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The Hodge theory of maps

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5 Lectures at the Hodge Theory Summer School
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by Luca Migliorini and Mark Andrea de Cataldo

Notes for Lectures 4 and 5 delivered by M. de Cataldo

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Abstract

These are the lecture notes from my two lectures 4 and 5. To get an idea of what you will find in them, parse the table of contents. Caveat emptor (buyer beware, no refunds): the lectures have a very informal flavor to them and the notes reflect this fact. There are plenty of exercises and some refererences so you can start looking things up on your own. My book [5] contains some of the notions discussed here, as well as some amplifications.

Contents

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1 Sheaf cohomology and all that (a minimalist approach)

SC

1. We say that a sheaf of Abelian groups I on a topological space X is *injective* if

   the Abelian-group-valued functor on sheaves $\mathrm{Hom}(-, I)$ is exact.

2. **Exercise.**

   (a) Verify that for every sheaf F, the functor $\mathrm{Hom}(-, F)$ is exact on one side, but, in general, not on the other.

   (b) The injectivity of I is equivalent to the following:
   for every injection $F \to G$ and every map $F \to I$ there is a map $G \to I$ making the obvious diagram commute.

   (c) A short exact sequence $0 \to I \to A \to B \to 0$ splits.

   (d) If $0 \to A \to B \to C \to 0$ is exact and A is injective, then B is injective IFF C is.

   (e) A vector space over a field field k is an injective k-module.
   Reversing the arrows, you can define the notion of *projectivity* (for sheaves, modules over a ring, etc.). Show that free implies projective.

3. It is a fact that every Abelian group can be embedded into an injective Abelian group. Obviously, this is true in the category of vector spaces!

4. **Exercise.**

   Deduce form the embedding statememt above that every sheaf F can be embedded into an injective sheaf. (Hint: consider the sheaf $\Pi_{x \in X} F_x$ on X.)

5. By iteration of the embedding result, it is easy to show that given every sheaf F, there is an *injective resolution of* F, i.e. a long exact sequence

   $$0 \to F \xrightarrow{e} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} \dots$$

   s.t. each I is injecitve.

6. The resolution is not unique, but it is so in the homotopy category. Let us not worry about this. See [4].

7. Let $f : X \to Y$ be continuous and F be a sheaf on X.

   The *direct image sheaf* f_*F on Y is the sheaf (check this)

   $$U \mapsto F(f^{-1}(U)).$$

8. A *complex of sheaves* K is a

$$\ldots \to K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} \ldots$$

with $d^2 = 0$. We have the cohomology sheaves

$$\mathcal{H}^i(K) := \operatorname{Ker} d^i / \operatorname{Im} d^{i-1}$$

- recall that everything is defined as a presheaf and you must take the associated sheaf; the only excpetion is the kernel (check this) -.

A *map of complexes* $f : K \to L$ is a compatible system of maps $f^i : K^i \to L^i$. Compatible means that the obvious diagrams commute.

There is the induced map $\mathcal{H}^i(K) \to \mathcal{H}^i(L)$.

A *quasi-isomorphism (qis)* $f : K \to L$ is a map inducing isomorphisms on all cohomology sheaves.

The *translated complex* $K[d]$ has $K[d]^i : K^{d+i}$ with the same differentials (up to sign $(-1)^d$).

Note that $K[1]$ means moving the entries one step to the *left*.

An exact sequence of complexes is the obvious thing.

Later, I will mentione *distinguished triangles*:

$$K \to L \to M \xrightarrow{+} K[1]$$

You can mentally replace this with a short exact sequence

$$0 \to K \to L \to M \to 0$$

and this turns out to be ok.

9. The *direct image complex* Rf_*F associacted with (F, f) is "the" complex of sheaves on Y

$$Rf_*F := f_*I,$$

where $F \to I$ is an injective resolution as above.

This is well-definied up to unique isomorphism in the homotopy category. This is easy to verify (check it). For the basic definitions, and a proof of this fact, see [4].

