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Period Domains and Period Mappings

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PERIOD DOMAINS AND PERIOD MAPPINGS

Contents

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Introduction and Review

The aim of these lectures is to develop a working understanding of the notions of period domain and period mapping, along with familiarity with basic examples thereof. The fundamental references are [9] and [10]. We will not give specific references to these, but essentially everything below is contained in or derivative of these articles. Three general references are [11], [14], and [3]. There was not time to put in the figures (my apologies). See the first chapter of [3]. There is space to copy them in by hand, which is a good exercise in itself.

In previous lectures you have studied the notion of a polarized Hodge structure H of weight k over the integers. To recapitulate, H is a a triple $(H_{\mathbb{Z}}, \oplus H^{p,q}, Q)$, where (a) $H_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module, (b) $\oplus H^{p,q}$ is a direct sum decomposition of the complex vector space $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ satisfying $\overline{H^{p,q}} = H^{q,p}$ where $p+q=k$, (c) $Q(x,y)$ is a non-degenerate bilinear form which is symmetric for k even and anti-symmetric for k odd. The bilinear form is compatible with the direct sum decomposition (the *Hodge decomposition*) in the following way:

$$Q(x,y) = 0 \text{ if } x \in H^{p,q} \text{ and } y \in H^{r,s},\ (r,s) \neq (q,p). \tag{1}$$

$$i^{p-q} Q(x, \bar{x}) > 0 \text{ for } x \in H^{p,q} - \{0\}. \tag{2}$$

The compatibility relations are the *Riemann bilinear relations.* The *Weil operator* is the linear transfomration $C : H_{\mathbb{C}} \longrightarrow H_{\mathbb{C}}$ such that $C(x) = i^{p-q}x$. It is a real operator, that is, it restricts to a real linear transformation of $H_{\mathbb{R}}$. The expression

$$h(x,y) = Q(Cx, \bar{y})$$

defines a positive hermitian form.

A Hodge structure can also be defined by a filtration. Let

$$F^p = \bigoplus_{a \geq p} H^{a,b}.$$

Then one has the decreasing filtration $\cdots \supset F^0 \supset F^1 \supset F^2 \supset \cdots$. It satisfies

$$H = F^p \oplus \overline{F^{k-p+1}}$$

for a Hodge structure of weight k. Conversely, a filtration satisfying this property defines a Hodge structure of weight k.

The motivating example of a polarized Hodge structure of weight k is the primitive k-th cohomology of a projective algebraic manifold of dimension k. Before defining primitivity, let us consider the simplest nontrivial example, the first cohomology of a compact Riemann surface M. An *abelian differential* is a one-form which in a local analytic coordinate z can be written as

$\phi = f(z)dz$, where $f(z)$ is a holomorphic function. The exterior derivative of such a form is

$$d\phi = \frac{\partial f}{\partial z} dz \wedge dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz \tag{3}$$

The first term vanishes since $dz \wedge dz = 0$. The second term vanishes since f is holomorphic; here we have used the Cauchy-Riemann equation. It follows that abelian differentials are closed, and so represent cohomology classes. The space of abelian differentials of a compact Riemann surface spans the subspace of classes of type $(1,0)$, that is, the subspace $H^{1,0}(M)$. The Hodge structure is given by that space and $H^{0,1}(M)$, which is spanned by the the complex conjugates of the abelian differentials. Thus one has

$$H^1(M,\mathbb{C}) = H^{1,0}(M) \oplus H^{0,1}(M) = \{f(z)dz\} \oplus \{g(\bar{z})d\bar{z}\},$$

where $g(\bar{z})$ is anti-holomorphic.

The cohomology class of a nonzero abelian differential is nonzero. To see this, consider the integral

$$\sqrt{-1}\int_M \phi \wedge \bar{\phi}. \tag{4}$$

Let $\{T_\alpha\}$ be a triangulation of M that is so fine that each closed triangle T_α is contained in a coordinate neighborhood with coordinate z_α. Thus on T_α, $\phi = f_\alpha(z_\alpha)$. Therefore the above integral is a sum of terms

$$\sqrt{-1}\int_{T_\alpha} |f_\alpha(z_\alpha)|^2 dz_\alpha \wedge d\bar{z}_\alpha. \tag{5}$$

Now $\sqrt{-1}dz_\alpha \wedge d\bar{z}_\alpha = 2dx_\alpha \wedge dy_\alpha$, where $z_\alpha = x_\alpha + \sqrt{-1}y_\alpha$. The form $dx_\alpha \wedge dy_\alpha$ is the volume form in the natural orientation determined by the complex structure: rotation by 90° counterclockwise in the x_α-y_α plane. Thus the integral (4) is a sum of positive terms (5). We conclude that if $\phi \neq 0$, then

$$\sqrt{-1}\int_M \phi \wedge \bar{\phi} > 0. \tag{6}$$

This is the second Riemann bilinear relation; it implies that the cohomology class of ϕ is nonzero.

Consider a second abelian differential $\psi = \{g_\alpha(z_\alpha)dz_\alpha\}$. Then

$$\sqrt{-1}\int_M \phi \wedge \psi = 0, \tag{7}$$

since this integral is a sum of integrals with integrands $f_\alpha(z_\alpha)g_\alpha(z_\alpha)dz_\alpha \wedge dz_\alpha = 0$. This is the first Riemann bilinear relation. We conclude that *The first cohomology group of a compact orientable Riemann surface is a polarized Hodge structure of weight one.*

Let us now return to the issue of primitivity. The k-th cohomology of any projective algebraic manifold is a Hodge structure of weight k. The cohomology ring of such a manifold is Hodge structure which is a direct sum of pure Hodge structures of various weights. The cohomology ring can be reconstructed from its *primitive part*. To define it, let

$$L : H^k(M) \longrightarrow H^{k+2}(M) \tag{8}$$

be the operator defined by $L(x) = \omega \wedge x$, where ω is the positive $(1,1)$ form which represents the Kälher class, that is, the hyperplane class. The adjoint of the operator L is the operator defined by

$$\langle L(x), y \rangle = \langle x, \Lambda(y) \rangle\,, \tag{9}$$

where $\langle x, y \rangle$ is the cup product. A cohomology class x is *primitive* if $\Lambda(x) = 0$. We write the primitive cohomolology as $H^k(M)_o$ or $H^k(M)_{prim}$. When M is a variety of complex dimension d, where d is odd,

$$H^d(M)_o = H^d(M).$$

The case of Riemann surfaces illustrates this phenomenon: $H^1(M)$ is primitive. When d is even,

$$H^d(M) = \{x \in H^d(M) \mid L(x) = 0\}.$$

For example, in the case of an algebraic surface, where $d = 2$, the primitive cohomology is the orthogonal complement of the hyperplane class. In general the cohomology is the direct sum of the sub-Hodge structures $L^i H^j(M)_o$.

Example 1. *Consider an algebraic surface $M \subset \mathbb{P}^3$. Let ω be the Kähler form on $\mathbb{P}^3$, that is the form given by $\sqrt{-1}\partial\bar\partial \log ||Z||^2$, where Z is the vector of homogeneous coordinates. The positive $(1,1)$ form ω represents the cohomology class of a hyperplane section of M. One finds that*

$$\int_M \omega^2 > 0.$$

On the other hand, consider a holomorphic two-form ϕ, given in local coordinates by $f(z,w)dz \wedge dw$. Note that

$$i^2 \int_U |f|^2 dz \wedge dw \wedge d\bar z \wedge d\bar w = -\int_U |f|^2 (idz \wedge d\bar z) \wedge (idw \wedge d\bar w) < 0$$

Thus the hermitian form $h(x,y)$ defined above takes different signs on the span of ω and on $H^{2,0}(M)$. Moreover, the $(2,0)$ cohomology of M is primitive, since if x has type $(2,0)$, then $L(x)$ has type $(3,1)$, which is necessarily zero. Thus h cannot be definite on $H^2(M)$. It is positive on the hyperplane class and (as it turns out) negative on the primitive cohomology.

1. Period domains and monodromy

Fix a lattice $H_{\mathbb{Z}}$, a weight k, a bilinear form Q, and a vector of Hodge numbers $h = (h^{p,q}) = (\dim H^{p,q})$. For example, we might take $H_{\mathbb{Z}} = \mathbb{Z}^{2g}$, the weight to be one, and Q to be the standard symplectic form, and $h = (g, g)$, where $h^{1,0} = g$ and $h^{0,1} = g$. This is the case of Riemann surfaces. Fix a lattice $\mathbb{Z}^n$, where $n = \text{rank } H_{\mathbb{Z}}$ and a bilinear form Q_0 on $\mathbb{Z}^n$ isometric to Q. A *marked Hodge structure* (H, m) is a Hodge structure H on $H_{\mathbb{Z}}$ together with an isometry $m : \mathbb{Z}^n \longrightarrow H_{\mathbb{Z}}$. A marked Hodge structure determines a distinguished basis $\{m(e_i)\}$ of $H_{\mathbb{Z}}$, and such a distinguished basis in turn determines a marking.

Let D be the set of all marked Hodge structures on $H_{\mathbb{Z}}$ polarized by Q with Hodge numbers h. This is the *period domain* with the given data. We will show in a moment that this set is a complex manifold. Below we study important special cases of period domains, beginning with elliptic curves and Riemann surfaces of higher genus, then progressing to period domains of higher weight. For domains of higher weight, we will discover a new phenomenon, *Griffiths transversality*, which plays an important role.

Let Γ be the group of isometries of $H_{\mathbb{Z}}$ relative to Q. This is an arithmetic group – a group of integral matrices defined by algebraic equations – which acts on D. It turns out that the action is properly discontinuous, so that the quotient D/Γ, the *period space*, is defined as an analytic space. Whereas D parametrizes marked Hodge structures, D/Γ parametrizes isomorphism classes of Hodge structures. The period space is the quotient of a certain complex manifold D by the action of a discrete group Γ. As such it is an analytic space with mild singularities (quotient singularities). Such spaces are called *orbifolds* or *V-manifolds*. Every point of such an object has a neighborhood which is the quotient of an open set in $\mathbb{C}^n$ by a the action of a finite group. When the group is trivial, one has a manifold. In general, open sets of orbifolds are parametrized by open sets in $\mathbb{C}^n$, but the parametrization may be n-to-1 with $n > 1$. In addition, a special phenomenon occurs in dimension one. Locally, the only possible orbifold structure is the quotient of a disk by a cyclic group. The quotient of the unit disk Δ by the group μ_n of n-th roots of unity is homeomorphic to the unit disk. Indeed, the map $f(z) = z^n$ identifies Δ/μ_n with Δ. The map $f : \Delta \longrightarrow \Delta$ descends to a bijection $\Delta/\mu_n \longrightarrow \Delta$. This phenomenon, where orbifolds are also manifolds, is special to dimension one.

Note that the group Γ may be viewed as the subgroup of matrices with integer entries in an orthogonal or symplectic group.

