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Introductory Examples

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Introductory Examples

The basic idea of Hodge theory is that the cohomology of an algebraic variety has more structure than one sees when viewing the same object as a “bare” topological space. This extra structure helps us understand the geometry of the underlying variety, and it is also an interesting object of study in its own right. Because of the technical complexity of the subject, in this chapter, we look at some motivating examples which illuminate and guide our study of the complete theory. We shall be able to understand, in terms of specific and historically important examples, the notions of Hodge structure, period map, and period domain. We begin with elliptic curves, which are the simplest interesting Riemann surfaces.

1.1 Elliptic Curves

The simplest algebraic variety is the Riemann sphere, the complex projective space $\mathbb{P}^1$. The next simplest examples are the branched double covers of the Riemann sphere, given in affine coordinates by the equation

$$y^2 = p(x),$$

where $p(x)$ is a polynomial of degree d. If the roots of p are distinct, which we assume they are for now, the double cover C is a one-dimensional complex manifold, or a Riemann surface. As a differentiable manifold it is characterized by its genus. To compute the genus, consider two cases. If d is even, all the branch points are in the complex plane, and if d is odd, there is one branch point at infinity. Thus the genus g of such a branched cover C is $d/2$ when d is even and $(d-1)/2$ when d is odd. These facts follow from Hurwitz's formula, which in turn follows from a computation of Euler characteristics (see Problem 1.1.2). Riemann surfaces of genus 0, 1, and 2 are illustrated in Fig. 1. Note that if $d = 1$ or $d = 2$, then C is topologically a sphere. It is not hard to prove that it is also isomorphic to the Riemann sphere as a complex manifold.

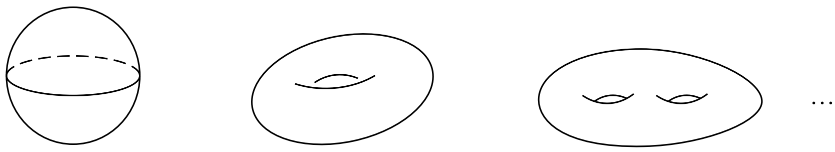

Figure 1. Riemann surfaces.

Now consider the case $d = 3$, so that the genus of C is 1. By a suitable change of variables, we may assume the three roots of $p(x)$ to be 0, 1, and λ, where $\lambda \neq 0, 1$:

$$y^2 = x(x-1)(x-\lambda). \tag{1.1.1}$$

We shall denote the Riemann surface defined by this equation (1.1.1) by $\mathcal{E}_\lambda$, and we call the resulting family the Legendre family. As topological spaces, and even as differentiable manifolds, the various $\mathcal{E}_\lambda$, are all isomorphic, as long as $\lambda \neq 0, 1$, a condition which we assume to be now in force. However, we shall prove the following:

Theorem 1.1.2 *Suppose that* $\lambda \neq 0, 1$. *Then there is an* $\epsilon > 0$ *such that for all* λ' *within distance* ϵ *from* λ, *the Riemann surfaces* $\mathcal{E}_\lambda$ *and* $\mathcal{E}_{\lambda'}$ *are not isomorphic as complex manifolds.*

Our proof of this result, which guarantees an infinite supply of essentially distinct elliptic curves, will lead us directly to the notions of period map and period domain and to the main ideas of Hodge theory.

The first order of business is to recall some basic notions of Riemann surface theory so as to have a detailed understanding of the topology of $\mathcal{E}_\lambda$, which for now we write simply as $\mathcal{E}$. Consider the multiple-valued holomorphic function

$$y = \sqrt{x(x-1)(x-\lambda)}.$$

On any simply connected open set which does not contain the branch points $x = 0,\ 1,\ \lambda,\ \infty$, it has two single-valued determinations. Therefore, we cut the Riemann sphere from 0 to 1 and from λ to infinity, as in Fig. 2. Then analytic continuation of y in the complement of the cuts defines a single-valued function. We call its graph a "sheet" of the Riemann surface. Note that analytic continuation of y around δ returns y to its original determination, so δ lies in a single sheet of $\mathcal{E}$. We can viewit as lying in the Riemann sphere

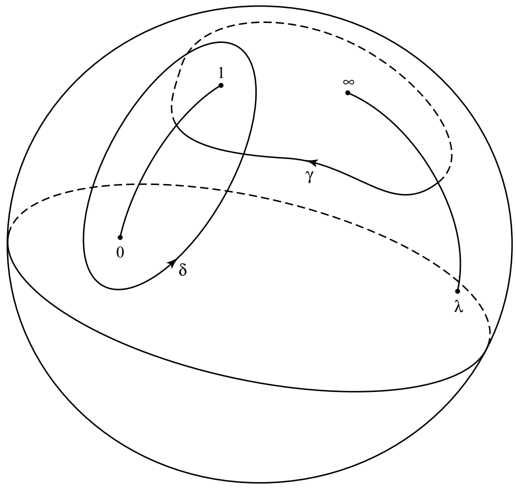

Figure 2. Cuts in the Riemann sphere.

itself. But when we analytically continue along γ, we pass from one sheet to the other as we pass the branch cut. That path is therefore made of two pieces, one in one sheet and one in the other sheet.

Thus the Riemann surface of y consists of two copies of the Riemann sphere minus the cuts, which are then "cross-pasted": we glue one copy to the other along the cuts but with opposite orientations. This assembly process is illustrated in Fig. 3. The two cuts are opened up into two ovals, the opened-up Riemann sphere is stretched tolook like the lower object in the middle,

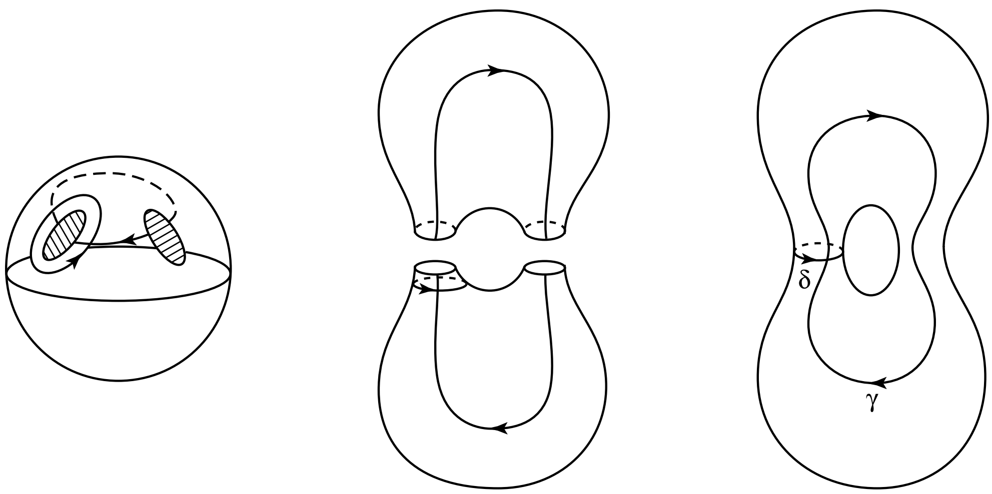

Figure 3. Assembling a Riemann surface.

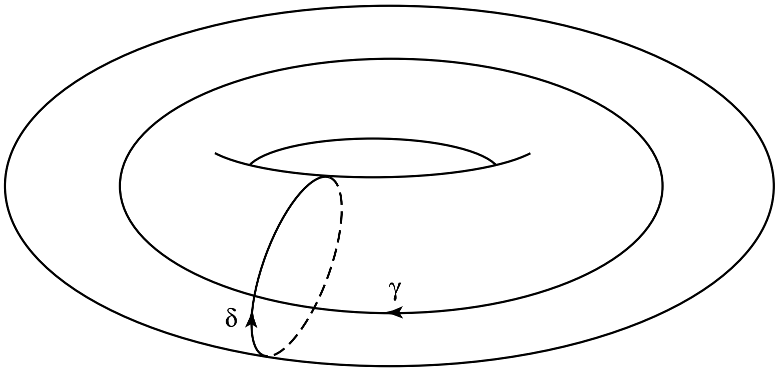

Figure 4. Torus.

a second copy is set above it to represent the other sheet, and the two sheets are cross-pasted to obtain the final object.

The result of our assembly is shown in Fig. 4.

The oriented path δ indicated in Fig. 4 can be thought of as lying in the Riemann sphere, as in Fig. 2, where it encircles one branch cut and is given parametrically by

$$\delta(\theta) = 1/2 + (1/2 + k)e^{i\theta}$$

for some small k. The two cycles δ and γ are oriented oppositely to the x and y axes in the complex plane, and so the intersection number of the two cycles is

$$\delta \cdot \gamma = 1.$$

We can read this information off either Fig. 3 or Fig. 2. Note that the two cycles form a basis for the first homology of $\mathcal{E}$ and that their intersection matrix is the standard unimodular skew form,

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

With this explanation of the homology of our elliptic curve, we turn to the cohomology. Recall that cohomology classes are given by linear functionals on homology classes, and so they are given by integration against a differential form. (This is de Rham's theorem – see Theorem 2.1.1). In order for the line integral to be independent of the path chosen to represent the homology class, the form must be closed. For the elliptic curve $\mathcal{E}$ there is a naturally given differential one-form that plays a central role in the story we are recounting. It is defined by

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}. \tag{1.1.3}$$

As discussed in Problem 1.1.1, this form is holomorphic, that is, it can be written locally as

$$\omega = f(z)dz,$$

where z is a local coordinate and $f(z)$ is a holomorphic function. In fact, away from the branch points, x is a local coordinate, so this representation follows from the fact that $y(x)$ has single-valued holomorphic determinations. Because f is holomorphic, ω is closed (see Problem 1.1.7). Thus it has a well-defined cohomology class.

Now let δ^* and γ^* denote the basis for $H^1(\mathcal{E};\mathbb{Z})$ which is dual to the given basis of $H_1(\mathcal{E};\mathbb{Z})$. The cohomology class of ω can be written in terms of this basis as

$$[\omega] = \delta^* \int_\delta \omega + \gamma^* \int_\gamma \omega.$$

In other words, the coordinates of $[\omega]$ with respect to this basis are given by the indicated integrals. These are called the *periods* of ω. In the case at hand, they are sometimes denoted A and B, so that

$$[\omega] = A\delta^* + B\gamma^*. \tag{1.1.4}$$

The expression (A, B) is called the *period vector* of $\mathcal{E}$.