10. If C is a *bounded below* complex of sheaves on X, i.e. with $\mathcal{H}^i(K) = 0$ $\forall i \ll 0$, then C admits an *injective resolution*, i.e. a qis $C \to I$, where each entry I^j is injective.

Again, this is well-defined up to unique isomorphism in the homotopy category.

Rf_* is a "derived functor." However, this notion and the proof of this fact require to plunge in the derived cateogory. See [7].

11. We can thus define the *derived direct image complex*

$$Rf_*C := f_*I.$$

12. Define the *(hyper)cohomology groups of (X with coefficients in) C* as follows:

    take the unique map $c : X \to p$ (a point);

    take the complex of global sections $Rc_*C = c_*I = I(X)$;

    set

$$H^i(X,C) := H^i(I(X)).$$

13. **Exercise.**

    Use the homotopy statements to formulate and prove that these groups are well-defined (typically, this means: unique up to unique isomorphism; make this precise).

14. The *direct image sheaves of C wrt to f* are

$$R^if_*C := \mathcal{H}^i(Rf_*C) := \mathcal{H}^i(f_*I), \qquad i \in \mathbb{Z}.$$

15. **Exercise.**

    Prove that the sheaf R^if_*C is the sheaf associated with the presheaf

$$U \mapsto H^i(f^{-1}(U), C).$$

    (See [8])

    Remark that for every $y \in Y$ there is a natural map (it is called the base change map)

$$(R^if_*C)_y \longrightarrow H^i(X_y, C_{|X_y}).$$

    Give examples where this map is not an isomorphism/injecitve/surjective.

16. It is a fact that if I on X is injective, then f_*I on Y is injective.

    A nice proof of this fact uses the fact that the pull-back functor f^* on sheaves is the left adjoint to f_*, i.e. (cf. [7])

$$\mathrm{Hom}(f^*F, G) = \mathrm{Hom}(F, f_*G).$$

17. **Exercise**.

    Use the adjunction property to prove that I injective implies f_*I injective.

    Observe that the converse does not hold.

    Observe that if I is injective, then, in general, the pull-back f^*I is not injective. Find classes of maps for which the conclusion holds.

18. **Exercise.**

Use that f_* preserves injectives to deduce that

$$H^i(X, C) = H^i(Y, Rf_*C).$$

19. It is a fact that, on good spaces, the cohomology defined above with for the constant sheaf $\mathbb{Z}_X$ is the same as the one defined using Cech and singular cohomologies:

$$H^i(X, \mathbb{Z}_X) = H^i(X, \mathbb{Z}) = \check{H}^i(X, \mathbb{Z}).$$

20. **Exercise.**

(a) Let $j : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ be the open immerison. Determine the sheaves $R^q j_* \mathbb{Z}$.

20b (b) (DT!)
Let $X = Y = \mathbb{C}$, $X^* = Y^* = \mathbb{C}^*$, let $f : \mathbb{C} \to \mathbb{C}$ be the holomorpic map $z \mapsto z^2$, and let $f^* : \mathbb{C}^* \to \mathbb{C}^*$ be the restriction.
Show that $R^i f_* \mathbb{Z}_X = 0$, $\forall i > 0$. Ditto for f^*.
Show that there is a split short exact sequence of sheaves of vector spaces (if you use $\mathbb{Z}$-coefficients, there is no splitting)

$$0 \longrightarrow \mathbb{Q}_Y \longrightarrow f_* \mathbb{Q}_X \longrightarrow Q \longrightarrow 0$$

and study determine the stalks of Q.
Do the same over Y^*. The sequence is one of locally constant sheaves and Q is the locally constant sheaf with stalk at a fixed point $x \in X$ endowed with the automorphism multiplication by -1 (explain what this means).

(c) Show that on a good connected space X a locally constant sheaf L (a.k.a. *local system*) yields a representation of the fundamental group $\pi_1(X, x)$ in the group $A(L_x)$ of automorphisms of the stalk L_x at a pre-fixed point $x \in X$, and viceversa. (Hint: consider the quotient $(\tilde{X} \times L_x)/\pi_1(X, x)$ under a suitable action).