Consider now a family of algebraic varieties $\{X_s \mid s \in S\}$. Let $\Delta \subset S$ be the *discriminant locus* — the set of points in S where the fiber X_s is singular. We assume this to be a proper (Zariski) closed subset. The map f that associates to a point $s \in S - \Delta$ the class of the Hodge structure of $H^k(X_s)$ in D/Γ is called the *period map.* It is a map with quite special properties; in particular, it is holomorphic, it is the quotient of a holomorphic map of $\tilde{f} : \tilde{S} \longrightarrow D$, where $\tilde{S}$ is the universal cover, and its behavior as one approaches the discriminant locus is controlled by the *monodromy representation.* The monodromy reprentation is a homomorphis

$$\rho : \pi_1(S, o) \longrightarrow \text{Aut} H^k(Xo)$$

which is equivariant in the sense that

$$\tilde{f}(\gamma x) = \rho(\gamma)\tilde{f}(x),$$

where $\tilde{f}$ is the"lift" of f.

To define this monodromy representation, consider first a family of algebraic varietes X/Δ, that is, a map $f : X \longrightarrow \Delta$, where Δ is the a disk of unit radius and every point of Δ except the origin is a regular value. Thus the fibers $X_s = f^{-1}(s)$ are smooth for $s \neq 0$. However, X_0 is in general singular. Let $\xi = \partial/\partial\theta$ be the angular vector field. It defines a flow $\phi(\theta)$ where $\phi(\theta)$ is the diffeomorphism "rotation counterclockwise by θ radians. Note that $\phi(\theta_1)\phi(\theta_2) = \phi(\theta_1 + \theta_2)$: the flow is a one-parameter family of diffeomorphisms such that $\phi(2\pi)$ is the identity. Over the punctured disk $\Delta^* = \Delta - \{0\}$, one may construct a vectorfield η which lifts ξ in the sense that $f_*\eta = \xi$. Let $\psi(\theta)$ be the associated flow. It satisfies $f \circ \psi(\theta) = \psi(\theta)$ and $\psi(\theta_1)\phi(\theta_2) = \phi(\theta_1 + \theta_2)$ Let $T = \psi(2\pi)$. This transformation, the monodromy transformation, is not usually the identity map, even when considered on the level of homology, which is usually how it is viewed.

More generally, one proceeds as follows. Let X/S be a family of varieties with discriminant locus Δ. Let γ be a loop in $S - \Delta$. It is given parametrically by a map $\gamma(t) : [0, 1] \longrightarrow S - \Delta$. Consider the "cylinder" $f^{-1}(\gamma([0, 1])$ over the "circle" $\gamma([0, 1])$. Let η be a vectorfield defined on this cylinder such that $f_*\eta = \partial/\partial t$. Let $\psi(t)$ be the corresponding flow, and let $\rho(\gamma) = \psi(1)$ considered in homology: using the flow, we push the cycles around the cylinder from the fiber $X_{\gamma(0)}$ back to the same fiber. This map is not necessarily the idenity, though it is if γ is homotopic to the identity. There results a homomorphism

$$\pi_1(S - \Delta, o) \longrightarrow Aut(H^k(X_o, \mathbb{Z})).$$

This is the *monodromy representation*.

As the simplest example of a monodromy representation, let M be the Moebius band. It is a bundle over the circle with fiber which can be identified with the interval $[-1, 1]$. Let $f : M \longrightarrow S^1$ be the projection. Consider also the boundary of the Moebius band, ∂M. The fiber of $f : \partial M \longrightarrow S^1$ is the two point space $f^{-1}(\theta) \cong \{-1, +1\}$. What is the monodromy representation for $\partial M \longrightarrow S^1$? It is the nontrivial map $\pi_1(S^1, 0) \longrightarrow Aut\, H_0(\{-1, +1\})$, which can be identified with the map $\mathbb{Z} \longrightarrow \mathbb{Z}/2$. It is generated by the permutation which interchanges $+1$ and -1. Quite often monodromy representations are (nearly) surjective and have large kernels. See [2] and [8].

Figure: Moebius band

Exercise 1. *Consider the family $M \longrightarrow S^1$ where M is the Moebius strip. What is the monodromy on $H_1(f^{-1}(0), \partial f^{-1}(0)) \cong H_1([-1,+1], \{-1,+1\})$?*

A more significant example, which will we will study in more detail in the next section, concerns the family of elliptic curves $y^2 = (x^2 - t)(x - 1)$. For $|t| < 1/3$, (for example) this family is smooth. The fiber over the origin is a cubic curve with one node. Consider as homology basis the positively oriented circle δ of radius $1/2$. It encircles the two branch point at $\pm\sqrt{t}$. Let γ be the path that runs from ∞ to the branch point between $-\sqrt{t}$ and $+\sqrt{t}$. When it meets the cut, it travels upwards to $\sqrt{-2/3}$, after which it makes a large rightward arc before traveling back to the base point at infimity. By drawing picutres at $\theta = 0, \pi/2$, and π, ones sees that (s) $T(\delta) = \delta$; (b) $T(\gamma) = \gamma + \delta$. Thus the matrix of T is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

Figure: Dehn twist

Exercise 2. *What is the monodromy for the family $y^2 = x(x-1)(x-t)$ near $t = 0, 1, \infty$?*

2. Elliptic curves

Let us now pass from the very general to the very concrete. We will determine the period domain and period space for Hodge structures of elliptic curves, and we will study several natural period mappings associated to families of elliptic curves. This discussion will provide a guide to understanding period domains for arbitrary Hodge structures of weight one. Once we do this, we will consider the non-classical situation, that of Hodge structures of higher weight.

Let $\mathcal{E}$ be an *elliptic curve*, that is, a Riemann surface of genus one. Such a surface can be defined by the affine equation $y^2 = p(x)$, where $p(x)$ is a cubic polynomial with distinct roots. Then $\mathcal{E}$ is a double cover of the Riemann sphere $\mathbb{CP}^1$ with branch points at the roots of p and also at infinity. A homology basis $\{\delta, \gamma\}$ for $\mathcal{E}$ is pictured below.

Figure: Homology basis for $\mathcal{E}$

The intersection matrix for the the indicated homology basis is the "standard symplectic form,"

$$Q_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Note that it is unimodular, i.e., has determinant 1. This is a reflection of Poincaré duality. Let $\{\delta^*, \gamma^*\}$ be the basis which is dual to the standard symplectic basis, i.e., $\delta^*(\delta) = \gamma^*(\gamma) = 1$, $\delta^*(\gamma) = \gamma^*(\delta) = 0$. As noted above, such a basis determines a marking of $H^1(M)$.

For a basis of $H^{1,0}(S)$, take the differential form

$$\omega = \frac{dx}{y} = \frac{dx}{\sqrt{p(x)}}.$$

It is obvious that ω is holomorphic on the part of S above the the complement of the set of zeros of $p(x)$, plus the point at infinity. A calculation in local coordinates shows that ω is holomorphic at those points as well. (Exercise).

At this point we can give an approximate answer to the question "what is the period domain for Hodge structures of elliptic curves?" A marking $m : \mathbb{Z}^2 \longrightarrow H^1(M, \mathbb{Z})$ determines a subspace $m^{-1}(H^{1,0}(M)) \subset \mathbb{C}^2$. That is, a marked Hodge structure determines a one-dimensional subspace of $\mathbb{C}^2$. The set of one-dimensional subspaces of a two-dimensional vector space is a Grassman variety. In this case it is just complex projective space of dimension one. Thus the set of marked Hodge structures D of weight one with $h^{1,0} = 1$ can be identified with a subset of $\mathbb{CP}^1$.

It is natural to ask: is $D = \mathbb{CP}^1$? The discussion below shows that it is not. Nonetheless, $\mathbb{CP}^1$ does play a special role. Every period domain has its so-called compact dual $\check{D}$, and in the present case, $\check{D} = \mathbb{CP}^1$. The compact dual is a compact complex manifold with a number of special properties, and the period domain D is an open subset of it.

To answer the question of which part of $\mathbb{CP}^1$ corresponds to polarized Hodge structures, consider the vector of integrals

$$(A, B) = \left(\int_\delta \omega, \int_\gamma \omega \right)$$

These are the *periods* of ω. We refer to them as the A-period and the B-period. Note that

$$\omega = A\delta^* + B\gamma^*.$$

The second Riemann bilinear relation is the statement

$$i \int_S \omega \wedge \overline{\omega} > 0.$$

Substituting the expression for ω in terms of the dual basis and using

$$(\delta^* \cup \gamma^*)[S] = 1,$$

we find that

$$i(A\bar{B} - B\bar{A}) > 0. \tag{11}$$

It follows that $A \neq 0$, and $B \neq 0$. Therefore the *period ratio* $Z = B/A$ is defined. The period ratio depends only on the choice of marking. It is therefore an invariant of the marked Hodge structure (H, m).

The ratio Z can be viewed as the B-period of the unique cohomology class in $H^{1,0}$ whose A-period is one:

$$(A, B) = (1, Z).$$

From (11), it follows that Z, the *normalized period*, has positive imaginary part. Thus to the marked Hodge structure (H^1, m) is associated a point in the upper half plane,

$$\mathcal{H} = \{z \in \mathbb{C} \mid \Im z > 0\}.$$

Consequently there is a map

$$\{\text{Marked Hodge Structures}\} \longrightarrow \mathcal{H}. \tag{12}$$

This map has an inverse given by

$$Z \in \mathbb{C} \mapsto \mathbb{C}(\delta^* + Z\gamma^*).$$

Thus (12) is an isomorphism:

$$D = \{\text{Marked Hodge Structures}\} \cong \mathcal{H}.$$

To see how D sits inside the Grassmannian $\mathbb{CP}^1$, let $[A, B]$ be homogeneous coordinates for projective space. Identify $\{[A, B] \mid A \neq 0\}$ with the complex line $\mathbb{C}$ via $[A, B] \mapsto B/A$. Then $\mathbb{CP}^1$ is identified with the one-point compactification of the complex line, where the point at infinity corresponds to the point of $\mathbb{CP}^1$ with homogeneous coordinates $[0, 1]$. The inclusion of D in $\check{D}$ can then be identified with the composition of maps

$$\mathcal{H} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$$

The upper half plane can be thought of as the part of the northen hemisphere strictly above the equator, which in turn can be thought of as the one-point compactification of the real line. This is no surprise, since the upper half plane is biholomorphic to the unit disk.

