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Hodge Domains and Automorphic Cohomology

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HODGE DOMAINS
AND AUTOMORPHIC COHOMOLOGY

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I. Background material

I.A. Mumford-Tate groups and domains.

Notations:

- V is a $\mathbb{Q}$-vector space
- $Q : V \otimes V \to \mathbb{Q}$ is a non-degenerate form with $Q(v,w) = (-1)^n Q(w,v)$
- $G = \mathrm{Aut}(V, Q) \subset \mathrm{GL}(V)$
- $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$, viewed as real Lie group
- $S^1 \subset \mathbb{S}$ is the maximal compact subgroup $\{z \in \mathbb{C} : |z| = 1\}$

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[1]The terminology and notations are the same as those in the lectures in the summer school. These lectures are available at http://cdsagenda5.ictp.trieste.it/ full_display.php?ida=a09153.

Definitions. (i) A *Hodge structure* $(V, \widetilde{\varphi})$ is given by

$$\widetilde{\varphi} : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}}).$$

(ii) A *polarized Hodge structure of weight n* (V, Q, φ) is given by

$$\varphi : S^1 \to G(\mathbb{R})$$

such that the characters of φ lie in $[-n, n]$.

Remarks. In (i) we have that over $\mathbb{Q}$

$$V = \oplus V^{(n)}$$

where $\widetilde{\varphi}\big|_{\mathbb{Q}^*}$ acts on $V^{(n)}$ by $\widetilde{\varphi}(r) = r^n \, \mathrm{id}_{V^{(n)}}$ where n is the weight. In this talk we will only deal with pure Hodge structures of weight n.

In (ii) we have that $V_{\mathbb{C}} = \underset{p+q=n}{\oplus} V_{\varphi}^{p,q}$ where the $V_{\varphi}^{p,q}$ are eigenspaces for $\varphi(S^1)$ where for $v \in V_{\varphi}^{p,q}$, $\varphi(z)v = z^{p-q}v$ $(= z^p \bar{z}^q v)$.

Definitions. (i) The *Mumford-Tate group* $M_{\widetilde{\varphi}}$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that $\widetilde{\varphi}(\mathbb{S}) \subset M_{\widetilde{\varphi}}(\mathbb{R})$. (ii) The *Mumford-Tate group* M_{φ} is the smallest $\mathbb{Q}$-algebraic subgroup of G such that $\varphi(S^1) \subset G(\mathbb{R})$.

Remarks. $M_{\widetilde{\varphi}}$ is usually called the Mumford-Tate group and M_{φ} the restricted Mumford-Tate group. In this talk we shall be concerned exclusively with M_{φ} and shall refer to it as the Mumford-Tate group. In the tensor algebra $T^{\bullet,\bullet} = \underset{k,l \geqq 0}{\oplus} V^{\otimes^k} \otimes \check{V}^{\otimes^l}$ it is defined by the condition to fix pointwise the algebra of Hodge tensors $\mathrm{Hg}_{\varphi}^{\bullet,\bullet} \subset T^{\bullet,\bullet}$.

Let D be the period domain consisting of all polarized Hodge structures (V, Q, φ) with given Hodge numbers $h^{p,q} = \dim V_{\varphi}^{p,q}$. Fixing a reference point $\varphi \in D$, we have

$$D \cong G(\mathbb{R})/H_{\varphi}$$

where H_{φ} is the compact isotropy group of φ.

Definition. The *Mumford-Tate domain*

$$D_{M_{\varphi}} \subset D$$

is the $M_{\varphi}(\mathbb{R})$-orbit of φ.

Remarks. The component $D_{M_{\varphi}}^{o}$ of $D_{M_{\varphi}}$ through φ consists of the component through φ of the set of all $\varphi' \in D$ with $\mathrm{Hg}_{\varphi'}^{\bullet,\bullet} \supseteq \mathrm{Hg}_{\varphi}^{\bullet,\bullet}$. It is a homogeneous complex manifold

$$D_{M_{\varphi}} \cong M_{\varphi}(\mathbb{R})/H_{M_{\varphi}}$$

where $H_{M_{\varphi}} = H_{\varphi} \cap H(\mathbb{R})$ is the compact centralizer of the circle $\varphi(S^1) \subset M_{\varphi}(\mathbb{R})$.

I.B. **The structure theorem for global variations of Hodge structure.** We assume that $V = V_{\mathbb{Z}} \otimes \mathbb{Q}$ and let

$$\Phi : S \to \Gamma \backslash D$$

be a global variation of Hodge structure. This means

- S is a smooth, quasi-projective variety;
- Φ is a local liftable, holomorphic mapping that satisfies the infinitesimal period relation;
- $\rho : \pi_1(S, s_0) \to G_{\mathbb{Z}}$ is the monodromy representation with image the *monodromy group* Γ.

The Mumford-Tate group M_{Φ} of the variation of Hodge structure may be defined as follows: Suppose that $s_0 \in S$ is a very general point and identify the fibre of the flat bundle $\mathbb{V}_{\mathbb{C}} = \widetilde{S} \times_{\pi_1(S, s_0)} V_{\mathbb{C}}$ at s_0 with $V_{\mathbb{C}}$. Then $M_{\Phi} \subset G$ is the subgroup fixing the subspace $\mathrm{Hg}_{s_0}^{\bullet, \bullet}(\mathbb{V}_{s_0})$, this being the algebra of Hodge tensors at s_0 that as a subspace gives a sub-variation of Hodge structure. Then monodromy acts on this subspace as a finite group, and by passing to a finite covering of S and taking the induced variation of Hodge structure we may assume that $\mathrm{Hg}_{s_0}^{\bullet, \bullet}(\mathbb{V}_{s_0})$ is pointwise fixed by monodromy. This gives the inclusion $\Gamma \subset M_{\Phi}$, and then denoting by $\overline{\Gamma}^{\mathbb{Q}}$ the $\mathbb{Q}$-Zariski closure of Γ we have