(d) Use the principle of analytic continuation and the monodromy theorem (cf. [9]) to prove that every local system on a simply connected space is constant (trivial representation).

(e) Give an example of a local system that is not semisismple. (simple:= irreducible:= no non trivial subobject; semisimple := direct sum of simples). (Hint: consider, for example, the standard 2×2 unipotent matrix.)
Note that the matrix in th eHint is the one of the Picard-Lefschetz transformation associated with the degeneration of a one-parameter family of elliptic plane cubic curves to a rational cubic curve with a node; in other words it is the monodromy of the associated non trivial! fiber bundle over a punctored disk with fiber $S^1 \times S^1$).

(f) Given a fiber bundle, e.g. a smooth proper map (see the Erhesmann Lemma $f : X \to Y$, with fiber X_y, prove that the direct image sheaf is locally constant with typical stalk

$$(R^i f_* \mathbb{Z}_X)_y = H^i(X_y, \mathbb{Z}).$$

(g) Show that the Hopf bundle $h : S^3 \to S^2$, with fiber S^1, is not (isomorphic to) a trivial bundle, though the local systems $R^i h_* \mathbb{Z}_{S^3}$ are trivial on S^2.

Do the same for $k : S^1 \times S^3 \to S^2$. Verify that you can turn the above into a proper holomorphic submersion of compact complex manifolds $k : S \to \mathbb{CP}^1$ (see the Hopf surface in [1])

Show that the Deligne theorem on the degeneration for smooth projective maps cannot hold for the Hopf map above. Deduce that this is an example of a map in complex geometry for which the DT does not hold.

(h) Show that if a map f is proper and with finite fibers (e.g. a finite topological covering, a branched covering, the normalization of a complex space, for example of a curve, the embedding of a closed subvariety etc.), then $R^i f_* F = 0$ for every $i > 0$ and every sheaf F.

Compute $f_* \mathbb{Z}$ in your examples.

2 The intersection cohomology complex

ic

We are going to define and "calculate" the intersection complex I_X of a variety of dimension d with one isolated singularity:

$$Y = Y_{reg} \coprod Y_{sing}, \qquad U := Y_{reg}, \qquad Y_{sing} = \{p\},$$

$$U \xrightarrow{j} Y \xleftarrow{i} p.$$

I_X, and its twisted version $I_X(L)$, can be defined for any variety.

1. Recall that given a complex K the *a-th truncated complex* $\tau_{\leq a}K$ is the subcomplex C with the following entries:

   $$C^b = K^b, \ \forall b < a, \qquad C^a = \operatorname{Ker} d^a, \qquad C^b = 0, \ \forall b > a.$$

   The single most important property is that

   $$\mathcal{H}^b(\tau_{\leq a}K) = \mathcal{H}^b(K), \qquad \forall b \leq a, \qquad \text{zero otherwise.}$$

2. Let Y be as above. Define the *intersection cohomology complex* (with $\mathbb{Z}$-coefficients, for example) as follows

   $$I_Y := \tau_{\leq d-1} Rj_* \mathbb{Z}_U.$$

3. **Toy model.**

   Let $Y \subseteq \mathbb{C}^3$ be the affine cone over an elliptic curve $E \subseteq \mathbb{CP}^2$.

   $R^0 j_* \mathbb{Z}_U = \mathbb{Z}_Y$ (recall that we always have $R^0 f_* = f_*$).

   As to the others we observe that U is the $\mathbb{C}^*$-bundle of the hyperplane line bundle H on E, i.e. the one induced by the hyperplane bundle on $\mathbb{CP}^2$. By choosing a metric, we get the unit sphere (here S^1) bundle U' over E. Note that U' and U have the same homotopy type. The bundle $U' \to E$ is automatically an oriented S^1-bundle. The associated Euler class $e \in H^2(E, \mathbb{Z})$ is the first Chern class $c_1(H)$.