From the periods of ω we are going to construct an invariant that can detect changes in the complex structure of $\mathcal{E}$. In the best of all possible worlds this invariant would have different values for elliptic curves that have different complex structures. The first step toward constructing it is to prove the following.

Theorem 1.1.5 *Let $H^{1,0}$ be the subspace of $H^1(\mathcal{E};\mathbb{C})$ spanned by ω, and let $H^{0,1}$ be the complex conjugate of this subspace. Then*

$$H^1(\mathcal{E};\mathbb{C}) = H^{1,0} \oplus H^{0,1}.$$

The decomposition asserted by this theorem is the *Hodge decomposition* and it is fundamental to all that follows. Now there is no difficulty in defining the $(1, 0)$ and $(0, 1)$ subspaces of cohomology: indeed, we have already done this. The difficulty is in showing that the defined subspaces span the cohomology, and that (equivalently) their intersection is zero. In the case of elliptic curves, however, there is a quite elementary proof of this fact. Take the cup product of (1.1.4) with its conjugate to obtain

$$[\omega] \cup [\bar{\omega}] = (A\bar{B} - B\bar{A})\delta^* \cup \gamma^*.$$

Multiply the previous relation by $\mathrm{i} = \sqrt{-1}$ and use the fact that $\delta^* \cup \gamma^*$ is the fundamental class of $\mathcal{E}$ to rewrite the preceding equation as

$$\mathrm{i} \int_{\mathcal{E}} \omega \wedge \bar{\omega} = 2\,\mathrm{Im}(B\bar{A}).$$

Now consider the integral above. Because the form ω is given locally by $f\,dz$, the integrand is locally given by

$$\mathrm{i}|f|^2 dz \wedge d\bar{z} = 2|f|^2 dx \wedge dy,$$

where $dx \wedge dy$ is the natural orientation defined by the holomorphic coordinate, that is, by the complex structure. Thus the integrand is locally a positive function times the volume element, and so the integral is positive. We conclude that

$$\mathrm{Im}(B\bar{A}) > 0.$$

We also conclude that neither A nor B can be 0 and, therefore, that the cohomology class of ω cannot be 0. Consequently the subspace $H^{1,0}(\mathcal{E})$ is nonzero.

Because neither A nor B can be 0 we can rescale ω and assume that $A = 1$. For such "normalized" differentials, we conclude that *the imaginary part of the normalized B-period is positive*:

$$\mathrm{Im}\, B > 0. \tag{1.1.6}$$

Now suppose that $H^{1,0}$ and $H^{0,1}$ do not give a direct sum decomposition of $H^1(\mathcal{E};\mathbb{C})$. Then $H^{1,0} = H^{0,1}$, and so $[\bar{\omega}] = \lambda[\omega]$ for some complex number λ. Therefore

$$\delta^* + \bar{B}\gamma^* = \lambda(\delta^* + B\gamma^*).$$

Comparing coefficients, we find that $\lambda = 1$ and then that $B = \bar{B}$, in contradiction with the fact that B has a positive imaginary part. This completes the proof of the Hodge theorem for elliptic curves, Theorem 1.1.5.

An Invariant of Framed Elliptic Curves

Now suppose that $f : \mathcal{E}_\mu \longrightarrow \mathcal{E}_\lambda$ is an isomorphism of complex manifolds. Let ω_μ and ω_λ be the given holomorphic forms. Then we claim that

$$f^*\omega_\lambda = c\,\omega_\mu \tag{1.1.7}$$

for some nonzero complex number c. This equation is certainly true on the level of cohomology classes, although we do not yet know that c is nonzero.

However, on the one hand,

$$\int_{[\mathcal{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = |c|^2 \int_{[\mathcal{E}_\mu]} \omega_\mu \wedge \bar{\omega}_\mu,$$

and on the other,

$$\int_{[\mathcal{E}_\mu]} f^*\omega_\lambda \wedge f^*\bar{\omega}_\lambda = \int_{f_*[\mathcal{E}_\mu]} \omega_\lambda \wedge \bar{\omega}_\lambda = \int_{[\mathcal{E}_\lambda]} \omega_\lambda \wedge \bar{\omega}_\lambda.$$

The last equality uses the fact that an isomorphism of complex manifolds is a degree-one map. Because $\mathrm{i}\,\omega_\lambda \wedge \bar{\omega}_\lambda$ is a positive multiple of the volume form, the integral is positive and therefore

$$c \neq 0. \tag{1.1.8}$$

We can now give a preliminary version of the invariant alluded to above. It is the ratio of periods B/A, which we write more formally as

$$\tau(\mathcal{E}, \delta, \gamma) = \frac{\int_\gamma \omega}{\int_\delta \omega}.$$

From Eq. (1.1.6) we know that τ has a positive imaginary part. From the just-proved proportionality results (1.1.7) and (1.1.8), we conclude the following.

Theorem 1.1.9 *If $f : \mathcal{E} \longrightarrow \mathcal{E}'$ is an isomorphism of complex manifolds, then $\tau(\mathcal{E}, \delta, \gamma) = \tau(\mathcal{E}', \delta', \gamma')$, where $\delta' = f_*\delta$ and $\gamma' = f_*\gamma$.*

To interpret this result, let us define a *framed elliptic curve* $(\mathcal{E}, \delta, \gamma)$ to consist of an elliptic curve and an integral basis for the first homology such that $\delta \cdot \gamma = 1$. Then we can say that "if framed elliptic curves are isomorphic, then their τ-invariants are the same."

Holomorphicity of the Period Mapping

Consider once again the Legendre family (1.1.1) and choose a complex number $a \neq 0,\ 1$ and an $\epsilon > 0$ which is smaller than both the distance from a to 0 and the distance from a to 1. Then the Legendre family, restricted to λ in the disk of radius ϵ centered at a, is trivial as a family of differentiable manifolds. This means that it is possible to choose two families of integral homology cycles δ_λ and γ_λ on $\mathcal{E}_\lambda$ such that $\delta_\lambda \cdot \gamma_\lambda = 1$. We can "see" these cycles by modifying Fig. 2 as indicated in Fig. 5. A close look at Fig. 5 shows that we

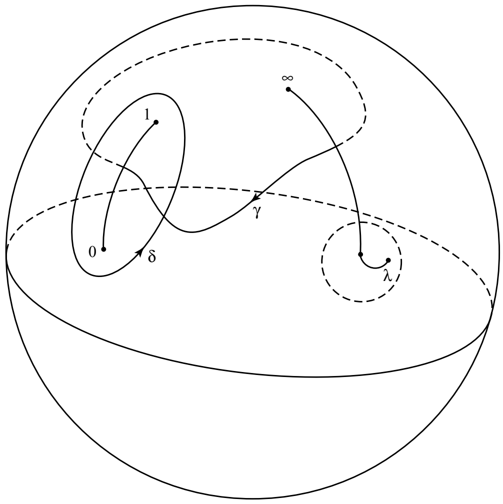

Figure 5. Modified cuts in the Riemann sphere.

can move λ within a small disk Δ without changing either δ_λ or γ_λ. Thus we can view the integrals defining the periods A and B as having constant domains of integration but variable integrands.

Let us study these periods more closely, writing them as

$$A(\lambda) = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}, \qquad B(\lambda) = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

We have suppressed the subscript on the homology cycles in view of the remarks made at the end of the previous paragraph. The first observation is the following.

Proposition 1.1.10 *On any disk Δ in the complement of the set $\{0, 1, \infty\}$, the periods of the Legendre family are single-valued holomorphic functions of λ.*

The proof is straightforward. Since the domain of integration is constant, we can compute $\partial A/\partial\bar{\lambda}$ by differentiating under the integral sign. But the integrand is a holomorphic expression in λ, and so that derivative is 0. We conclude that the period function $A(\lambda)$ is holomorphic, and the same argument applies to $B(\lambda)$.

Notice that the definitions of the period functions A and B on a disk Δ depend on the choice of a symplectic homology basis $\{\delta, \gamma\}$. Each choice of basis gives a different determination of the periods. However, if δ' and γ' give a different basis, then

$$\begin{aligned}\delta' &= a\delta + b\gamma\\ \gamma' &= c\delta + d\gamma,\end{aligned}$$

where the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has determinant 1. The periods with respect to the new basis are related to those with respect to the old one as follows:

$$\begin{aligned}A' &= aA + bB\\ B' &= cA + dB.\end{aligned}$$

Thus the new period vector (A', B') is the product of the matrix T and the old period vector (A, B). The τ-invariants are related by the corresponding fractional linear transformation:

$$\tau' = \frac{d\tau + c}{b\tau + a}.$$

The ambiguity in the definition of the periods and of the τ invariant is due to the ambiguity in the choice of a homology basis. Now consider a simply connected open set U of $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ and a point λ_0 and λ of U. The choice of homology basis for $\mathcal{E}_{\lambda_0}$ determines a choice of homology basis for all other fibers $\mathcal{E}_\lambda$. Thus the periods $A(\lambda)$ and $B(\lambda)$ as well as the ratio $\tau(\lambda)$ are single-valued holomorphic functions on U. On the full domain $\mathbb{P}^1 \backslash \{0, 1, \infty\}$, however, these functions are multivalued.

We can now state a weak form of Theorem 1.1.2.

Theorem 1.1.11 *The function τ is nonconstant.*

If τ, defined on a simply connected open set U, is a nonconstant holomorphic function then its derivative is not identically zero. Therefore its derivative has at most isolated zeroes. For a randomly chosen point, different from one of these zeroes, τ is a locally injective function.

There are at least two ways to prove that τ is nonconstant. One is to compute the derivative directly and to show that it is nonzero. The other is to show that τ tends to infinity as λ approaches infinity along a suitable ray in the complex plane. We give both arguments, beginning with an analysis of τ along a ray.