Having identified the period domain D with the upper half-plane, let us identify the period space D/Γ. The key question is: what is the group of transformations that preserves the lattice $H^1_{\mathbb{Z}}$ and the bilinear form Q? The answer is clear: the group of 2×2 integral symplectic matrices. This is a group which acts transitively on markings. Let M be such a matrix, and consider the equation

$${}^tMQ_0M = Q_0.$$

Set

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The above matrix equation is equivalent to the single scalar equation $ad - bc = 1$, that is, to the condition $\det M = 1$. Thus the group Γ, which *is* an integral symplectic group, is also the group of integer matrices of determinant 1, that is the group $SL(2, \mathbb{Z})$. This group acts on complex projective space by fractional linear transformations. Indeed, we have

$$(1, Z)M = (a + cZ, b + dZ) \equiv (1, (b + dZ)/(a + cZ)),$$

so the action on normalized period matrices is by

$$Z \mapsto \frac{b + dZ}{a + cZ}.$$

The action on $\mathcal{H}$ is properly discontinuous: that is, for a compact set $K \subset \mathcal{H}$, there are only finitely many group elements g such that $gK \cap K \neq \emptyset$. Consequently the quotient $\mathcal{H}/\Gamma$ is a Hausdorff topological space. It is even more: in general an analytic manifold with a natural orbifold structure. To conclude, we have found that

$$\{\text{Isomorphism Classes of Hodge Structures}\} \cong D/\Gamma \cong \mathcal{H}/\Gamma.$$

To see what sort of an object is the period space, note that a fundamental domain for Γ is given by the set

$$\mathcal{F} = \{z \in \mathcal{H} \mid |\Re(z)| \leq 1/2, |z] \geq 1\}.$$

which is pictured below. The domain $\mathcal{F}$ is a triangle with two real vertices at ω and $-\bar{\omega}$, where

$$\omega = \frac{-1+\sqrt{-3}}{2}$$

is a primitive cube root of unity, and with one ideal vertex at infinity. The group Γ is generated by the element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

corresponding to the fractional linear transformation

$$S(Z) = -1/Z,$$

and the element

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

corresponding to the fractional linear transformation

$$T(Z) = Z + 1.$$

The period space D/Γ is the same as the fundamental domain $\mathcal{F}$ with the identifications determined by S and T applied. The identification defined by S glues the right and left sides of $\mathcal{F}$ to make a cylinder. The points ω and $-1/\omega = -\bar{\omega}$ are identified by T. The map S glues one side of the edge of the cylinder to the other: the arc from ω to i is identified with the arc from $-1/\omega$ to i. Topologically, the result is a disk. As an orbifold it can be identified with the complex line with two special points corresponding to ω and i. From one point of view this is because ω and i are fixed points for the action of Γ, of order 6 and 2, respectively. From another point of view, there is a meromorphic function $j(z)$, the quotient of modular forms of weight 12, which is invariant under the action of Γ and which gives a bijective holomorphic map $\mathcal{H}/\Gamma \longrightarrow \mathbb{C}$.

The modular forms are defined as follows.

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^4},$$

$$g_3 = 140 \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^6},$$

and

$$\Delta = g_2^3 - 27g_3^2.$$

The function $\Delta(\tau)$ is the *discriminant.* It vanishes if and only if the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3$$

is singular. The j-invariant is the quotient

$$j(\tau) = \frac{1728 g_2^3}{\Delta}$$

Thus $j(\tau) = 0$ for the elliptic curve with $g_2 = 0$. By a change of coordinates, we may assume that the curve has equation $y^2 = x^3 - 1$. This is elliptic curve with branch points at infinity and the cube roots of unity. It has an automorphism of order six, given by $(x, y) \mapsto (\omega x, -y)$.

Figure: Fundamental domain

Exercise 3. *Show that the group $G = SL(2, \mathbb{R})$ acts transitively on the upper half plane by fractional linear transformations. Show that the isotropy group $K = \{g \in G \mid g.i = i\}$ is isomorphic to the unitary group $U(1)$. Then show $\mathcal{H} \cong G/K$. The period space is then $G_{\mathbb{Z}} \backslash G/K$, where $G_{\mathbb{Z}}$ is the set of integer valued points of G.*

3. Period mappings: an example

Let us consider now the family of elliptic curves $\mathcal{E}_t$ given by

$$y^2 = x(x-1)(x-t)$$

The parameter space for this family is the extended complex line, $\mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$. The fibers $\mathcal{E}_t$ are smooth for $t \neq 0, 1, \infty$. The set of points $\Delta = \{0, 1, \infty\}$ is the *discriminant locus* of the family. Thus the family $\{\mathcal{E}_t\}$ is smooth when restricted to $\mathbb{CP}^1 - \Delta$.

Fix a point t_0 in the complement of the discriminant locus, and let U be a coordinate disk centered at t_0 which lies in the complement of the discriminant locus. Then a homology basis $\{\delta, \gamma\}$ for $\mathcal{E}_t$ exists which lies in the inverse image of the complement of U. See the figure below. Therefore the periods for $\mathcal{E}_t$ take the form

$$A_t = \int_\delta \frac{dx}{\sqrt{x(x-1)(x-t)}} \qquad B_t = \int_\gamma \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

The domains of integration are fixed and the integrands depend holomorphically on the parameter t. Therefore A_t and B_t are holomorphic functions on U, as is the ratio $Z_t = B_t/A_t$. Thus the period map, which so far we have defined only locally, that is, on U, is a holomorphic function with values in $\mathcal{H}$.

Figure: Homology basis for $\mathcal{E}_t$

One approach to defining the period map globally is to consider its full analytic continuation, which will be a function from the universal cover of the parameter space minus the idiscriminant locus to the period domain, in this case the upper half plane. The branches of the analytic continuation correspond to different markings of the fibers $\mathcal{E}_t$, that is, to different homology bases. Consider therefore the composed map

$$\{\text{Universal cover of } \mathbb{CP}^1 - \Delta\} \longrightarrow \mathcal{H} \longrightarrow \Gamma\backslash\mathcal{H}.$$

This map is holomorphic, and it is also invariant under the action of covering transformations on the left. Thus we obtain a holomorphic map

$$\mathbb{CP}^1 - \Delta \longrightarrow \Gamma\backslash\mathcal{H}.$$

This is the period map.

The family of elliptic curves just discussed has three singular fibers. Are there nontrivial families with fewer fibers? Consider a family with just two singular fibers, which we may take to be at zero and infinity. Thus the parameter space for the smooth fibers is $\mathbb{C}^*$, the complex line with the origin removed. The universal cover of $\mathbb{C}^*$ is the complex line, and the covering map $\mathbb{C} \longrightarrow \mathbb{C}^*$ is given by $\exp 2\pi i z$. Thus the lift of the period map to the universal cover is a holomorphic map $\mathbb{C} \longrightarrow \mathcal{H}$. Now the upper half plane is biholomorphic to the unit disk (exercise: verify this). Consequently the lift is in essence a bounded entire function. Such functions are constant. This means that if s and t are nonzero complex numbers, then $\mathcal{E}_s \cong \mathcal{E}_t$ for a family of elliptic curves with at most two singular fibers.

As a related application, suppose that one has a family of elliptic curves parametrized by the unit disk minus the origin, which we write as Δ^*. Suppose further that the monodromy transformation is trivial. This means that analytic continuation defines a period map $f : \Delta^* \longrightarrow \mathcal{H}$. Again using the fact that $\mathcal{H}$ is biholomorphic to the disk, we apply the Riemann removable singularity theorem to conclude that the period map extends to a holomorphic map $f : \Delta \longrightarrow \mathcal{H}$. The point $f(0)$ corresponds to the Hodge structure of a a smooth elliptic curve. From this we conclude that the family of elliptic curves on the punctured disk is the restriction of a family of elliptic curves on the disk.

Slight technical problem: fine versus coarse moduli space?

3.1. **Asymptotics of the period map.** Let us examine the behavior of the period map for the family $y^2 = x(x-1)(x-t)$ as t approaches infinity along the ray $[2, \infty)$ on the real axis. Let δ and γ be as above, and note that

$$\int_\delta \frac{dx}{\sqrt{x(x-1)(x-t)}} \sim \int_\delta \frac{dx}{x\sqrt{-t}} \sim \frac{2\pi}{\sqrt{t}}$$

when t is large. By deforming the path of integration over γ, we find that

$$\int_\gamma \frac{dx}{\sqrt{x(x-1)(x-t)}} = -2\int_0^t \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

The difference between the last integrand and the integrand $1/x(x-t)$ is $1/2x^2 +$ higher powers of x^{-1} The integral of the latter expression is asymptotically neglible. The residual integral can be computed exactly:

$$-2\int_0^t \frac{dx}{\sqrt{x(x-1)(x-t)}} = \frac{4}{\sqrt{t}} arctan \frac{\sqrt{1-t}}{\sqrt{t}} \sim \frac{2\sqrt{-1}}{\sqrt{t}} \log t.$$

Thus the period ratio Z satisfies the asymptotic expansion

$$Z(t) \sim \frac{\sqrt{-1}}{\pi} \log t.$$

This behavior is typical, though much harder to prove. The dominant term in a period is of the form $t^a(\log t)^b$, where a is a rational number such that $(2\pi a)$ is a root of unity. The exponent b is the index of nilpotence of $\gamma^m - 1$, plus one, where $\lambda^m = 1$ for all λ. Here we use that fact, to be proved later, that the eigenvalues of T are roots of unity, so that such an m exists. In our example, $m = 1$ and $b = 2$, since $(T-1)^2 = 0$.

Although it is not yet apparent, the asymptotic behavoir of the period map is controlled by the index of nilpotence of $\gamma^m - 1$. Here m is an integer such that the eigenvalues λ of γ satifsy $\lambda^m = 1$. Then the eigenvalues of γ^m are one, and γ^m is nilpotent. For algebraic curves, $(\gamma^m - 1)^2 = 0$, for surfacs $(\gamma^m - 1)^3 = 0$, etc. Of course the index of nilpotence can be less than maximal in special cases.

Exercise 4. *(a) Consider the family of zero-dimensonal varities X_t defined by $z^p = t$. Describe the monodromy on $H_0(X_t)$. (b) [Harder] Consider the family $z^p + w^q = t$. Describe the monodromy on $H_0(X_t)$. (See also* [13]*).*

Exercise 5. *Consider the family of ellpitc curves $\{\mathcal{E}_t\}$ defined by $y^2 = x^3 - 1$. What is the monodromy representation? What is the period map? Describe the singular fibers.*

4. Hodge structures of weight one

Let us now study the period domain for polarizd Hodge structures of weight one. One source of such Hodge structures is the first cohomology of Riemann surfaces. Another is the first cohomology of abelian varieties. An abelian variety is a compact complex torus that is also a projective algebraic variety. Elliptic curves are abelian varieties of dimension one.

It is natural to ask if all weight one Hodge structures come from Riemann surfaces. This is the case for elliptic curves, since the moduli space (the space of isomorphism classes) of elliptic curves has dimension one, the same as the dimension of the period space. For genus greater than one, the dimension of the moduli space is $3g - 3$. Thus the space of Hodge structures of which come from such Riemann surfaces is a space of dimension at most $3g - 3$ for $g > 1$. The Torelli theorem states that the period map is injective. This means that the map which associates to a point in the moduli space its corresponding Hodge structure is injective. Consequently the space of Hodge structures coming from Riemann surfaces of genus greater than one has exactly dimension $3g - 3$.