4. **Exercise.**

   (You will find all you need in [3].) Use the spectral sequence for this oriented bundle (here it is just the Wang sequence) to compute the groups

   $$H^i(U', \mathbb{Z}) = H^i(U, \mathbb{Z}).$$

   Answer: (caution: the answer below is for $\mathbb{Q}$-coefficients only!: work this situation out and keep track of the torsion)

   $$H^0(U) = H^0(E), \quad H^1(U) = H^1(E), \quad H^2(U) = H^1(E), \quad H^3(U) = H^2(E).$$

   Deduce that, with $\mathbb{Q}$-coefficents (work out the $\mathbb{Z}$ case as well), we have that I_Y has only two non zero cohomology sheaves

   $$\mathcal{H}^0(I_Y) = \mathbb{Q}_Y, \qquad \mathcal{H}^1(I_Y) = H^1(E)_p \ \text{(skyscraper at } p\text{)}.$$

5. **Exercise.**

   Compute I_Y for $Y = \mathbb{C}^d$, with p the origin.

   Answer: $I_Y = \mathbb{Q}_Y$ (here $\mathbb{Z}$-coefficients ok).

6. The above result is general:

   if Y is nonnsingular, then $I_Y = \mathbb{Q}_Y$ ($\mathbb{Z}$ ok);

   if Y is the quotient of a nonsingular variety by a finite group action, then $I_Y = \mathbb{Q}_Y$ ($\mathbb{Z}$ coefficients, KO!).

7. Let L be a local system on U. Define

$$I_Y(L) := \tau_{\leq d-1} R j_* L.$$

   Note that, and this is a general fact, that

$$\mathcal{H}^0(I_Y(L)) = j_* L.$$

8. Useful notation: $j_! L$ is the sheaf on Y which agrees with L on U and has stalk zero at p.

9. **Exercise.**

   (a) Let C be a singular curve. Compute I_C.
   Answer: let $f : \hat{C} \to C$ be the normalization. Then $I_C = f_* \mathbb{Z}_{\hat{C}}$.

   (b) Let things be as in §1, Exercise 20b. Let $L = (f_* \mathbb{Z}_X)_{|Y^*}$ and $M := Q_{|Y^*}$. Compute

$$I_Y(L), \qquad I_Y(M).$$

   (c) Let U be as in the toy model. Determine $\pi_1(U)$. Classify local systems of low ranks on U. Find some of their $I_Y(L)$'s.

   (d) Let $f : C \to D$ be a branched cover of nonsingular curves. Let $f^o : C^o \to D^o$ be the corresponding topological cover.
   Prove that $L := f^o_* \mathbb{Q}_{C^o}$ is semisimple ($\mathbb{Z}$-coefficients is KO!, even fo rthe identity!).
   Determine $I_D(L)$ and describe its stalks. (Try higher dimensions.)

3 Verdier duality

vd

1. Let M^m be an oriented manifold. We have Poincaré duality:

$$H^i(M, \mathbb{Q}) \simeq H_c^{m-i}(M, \mathbb{Q})^*.$$

2. **Exercise.**

   Find compact and non-compact examples of the failure of Poincaré duality for singular complex varieties.

3. Verdier duality (which we do not define here; see [7]) achieves the following.

   Given a comple of sheaves K on Y, we get its *Verdier dual* K^*, with, for every open $U \subseteq Y$

$$H^i(U, K^*) = H_c^{-i}(U, K)^*.$$

   Note that $H_c^i(Y, K)$ is defined the same way as $H^i(Y, K)$, except that we take global sections with compact supports.