Asymptotics of the Period Map

Let us show that τ is a nonconstant function of λ by showing that τ approaches infinity along the ray $\lambda > 2$ of the real axis. Indeed, we will show that $\tau(\lambda)$ is asymptotically proportional to $\log\lambda$. To see this, assume $\lambda \gg 2$, and observe that

$$\int_\delta \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} \sim \int_\delta \frac{dx}{x\sqrt{-\lambda}} = \frac{2\pi}{\sqrt{\lambda}}.$$

By deforming the path of integration, we find that

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = -2\int_1^\lambda \frac{dx}{\sqrt{x(x-1)(x-\lambda)}}.$$

The difference between the last integral and $1/x(\sqrt{x-\lambda})$ is $1/(2x^2) +$ higher powers of $1/x$, an expression with asymptotically negligible integral $-\lambda^{-1}/4 +$ higher powers of λ^{-1}. The residual integral,

$$-2\int_1^\lambda \frac{dx}{x\sqrt{(x-\lambda)}},$$

can be computed exactly:

$$-2\int_1^\lambda \frac{dx}{x\sqrt{x-\lambda}} = \frac{4}{\sqrt{\lambda}}\arctan\frac{\sqrt{1-\lambda}}{\sqrt{\lambda}} \sim \frac{2\mathrm{i}}{\sqrt{\lambda}}\log\lambda.$$

Thus one finds

$$\tau(\lambda) \sim \frac{\mathrm{i}}{\pi}\log\lambda, \tag{1.1.12}$$

as claimed. Note also that $\tau(\lambda)$ has a positive imaginary part, as asserted in (1.1.6).

Derivative of the Period Map

We now prove the strong form of Theorem 1.1.2 by showing that $\tau'(\lambda) \neq 0$ for $\lambda \neq 0,\ 1$ for any determination of τ. To this end, we write the holomorphic differential ω_λ in terms of the dual cohomology basis $\{\delta^*,\ \gamma^*\}$:

$$\omega_\lambda = A(\lambda)\delta^* + B(\lambda)\gamma^*.$$

The periods are coefficients that express ω_λ in this basis, and the invariant $\tau(\lambda)$ is an invariant of the line spanned by the vector ω_λ. The expression

$$\omega'_\lambda = A'(\lambda)\delta^* + B'(\lambda)\gamma^*$$

is the derivative of the cohomology class ω_λ with respect to the "Gauss–Manin connection." This is by definition the connection on the bundle of cohomology vector spaces

$$\bigcup_{\lambda \in \mathbb{P}^1 \setminus \{0,1,\infty\}} H^1(\mathcal{E}_\lambda)$$

with respect to which the classes δ^*, γ^* are (locally) constant. Then

$$[\omega_\lambda] \cup [\omega'_\lambda] = (AB' - A'B)\, \delta^* \cup \gamma^*.$$

However, $\tau' = 0$ if and only if $AB' - A'B = 0$. Thus, to establish that $\tau'(\lambda) \neq 0$, it suffices to establish that $[\omega_\lambda] \cup [\omega'_\lambda] \neq 0$.

Now the derivative of ω_λ is represented by the meromorphic form

$$\omega'_\lambda = \frac{1}{2} \frac{dx}{\sqrt{x(x-1)(x-\lambda)^3}}. \tag{1.1.13}$$

This form has a pole of multiplicity two at $p = (\lambda, 0)$. To see this, note that at the point p, the function y is a local coordinate. Therefore the relation $y^2 = x(x-1)(x-\lambda)$ can be written as $y^2 = u(y)\lambda(\lambda-1)(x-\lambda)$ where $u(y)$ is a holomorphic function of y satisfying $u(0) = 1$. Solving for x, we obtain $x = \lambda +$ terms of order ≥ 2 in y. Then setting $p(x) = x(x-1)(x-\lambda)$, we have

$$\omega = \frac{dx}{y} = \frac{2dy}{p'(x)} \sim \frac{2dy}{\lambda(\lambda-1)},$$

where $a \sim b$ means that a and b agree up to lower-order terms in y. Using (1.1.13), the previous expression, and the expansion of x in terms of y, we find

$$\omega' = \frac{1}{2} \frac{dx}{y(x-\lambda)} \sim \frac{dy}{\lambda(\lambda-1)(x-\lambda)} \sim \frac{dy}{y^2} + \text{a regular form}. \tag{1.1.14}$$

To explain why such a form represents a cohomology class on $\mathcal{E}$, not just on $\mathcal{E}\setminus\{p\}$, we first note that its residue vanishes. Recall that the residue of ϕ is defined as

$$\frac{1}{2\pi \mathrm{i}} \int_{C_p} \phi = \operatorname{res}(\phi)(p),$$

where C_p is a small positively counterclockwise–oriented circle on S centered at p. Next, note that the residue map in fact is defined on the level of cohomology classes (just apply Stokes' theorem). In fact the resulting map "res" is the coboundary map from the exact sequence of the pair $(\mathcal{E}, \mathcal{E}\backslash\{p\})$,

$$0 \longrightarrow H^1(\mathcal{E}) \longrightarrow H^1(\mathcal{E}\backslash\{p\}) \xrightarrow{\text{res}} H^2(\mathcal{E}, \mathcal{E}\backslash\{p\}),$$

provided we identify the third vector space with $\mathbb{C}$ using the isomorphism

$$H^2(\mathcal{E}, \mathcal{E}\backslash\{p\}) \cong H^0(\{p\}) \cong \mathbb{C}.$$

See Problem 1.1.10, where this sequence is discussed in more detail.

Observe that the sequence above is the simplest instance of the so-called Gysin sequence for a smooth hypersurface (here just the point p) inside a smooth variety (here the curve $\mathcal{E}$). The Gysin sequence is at the heart of many calculations and is treated in detail in Section 3.2.

From the Gysin sequence we see that ω'_λ represents a cohomology class on $\mathcal{E}$, not just on $\mathcal{E}\backslash\{p\}$. We now claim that

$$\int_{\mathcal{E}} [\omega] \cup [\omega'] = \frac{4\pi \mathrm{i}}{\lambda(\lambda - 1)}. \tag{1.1.15}$$

By establishing this formula we will complete the proof that $\tau'(\lambda) \neq 0$ for $\lambda \neq 0,\ 1$. To do this, first observe that the formula (1.1.14) implies that $\omega' + d(1/y)$ has no pole at p. To globalize this computation, let U be a coordinate neighborhood of p on which $|y| < \epsilon$, and let $\rho(z)$ be a smooth function of $|z|$ alone which vanishes for $|z| > \epsilon/2$, which is identically one for $|z| < \epsilon/4$, and which decreases monotonically in $|z|$ on the region $\epsilon/4 < |z| < \epsilon/2$. Then the form

$$\tilde{\omega}' = \omega' + d(\rho(y)/y)$$

lies in the same cohomology class on $\mathcal{E}\backslash\{p\}$ as does ω'. By construction, it extends to a form on $\mathcal{E}$ and represents the cohomology class of ω' there. Because ω and $\tilde{\omega}'$ are both holomorphic one-forms on the complement of U,

$$\int_{\mathcal{E}} \omega \wedge \tilde{\omega}' = \int_U \omega \wedge d(\rho/y).$$

Because $\omega \wedge d(\rho/y) = -d(\rho\omega/y)$, Stokes' theorem yields

$$\int_U d\frac{\rho\omega}{y} = -\int_{|y|=\frac{\epsilon}{4}} \frac{\rho\omega}{y} = -\int_{|y|=\frac{\epsilon}{4}} \frac{\omega}{y}.$$

A standard residue calculation of the line integral then yields

$$\int_{|y|=\epsilon/4} \frac{\omega}{y} = \frac{4\pi \mathrm{i}}{\lambda(\lambda - 1)}.$$

This completes the proof.

Picard–Fuchs Equation

In computing the derivative of the period map, we proved that the form ω and its derivative ω' define linearly independent cohomology classes. Therefore the class of the second derivative must be expressible as a linear combination of the first two classes. Consequently there is a relation

$$a(\lambda)\omega'' + b(\lambda)\omega' + c(\lambda)\omega = 0 \tag{1.1.16}$$

in cohomology. The coefficients are meromorphic functions of λ, and on the level of forms the assertion is that the left-hand side is exact on $\mathcal{E}_\lambda$. Let ξ be a one-cycle and set

$$\pi(\lambda) = \int_\xi \omega.$$

Then (1.1.16) can be read as a differential equation for the period function

$$a\pi'' + b\pi' + c\pi = 0.$$

One can determine the coefficients in this expression. The result is a differential equation with regular singular points at 0, 1, and ∞:

$$\lambda(\lambda - 1)\pi'' + (2\lambda - 1)\pi' + \frac{1}{4}\pi = 0. \tag{1.1.17}$$

Solutions are given by hypergeometric functions (see [45], Section 2.11). To find the coefficients a, b, c above, we seek a rational function f on $\mathcal{E}_\lambda$ such that df is a linear combination of ω, ω', and ω'' whose coefficients are functions of λ. Now observe that

$$\begin{aligned} \omega' &= \frac{1}{2}x^{-1/2}(x-1)^{-\frac{1}{2}}(x-\lambda)^{-\frac{3}{2}}\,dx \\ \omega'' &= \frac{3}{4}x^{-\frac{1}{2}}(x-1)^{-\frac{1}{2}}(x-\lambda)^{-\frac{5}{2}}\,dx. \end{aligned}$$

Thus it is reasonable to consider the function

$$f = x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}(x-\lambda)^{-\frac{3}{2}}.$$

Indeed,

$$df = (x-1)\omega' + x\omega' - 2x(x-1)\omega'',$$

which is a relation between ω' and ω''. This is progress, but the coefficients are not functions of λ. Therefore consider the equivalent form

$$\begin{aligned} df &= [(x-\lambda) + (\lambda - 1)]\omega' + [(x-\lambda) + \lambda]\omega' \\ &\quad - 2[(x-\lambda) + \lambda][(x-\lambda) + (\lambda - 1)]\omega'' \end{aligned}$$

and use the relations

$$(x-\lambda)\omega' = \frac{1}{2}\omega, \qquad (x-\lambda)\omega'' = \frac{3}{2}\omega'$$

to obtain

$$-\frac{1}{2}df = \frac{1}{4}\omega + (2\lambda - 1)\omega' + \lambda(\lambda - 1)\omega''.$$

This completes the derivation.