As we shall see shortly, the space of Hodge structures of genus g, that is, with $\dim H^{1,0} = g$ has dimension $g(g + 1)/2$. It follows that the dimension of the space of Hodge structures of genus g is larger than the dimension of the space of Hodge structures coming from Riemann surfaces of genus g for $g > 3$.

One can invert the construction of a polarized Hodge structure given an abelian variety. If H is a Hodge structure of weight one, define the quotient

$$J(H) = \frac{H_{\mathbb{C}}}{H^{1,0} + H_{\mathbb{Z}}}$$

Exercise 6. *$J(H)$ is a compact complex torus whose first cohomology is isomorphic to H as a Hodge structure.*

Remark 1. *If H is a polarized Hodge structure, one can show more: $J = J(H)$ is a projective algebraic variety. The idea is as follows. The polarizing form Q is an element of $\Lambda^2 H_{\mathbb{Z}}$. One may view it as an element ω_Q of $H^2(J, \mathbb{Z})$. The first Riemann bilinear relation shows that ω_Q has type $(1, 1)$. The second Riemann bilinear relation is equivalent to the statement that ω_Q is represented by a positive $(1, 1)$ form. The exponential sheaf sequence* *shows that ω_Q is the first Chern class of a holomorphic line bundle L. The positivity of the first Chern class allows one to apply Kodaira's theorem, which implies that sections of some power of L give a projective imbedding of J.*

Let us now describe the period space D for polarized Hodge structures of weight one and genus $g = \dim H^{1,0}$. This period space depends on the choice of a skew form Q. For now we assume that

$$A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is he $g \times g$ identity matrix. Let $\{\delta^j, \gamma^j\}$, where j runs from be a basis for $H_{\mathbb{Z}}$. Write $e_i = \delta_i$ and $e_{g+i} = \gamma_i$. We assume the matrix of inner products (e_i, e_j) to be Q. Let $\{\phi_i\}$ be a basis for $H^{1,0}$, where i runs from 1 to g. Then

$$\phi_i = \sum_j A_{ij}\delta^j + \sum_j B_{ij}\gamma^j.$$

The $g \times 2g$ matrix

$$P = (A, B)$$

is called the *period matrix* of the Hodge structure with respect to the given bases. The basis for $H_{\mathbb{Z}}$ defines a marking m of the Hodge structure, and $S = m^{-1}H^{1,0} \subset \mathbb{C}^{2g}$ is the row space of the period matrix. Thus the period domain D is a subset of the Grassmannian of g-planes in complex $2g$-space. The Riemann bilinear relations impose restrictions on these subspaces. The first Riemann bilinear relation gives a set of equalities that express the fact that $Q(v, w) = 0$ for any two vectors in S. The second Riemann bilinear relation gives a set of inequalities: $\sqrt{-1}Q(v, \bar{v}) > 0$ for any nonzero vector in S.

To understand both the equations and the inequalities, we first show that A is a nonsingular matrix. If that is the case, we can change the basis of $H^{1,0}$ by replacing ϕ_i with the sum $A^{ij}\phi_j$, where (A^{ij}) is the inverse of A. The period matrix then takes the form

$$P = (I, A^{-1}B) = (I, Z).$$

The first bilinear relation implies that Z is symmetric. The second bilinear relation implies that Z has positive-definite imaginary part. From this we conclude that D is an open subset of a complex Euclidean space of dimension $g(g+1)/2$. It is a generalization of the upper half space in $\mathbb{C}$, the so-called *Siegel upper half space* of genus g, written $\mathcal{H}_g$.

To show that A is nonsingular, let $\phi = v_m\phi_m$ be an arbitrary nonzero element of $H^{1,0}$. Then

$$\sqrt{-1}Q(\phi, \bar{\phi}) > 0.$$

Therefore

$$\sqrt{-1}Q(v_m A_{mj}\delta^j + v_m B_{mj}\gamma^j, \bar{v}_n \bar{A}_{nk}\delta^k + \bar{v}_n \bar{B}_{nk}\gamma^k) > 0,$$

and so

$$\sqrt{-1}(v_m \bar{v}_n A_{mj} \bar{B}_{nj} - v_m \bar{v}_n B_{mj} \bar{A}_{nj}) > 0. \tag{13}$$

Let

$$H_{mn} = \sqrt{-1}(A_{mj} \bar{B}_{nj} - B_{mj} \bar{A}_{nj}).$$

Then

$$H = \sqrt{-1}(AB^* - BA^*),$$

where B^* is the Hermitian conjugate of B. Then (14) asserts that H is a positive hermitian matrix. If A were singular, there would exist a vector v such that $vA = 0$, in which case we would also have $A^* \bar{v} = 0$, so that $vH\bar{v} = 0$, a contradiction. Thus A is nonsingular, as claimed.

At this point we know that the period matrix can be brought to the normalized form $P = (I, Z)$, where for the moment Z is an arbitrary $g \times g$ matrix. Let us apply the first Riemann bilinear relation to elements $\phi = v_m \phi_m$ and $\psi = w_n \phi_n$ of $H^{1,0}$. One finds that

$$v_m w_n A_{mj} B_{nj} - v_m w_n B_{mj} A_{nj} = 0. \tag{14}$$

Setting A to the identity matrix, this simplifies to

$$v_m w_n B_{nm} - v_m w_n B_{mn} = 0, \tag{15}$$

which can be written

$$v(B - {}^tB)w = 0$$

for arbitrary v and w. Therefore B, that is, Z, is a symmetric matrix, as claimed. The fact that H is postive-definite is now equivalent to the statement that Z has positive-definite imaginary part. To conclude:

$$\mathcal{H}_g = \{Z \mid Z \text{ is a complex symmetric matrix with positive imaginary part}\}.$$

Remark 2. *As in the case of $\mathcal{H} = \mathcal{H}_1$, there is a group-theoretic description. Let $G = Sp(g, \mathbb{R})$ be the group of real $2g \times 2g$ matrices which preserve the form Q. This is the real symplectic group. Fix a polarized Hodge structure H. Let g be an element of the isotropy group K of this "reference" Hodge structure. The element g preserves this Hodge structure. Then g is determined by its restriction to $H^{1,0}$. The restriction to this subspace preserves the positive hermitian form $\sqrt{-1}Q(v, \bar{w})$. Therefore K is the unitary group of $H^{1,0}$. Since G acts transitively on $\mathcal{H}_g$ (Exercise!), we find that*

$$\mathcal{H}_g \cong G/K = Sp(g, \mathbb{R})/U(g).$$

The group which permutes the markings is $G_{\mathbb{Z}} = Sp(g, \mathbb{Z})$. Thus the period space is

$$\Gamma \backslash \mathcal{H}_g = G_{\mathbb{Z}} \backslash G/K = Sp(g, \mathbb{Z}) \backslash Sp(g, \mathbb{R})/U(g).$$

Exercise 7. *Compute the dimensions of the Lie groups $Sp(g, \mathbb{R})$ and $U(g)$.*

Remark 3. *Let G be a non-compact Lie simple Lie group and let K be a maximal compact subgroup. The quotient $D = G/K$ is a* homogeneous space. *By homogeneous we mean that there is a transitive group action. The group is G. Spaces of this kind are a special kind of homogeneous space known as a* symmetric space. *These spaces carry a G-invariant Riemannian metric, and at each point x of D there is an isometry i_x which fixes x and which acts as -1 on the tangent space. That symmetry is given by an element of K which can be identified with $-I$, where I is the identity matrix, when G and K are identified with matrix groups in a suitable way. The Siegel upper half space is an example of a* Hermitian *symmetric space. This is a symmetric space which is also a complex manifold. For such spaces the isotropy group contains a natural subgroup isomorphic to the circle group $U(1)$. Its action gives the complex structure tensor, for then isometries representing multiplication by a unit complex number are defined. In particular, an isometry representing rotation counterclockwise through an angle of* $90°$ *is defined at each point.*

Remark 4. *Let G/K be a hermitian symmetric space, and let $\Gamma = G_{\mathbb{Z}}$. A theorem of Baily-Borel shows that $M = G_{\mathbb{Z}}\backslash G/K$ has s projective imbedding. Thus, like the quotient of the upper half plane by the action of $SL(2,\mathbb{Z})$, these spaces are quasiprojective algebraic varieties. By this we mean that they are of the form "a projective variety minus a projective subvariety."*

5. Hodge structures of weight two

New phenomena arise when one considers Hodge structures of weight greater than one. All of the new phenomena present themselves in the weight two case. Since we can study without struggling with notational issues, we consider this case first. Polarized Hodge structures of weight two arise in nature as the primitive second cohomology of an algebraic surface M. In this case primitivity has a simple meaning: orthogonal to the hyperplane class. Such a Hodge structure has the form

$$H^2_0 = H^{2,0} \oplus H^{1,1}_0 \oplus H^{0,2}.$$

We remarked earlier that classes of type $(2,0)$ are primitive for reasons of type. The same is of course true for classes of type $(0,2)$. In this case, the Hodge filtration is $F^2 = H^{2,0}$, $F^1 = H^{2,0} \oplus H^{1,1}_0$, $F^0 = H^2_0$. By the first Riemann bilinear relation, the orthogonal complement of F^2 is F^1. Thus the data F^2 and Q determine the polarized Hodge structure. Let us set $p = \dim H^{2,0}$ and $q = \dim H^{1,1}_0$. Then a marked Hodge structure determines a subspace $S = m^{-1}F^2$ of dimension p in $\mathbb{C}^{2p+q}$. The period domain D whose description we seek is therefore a subset of the Grassmannian of p-planes in $2p+q$-space. As in the case of weight one structures, the first and second bilinear relations impose certain equalities and inequalities. As a result, D will be an open subset of a certain closed submanifold $\check{D}$ of the

Grassmannian. The submanifold $\check{D}$ is the set of isotropic spaces of dimension p: the set of p- dimensional subspaces S on which Q is identically zero. The open set D is defined by the requirement that $Q(v, \bar{v}) < 0$ for all nonzero vectors in S.

As before, the period domain admits a group-theoretic description: it is a certain kind of complex homogeneous space. Let G be the special orthogonal group of the vector space $H_{\mathbb{R}}$ endowed with the symmetric bilinear form Q. We will show in a moment that this group acts transitively on marked Hodge structures. Assuming this for the moment, consider some particular marked Hodge structure. An element g of G which leaves this Hodge structure invariant restricts to the subspaces of the Hodge decomposition. Since g is a real linear transformation, its restriction to $H^{2,0}$ determines its restriction to $H^{0,2}$: just take the conjugate. The restriction to $H^{2,0}$ also preserves the negative hermitian form $Q(v, \bar{w})$. Similarly, the restriction of g to $H^{1,1}_0$ preserves the negative-definite form $Q(v, w)$. Let V be the isotropy subgroup of G: the subgroup whose elments leave the particular marked Hodge structure fixed. We have just defined a map $V \longrightarrow U(p) \times SO(q)$ via $g \mapsto (g|H^{2,0}, g|H^{1,1}_0)$. This map is an isomorphism (exercise). Thus we find that

$$D \cong G/V.$$

The easiest way to see the complex structure on D is realize that it is an open set of $\check{D}$, which in turn is an algebraic submanifold of the Grassmannian. Since G acts by holomorphic transformations, the complex structure defined group-theoretically agrees with the one defined naively.