   The formation of $K*$ is functorial in K:

$$K \longrightarrow L, \qquad L^* \longrightarrow K^*,$$

   and satisifies

$$K^{**} = K, \qquad (K[l])^* = K^*[-l]$$

4. **Exercise.**

   Recall the definition of the translation functor $[m]$ on complexes and the one of H^i and H_c^i and show that

$$H^i(Y, K[l]) = H^{i+l}(Y, K), \qquad H_c^i(Y, K[l]) = H_c^{i+l}(Y, K).$$

5. It is a fact that, for the oriented manifold M^m we have

$$\mathbb{Q}_Y^* = \mathbb{Q}_Y[m]$$

   so that we get Poincaré duality (verify this!).

   If M is not oriented, you get something else. See [3] (look for "densities"); see [10] (look for "sheaf of orientations"); [9], look for "Borel-Moore cycles" and the resulting complex of sheaves (this maybe somewhere in [2].)

6. One of th emost important proeprties of I_Y is its self-duality, which we express as follows (the translation is just for comodity): first set

$$IC_Y := I_Y[d]$$

(we have translated the complex I_Y, which had non zero cohomology sheaves only in degrees $[0, d-1]$, to the LEFT by d units, so that the corresponding interval is now $[-d, -1]$), then we have that

$$IC_Y^* = IC_Y.$$

7. **Exercise.**

   Use the toy model to verify that the equality holds (in that case) at the level of cohomology sheaves by verifying that (here V is a "typical" neighborhood of p)

   $$\mathcal{H}^i(IC_Y)_p = H^i(V, IC_V) = H_c^{-i}(V, IC_V)^*.$$

   (To do this, you will need to compute $H_c^i(U)$ as you did $H^i(U)$; be careful though about using homotopy types and H_c!). (You will find the following distinguished triangle useful - recall we can view them as short exact sequences, and as such, yielding a long exact sequence of cohomology groups, with or without supports - :

   $$\mathcal{H}^0(I_Y) \longrightarrow I_Y \longrightarrow \mathcal{H}^1(I_Y)[-1] \xrightarrow{+};$$

   you will also find useful the following long exact sequence

   $$\ldots \longrightarrow H_c^a(U) \longrightarrow H_c^a(Y) \longrightarrow H_c^a(p) \longrightarrow H_c^{a+1}(U) \longrightarrow \ldots$$

8. Define the *intersection cohomology groups of* Y as

   $$IH^i(Y) = H^i(Y, I_Y), \qquad IH_c^i(Y) = H_c^i(Y, I_Y).$$

   The original definition is more geometric and involves chains and boundaries, like in the early day of homology.

9. Since $IC_Y^* = IC_Y$, we get that

   $$H^i(Y, IC_Y) = H_c^{-i}(Y, IC_Y)^*.$$

   Using $IC_Y = I_Y[d]$, we get, by Verdier duality

   $$H^i(Y, I_Y) = H_c^{2n-i}(Y, I_Y)^*,$$

   and we immediately deduce *Poincaré duality for intersection cohomology groups* on an arbitrarily singular complex algebraic variety/space:

   $$IH^i(Y, I_Y) = IH_c^{2d-i}(Y, I_Y)^*.$$

vdrf

10. Variant for twisted coefficients.

If $Y^o \subseteq Y_{reg} \subseteq Y$ and L is a local system on Y^o, we have $I_Y(L)$, its translated $IC_Y(L)$ and one has

$$IC_Y(L)^* = IC_Y(L^*).$$

There is the corrisponding duality statement for the groups $IH^i(Y, I_Y(L))$ etc.:

$$IH^i(Y, I_Y(L^*)) = IH_c^{2d-i}(Y, I_Y(L))^*,$$

where of course L^* is the dual local system.

11. **Exercise.**

Define the dual local system L^* of a local system L as the sheaf of germs of sheaf maps $L \to Q_Y$.

(a) Show that it is a local system and that there is a pairing (map of sheaves)

$$L \otimes_{\mathbb{Q}_Y} L^* \longrightarrow \mathbb{Q}_Y$$

inducing identifications

$$(L_y)^* = (L^*)_y.$$

(Recall that the tensor product is defined by taking the sheaf associated with the presheaf tensor product $U \mapsto L(U) \otimes \mathbb{Q}_U(U) L^*(U))$.