One can find power series solutions of (1.1.17) which converge in a disk Δ_0 around any $\lambda_0 \neq 0, 1$. Analytic continuation of the resulting function produces a multivalued solution defined on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. Now let π_1 and π_2 be two linearly independent solutions defined on Δ_0, and let γ be a loop in $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ based at λ_0. Let π_i' be the function on Δ_0 obtained by analytic continuation of π_i along γ. Because the π_i' are also solutions of the differential equation (1.1.17), they can be expressed as linear combinations of π_1 and π_2:

$$\begin{pmatrix} \pi_1' \\ \pi_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}.$$

The indicated matrix, which we shall write as $\rho(\gamma)$, depends only on the homotopy class of α, and is called the *monodromy matrix*. We determine this matrix in the next section using a geometric argument. For now we note that the map that sends α to $\rho(\alpha)$ defines a homomorphism

$$\rho : \pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, \lambda_0) \longrightarrow \mathrm{GL}(2, \mathbb{C}).$$

It is called the *monodromy representation* and its image is called the *monodromy group*.

The Local Monodromy Representation

To better understand the monodromy representation, consider the family of elliptic curves $\mathcal{E}_s$ defined by

$$y^2 = (x^2 - s)(x - 1).$$

The fiber $\mathcal{E}_0$, given by $y^2 = x^2(x-1)$ has a node at $p = (0, 0)$. As s approaches 0, the fiber undergoes the changes pictured in Fig. 6. A copy of the loop δ is slowly contracted to a point, producing the double point at p. Note that in the limit of $s = 0$, the cycle δ is homologous to 0.

Now restrict this family to the circle $|s| = \epsilon$ and consider the vector field $\partial/\partial\theta$ in the s-plane. It lifts to a vector field ξ on the manifold

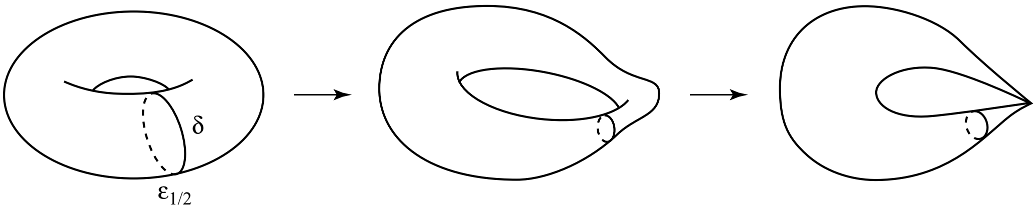

Figure 6. Degeneration of an elliptic curve.

$M = \{(x, y, s) \mid y^2 = (x^2 - s)(x - 1)\}$ which fibers over the circle via $(x, y, s) \mapsto s$. By letting the flow which is tangent to ξ act for time ϕ, one defines a diffeomorphism g_ϕ of the fiber at $\theta = 0$ onto the fiber at $\theta = \phi$. This is illustrated in Fig. 7. (We think of a fluid flow transporting points of $\mathcal{E}_0$ to points of $\mathcal{E}_\phi$, with streamlines tangent to the vector field.)

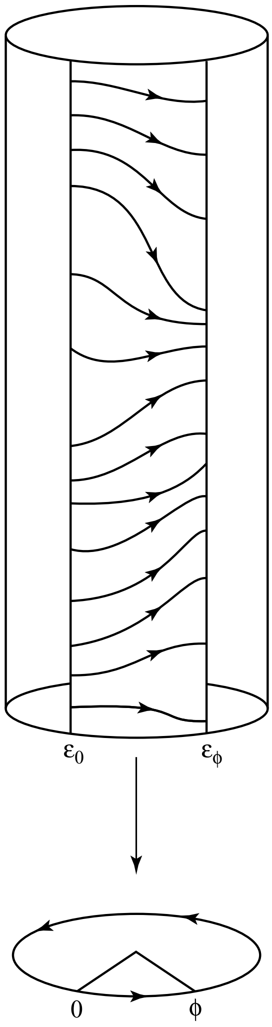

Figure 7. Diffeomorphism $g_\phi : \mathcal{E}_0 \longrightarrow \mathcal{E}_\phi$.

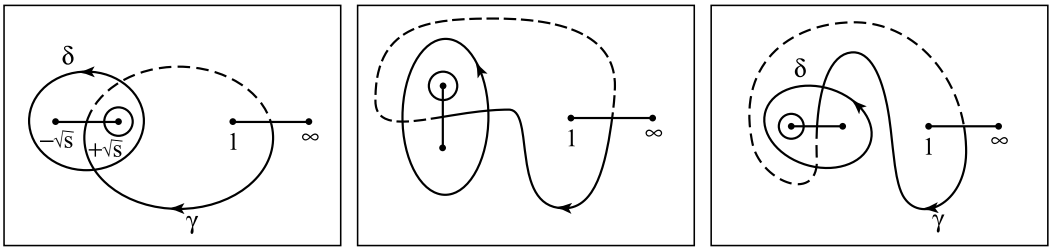

Figure 8. Picard–Lefschetz transformation.

Now consider the diffeomorphism $g_{2\pi}$. It carries the fiber at $\theta = 0$ to itself and therefore defines a map T on the homology of the fiber which depends only on the homotopy class of $g_{2\pi}$. This is the *Picard–Lefschetz transformation* of the degeneration $\mathcal{E}_\lambda$.

Through a careful study of the pictures in Fig. 8, we obtain the matrix of T in the "standard" basis $\{\delta, \gamma\}$. Because the matrix is not the identity, we conclude that the diffeomorphism is not homotopic to the identity map.

The left panel in Fig. 8 represents the cycle δ and γ on the fiber $\mathcal{E}_s$ for $s = r$ for some small r. The middle panel in Fig. 8 shows how the flow has mapped these cycles to $\mathcal{E}_s$ with $s = re^{\pi \mathrm{i}}$. The right panel shows the result for $s = re^{2\pi \mathrm{i}}$.

It is clear from the pictures in Fig. 8 that $T(\delta) = \delta$. To determine $T(\gamma) = \gamma'$, we observe that $\gamma' = a\delta + b\gamma$, and we compute intersection numbers as follows. Using the sign convention as explained above in relation to Fig. 4, we claim that $\gamma' \cdot \delta = -1$, $\gamma' \cdot \gamma = +1$. The former is clear from the right picture in Fig. 8, while the latter follows by superimposing the left and right pictures in Fig. 8. Thus $b = 1$ and $a = 1$, and so

$$T(\gamma) = \gamma + \delta.$$

The matrix of T relative to the basis $\{\delta, \gamma\}$ is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{1.1.18}$$

Equivalently, we have the Picard–Lefschetz formula

$$T(x) = x - (x \cdot \delta)\delta \tag{1.1.19}$$

for an arbitrary homology cycle x.

The Picard–Lefschetz formula is valid in great generality: it holds for any degeneration of Riemann surfaces acquiring a node where the local analytic equation of the degeneration is $y^2 = x^2 - s$. For such a degeneration the cycle

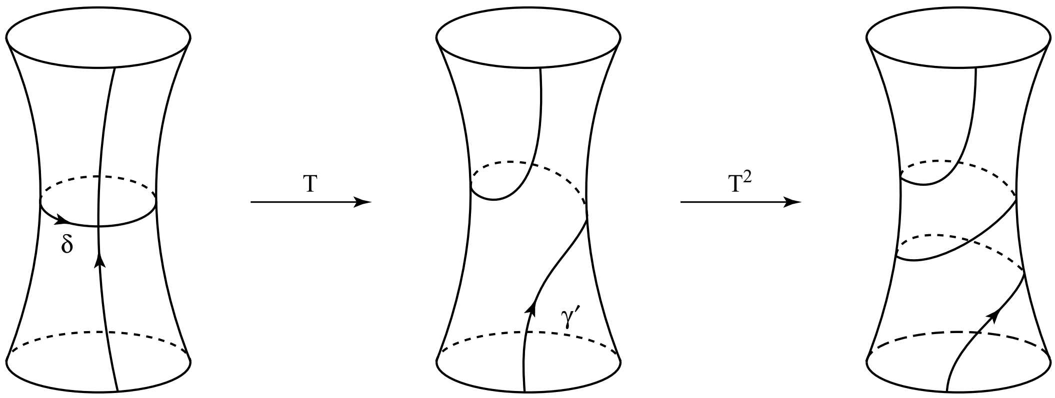

Figure 9. Dehn twist.

δ is the one that is "pinched" to obtain the singular fiber, as in Fig. 6. This is the so-called vanishing cycle: under the inclusion of $\mathcal{E}_s$ into the total space of the degeneration, δ is homologous to 0. In a neighborhood of the vanishing cycle the Picard–Lefschetz diffeomorphism $g_{2\pi}$ acts as in Fig. 9: it is a so-called *Dehn twist*.

The Global Monodromy Representation

The Picard–Lefschetz transformation determines the *local monodromy representation* $\rho : \pi_1(\Delta^*, p) \longrightarrow \mathrm{GL}(2, \mathbb{C})$ for a family of Riemann surfaces defined on the punctured disk $0 < |s| < \epsilon$, where the fiber at $s = 0$ has a node. Let us now determine the *global monodromy transformation* for the Legendre family $y^2 = x(x - 1)(x - \lambda)$. This is a representation

$$\rho : \pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\}, p) \longrightarrow \mathrm{GL}(2, \mathbb{C}).$$

As a first step, consider the degeneration $\lambda \to 0$. If λ moves clockwise around the circle of radius $r < 1$ centered at 0, then the slit connecting the branch cuts $x = \lambda$, and $x = 0$ turns through a full circle. Comparing with Fig. 8, where the slit makes a half turn, we see that the monodromy transformation for $\lambda \to 0$ is the square of the matrix T in (1.1.18).

Now fix a base point p and choose generators a and b for the fundamental group of the parameter space as in Fig. 10.

