology,* Universitext. Springer-Verlag, Berlin, 2004.

[4] P. Griffiths, ed.: *Topics in transcendental algebraic geometry,* Ann. of Math. Studies **106**, Princeton University Press, Princeton 1984.

[5] B. Iversen: *Cohomology of Sheaves*, Universitext, Springer-Verlag, Berlin Heidelberg 1986.

[6] M. Kashiwara and P. Schapira: *Sheaves on manifolds*, Grundlehren der mathematischen Wissenschaften. Vol. **292**, Springer-Verlag, Berlin Heidelberg 1990.

[7] C. Peters, J. Steenbrink: *Mixed Hodge Structures,* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics, Vol.**52** Springer Verlag Berlin Heidelberg 2008.

[8] C. Voisin: *Hodge theory and complex algebraic geometry,* I, II. Cambridge Studies in Advanced Mathematics, **76, 77**. Cambridge University Press, Cambridge, 2003.