(b) If L is given by the representation $r : \pi_1(Y, y) \to A(L_y)$, find an expression for a representation associated with L^*. (Hint: inverse-transpose.)

12. *Verdier duality and Rf_* for a proper map.*

It is a fact that if f is proper, then

$$(Rf_*C)^* = Rf_*C^*.$$

We apply this to $IC_Y(L)^* = IC_Y(L^*)$ and get

$$(Rf_*IC_Y(L))^* = Rf_*IC_Y(L^*).$$

In particular,

$$Rf_*IC_Y \qquad \text{is self-dual.}$$

4 The decomposition theorem (DT)

dt

1. Let $f : X \to Y$ be a *proper* map of algebraic varieties and L be a *semisimple* (= direct sum of simples; simple = no nontrivial subobject) local system with $\mathbb{Q}$-coefficients (most of what follows fails with coefficinets not in a field of characteristic zero) on an dense open set $X^o \subseteq X_{reg} \subseteq X$.

   Examples include:

   X is nonsingular, $L = \mathbb{Q}_X$; then $I_X(L) = I_X = \mathbb{Q}_X$;

   X is singular, $L = \mathbb{Q}_{X_{reg}}$; then $I_X(L) = I_X$.

2. **Decomposition theorem.**

   There is a splitting in the derived category of sheaves on Y:

$$Rf_* \, I_X(L) \simeq \bigoplus_{b \in B} I_{\overline{Z_b}}(L_b)[l_b]$$

   where:

   B is a finite set of indices,

   $Z_b \subseteq Y$ is a collection of locally closed nonsingular subvarieties,

   L_b is a semisimple local system on Z_b, and

   $l_b \in \mathbb{Z}$.

3. The case where we take $I_X = I_X(L)$ is already important.

   Even is X and Y are smooth, we must dealt with I_Z's on Y, i.e. we cannot have a direct sum of shifted sheaves for example.

   Deligne's theorem (1968), including the semisimplicity statement (1972) for proper smooth maps of smooth varieties is a special case

$$Rf_* \, \mathbb{Q}_X \simeq \bigoplus_{i \geq 0} R^i f_* \mathbb{Q}_X[-i], \qquad I_Y(R^i f_* \mathbb{Q}_X) = R^i f_* \mathbb{Q}_X.$$

symm

4. **Exercise.**

   Show that, using the self-duality of IC_Y, the rule $(K[l])^* = K^*[-l]$, the DT above, and the fact that $IC_T = I_T[\dim T]$ to show that we have that the DT can also expressed in the following more symmetric form, where r is a uniquely determined nonnegative integer:

$$Rf_* IC_X \simeq \bigoplus_{i=-r}^{r} P^i[-i]$$

   where each P^i is a direct sum of some of the $IC_{\overline{Z_b}}$ appearing above *without furhter translations* [?]*!* and

$$(P^i)^* = P^{-i}, \qquad \forall i \in \mathbb{Z}.$$

Try this first in the case of smooth proper maps, where $Rf_*\mathbb{Q}_X = \oplus R^i f_*[-i]\mathbb{Q}_X$. This helps getting used to the change of indexing scheme as you go from I_Y to $IC_Y = I_Y[d]$.

5. **Exercise.**

   (a) Go back to all the examples we met earlier, see what the DT says there and study the summands.

   (b) Argue that the DT cannot possibly hold for the Hopf surface map $h : S \to \mathbb{CP}^1$ met earlier. (Hint: it is not an algebraic surface; still the DT could still hold; it is the classical example of a non Kähler manifold; still the DT could hold; it has first Bettin number $b_1(S) = 1$ and, finally, this prohibiths the conclusion of the DT from being true for this map.)

   Note that this is a proper holomorphic submersion of compact complex manifolds, that the target and the fibers are projective, yet DT fails!.

stc (c) Let $f : X \to C$ be a proper map with connected fibers, X a nonsingular algebraic surface, C a nonsingular curve.