Let $A = \rho(a)$ and $B = \rho(b)$ be the monodromy matrices relative to the basis indicated in the left picture of Fig. 8. From the discussion in the previous paragraph, we have

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

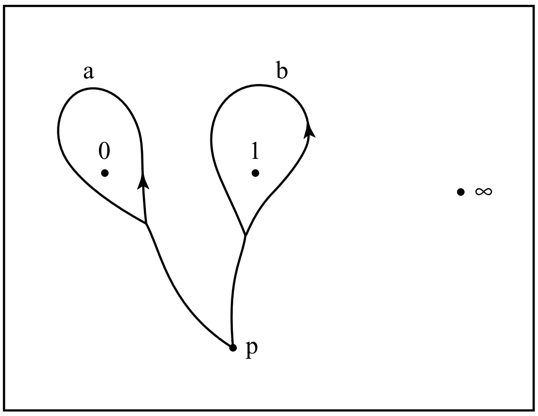

Figure 10. Parameter space for the Legendre family.

We claim that

$$B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

To see that this is so, consider the degeneration $\lambda \to 1$ and recall, as shown in Fig. 11, how the standard homology basis is defined relative to the standard branch cuts.

The set of branch cuts in Fig. 11 is ill adapted to computing the monodromy matrix of the degeneration $\lambda \to 1$. Instead we consider the cuts in Fig. 12. The first frame gives the standard homology basis relative to this set of cuts. The second frame shows the result of rotating the branch slit connecting λ and 1 through a half circle. The cycle δ' is obtained by dragging γ along with this

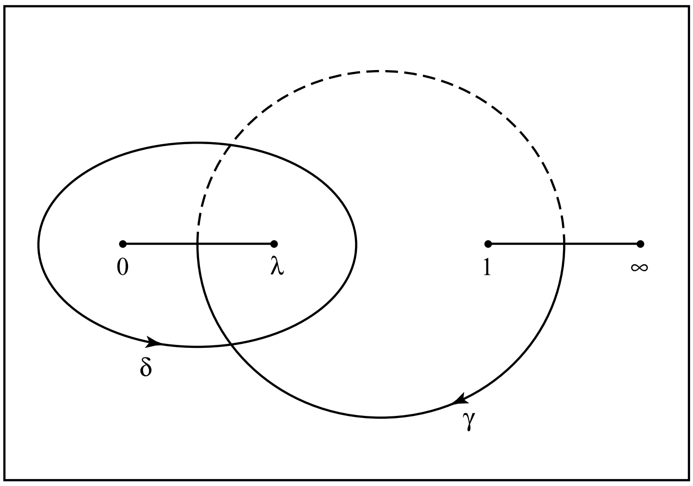

Figure 11. Standard homology basis.

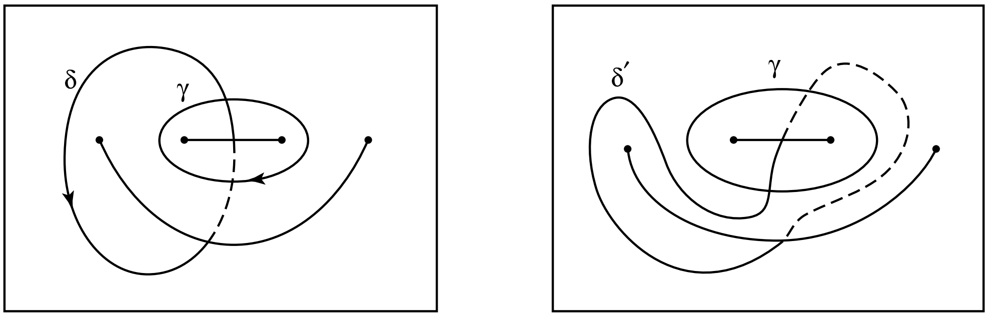

Figure 12. Monodromy for $\lambda \to 1$.

rotation via the Picard–Lefschetz diffeomorphism. Computing intersection numbers, we find $\delta' = \delta - \gamma$. Thus the Picard–Lefschetz transformation is

$$S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Note that this matrix can also be computed using (1.1.19), taking note of the fact that the vanishing cycle is γ. The monodromy transformation $\rho(b)$ is therefore given by S^2, which is the indicated matrix B.

Now let $\Gamma = \rho(\pi_1)$ denote the monodromy group. According to the preceding discussion, it is the group generated by the matrices A and B:

$$\Gamma = \langle A, B \rangle .$$

The given matrices are congruent modulo 2 to the identity matrix, so every matrix in Γ has this property. Let $\Gamma(N)$ be the set of matrices in $SL(2, \mathbb{Z})$ which is congruent to the identity modulo N. It is a normal subgroup of finite index, and what we have seen is that Γ is a subgroup of $\Gamma(2)$. We now assert the following result, which completely describes the global monodromy representation.

Theorem 1.1.20

(a) $\pi_1(\mathbb{P}^1 \backslash \{0, 1, \infty\})$ *is a free group on two generators.*
(b) The monodromy representation is injective.
(c) The image of the monodromy representation is $\Gamma(2)$.
(d) $\Gamma(2)$ *has index six in* $SL(2, \mathbb{Z})$.

Proof. The proof of (a) is standard, since $\mathbb{P}^1 \backslash \{0, 1, \infty\}$ is homotopy equivalent to the space obtained by joining two circles together at a point. The proof of (d) is an easy exercise. For the proof of (b), observe that monodromy matrices $\rho(\gamma)$ operate as linear fractional transformations on the part of the

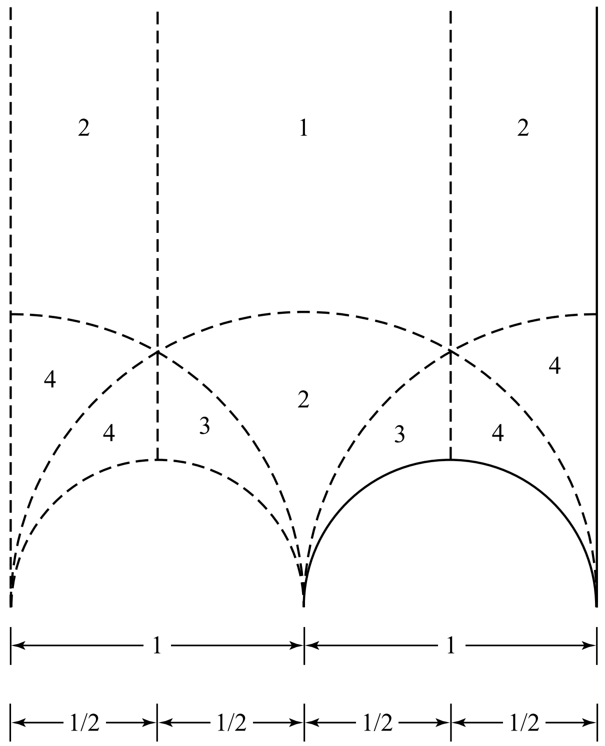

Figure 13. Fundamental domain for Γ and Γ(2).

complex plane with a positive imaginary part. A fundamental domain for this action is given by the region indicated in Fig. 13. It is an ideal quadrilateral, with sides formed by two semicircles with endpoints on the real axis and two vertical rays with an endpoint on the real axis.

For our purposes it is better to look at this fundamental domain in the disk model of hyperbolic space. It is then the central quadrilateral in Fig. 14. The transformations A and B as well as their inverses act as reflections in the sides of the quadrilateral, and repeated applications of these transformations tile the disk by quadrilaterals congruent to the given one. Now take the center E of the given quadrilateral and consider the set of points

$$V = \{g(E) \mid g \in \Gamma\}.$$

Join two points x and y by a geodesic if $y = gx$ where $g = A^{\pm 1}$ or $g = B^{\pm 1}$ – that is, if g is a standard generator of Γ or its inverse. The union of all these geodesics is the three $\mathcal{T}$, part of which is illustrated by the dotted lines in Fig. 13. Let $\Gamma(\mathcal{T})$ be the group of automorphisms of $\mathcal{T}$ defined by elements of Γ. Thus we have the composition

$$\pi_1 \xrightarrow{\rho} \Gamma \xrightarrow{\sigma} \Gamma(\mathcal{T}).$$

Consider now an element γ of π_1. It is a word spelled with the letters $a^{\pm 1}$, $b^{\pm 1}$. By considering the action of $\rho(\gamma)$ on $\mathcal{T}$ – indeed, by considering the

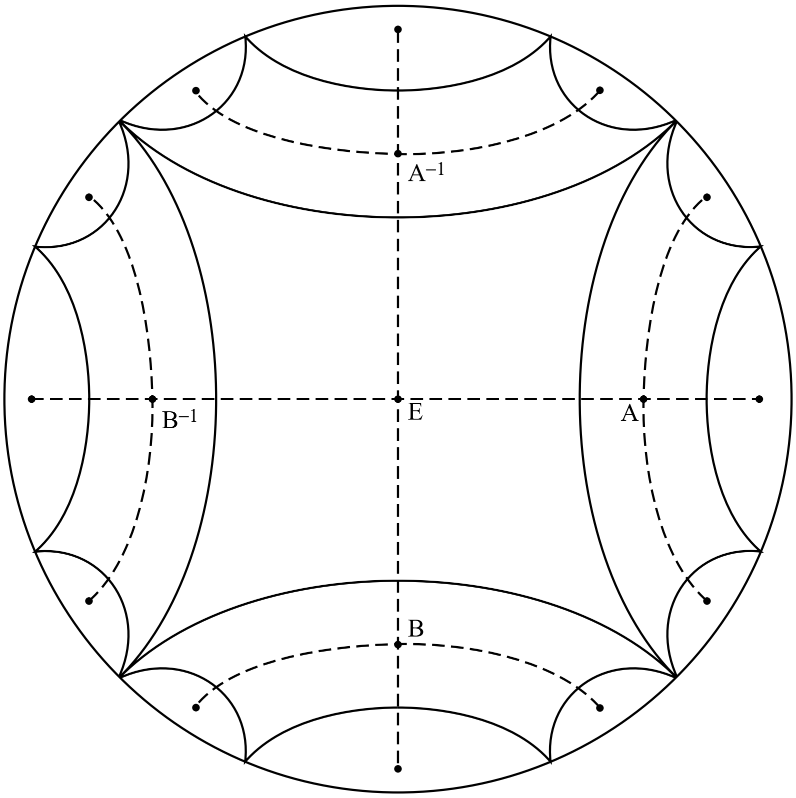

Figure 14. Fundamental domains for $\Gamma(2)$ – disk model.

position of $\rho(\gamma)(E)$ – one finds that $\rho(\gamma) \neq 1$ if $\gamma \neq 1$. See Problem 1.1.13 for more details.