We now encounter the first major difference with classical Hodge theory. In general the isotropy group V, while compact, is not maximal compact. Indeed, for weight two structures, a maximal compact subgroup K has the form $SO(2p) \times SO(q)$. An element of K is an element of G which preserves (a) the subspace $H^{1,1}_0$ and the subspace $H^{2,0} \oplus H^{0,2}$. The latter is the complexification of the set of points left invariant by conjugation. The restriction of Q to this subpace is a negative definite form of rank $2p$. Thus K is isomorphic to $S(O(2p) \times O(q))$. Only in the case $p = 1$ do we have $V = K$. In that case D is a hermitian symmmetric space, and it can be realized as a bounded domain in complex euclidean space. In all other cases D is not Hermitian symmetric, and it is not an open subset of complex Euclidean space. Indeed, D abounds in compact complex subvarieties of positive dimension, namely K/V and its tranlsates by G. It is worth mentioning that (a) the Hermitian symmetric case occurs for K3 surfaces, hence their prominent role in algebraic geometry, and (b) in general (all weights) the map $G/V \longrightarrow G/K$ has target a symmetric space, and the source is a complex manifold. However, the map is not holomorphic, even when G/K happens to be hermitian- symmetric.

Despite the nonclassical nature of D in higher weight, there are two facts that make period domains and period mappings useful tools of study. The first is that the period map is holomorphic. The second is that it satisfies a differential equation which forces it, unlike an arbitrary holomorphic map into a period domain, to behave as if it were mapping into a bounded domain. This differential equation, which we shall explain presently, is *Griffiths transversality.*

Let us begin with the definition of the period map, which we do in complete generality. Let $f : X \longrightarrow S$ be a smooth family of algebraic varieties. Let $U \subset S$ be a contractible open set with distinguished point o. Then the restriction of f to $f^{-1}(U)$ is a family differentiably isomorphic to $X_o \times U \longrightarrow U$, where the map is projection on the second factor. Let $\{\delta^i\}$ be a basis for $H^k(X_o, \mathbb{Z})_0$. Because of the product structure, this basis defines a basis for $H^k(X_s, \mathbb{Z})_0$ for all s in U. Consequently we have a family of markings $m_s : \mathbb{Z}^n \longrightarrow H^k(X_s, \mathbb{Z})$. Let $F^p_s = m_s^{-1}(F^p H^k(X_s)_0)$. Thus is defined a family of filtrations F^p_s of $\mathbb{C}^n$. This is the local period map $p : U \longrightarrow D$. If one admits that the period mapping is holomorphic, one defines the period mapping from the universal cover of S to D as the full analytic continuation of such a local period map. Thus one has first

$$p : \tilde{S} \longrightarrow D$$

and subsequently the quotient map

$$p : S \longrightarrow \Gamma \backslash D.$$

However, there is another argument. Lift the family X/S to a family $X/\tilde{S}$. Then the bundle of cohomology groups of fibers is trivial, i.e, is isomorphic to $H^k(X_o, \mathbb{Z}) \times \tilde{S}$. Therefore there is a marking for the cohomology of the family pulled back. Using this marking, which makes to no reference to the as yet unproved holomorphicity of the period map, we construct a period map $p : \tilde{S} \longrightarrow D$.

We will show that the period map is holomorphic in the special case of surfaces in $\mathbb{P}^3$. The method of proof, which relies on the Griffiths-Poincaré residue, works for the case of hypersurfaces in $\mathbb{P}^n$. Using the residue calculus, we will also establish Griffiths transversality in this case.

Exercise 8. *(a) Determine the groups G, V, and K for a Hodge structure of weight three. (b) Do the same for weight four.*

6. Poincaré residues

Let X_i be homogeneous coordinates on $\mathbb{C}^{n+1}$, where i ranges from 0 to n. Consider the n-form

$$d(X_1/X_0) \wedge \cdots \wedge d(X_n/X_0) = \frac{1}{X_0^{n+1}} \sum_{i=0}^{n} X_i dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n$$

where we view the X_i/X_0 as affine coordinates on the open set $U_0 = \{[X] \mid X_0 \neq 0\}$. Note that this form is homogeneous of degree zero. An object is homogeneous of degree d if it is multiplied by λ^d when each X_i is multiplied by λ. This form is meromorphic on $\mathbb{P}^n$ with a pole of order $n+1$ on the hyperplane $X_0 = 0$.

Let

$$\Omega = \sum_{i=0}^{n} X_i dX_0 \wedge \cdots \wedge \widehat{dX_i} \wedge \cdots \wedge dX_n.$$

It is an expression which is homogeneous of degree $n+1$, and may be viewed as a holomorphic section of $\Omega^n(n+1) \cong \mathcal{O}$ on $\mathbb{P}^n$. Meromorphic n-forms on $\mathbb{P}^n$ are expressions of the form

$$\Omega_A = \frac{A\Omega}{Q^r},$$

where A and Q are homogeneous polynomials such that Ω_A is homogenous of degree zero. Consider, for example, the case of a hypersurface of degree d, defined by $Q = 0$, in $\mathbb{P}^3$. The degree of Ω is 4, so meromorphic forms are given by

$$\Omega_A = \frac{A\Omega}{Q},$$

where $\deg A = d - 4$. It is easy to count the dimension of the space of meromorphic forms of this kind: the dimension of the space of homogeneous polynomials of degree d in $n+1$ variable is the binomial coefficient $\binom{d+n+1}{n}$. As a mnemonic device, remember that for $d = 1$, the answer is $n+1$, the number of homogeneous coordinates, and that the result is a polynomial of degree n in d. (Exercise: prove all these statements).

We come now the Poincaré residue. The forms Ω_A for fixed Q define cohomology classes of degree n on $\mathbb{P}^n - X$, where X is the locus $Q = 0$. Grothendieck's algebraic de Rham theorem applied to this case says that the cohomology of $\mathbb{P}^n - X$ is generated by the classes of the Ω_A. We assume here that X is smooth.

There is a purely topological construction

$$res : H^n(\mathbb{P}^n - X) \longrightarrow H^{n-1}(X)$$

which is defined as follows. Given a cycle γ of dimension $n-1$ on X, let $T_\epsilon(\gamma)$ denote the tube of radius epsilon around it relative to some Riemannian metric. If ϵ is sufficiently small, then the boundary of this tube lies in $\mathbb{P}^n - X$. Define

$$res(\alpha)(\gamma) = \frac{1}{2\pi i}\alpha(\partial T_\epsilon(\gamma)).$$

We call this the *topological residue* of α. For small ϵ the region inside the $T_{\epsilon'}$ but outside $T_{\epsilon''}$ for $\epsilon'' < \epsilon' < \epsilon$ is bounded by smooth, nonintersecting tubes. The fact α is closed gives (by Stokes theorem) that

$$\alpha(\partial T_{\epsilon'}(\gamma)) = \alpha(\partial T_{\epsilon''}(\gamma)).$$

Thus the topological residue is independent of the tube chosen so long as it is small enough. We can also write

$$res(\alpha)(\gamma) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \alpha(\partial T_\epsilon(\gamma)),$$

even though the sequence is in fact constant.

The residue map fits into the exact sequence

$$\cdots \longrightarrow H^n(\mathbb{P}^n) \longrightarrow H^n(\mathbb{P}^n - X) \xrightarrow{res} H^{n-1}(X) \xrightarrow{G} H^{n+1}(\mathbb{P}^n) \longrightarrow \cdots$$

where G is the *Gysin* map. It is the transpose via Poincaré duality of the $H_k(X) \longrightarrow H_k(\mathbb{P}^n)$. The kernel of the Gysin map is the primitive cohomology. Thus there is a surjection

$$H^n(\mathbb{P}^n - X) \xrightarrow{res} H^{n-1}(X)_o \longrightarrow 0$$

When α is represented by a meromorphic form Ω_A, we would like to represent it by a differential form on X. To this end we will define an *analytic residue* for forms with pole of order one. We will then compare the analytic and topological residues, concluding that they are the same on the level of cohomology.

Let us consider first the local version of the the analytic residue, where projective space is replaces by a coordinate neighborhood in $\mathbb{C}^n$ and Ω_A is replaced by the expression $\alpha = adz_1 \wedge \cdot \wedge dz_n/f$, where the hypersurface is defined by the holomorphic equation $f(z_n) = 0$. Note that

$$df = \sum_i \frac{\partial f}{\partial z_i} dz_i.$$

If $f(z) = 0$ defines a smooth hypersurface, then at least one of its partial derivatives is nonvanishing ateach point of $f = 0$. Assume that the partial $\frac{\partial f}{\partial z_n}$ is not zero at a point of $f = 0$ by shrinking the neighborhood if necessary, we may assume that this partial derivative is nonzero throughout the neighborhood. Multiply df by $dz_1 \wedge \cdots \wedge dz_{n-1}$. Then

$$dz_1 \wedge \cdots dz_{n-1} \wedge df = \frac{\partial f}{\partial z_n} dz_1 \wedge \cdots \wedge dz_n.$$

Thus

$$\frac{adz_1 \wedge \cdots \wedge dz_n}{f} = \frac{adz_1 \wedge \cdots \wedge dz_{n-1}}{\partial f/\partial z_n} \wedge \frac{df}{f}$$

Define the analytic residue to be the coefficient of df/f, restricted to $f = 0$. Thus

$$Res\left(\frac{adz_1 \wedge \cdots \wedge dz_n}{f}\right) = \frac{adz_1 \wedge \cdots \wedge dz_{n-1}}{\partial f/\partial z_n} \Big|_{f=0}.$$

Since $\partial f/\partial z_n$ is nonzero on $f = 0$, we see that the analytic residue of a meromorphic form with a pole of order one is a holomorphic form.