   Let C^o be the set of regular values, $\Sigma := C \setminus C^o$ (it is a fact that it is finite). Let $f^o : X^o \to C^o$ and $j : C^o \to C$ be the obvious maps.

   Deligne's theorem applies to f^o and is a statement on C^o: show that it take the following form

$$Rf^o_*\mathbb{Q}_{X^o} \simeq \mathbb{Q}_{Y_o} \oplus R^i f^o_*\mathbb{Q}_{X^o}[-1] \oplus \mathbb{Q}_{C^o}[-2].$$

   Show that the DT takes the form (let $R^1 := R^1 f^o_*\mathbb{Q}_{X^o}$)

$$Rf_*\mathbb{Q}_X \simeq \mathbb{Q}_C \oplus j_*R^1[-1] \oplus \mathbb{Q}_C[-2] \oplus V_\Sigma[-2],$$

   where V_Σ is the skyscraper sheaf on the finite set Σ with stalk at each $\sigma \in \Sigma$ a vector space V_σ of rank equal to the number of irreducible components of $f^{-1}(\sigma)$ *minus one*.

   Find a more canonical description of V_σ as a quotient of $H^2(f^{-1}(\sigma))$.

   Note that this splitting contains quite a lot of information. Extract it:

   - the only feature of $f^{-1}(\sigma)$ that contributes to $H^*(X)$ is its number of components; if this is one, there is no contribution, no matter how singular (including multiplicities) the fiber;
   - let $c \in C$, let Δ be a small disk around c, let $\eta \in \Delta^*$ be a regular value;
     we have the bundle $f^* : X_{\Delta^*} \to \Delta^*$ with typical fiber X_η;
     we have the (local) monodromy for this bundle: i.e. R^i is a local system, i.e. $\pi_1(\Delta^*) = \mathbb{Z}$ acts on $H^i(X_\eta)$;
     denote by $R^{1^{\pi_1}} \subseteq R^1_\eta$ the invariants of this (local) action;
     show the following general fact for local systems L on a good connected space Y: the invariants of the local system $L_y^{\pi_1(Y,y)} = H^0(Y, L)$;

there are the natural map restriction maps

$$H^1(X_\eta) \supseteq H^1(X_\eta)^{\pi_1} \xleftarrow{r} H^1(f^{-1}(\Delta) \xrightarrow{\simeq} H^1(X_c);$$

use the DT above to deduce that r is *surjective* - this is the celbrated *local invariant cycle theorem*: all local invariant classes come from X_Δ -; it comes for free from the DT.

Finally observe, that in this case, we indeed have $Rf_*\mathbb{Q}_X \simeq \oplus R^i f_* \mathbb{Q}[-i]$ (but you should view this as a coincidence due to the low dimensions).

(d) Write down the DT for a projective bundle over a smooth variety.

(e) Ditto for the blowing up of a nonsingular subvariety of a nonsingular variety.

(f) Let Y be a 3-fold with an isolated singularity at $p \in Y$. Let $f : X \to Y$ be a resolution of the singularities of Y: f is an isomoprphism over $Y \setminus p$.

3f i. Assume $\dim f^{-1}(p) = 2$; show, using the symmetries expressed by Exercise 4, that the DT takes the following form:

$$Rf_*\mathbb{Q}_X = IC_Y \oplus V_p[-2] \oplus W_p[-4],$$

where $V_p = W_p^*$ are skyscraper sheaves with dual stalks.
Hint: use $H_4(X_p) \neq 0$ (why is this true?) to infer, using that $\mathcal{H}^4(I_X) = 0$, that one must have a summand contributing to $R^4 f_* \mathbb{Q}$ etc.
Deduce that the irreducible components of top dimension 2 of X_p yield linearly indepedent cohomology classes in $H^2(X)$.

ii. Assume that $\dim f^{-1}(p) \leq 1$. Show that we must have

$$Rf_*\mathbb{Q}_X = I_Y.$$

Note that this is remarkable and highlights a general principle: the proper algebraic maps are restricted by the fact that the topology of Y, impersonated by I_Y, restricts the topology of X. Using C^∞ maps, even real algebraic maps, you meet no such general restrictions.