For the proof of (c), the main idea is to compare the fundamental domain for the action of $\Gamma = SL(2, \mathbb{Z})$ with that of $\Gamma(2)$. A fundamental domain of the latter is made of six copies of the fundamental domain of the former, as indicated in Fig. 13. See [45] for more details. Q.E.D.

Remark 1.1.21 In the preceding example the kernel of the monodromy representation was trivial and its image was of finite index. The latter is typical [14], but the former is not. Thus, while the kernel of the monodromy representation for cubic curves is finite [153], it is "large" for most families of hypersurfaces [34]. The notion "large" can be made precise; in particular, large groups are infinite and in fact contain a nonabelian free group.

Monodromy of the Picard–Fuchs Equation

Let us now return to the problem of understanding the monodromy of the period functions $\pi_i(\lambda)$ which solve the Picard–Fuchs equation (1.1.17). Analytic continuation of a solution

$$\pi(\lambda) = \int_{\xi} \omega$$

along a loop α transforms the solution into

$$\pi'(\lambda) = \int_{\rho(\xi)} \omega.$$

Thus the monodromy representation for solutions of the Picard–Fuchs equation is the same as the geometric monodromy representation. In particular, the representation must take values in the group of matrices with integer coefficients. In the case of the Legendre family, it is precisely the group $\Gamma(2)$ described in Theorem 1.1.20.

Problems

1.1.1. Show that the differential (1.1.3) is holomorphic on $\mathcal{E}$.

1.1.2. Show that the Euler characteristic of the Riemann sphere is two. Compute the Euler characteristic of the Riemann sphere with d points deleted. Let k be a divisor of d. Then there is a k-fold unbranched cover of the Riemann sphere defined by the equation $y^k = p(x)$, where p is a polynomial of degree d. Compute the Euler characteristic of this unbranched cover. Then compute the Euler characteristic of the corresponding k-fold branched cover, defined by the same equation. Finally, compute the genus of that branched cover.

1.1.3. Show that the only singular fibers of the Legendre family $y^2 = x(x-1)(x-\lambda)$ are at $\lambda = 0,\ 1$. Consider next the family of elliptic curves $x^3 + y^3 + z^3 + \lambda xyz$. What are its singular fibers?

1.1.4. Consider the family of elliptic curves $\mathcal{E}_{a,b,c}$ defined by $y^2 = (x-a)(x-b)(x-c)$. What is the locus in $\mathbb{C}^3 = \{(a, b, c)\}$ of the singular fibers (the discriminant locus)? More difficult: describe the monodromy representation.

1.1.5. Consider the family of elliptic curves $\mathcal{E}_{A,B,C}$ defined by $y^2 = x^3 + Ax^2 + Bx + C$. What is the locus in $\mathbb{C}^3 = \{(A, B, C)\}$ of the singular fibers? More difficult: describe the monodromy representation.

1.1.6. Consider the compact Riemann surface M with affine equation $y^2 = p(x)$ where p has degree two. Show that M is isomorphic as a complex manifold (or as an algebraic curve) to the Riemann sphere.

1.1.7. A holomorphic one-form on a Riemann surface is a differential one-form which is locally given by $f(z)dz$, where z is a local holomorphic coordinate. Show that such a form is closed. Formulate and investigate the analogous assertion(s) for holomorphic forms on a complex manifold of complex dimension 2.

1.1.8. Prove the following identities: $\arccos u = \mathrm{i}\log(u + \sqrt{u^2 - 1})$, $\arccos u = \arctan(\sqrt{(1 - u^2)}/u)$, $\arctan u = \arccos(1/\sqrt{1 + u^2})$. Then show that all the integrals and estimates leading up to (1.1.12) hold as asserted.

1.1.9. Consider the family of elliptic curves defined by $y^2 = (x^2 - s)(x - 1)$. Find the asymptotic form of the period map $\tau(s)$ as s approaches 0. Comment on the relation between what you find and the asymptotic form in (1.1.12).

1.1.10. Let S be a Riemann surface, and let $A \subset S$ be a nonempty finite set. Show that there is an exact sequence

$$0 \longrightarrow H^1(S) \longrightarrow H^1(S \backslash A) \xrightarrow{\text{res}} H^0(A) \longrightarrow H^0(S) \longrightarrow 0.$$

Note that elements of $H^0(A)$ are linear functionals on the vector space spanned by the points of A. They can be viewed as the pointwise residue as defined previously, and they can be combined to form the globally defined map "res". Your argument should show that the above sequence is defined on the level of integral cohomology.

1.1.11. Consider the degeneration of elliptic curves $\mathcal{E}_t$ defined by $y^2 = x^3 - t$. Find all values of t for which $\mathcal{E}_t$ is singular. By drawing a series of pictures of branch cuts, show that the monodromy transformation for $t = 0$ has order six, and find the corresponding matrix.

1.1.12. Let $\{\mathcal{E}_t\}$ be a family of elliptic curves with just two singular fibers, one at $t = 0$, the other at $t = \infty$. Show that the complex structure of $\mathcal{E}_t$ does not vary.

1.1.13. Let F be the free group with two generators a and b. Assign a graph $\mathcal{T}(F)$ to this group by letting the vertices be the elements of F, i.e., the words in a and b. We connect the vertices represented by w and wa by an edge, and we likewise draw an edge between w and wa^{-1}, between w and wb, and between w and wb^{-1}. No other vertices are connected. This defines also an action of F on $\mathcal{T}(F)$. Show that $\mathcal{T}(F)$ is a tree (compare the graph with the tree of Fig. 14). Show that the action of F on $\mathcal{T}(F)$ is free and faithful, i.e., if some word $w \in F$ fixes a vertex, then $w = 1$.

1.2 Riemann Surfaces of Higher Genus

Let us now consider the Hodge theory and period mapping for Riemann surfaces of genus bigger than 1, as illustrated in Fig. 15. The cycles δ_i and γ_i

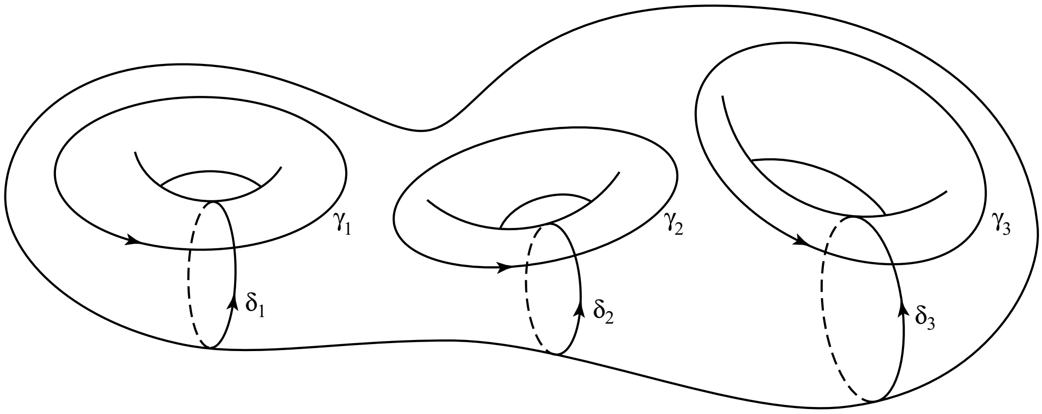

Figure 15. Riemann surface of genus 3.

form a “standard basis” for the first homology. For such a basis the intersection matrix has the form

$$J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix},$$

where $\mathbb{1}_g$ is the $g \times g$ identity matrix. One can define a Riemann surface S of this kind by the equation

$$y^2 = (x - t_1)(x - t_2) \cdots (x - t_n), \tag{1.2.1}$$

where $n = 2g + 2$. However, for $g > 2$, there are Riemann surfaces which are not given by such equations (see Problem 1.2.1). Those which have an equation (1.2.1) are called *hyperelliptic Riemann surfaces*.

So far we have used a topological definition of the genus – the number of handles, which we can compute from the Euler characteristic. For another point of view, consider the vector space $\Omega(S)$ of holomorphic one-forms. These, which we have encountered already in the case of elliptic curves, are differentials which can be written locally as $f(z)\,dz$ where $f(z)$ is a holomorphic function. Riemann's contribution to the Riemann–Roch theorem can be succinctly written as

$$\dim\, \Omega(S) = g(S). \tag{1.2.2}$$

This formula implies, as in the case of an elliptic curve, that the complex 1-cohomology of S decomposes as discussed in Theorem 1.1.5:

$$H^1(S; \mathbb{C}) = H^{1,0} \oplus H^{0,1}.$$

The first summand is the space of holomorphic differentials, and

$$H^{0,1} = \overline{H^{1,0}}$$

is the complex conjugate space, where the conjugation comes from the isomorphism

$$H^1(S;\mathbb{C}) = H^1(S,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Formally this means that H^1 carries a *Hodge structure of weight 1*, and the above decomposition is the *Hodge decomposition*.

Coming back to the equality (1.2.2), we note that it is not at all obvious in general. What is easy (see Problem 1.2.2) is the inclusion $\Omega(S) \oplus \overline{\Omega(S)} \subset H^1_{\mathrm{DR}}(S;\mathbb{C})$, whence comes an inequality. However, for hyperelliptic Riemann surfaces the 1-forms

$$\omega_i = \frac{x^i dx}{y}, \quad i = 0, \ldots, g-1 \tag{1.2.3}$$

are independent. Indeed, given any polynomial $p(x) = v_1 + v_2 x + v_3 x^2 + \cdots + v_g x^{g-1}$ of degree $\leq (g-1)$, the one-form $p(x)dx/y$ is 0 if and only if p is the zero polynomial. So, in view of the preceding, the forms (1.2.3) give a basis for $\Omega(S)$. In this case the Hodge decomposition for H^1 therefore follows.