Let $\{U_\beta\}$ be a coordinate cover of a neighborhood of $T_\epsilon(X)$ which restricts to a coordinate cover of X. We may assume that X is locally given by $z_n^\beta = 0$. Let $\{\rho_\beta\}$ be a partition of unity subordinate to the given cover. Then

$$res(\alpha)(\gamma) = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \sum_\beta \int_{\partial T_\epsilon(\gamma)} \rho_\beta a_\beta dz_1^\beta \wedge \cdots \wedge dz_{n-1}^\beta \frac{dz_n^\beta}{z_n^\beta}$$

Use Fubini's theorem to integrate first with respect to z_n^β and then with respect to the other variables. To evaluate the integral in z_n^β, define for any function $g(z_1^\beta, \ldots, z_n^\beta)$, the quantity

$$[g]_\epsilon(z_1^\beta, \ldots z_{n-1}^\beta) = \frac{1}{2\pi} \int_0^{2\pi} g(z_1^\beta, \ldots, z_{n-1}^\beta, \epsilon e^{2\pi i\theta_n^\beta}) d\theta_n^\beta$$

Then $[g]_\epsilon$ is an average value satisfying

$$\lim_{\epsilon\to 0} [g(z_1, \ldots, z_{n-1})]_\epsilon = g(z_1, \ldots, z_{n-1}, 0).$$

Then one has

$$res(\alpha)(\gamma) = \sum_\beta \int_\gamma (\rho_\beta a_\beta)(z_1^\beta, \ldots, z_{n-1}^\beta, 0) dz_1^\beta \wedge \cdots \wedge dz_{n-1}^\beta$$

The integral on the right is the integral of the analytic residue over γ. To summarize,

$$res(\alpha)(\gamma) = Res(\alpha)(\gamma).$$

Thus the two residue maps are the same on the level of cohomology classes.

Recall what Grothendieck's algebraic de Rham theorem says: the cohomology of $H^n(\mathbb{P}^n - X)$ is spanned by the classes $A\Omega/Q^r$. Thus there is a filtration of the cohomology by order of pole: define

$$P^r H^n(\mathbb{P}^n - X) = \{\beta \mid \beta = [A\Omega/Q^k], \text{ where } k \le r\}$$

One has the increasing filtration $P^1 \subset P^2 \subset \cdots \subset P^{n+1} = H^n(\mathbb{P}^n - X)$. This filtration maps to a filtration of the primitive cohomology. From the discussion above, we see that

$$res P^1 H^n(\mathbb{P}^n - X) = F^{n-1} H^{n-1}(X)_o.$$

Somewhat more elaborate arguments, which we shall sketch in a moment, show that

$$res P^r H^n(\mathbb{P}^n - X) = F^{n-r} H^{n-1}(X)_o.$$

Thus the residue maps the increasing filtration by order of pole to the decreasing Hodge filtration.

7. Properties of the period mapping

We will now establish some properties of the period mapping for hypersurfaes. All properties stated hold in general (see Eduardo Cattani's lectures).

Theorem 1. *The period mapping is holomorphic.*

Let $\{F^p(t)\}$ be a decreasing filtration with parameter t. Let $v_i(t)$ be a basis of $F^p(t)$ To first order, $v_i(t) = v_i(0)+t\dot{v}_i(0)$. Define the homomorphism

$$\Phi_p : F^p(0) \longrightarrow H_{\mathbb{C}}/F^p(0)$$

by

$$\Phi_p(v_i(0)) = \dot{v}_i(0) \bmod F^p(0).$$

This homomorphism is zero if and only if $F^p(t)$ is constant to first order as a map into the Grassmannian of k-planes in $H_{\mathbb{C}}$, where $k = \dim F^p$. This homomorphism is the velocity vector of the curve $F^p(t)$ in the Grassmannian. The complex structure of D is inherited from the complex structure of the product of Grassmannians in which it imbeds by the map that sends a filtration to a vector of subspaces with components $F^p \in Grass(\dim F^p, H_{\mathbb{C}})$. Thus it is sufficient to compute the derivative of $F^p(t)$ viewed as a curve in the Grassmannian.

We do this only for $F^nH^n(X)$, where X is a hypersurface in $\mathbb{P}^{n+1}$, but make some remarks about the general case. To that end, let $Q(t) = Q + tR$ be a pencil of equations defining a pencil of hypersurfaces. Let A be a polynomial such that $A\Omega/Q(t)$ defines a meromorphic form on $\mathbb{P}^{n+1}$. The $resA\Omega/Q(t)$ is a curve of vectors in $F^n(t)$. Let γ_i be a basis for $H_n(X_0, \mathbb{Z})$, where $Q(t) = 0$ defined X_t. For a given ϵ, there is a δ such that the tubes $\partial T_\epsilon(\gamma_i)$ form a basis for $H^{n+1}(\mathbb{P}^{n+1} - X_t)$ so long as $|t| < \delta$. Thus coordinates of $resA\Omega/Q(t)$ are give by the integrals (periods)

$$I_k(t) = \int_{\partial T_\epsilon(\gamma_i)} \frac{A\Omega}{Q(t)}.$$

Because the domain of integration is fixed, as in our example of elliptic curves, derivatives of $I_k(t)$ can be computed by differentiation under the integral sign. Since $Q(t)$ depends holomorphically on parameters, the derivative of $I_k(t)$ with respect to $\bar{t}$ is zero. That is, it satisfies the Cauchy-Riemann equation, and so $F^n(t)$ is holomorphic. The same argument applies to $F^p(t)$ once one establishes compatibility of the filtrations by order of pole and by Hodge level.

The next assertion is Griffiths transversality.

Theorem 2. *Let $\{F^p(t)\}$ be a family of Hodge filtrations coming from a family of projective algebraic manifolds. Then Φ_p takes values in F^{p-1}/F^p.*

There are two parts to the proof in the case of hypersurfaces. The first is an observation from calculus. Let

$$I_k(t) = \int_{\partial T_\epsilon(\gamma_i)} \frac{A\Omega}{Q(t)^r}.$$

where here we allow poles of arbitrary order. Then

$$\frac{\partial I_k(t)}{\partial t}\Big|_{t=0} = -(r+1)\int_{\partial T_\epsilon(\gamma_i)} \frac{\dot{Q}(0)A\Omega}{Q(0)^{r+1}}.$$

Thus differentiating a meromorphic differential moves it by just one step in the pole filtration. There is, of course, a subtlety. What is important is the cohomology class of the differential. It could happen that a meromorphic differential with a pole of order r is cohomologous to one with a pole of smaller order.

The second part is the compatibility between the filtration by order of pole and the Hodge filtration. We have established it only for hypersurfaces and only for $F^nH^n(X)$. However, given that compatiblility, we see that Φ_p takes values in F^{p-1}/F^p.

Because of the distinction between the order of pole of a meromorphic differential and minimum order of pole of a meromorphic form in the cohomology class of a differential, it can be a somewhat delicate matter to detemine whether $\Phi_p \neq 0$, that is, whether the derivative of the period mapping is nonzero. We address this issue in the next section.

8. The Jacobian ideal and the local Torelli theorem

Let us now investigate the question of whether the cohomology class of a meromorphic form can be represented by one of lower pole order. An answer to this question will lead to a proof of the following result [**?**], which is the local Torelli theorem of Griffiths.

Theorem 3. *The period map for hypersurfaces of degree d in $\mathbb{P}^n$ is locally injective for $d > 2$ and $n > 1$, except for the case of cubic surfaces.*

To efficiently study differential forms of all degree on $\mathbb{P}^{n+1} - X$, we introduce a convenient calculus. Let $dV = dX_0 \wedge \cdots \wedge dX_{n+1}$ be the "holomorphic volume form on $\mathbb{C}^{n+2}$. Let

$$E = X_k i\left(\frac{\partial}{\partial X_k}\right)$$

be the Euler vector field. It is an operator which is homogeneous of degree zero. Then

$$\Omega = i(E)dV,$$

where $i(E)$ means interior multiplication by the vectorfield E. Apply the operator $i(E)$ to the identity

$$dQ \wedge dV = 0,$$

to obtain

$$(\deg Q) Q dV = dQ \wedge \Omega.$$

If we use "$\equiv$" to mean "up to addition of an expression which is a multiple of Q," then this relation reads

$$dQ \wedge \Omega \equiv 0.$$

Apply $K_\ell = i(\partial/\partial X_\ell)$ to the proceeding to obtain

$$Q_\ell \Omega \equiv dQ \wedge \Omega_\ell,$$

where

$$\Omega_\ell = K_\ell \Omega$$

Now consider a general meromorphic n-form on $\mathbb{P}^{n+1} - X$. It can be written as

$$\sum_\ell \frac{A_\ell K_\ell \Omega}{Q^r}.$$

The exterior derivative of such an expression is simple if we ignore all terms of lower pole order:

$$d \sum_\ell \frac{A_\ell K_\ell \Omega}{Q^r} \equiv -(r+1) \sum_\ell \frac{A_\ell dQ \wedge K_\ell \Omega}{Q^{r+1}}$$

One also has the identity

$$K_\ell dQ + dQ K_\ell = Q_\ell.$$

Therefore

$$d \sum_\ell \frac{A_\ell K_\ell \Omega}{Q^r} \equiv (r+1) \sum_\ell \frac{A_\ell K_\ell dQ \wedge \Omega}{Q^{r+1}} - (r+1) \sum_\ell \frac{A_\ell Q_\ell \Omega}{Q^{r+1}}$$

But $dQ \wedge \Omega \equiv 0$, so that

$$d \sum_\ell \frac{A_\ell K_\ell \Omega}{Q^r} \equiv -(r+1) \sum_\ell \frac{A_\ell Q_\ell \Omega}{Q^{r+1}}$$

Define the *Jacobian ideal* to be the ideal generated by the partials of Q. Then we have the following result:

Proposition 1. *Let Ω_A be a meromorphic pole with a form which has a pole of order r. It is cohomologous to a form with a pole of order one less if A is in the Jacobian ideal of Q.*

As a first consequence of this result, we compute the dimension of $H^{n,0}(X)$. This space is spanned by residues of meromorphic forms $A\Omega/Q$. The degree of the numerator is $d-(n+2)$. The Jacobian ideal is generated in degree $d-1$. Thus the residue of Ω_A is nonzero if A is nonzero, just as in the case of abelian differentials. We conclude that

$$\dim H^{n,0}(X) = \binom{d-1}{n+1}.$$

This expression is a polynomial in d of degree $n+1$ with integer coefficients.

We can now prove the local Torelli theorem of Griffiths. Consider first a hypersurface X of degree d in $\mathbb{P}^{n+1}$. The space $F^nH^n(X)$ is nonzero under the hypotheses of the theorem and is spanned by residues of meromorphic forms $A\Omega/Q$, where $\deg A = d-(n+2)$. The derivative of such a form at $t=0$ for the pencil $Q+tR$ has the form

$$-\frac{RA\Omega}{Q^2}.$$

The numerator has degree $2d-(n+1)$. To proceed, we call upon an important fact from commutative algebara that holds whenever X is smooth:

Proposition 2. *Let R denote the polynomial ring in the $n+2$ variables of Q, where $Q=0$ is smooth. Let $S=R/J$ be the quotient ring. It is a finite-dimensional graded $\mathbb{C}$-algebra. Its component of top degree has degree $t=(d-1)(n+2)$ and dimension one. Let $\phi : S^t \longrightarrow \mathbb{C}$ be any nonzero map. Consider the composition*

$$S^a \times S^{t-a} \longrightarrow S^t \xrightarrow{\phi} \mathbb{C}.$$

where the first map sends classes represented by polynomials A and B to the class of AB. The composition is a perfect pairing.