5 The relative hard Lefschetz and the hard Lefschetz for intersection cohomology groups

rhl

1. Let $f : X \to Y$ be a projective smooth map of nonsingular varieties and $\ell \in H^2(X, \mathbb{Q})$ be the first Chern class of a line bundle on X which is ample (Hermitean positive) on every X_y.

   We have the iterated cup product map

   $$\ell^i : R^j f_* \mathbb{Q}_X \longrightarrow R^{j+2i} f_* \mathbb{Q}_X.$$

   We have the *hard Lefschetz theorem* for the iterated cup product action of $\ell_y \in H^*(X_Y, \mathbb{Q})$: let $d = \dim X_y$, then

   $$\ell_y^i : R^{d-i} f_* \mathbb{Q}_X \xrightarrow{\simeq} R^{d+i} f_* \mathbb{Q}_X.$$

   We view what above as the *relative hard Lefschetz theorem for smooth proper maps.*

2. Recall the symmetric form of the DT $Rf_* \simeq \oplus_i P^i[-i]$. It is a formality to show that we get iterated cup rpoduct maps

   $$\ell^i P^j \to P^{j+2i}.$$

   The *relative hard Lefschetz theorem* is the statement that

   $$\ell^i : P^{-i} \simeq P^i.$$

   Note that Verdier duality shows that $P^{-i} = (P^i)^*$. Verdier Duality holds in general, outside of algebraic geometry and holds, for example for the Hopf surface map $h : S \to \mathbb{CP}^1$.

   The RHL is a considerably deeper statement.

3. **Exercise.**

   (a) Make the statement of the RHL explicit in the example of a map from a surface to a curve (see §4, Exercise 5c.)

   (b) Ditto for §4, Exercise 5(f)i. (Hint: in this case you get $\ell : V_p \simeq W_p$).

4. The hard Lefschetz theorem in intersection cohomology: apply the rhl to $X \to$ *point*:

   let ℓ be the first Chern class of an ample line bundle on a projective variety X of dimension d, then

   $$\ell^i : IH^{d-i}(X, \mathbb{Q}) \xrightarrow{\simeq} IH^{d+i}(X, \mathbb{Q}).$$

5. Hodge-Lefscehtz package for intersection cohomology.

   Let X be a projective variety. Then the statements of (see [6] for these statements):

   the two Lefscehtz theorems, of the primitive Lefschetz decomposition, of the Hodge decomposition and of the Hodge-Riemann bilinear relations hold for $IH(X, \mathbb{Q})$.

6. **Exercise.**

   Let $f : X \to Y$ be a resolution of the singularities of a projective surface with isolated singularities (for simplicity; after you solve this, you may want to tackle the case when the singularities are not isolated).

   Show that the DT takes the form

   $$Rf_* \mathbb{Q}_X[2] = IC_Y \oplus V_\Sigma$$

   where Σ is the set of singularities of Y and V_Σ is skyscraper with fiber $V_\sigma = H^2(X_\sigma)$.

   Deduce that the fundamental classes E_i of the curves given by the irreducible components in the fibers are linearly independent.

   Use Poincar'é duality to deduce that the intersection form (cup product) matrix $||E_i \cdot E_j||$ on these classes is non degenerate.

   (Grauert proved a general theorem, valid in the analytic context and for a germ (Y, σ) that even shows that this form is negative definite).

   Show that the contribution $IH^*(Y)$ to $H^*(X)$ can be viewed as the space orthogonal to the span of the E_i's.

   Deduce that $IH^*(Y)$ sits inside $H^*(X)$ compatibly with the Hodge decomposition, i.e. $IH^j(Y)$ inherits a pure Hodge structure of weight j.

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