Let us now consider the period map for the Riemann surfaces $S = S_t$ given by Eq. (1.2.1), where $t = (t_1, \ldots, t_n)$. To this end we fix a standard basis, and we denote the elements of the dual basis by δ^i, γ^i. Thus $\delta^i(\delta_j) = \delta^i_j$, where the last symbol is Kronecker's δ, equal to 0 if $i \neq j$ and equal to 1 if $i = j$. This basis, or "marking," gives an isomorphism

$$H^1(S;\mathbb{Z}) \xrightarrow{m} \mathbb{Z}^{2g}$$

which extends to an isomorphism

$$H^1(S;\mathbb{C}) \xrightarrow{m} \mathbb{C}^{2g}.$$

Then the subspace

$$m(H^{1,0}(S)) \subset \mathbb{C}^{2g} \tag{1.2.4}$$

defines a point in the Grassmannian of g-planes in $2g$-space. It depends on both S and the marking.

Let

$$U = \{(t_1, \ldots, t_n) \mid t_i \neq t_j\}$$

be the parameter space for the nonsingular Riemann surfaces S. It has the form

$$U = \mathbb{C}^n \backslash \Delta,$$

where the *discriminant hypersurface* Δ is the union of hyperplanes $t_i = t_j$. The set of cohomology groups $H^1(S_u;\mathbb{Z})$ for u in U forms a local system of $\mathbb{Z}$-modules of rank $2g$ which we denote by H^1_U. Let $\widetilde{U}$ be the universal cover of U, and consider the pullback of the local system, which we denote by $H^1_{\widetilde{U}}$. Because $\widetilde{U}$ is simply connected, this system is trivial. Thus there is an isomorphism m from it to the trivial local system $\mathbb{Z}^{2g} \times \widetilde{U}$. Let

$$m(\tilde{u}) : H^1(S_{\tilde{u}}) \longrightarrow \mathbb{Z}^{2g} \times \{\tilde{u}\} \longrightarrow \mathbb{Z}^{2g}$$

denote the isomorphism of the fibers over $\tilde{u}$, composed with the projection to the first factor. Thus $m(\tilde{u})$ is an isomorphism of $H^1(S_{\tilde{u}}, \mathbb{Z})$ with $\mathbb{Z}^{2g}$. Since H^1_U is a local system, one has the relation

$$m(\gamma \cdot \tilde{u}) = \rho(\gamma) m(\tilde{u}),$$

where ρ is the monodromy representation and $\gamma \cdot \tilde{u}$ is the action of $\pi(U, u_0)$ on $\widetilde{U}$ by covering transformations. It follows that formula (1.2.4) defines a map

$$\widetilde{\mathcal{P}} : \widetilde{U} \longrightarrow \mathrm{Grass}(g, 2g)$$

which satisfies the equivariance condition

$$\widetilde{\mathcal{P}}(\gamma \cdot \tilde{u}) = \rho(\gamma) \widetilde{P}(\tilde{u}). \tag{1.2.5}$$

Our aim is to understand this map in the same way that we understood the period map for elliptic curves. The first result is both important and easy to prove.

Theorem 1.2.6 *The period map is holomorphic.*

Proof. Observe that $\widetilde{\mathcal{P}}(S, m)$ is the same as the row space of the $g \times 2g$ matrix (A, B), where

$$A_{ij} = \int_{\delta_i} \omega_j$$

and

$$B_{ij} = \int_{\gamma_i} \omega_j.$$

Thus $\widetilde{\mathcal{P}}$ will be holomorphic if the integrals A_{ij} and B_{ij} are holomorphic functions of $t = (t_1, \ldots, t_n)$. We leave the proof of this fact as an exercise. The reader should adapt the argument used for elliptic curves to the hyperelliptic case. Q.E.D.

Let us now consider the values of the period map. We claim first, in analogy with the proof of Theorem 1.1.5, that A, the first half of the period matrix, is nonsingular. To see that this is so, consider a vector v such that $vA = 0$. Let $\omega = v_1\omega_1 + \cdots + v_g\omega_g$, and observe that

$$\int_{\delta_j} \omega = \sum_i \int_{\delta_j} v_i\omega_i = \sum_i v_i A_{ij} = (vA)_j.$$

Thus, v is a null vector of A if and only if all the integrals of ω over the δ_j vanish. Consequently the cohomology class of ω is a linear combination of the γ^j if v is a null vector,

$$[\omega] = \sum_j w_j\gamma^j$$

for some coefficients w_j. Then

$$\overline{[\omega]} = \sum_j \bar{w}_j\gamma^j.$$

However,

$$\gamma^i \cup \gamma^j = 0 \text{ for all } i \text{ and } j,$$

and so $[\omega] \cup \overline{[\omega]} = 0$. As in the case of elliptic curves,

$$([\omega] \cup \overline{[\omega]})[S] = \mathrm{i}\int_S \omega \wedge \bar{\omega} \geq 0.$$

Moreover, the integral is identically 0 if and only if ω is 0 as a one-form. In the case of a hyperelliptic Riemann surface,

$$\omega = \frac{p(x)dx}{y},$$

where $p(x) = v_1 + v_2x + v_3x^2 + \cdots + v_gx^{g-1}$. We have seen that such a holomorphic differential is 0 if and only if p is the zero polynomial, that is, if and only if v is the zero vector. Thus, if $vA = 0$, then $v = 0$, as required to prove the claim.

Now let

$$\tilde{\omega}_i = \sum_j A^{ij}\omega_j,$$

where A^{ij} is the ij component of the inverse of the matrix A. With the basis $\{\tilde{\omega}_i\}$ in place of the basis $\{\omega_i\}$, the matrix of A-periods is the identity. The nature of the matrix of B-periods is given by the following.

Theorem 1.2.7 *If the matrix of A-periods is the identity, then the matrix of B-periods is symmetric and has a positive definite imaginary part.*

Proof. The positivity assertion of the theorem mirrors the corresponding result (1.1.6) for elliptic curves; the symmetry statement is a new phenomenon. To prove symmetry, note that the two-form $\tilde{\omega}_i \wedge \tilde{\omega}_j$ vanishes identically because it has the local form $f(z)dz \wedge g(z)dz$ for some holomorphic functions f and g. Therefore,

$$\int_S \tilde{\omega}_i \wedge \tilde{\omega}_j = 0$$

for all i and j. Now write $\tilde{\omega}_i$ in terms of the standard cohomology basis:

$$[\tilde{\omega}_i] = \delta_i + \sum_k B_{ik}\gamma^k.$$

Because of our integral formula above, the cup product $[\tilde{\omega}_i] \cup [\tilde{\omega}_j]$ vanishes. However,

$$[\tilde{\omega}_i] \cup [\tilde{\omega}_j] = \left(\delta_i + \sum_k B_{ik}\gamma^k\right) \cup \left(\delta_j + \sum_\ell B_{j\ell}\gamma^\ell\right) = (B_{ij} - B_{ji})[S].$$

It follows that B is symmetric. For the positivity assertion, consider the abelian differential $\tilde{\omega} = v_1\tilde{\omega}_1 + \cdots + v_g\tilde{\omega}_g$. Then

$$\int_S \tilde{\omega} \wedge \bar{\tilde{\omega}} = ([\tilde{\omega}] \cup \overline{[\tilde{\omega}]})[S].$$

On one hand, the integral on the left is positive for nonzero $\tilde{\omega}$. On the other hand, the expression on the right can be evaluated by evaluating the cup product

$$\mathrm{i}\left(\sum_i v_i\delta_i + \sum_{ik} v_i B_{ik}\gamma^k\right) \cup \left(\sum_j \bar{v}_j\delta_j + \sum_{j\ell} \bar{B}_{j\ell}\bar{v}_j\gamma^\ell\right).$$

One obtains the identity

$$2\sum_{ij} v_i\bar{v}_j \operatorname{Im}(B_{ij}) = 2v \operatorname{Im}(B)^t\bar{v}[S].$$

Here v is a row vector and ${}^t v$ denotes its transpose. Thus B has a positive definite imaginary part, as claimed. Q.E.D.

Let $\mathfrak{h}_g$ denote the Siegel upper half-space of genus g: the set of symmetric complex matrices with a positive definite imaginary part. Then the period map takes values in this space viewed as a subset of the Grassmanian via the map

$$Z \mapsto \text{row space of } (\mathbb{1}_g, Z),$$

where $\mathbb{1}_g$ is the $g \times g$ identity matrix. The period map for the family (1.2.1) of hyperelliptic Riemann surfaces takes the form

$$\widetilde{\mathcal{P}} : \widetilde{U} \longrightarrow \mathfrak{h}_g.$$

Since a monodromy diffeomorphism associated with the path γ preserves the cup-product form, the monodromy matrices $\rho(\gamma)$ preserve the skew-symmetric form J. Thus the monodromy representation takes values in the group

$$\mathrm{Sp}(g, \mathbb{Z}) = \{M \in \mathrm{GL}(2g, \mathbb{Z}) \mid {}^t M J M = J\}.$$

This is the integer *symplectic group*. One may also consider $\mathrm{Sp}(g, \mathbb{R})$, the symplectic group with real coefficients. A matrix in the symplectic group can be decomposed into $g \times g$ blocks as

$$T = \begin{pmatrix} K & L \\ M & N \end{pmatrix}.$$

Such matrices operate on $g \times 2g$ matrices (A, B) via multiplication on the right,

$$(A, B) \mapsto (A, B)T,$$

and there is a corresponding action on the row spaces. One checks (Problem 1.2.4) that a symplectic matrix preserves the set of row spaces corresponding to matrices $(\mathbb{1}, Z)$, where Z is symmetric with a positive definite imaginary part. Thus the symplectic group also acts on $\mathfrak{h}_g$, namely, via

$$Z \mapsto T\langle Z\rangle \overset{\text{def}}{=} (K + ZM)^{-1}(L + ZN).$$

This is a kind of generalized fractional linear transformation which in the case of $g = 1$ reduces to the standard action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half-plane.

We want to establish some basic facts about this action. To begin, it acts transitively. This we can see by looking at the image of $\mathrm{i}\mathbb{1}_g$ under the map

$$T = \begin{pmatrix} \mathbb{1}_g & X \\ 0 & \mathbb{1}_g \end{pmatrix} \begin{pmatrix} {}^t W & 0 \\ 0 & W^{-1} \end{pmatrix}. \tag{1.2.8}$$

We find $T\langle \mathrm{i}\mathbb{1}_g\rangle = X + \mathrm{i}^t W W$. Since every positive definite Hermitian matrix Y can be written as $Y = {}^t W W$, this shows that the action is indeed transitive.