To prove the local Torelli theorem, suppose that R is such that the the derivative of every class Ω_A is zero. That is, RA lies in the Jacobian ideal for all A of degree $d-(n+2)$. Then RB is in the Jacobian ideal for all B of degree $t-d$. Under the stated hypotheses, the numbers $d-(n+2)$ and $t-d$ are non- negative. Because the pairing is perfect, it follows that R is in the Jacobian ideal. But an element of degree d in the Jacobian ideal has the form

$$R = \sum_{ij} A_{ij} X_i \frac{\partial Q}{\partial X_j}.$$

A vectorfield on $\mathbb{P}^n$ has the general form

$$\xi = \sum_{ij} A_{ij} X_i \frac{\partial}{\partial X_j}.$$

corresponding to the one-parameter group

$$I + tA \in GL(n+2, \mathbb{C})$$

Thus $R = \xi Q$ is tangent to the action of $PGL(n+1)$, so that to first order the pencil $Q + tR = 0$ is constant. This completes the proof.

9. The Horizontal Distribution - Distance Decreasing Properties

Let us examine Griffiths transversality in more detail. To this end it is useful to consider the Hodge structure on the Lie algebra of the group G which acts transitively on the period domain. Given any set of Hodge structures, the vector spaces that one can build from their underlying vector spaces carry natural Hodge structures. These constructions include: direct sum, tensor product, dual, and hom. (exercise: re-imagine the definitions). In particular, if H is a Hodge structure, then $End(H_{\mathbb{R}})$ carries a natural Hodge structure: if $\phi : H_{\mathbb{C}} \longrightarrow H_{\mathbb{C}}$ satisfies $\phi(H^{r,s}) \subset H^{r+p,s+q}$, we say that ϕ has type (p,q). Note that this Hodge stucture is of weight zero.

Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}_{\mathbb{C}}$ carries a Hodge decomposition inherited from the one on $End(H_{\mathbb{C}})$ defined by a given Hodge structure H. The Lie algebra of V is $\mathfrak{g}^{0,0}$. The holomorphic tangent space at H" can be identified with the subalgebra

$$\mathfrak{g} = \oplus_{p<0}\, \mathfrak{g}^{p,-p},$$

and one has

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{-} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{+},$$

where

$$\mathfrak{g}^{+} = \oplus_{p>0}\, \mathfrak{g}^{p,-p}$$

is the complex conjugate of $\mathfrak{g}^{-}$. The space $\mathfrak{g}^{-}$ is the holomorphic tangent space of D at H. this space is a V-module, it defines a homogeneous bundle isomorphic to the holomorphic tangent bundle. The subspace $\mathfrak{g}^{-1,1}$ is also a V-module. The homogeneous subbundle of the tangent bundle which it defines is the bundle of holomorphic tangent vectors which satisfy Griffiths transversality. We denote this bundel by $T_{hor}(D)$.

Exercise 9. *Show that $\mathfrak{g}^{-}$ is a Lie subalgebra. Describe it and the associated Lie group as explicitly as possible.*

Another decomposition of the Lie algebra is given by

$$\mathfrak{g}_{even} = \oplus_{p \text{ is even}} \mathfrak{g}^{-p,p}$$

and

$$\mathfrak{g}_{even} = \oplus_{p \text{ is even}} \mathfrak{g}^{-p,p}$$

Then

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{even} \oplus \mathfrak{g}_{odd} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}},$$

where $\mathfrak{k}$ is the Lie algebra of the maximal compact subgroup K of G containing V, and where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form.

It is only when $K = V$, that is, when $\mathfrak{g}_{even} = \mathfrak{g}^{0,0}$ that D is a Hermitian symmetric domaain. This is, for example, the case in weight two if and only if $h^{2,0} = 1$. This is because of the "accidental" isomorphisms between Lie groups in low dimensions. In this case the isomorphism is $SO(2, \mathbb{R}) \cong U(1)$.

The bundles $T_{hor}(D)$ has special properties relative to the holomorphic sectional curvatures of D. While we do not have the time to develop the theory here needed to explain this in full, here is a sketch. Let M be a surface in $\mathbb{R}^3$, x a point of M, and N a plane through x that contains the normal vector. N cuts a curve on M passing through x. That curve has an osculating circe — a circle which best approximates it. Let $K(N)$ be the reciprocal of the radius of that circle, taken with the correct sign: positive if the curve bends away from the normal vector, negative if it bend toward it. Call this the curvature of the curve $K(N)$. The extreme values of $K(N)$ occur for orthogonal planes. Their product is called the *Gaussian curvature* of M at x. Gauss showed that the Gaussian curvature is defined intrinsically, that is, by the metric on M. Now let M be any Riemannian manifold and P a plane in the tangent space of M at x. Let S_p be the surface consisting of geodesics emanating from x tangent to P. Let $K(P)$ be the Gauss curvature in the induced metric. Thus is a associated to every plane in the tangent space a number, the *sectional curvature*. Finally, suppose that M is a complex manifold. Then there is an endomorphism J of the tangent bundle whose square is minus the identity. This is the complex structure tensor. It gives a coherent notion of multiplication by $\sqrt{-1}$. A plane in the tangent space is a complex tangent plane if it is invariant under the action of J. The sectional curvature of these planes are called holomorphic sectional curvatures. The invariant metric on the Riemann sphere has constant holomorphic sectional curvature $+1$. For the invariant metric on the torus, the curvature is zero. On the unit disk or the upper half plane, it is -1. The unit ball (complex hyperbolic space) has an invariant metric with holomorphic sectional curvature -1. However, the Riemannian sectional curvatures are variable.

For period domains, the holomorphic sectional curvatures associated to vectors in $\mathfrak{g}^{-1,1}$ are negative. On the other hand, the sectional curvature for vectors tangent to fibers of the map $G/V \longrightarrow K/V$ are postive.

If $f : M \longrightarrow N$ is a a holomorphic map between complex manifolds with negative sectional curvatures, then $f^*ds^2_N \leq ds^2_M$. That is, horizontal maps decrease distances. The statement is NOT true for maps not tangent to the horizontal distribution.

Let us take the distance-decreasing property of period maps as a given and draw some consequences from it. The first illustrates the fact that period domains act with respect to horizontal holomorphic maps as holomorphic maps do with respect to bounded domains.

Proposition 3. *Let $f : \mathbb{C} \longrightarrow D$ be a holomorphic horizontal map. Then f is constant.*

For the proof, consider the Poincaré metric on the disk of radius R

$$ds_R^2 = \frac{R^2 dz d\bar{z}}{(R^2 - |z|^2)^2)}.$$

This is the metric of curvature -1. If ds_D^2 is the G-invariant metric on D, we have $f^* ds_D^2 \leq ds_R^2$. Notice that

$$ds_R^2(0) = \frac{dz d\bar{z}}{R^2}.$$

and that

$$f^* ds_D^2 = C dz d\bar{z}$$

for some $C > 0$. Then

$$C \leq \frac{1}{R^2}$$

for all $R > 0$. For R large enough, this is a contradiction.

Corollary 1. *Let $f : \mathbb{C}^* \longrightarrow \Gamma \backslash D$ be a period map. Then f is constant.*

Proof. For the proof, note that f has a lift $f : \mathbb{C} \longrightarrow D$. The lift must be constant. □

Corollary 2. *Let $X/\mathbb{P}^1$ be a family of algebraic hypersurfaces of degree at least three (or four in the case of dimension three). Then $X/\mathbb{P}^1$ has at least three singular fibers.*

Theorem 4. *(***Monodromy theorem***) Let γ be monodromy transformation for a period map $f : \Delta^* \longrightarrow \Gamma \backslash D$. Then γ is quasi-unipotent. That is there is are integers m and N such that $(\gamma^m - 1)^N = 0$.*

The theorem says that the eigenvalues of γ are roots of unity and that a suitable power of γ is nilpotent. For the proof, consider the lift $\tilde{f} : \mathcal{H} \longrightarrow D$, and use the Poincaré metric

$$ds_H^2 = \frac{dx^2 + dy^2}{y^2}.$$

The distance between $\sqrt{-1}n$ and $\sqrt{-1}n + 1$ is $1/n$, so that

$$d_D((\tilde{f}(\sqrt{-1}n), \tilde{f}(\sqrt{-1}n + 1)) \leq 1/n.$$

Write $\tilde{f}(\sqrt{-1}n) = g_n V$, for some $g_n \in G$. Then the above relation reads

$$d_D(g_n V, \gamma g_n V) \leq 1/n,$$

or

$$d_D(V, g_n^{-1} \gamma g_n V) \leq 1/n,$$

Consequently the sequence $\{g_n^{-1} \gamma g_n\}$ converges to the compact subgroup V. If the conjugacy class of γ has a limit point in a compact group, then its eigenvalues are of absolute value one. Because $\gamma \in G_{\mathbb{Z}}$, the eigenvalues are algebraic integers. Their conjugates are also eigenvalues. A theorem of Kronecker implies that the eigenvalue, which is an algebraic integer of absolute value one, is in fact a root of unity.

10. The Horizontal Distribution - Integral Manifolds

A subbundle of the holomorphic tangent bundle defines a *distribution*, that is, a field of subspaces at every point. An *integral manifold* of a distribution is a manifold everywhere tangent to the distribution. A distribution is said to be *involutive*, or *integrable* if whenever X and Y are vector fields tangent to the distribution, so is the Lie bracket $[X, Y]$. A theorem of Frobenius states that every point of an involutive distribution has a neighborhood U and coordinantes $z_1, \ldots, z_n$ such that the distribution, assumed to have dimension k at each point, is spanned by $\partial/\partial z_i$, $i = 1, \ldots, k$. The integral manifolds are then locally given by equations $z_i = c_i$ where $i = k+1, \ldots, n$. Thus the integral manifolds foliate the given manifold.

A distribution can be defined as the set of tangent vectors which are annihilated by a set of one-forms $\{\theta_i\}$. Consider the relation

$$d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y]).$$

From it we see that a distribution is involutive if and only if the $d\theta_i$ annihilate vectors tangent to it. Equivalently, the $d\theta_i$ are in the algebraic ideal generated by the forms θ_i.

Consider now the *contact* distribution, defined by the null space of the one-form

$$\theta = dz - \sum_{i=1}^{n} y_i dx_i. \tag{16}$$

Tnhis is the so-called *contact form.* Note that

$$d\theta = \sum_{i=1}^{n} dx_i \wedge dy_i$$

and that

$$\theta \wedge (d\theta)^n = \pm dx_1 \wedge \ldots \wedge dx_n \wedge dy_1 \wedge \ldots \wedge dy_n \wedge dz \neq 0.$$

Thus $\theta \wedge d\theta \neq 0$, and so the distribution defined by the contact form is not involutive. It follows that integral manifolds of the contact distribution are not of codimension 1. In fact, one can show that they are of dimension n, a fact that is already clear for $n = 1$. See [1] or [12].

It is easy to exhibit an n-dimensional integral manifold of the contact distribution. For any function $f(x_1, \ldots, x_n)$, consider the manifold M parametrized by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \frac{\partial f}{\partial x_i}, \ldots \frac{\partial f}{\partial x_n}, f).$$

Clearly it satisfies the contact equation.

A space of tangent vectors annihilated by the θ_i and such that pairs of tangent vectors are annihilated by the $d\theta_i$ is called an *integral element.* A basic question is: *if E is an integral element, is it tangent to an integral manifold V?* This is true of the contact distribution.

The reason for considering the contact distribution is that it provides a simple model for the Griffiths distribution, which is non-involutive whenever it is non-trivial. Some additional questions are: (a) what is the maximal dimension of an integral manifold? (b) can one characterize them? (c) what can one say about integral manifolds that are maximal with respect to inclusion? (d) what can one say about "generic" integral manifolds.

We study some of these questions in the case of weight two. To this end, choose a basis (Hodge frame) compatible with the Hodge decomposition $H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ and such that the intersection matrix for the bilinear form Q is

$$Q = \begin{pmatrix} 0 & 0 & I \\ 0 & -I & 0 \\ I & 0 & 0 \end{pmatrix}$$

Matrices of the form

$$g = \begin{pmatrix} I & 0 & 0 \\ X & I & 0 \\ Z & {}^tX & I \end{pmatrix}$$

where $Z + {}^tZ = {}^tXX$, satsify ${}^tgQg = Q$. They act on Hodge frames. The set of such matrices constitutes the unipotent group $G^- = \exp \mathfrak{g}^-$.

The *Maurer-Cartan* form for G^- is the form $\omega = g^{-1}dg$ where g is in block lower triangular form as above. It has the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ dX & 0 & 0 \\ W & {}^tdX & 0 \end{pmatrix}$$

where $W = dZ - {}^tXdX$ is skew-symmetric. A holomorphic tangent vector ξ is horizontal iff it is annihilated by W. The equation $W_{ij} = 0$ reads

$$dZ_{ij} = {}^tX_{ik}dX_{kj} = X_{ki}dX_{kj}$$

Thus the Griffiths distribution in weight two is given by a system of coupled contact equations.

In the case $h^{2,0} = 2$, the Maurer-Cartan form depends on a single form, $W_{12} = -W_{21}$. If we set $h^{2,0} = 2$, $h_o^{1,1} = q$, $X_{k1} = x_k$, $X_{k2} = y_k$, and $Z_{21} = z$, then the equation $W_{21} = 0$ reads

$$dz = y_i dx_i.$$

In this case the Griffiths distribution *is* the contact distribution! The maximum dimension of an integral manifold is q, and integral manifolds are given locally by $x \mapsto (x, \nabla z(x), z(x))$.

For $h^{2,0} = p > 2$, the behavior of the Griffiths distribution more complicated. To understand it better, consider the Maurer-Cartan ω form for G^- It is a $\mathfrak{g}^-$-valued 1-form which satisfies the integrability condition $d\omega - \omega \wedge \omega = 0$. In the weight two case,

$$d\omega = d\begin{pmatrix} 0 & 0 & 0 \\ dX & 0 & 0 \\ W & {}^t dX & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ dW & 0 & 0 \end{pmatrix},$$

where $W = 0$ on any integral element E. Thus on E, $d\omega = 0$, and by integrability, $\omega \wedge \omega = 0$ as well. Let X and Y be in E. Then

$$\omega \wedge \omega(X, Y) = [X, Y] = 0.$$

Consequently an integral element $E \subset \mathfrak{g}^{-1,1}$ satisfies $[E, E] = 0$. That is, *Integral elements are abelian subalgebras of $\mathfrak{g}^-$ contained in $\mathfrak{g}^{-1,1}$.* This result holds in arbitrary weight. Thus in general the integral elements are defined by quadratic equations.

Is every integral element tangent to an integral manifold? In general, the answer is "no." However, for the Griffiths distribution, the answer is "yes:" *given an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}^{-1,1}$, the manifold $V = \exp \mathfrak{a}$ is an integral manifold with tangent space $\mathfrak{a}$.*

The next question is: *On how many free parameters does an integral manifold depend?*

Two extreme cases are *rigidity* and *flexibility.* In the first case, an integral manifold through x with given tangent space is completely determined. Examples are given in [6]. These are precisely the examples in weight two, $h_o^{1,1}$ even of maximal dimension $h^{2,0}h^{1,1}/2$. and $h^{2,0} > 2$. In the second case, V with given tangent space depends on infinitely parameters. The contact system is an example: the parameter is an arbitrary smooth function f.

The integral manifold of maximal dimension of [6] is defined as follows. Fix a Hodge structure H with $\dim H^{2,0} = p$ and $\dim H^{1,1} = 2q$. Fix a complex structure on $H^{1,1}_{\mathbb{R}}$: an endomorphism J of this real vector space with $J^2 = -1$. Let $H^{1,1} = H^{1,1}_+ \oplus H^{1,1}_-$ be the eigenspace decomposition for J. Note that $H^{1,1}_+$ is totally isotropic: $Q(v, w) = 0$ for all $v, w \in H^{1,1}_+$. Let

$$\mathfrak{a} = Hom(H^{2,0}, H^{1,1}_+).$$

This is a subspace of $Hom(H^{2,0}, H^{1,1}) \cong \mathfrak{g}^{-1,1}$ of dimension pq.

We claim that $\mathfrak{a}$ is an abelian subspace. To see this, write

$$g(X,Z) = \begin{pmatrix} I & 0 & 0 \\ X & I & 0 \\ Z & {}^tX & 0 \end{pmatrix}$$

Then $[g(X,Z), g(X',Z')] = g(0, X^tX' - X'^tX)$. The matrix X^tX' is matrix of dot products of column vectors of X. Q.E.D.

The integral element $\mathfrak{a} = Hom(H^{2,0}, H^{1,1}_+)$ is tangent to the VHS $\exp \mathfrak{a}$. Let $V \subset G = SO(2p, 2q)$ be the group which preserves the reference Hodge structure and which commutes with J. Then $V \cong S(U(p) \times U(q))$. Let $H \cong U(p,q)$ be the subgroup of G which preserves J. Then V is an open subset of the orbit W of the reference Hodge structure under the action of H. Thus *W is a Hermitian symmetric space imbedded in D* as a closed, horizontal, complex manifold.

By choosing J artfully, one can ensure that there is an arithmetic group Γ operating on D with $H/\Gamma \cap H$ of finite volume and even compact. *The space $H/\Gamma \cap H$ is (quasi)-projective.*

Note. If $\dim H^{1,1}$ is odd, there is a VHS U of maximal dimension $pq+1$ which fibers over the unit disk whose fibers are $W's$ as described above. However, U, which is a kind of tube domain, does not admit a discrete group action with finite covolume. *Whether there is a quasi- projective example of dimension $pq+1$ is unknown.*

An integral element $\mathfrak{a}$ (aka infinitesimal variation of Hodge structure, aka abelian subspace) is *maximal* if whenever $\mathfrak{a}'$ is another integral element containing $\mathfrak{a}$, then $\mathfrak{a}' = \mathfrak{a}$. Integral elements of maximal dimension are maximal, but *the converse is not true.* For a geometric example, consider the integral elements that come from variations of Hodge structure of sufficiently high degree. The proof [5] is based on Donagi's symmetrizer construction.

We also give a "linear algebra" example [7]:

Theorem 5. *Let $\mathfrak{a}$ be an integral element for a weight two Hodge structure H. Suppose that $\mathfrak{a}$ is generic in the sense that there is a vector $v \in H^{2,0}$ such that $\mathfrak{a}(v) = H^{1,1}$. Then $\mathfrak{a}$ is isomorphic to $H^{1,1}$ and is maximal.*

Proof. Since the map $\mathfrak{a} \longrightarrow H^{1,1}$ defined by $X \mapsto X(e_1)$ is surjective, $\dim \mathfrak{a} \geq \dim H^{1,1} = q$. By a suitable change of basis, we can assume that there is a basis $e_1, \ldots, e_p$ for $H^{2,0}$ such that $\mathfrak{a}(e_1) = H^{1,1}$, a basis $M^1, \ldots, M^q$ for a subspace of $\mathfrak{a}$ and compatible basis $e^1, \ldots, e^q$ for $H^{1,1}$ such that $M^i(e_1) = e^j$. The condition $(A,B) = 0$ is the condition $A_i \cdot B_j - B_i \cdot A_j$ where A_i denotes the i-th column of A.

Let N be an element of $\mathfrak{a}$. We may subtract a linear combination of the M^i so that $N(e_1) = 0$, i.e., $N_1 = 0$.

On the one hand, $(M^k, N) = 0$. On the other hand,

$$(M^k, N)_{1j} = M_1^k \cdot N_j - N_1 \cdot M_j^k = e_k \cdot N_j = N_{kj}$$

Thus $N_{kj} = 0$ for all k, j. Thus N is in the span of the M^i. Q.E.D.

Theorem 6. *Let $\mathfrak{a}$ be a generic integral element for a weight two Hodge structure. Set $q = \dim H^{2,0}$. Then all integral manifolds tangent to $\mathfrak{a}$ are given by functions $f_2, \ldots f_q$ satisfying $[H_{f_i}, H_{f_j}] = 0$ where H_f is the Hessian of H.*

The proof is elementary and uses the matrix-valued contact system described earlier:

$$dZ_{ij} = {}^t X_{ik} dX_{kj} = X_{ki} dX_{kj}$$

Consider

$$dZ_{i1} = X_{ki} dX_{k1}$$

Then as usual, Z_{i1} is a function $f_i(X_{11}, X_{21}, \ldots, X_{q1})$.

The X_{ki} are determined by the contact equation:

$$X_{ki} = \frac{\partial f_i}{\partial X_{ki}}.$$

Thus one can determine the remaining functions Z_{ij}.

Question: do the functions Z_{ij} give a solution — the system is overdetermined. Compatibility: $dZ_{ij} = 0$.

This implies $[H_{f_i}, H_{f_j}] = 0$.

Example. Choose the f_i so that their Hessians are diagonal. Thus enforce

$$\frac{\partial^2 f_i}{\partial x_j \partial x_k} = 0 \qquad i \neq j.$$

In two variables this is the equation $\partial^2 g / \partial x \partial y = 0$, a form of the wave equation. A class of solutions is given by $g(x, y) = a(x) + b(y)$.

More generally,

$$f_i(x_1, \ldots, x_s) = \sum_j h_{ij}(x_j)$$

gives a very restricted but nonetheless infinite-dimensional family of solutions.

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