Consider next the orbit map

$$\begin{aligned} \pi : \mathrm{Sp}(g, \mathbb{R}) &\to \mathfrak{h}_g \\ T &\mapsto T\langle \mathrm{i}\mathbb{1}_g\rangle. \end{aligned}$$

We claim that this map is proper: any sequence $M_n \in \mathrm{Sp}(g)$ whose π-images $X_n + \mathrm{i}Y_n$ converge in $\mathfrak{h}_g$ has a convergent subsequence. From this it quite easy (see Problem 1.2.5) to show that $\mathrm{Sp}(g, \mathbb{Z})$ acts properly discontinuously on $\mathfrak{h}_g$, i.e., for any two compact sets $K_1, K_2 \subset \mathfrak{h}_g$, there are at most finitely many elements $\gamma \in \mathrm{Sp}(g, \mathbb{Z})$ such that $K_1 \cap \gamma K_2 \neq \emptyset$. Then a standard general result [35] asserts that the quotient of a complex manifold by a proper action action of a group is an *analytic space*: a (possibly singular) space on which is defined the notion of holomorphic function.

For the proof of the properness assertion, define T_n according to (1.2.8) so that $T_n = M_n U_n$ with $U_n \in \mathrm{U}(g) \cap \mathrm{Sp}(g)$, a compact group. Passing to a subsequence, we may assume that the U_n converge, and so it suffices to see that $\{T_n\}$ has a convergent subsequence. Since by assumption $X_n + \mathrm{i}Y_n$ converges to a point in $\mathfrak{h}_g$, the X_n converge and so it suffices to see that $\{{}^t W_n\}$ and $\{{}^t W_n^{-1}\}$ have convergent subsequences. The first is clear since ${}^t W_n W_n = Y_n$ converges and so $\{W_n\}$ is a bounded set. Replacing $\{W_n\}$ by a converging subsequence, put $W = \lim_{n\to\infty} W_n$. Then ${}^t W W > 0$ by assumption and so W is invertible; it follows that $W^{-1} = \lim_{n\to\infty} W_n^{-1}$, which completes the proof of our assertion.

The functional equation (1.2.5) asserts that the period map is equivariant as a map of $\widetilde{U}$ to $\mathfrak{h}_g$. Therefore, in light of the previous two paragraphs, there is a quotient map

$$\mathcal{P} : U \longrightarrow \mathfrak{h}_g / \mathrm{Sp}(g, \mathbb{Z}), \tag{1.2.9}$$

where, as noted, the right-hand side is an analytic space. It is definitely not a complex manifold because of the presence of fixed points of the action of $\mathrm{Sp}(g, \mathbb{Z})$ on $\mathfrak{h}_g$: see Problem 1.2.5. Indeed, the quotient has codimension two singularities, as in the model example of the quotient of $\mathbb{C}^2$ the group of transformations $(x, y) \mapsto (\pm x, \pm y)$. See Problem 1.2.6. Nonetheless, the notion of holomorphic function makes sense, and we have the following.

Theorem 1.2.10 *The period map* (1.2.9) *is a holomorphic map of analytic spaces.*

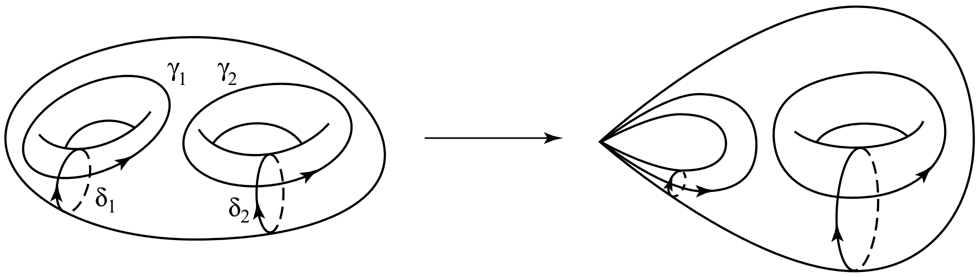

Figure 16. Degeneration.

Degenerations

Consider now the family of genus 2 Riemann surfaces S_t given by

$$y^2 = (x - a_1) \cdots (x - a_5)(x - t).$$

The normalized period matrix of the fibers S_t has the form

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

where all entries are multivalued holomorphic functions of t. Let us suppose that $a_1 = 0$, and let us examine the behavior of the period matrix for t near $t = 0$. If we use the standard bases illustrated in Fig. 16, then δ_1 is the vanishing cycle and the local monodromy transformation is given by

$$T(x) = x - 2(x \cdot \delta_1)\delta_1.$$

The factor of 2 comes from the fact that as t travels around a small circle centered at the origin, the branch slit connecting t to 0 makes a full turn: twice a half-turn, so twice the contribution of the vanishing cycle to monodromy. Following the same line of argument as used to establish (1.1.12), we find that

$$Z_{11}(t) = \frac{\mathrm{i}}{\pi} \log t + h(t),$$

where $h(t)$ is a holomorphic function, and where the remaining entries of the period matrix are holomorphic functions of t. Thus the multivaluedness of Z is of a very controlled sort.

Consider the linear transformation defined by $N(x) = (x \cdot \delta_1)\delta_1$. It is nilpotent and satisfies $T = 1 + 2N$. The results of the previous paragraph may be restated by saying that the matrix

$$\hat{Z}(t) = \exp\left(-\frac{\mathrm{i}\log t}{\pi} N\right) Z(t) \tag{1.2.11}$$

is single-valued and holomorphic in the punctured disk $0 < |t| < \epsilon$, and in fact is holomorphic in the disk $|t| < \epsilon$. Thus the period matrix itself can be written as

$$Z(t) = \exp\left(\frac{\mathrm{i}\log t}{\pi} N\right) \hat{Z}(t), \tag{1.2.12}$$

where $\hat{Z}(t)$ is holomorphic. This equation, which expresses the period matrix in terms of an exponential involving $\log t$, the logarithm of the Picard–Lefschetz transformation, and a holomorphic matrix, is a very special case of the *nilpotent orbit theorem* of Wilfried Schmid [201]. It is equivalent to the statement that in general the Picard–Fuchs equation has regular singular points.

Let us now inquire into the meaning of the entries of $\hat{Z}(0)$. Note first that if we replace the parameter t by ct, some nonzero constant c, we replace $\hat{Z}_{11}(0)$ by $\hat{Z}_{11}(0) + \mathrm{i}\log c/\pi$. Thus the value of $Z_{11}(0)$ has no significance. The remaining entries, $Z_{ij}(0)$ for $(i, j) \neq (1, 1)$, however, are well defined and equal to $\hat{Z}_{ij}(0)$. To interpret them, write the elements of the basis for the space of abelian differentials as

$$\omega_1(t) = \frac{dx}{\sqrt{x(x-t)(x-a)(x-b)(x-c)(x-d)}}$$

and

$$\omega_2(t) = \frac{x\,dx}{\sqrt{x(x-t)(x-a)(x-b)(x-c)(x-d)}}.$$

For $t = 0$ these expressions become

$$\omega_1(0) = \frac{dx}{x\sqrt{(x-a)(x-b)(x-c)(x-d)}}$$

and

$$\omega_2(0) = \frac{dx}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}.$$

Note that they make sense as possibly meromorphic differentials on the elliptic curve $\tilde{\mathcal{E}}$ defined by $y^2 = (x-a)(x-b)(x-c)(x-d)$. The Riemann surface $\tilde{\mathcal{E}}$ is the normalization of the algebraic curve $\mathcal{E} = S_0$ defined by $y^2 = x^2(x-a)(x-b)(x-c)(x-d)$ (see Problem 1.2.7).

The differential $\omega_1(0)$ is a differential of the third kind: it has simple poles at the two points corresponding to $x = 0$. Now consider the normalization of S_0, as illustrated in Fig. 17, and let $\tilde{\omega}$ be the normalized differential corresponding to $\omega_2(0)$: it is $\omega_2(0)$ divided by the integral of $\omega_2(0)$ over δ_2. We observe the following. First, the integral $Z_{22}(0)$ is a normalized period of $\omega_2(0)$:

$$Z_{22}(0) = \int_{\gamma_2} \tilde{\omega}.$$

Second, the integral $Z_{12}(0) = Z_{21}(0)$ is the normalized abelian integral for the divisor $p - q$:

$$Z_{12}(0) = \int_p^q \tilde{\omega}.$$

Each of these integrals, we emphasize, may be viewed as an integral on $\widetilde{\mathcal{E}}$. Thus the limiting values of entries of the period map can be interpreted as integrals on the normalization of the singular fiber of the degeneration, where that normalization has been marked in such a way that we "remember" what the singular fiber was. Indeed, if we glue p and q in $\widetilde{\mathcal{E}}$, the result is biholomorphic to S_0.

The integral $Z_{12}(0)$ is just the Abel–Jacobi class associated to the divisor $p - q$. This class determines the location of p and q on $\widetilde{\mathcal{E}}$ up to translation, and so determines S_0, given $\widetilde{\mathcal{E}}$. Note also that $Z_{22}(0)$ is the normalized period of $\widetilde{\mathcal{E}}$, and so it determines $\widetilde{\mathcal{E}}$. Therefore the limit period matrix determines the singular fiber S_0.

Generalizing Hodge Theory

Our study of the period map for a degeneration of Riemann surfaces $\{S_t\}$ leads us to ask whether it makes sense to take a limit of the Hodge structure $H^1(S_t)$, and whether it is possible to define a (suitably generalized) Hodge structure for the singular variety S_0. The answer to both questions is "yes." We shall take up the question of generalizing Hodge theory to singular varieties first, and then consider limits of Hodge structures.

Let us begin with an easy observation. Since the cohomology of S_0 (see Fig. 17) has rank 3, it cannot carry a Hodge structure of weight 1: these have even dimension in view of the relation $\overline{H^{1,0}} = H^{0,1}$. Nonetheless, the cohomology of S_0 carries considerable structure, both topological and complex analytic. To understand the topology, consider the normalization map $p : \widetilde{S}_0 \longrightarrow S_0$ and its induced map on cohomology,

$$H^1(S_0) \xrightarrow{p^*} H^1(\widetilde{S}_0).$$

It is easy to see that p^* is surjective, for example by showing that the corresponding map p_* on homology is injective (simply look at Fig. 17). Thus