$$\overline{\Gamma}^{\mathbb{Q}} \subseteq M_{\Phi}.$$

Being reductive, M_{Φ} splits into an almost direct product

$$M_{\Phi} = M_1 \times \cdots \times M_l \times A$$

of $\mathbb{Q}$-simple factors M_i and an abelian part A. We denote by $D_i \subset D$ the $M_i(\mathbb{R})$-orbit of a lift $\widetilde{\Phi}(s_0) \in D$ of $\Phi(s_0)$. Since $\widetilde{\Phi}(s_0)$ is a fixed point of $A(\mathbb{R})$ we will ignore the A-factor. We set $\Gamma_i = \Gamma \cap M_i$ and assume that Γ_i is non-trivial for $1 \leqq i \leqq k$ and is trivial for $k + 1 \leqq i \leqq l$.

Structure Theorem. *The D_i are homogeneous complex submanifolds of D and the variation of Hodge structure factors as*

$$\Phi : \underbrace{\Gamma_1 \backslash D_1 \times \cdots \times \Gamma_k \backslash D_k}_{} \times \underbrace{D_{k+1} \times \cdots \times D_l}_{}.$$

Φ is constant in the second set of factors, and for $1 \leqq i \leqq k$

$$\overline{\Gamma}_i^{\mathbb{Q}} = M_i.$$

Remark. Although it seems not to be known whether or not Γ_i is of finite index in $M_{i,\mathbb{Z}}$, their tensor invariants are the same. Because of the structure theorem we may say that Mumford-Tate domains are the basic objects as target spaces of period mappings.

I.C. **Hodge groups and domains.** A reductive $\mathbb{Q}$-algebraic group M may appear in different ways as a Mumford-Tate group. This suggests the following

Definitions. (i) A *Hodge representation* (V, ρ, φ) is given by a representation $\rho : M \to \mathrm{Aut}(V, Q)$ together with a circle $\varphi : S^1 \to M(\mathbb{R})$ such that, setting $\varphi_\rho = \rho \circ \varphi$, (V, Q, φ_ρ) gives a polarized Hodge structure. (ii) A *Hodge group* is a pair (M, φ) for which there is a Hodge representation (V, ρ, φ).

Remarks. For simplicity, we assume that M is semi-simple and ρ is faithful. Then Hodge representations and Hodge groups have been classified (cf. the lecture notes).

For a Hodge representation (V, ρ, φ) we denote by $D_{M_{\varphi_\rho}}$ the corresponding Mumford-Tate domain. We note that $(\mathfrak{m}, B, \mathrm{Ad}\,\varphi)$ is also a Hodge representation where $\mathfrak{m} \subset \mathfrak{g} \subset \mathrm{gl}(V)$ and the Cartan-Killing form $B : \mathfrak{m} \otimes \mathfrak{m} \to \mathbb{Q}$ is induced by Q on $\mathrm{gl}(V)$. We denote the corresponding Mumford-Tate domain simply by $D_{\mathfrak{m},\varphi}$.

Theorem. *As homogeneous complex manifolds, together with the exterior differential system given by the infinitesimal period relation,*

$$D_{M_{\varphi_\rho}} = D_{\mathfrak{m},\varphi}.$$

Definition. $D_{\mathfrak{m},\varphi}$ will be called a *Hodge domain*.

Because of the theorem, Hodge domains are the universal objects parametrizing families of polarized Hodge structures whose algebra of Hodge tensors contain a given algebra; namely the $\rho(M)$-invariants in $T^{\bullet,\bullet} = \underset{k,l \geqq 0}{\oplus} V^{\otimes k} \otimes \check{V}^{\otimes l}$. The Hodge domain is determined by the data (M, φ). It is a homogeneous complex manifold

$$D_{\mathfrak{m},\varphi} = M(\mathbb{R})/H_\varphi$$

where $H_\varphi = Z_{M(\mathbb{R})}(\varphi(S^1))$ is the compact centralizer of the circle $\varphi(S^1)$ in $M(\mathbb{R})$. If we denote

$$\mathfrak{m}^{-k,k} = \left\{ \begin{array}{l} \text{eigenspaces in } \mathfrak{m}_\mathbb{C} \text{ where} \\ \varphi(z) \text{ acts by } z^{-2k} \end{array} \right\}$$

then the $(1, 0)$ tangent space to $D_{\mathfrak{m},\varphi}$ at the identity coset is identified as (cf. the lecture notes)

$$T_e D_{\mathfrak{m},\varphi} = \underset{k>0}{\oplus} \mathfrak{m}^{-k,k}.$$

The infinitesimal period relation corresponds to $\mathfrak{m}^{-1,1}$, and the bracket

$$[\ ,\ ] : \Lambda^2 \mathfrak{m}^{-1,1} \to \mathfrak{m}^{-2,2}$$

gives the integrability conditions. We note that

> *Many different φ's may give the same homogeneous complex structures. Thus a Hodge domain is a homogeneous complex manifold* **with additional structure**.

II. GENERAL CONTEXT

Hodge representations, Hodge groups and Hodge domains occur in three contexts

- Algebraic geometry (variation of Hodge structure)
- Representation theory (discrete series and cuspidal automorphic representations)
- Arithmetic (L functions, Galois representations)

In the classical case of weight one Hodge structures, there is an extensive and rich interaction among these (theory of Shimura varieties). In higher weights the understanding of what this interaction might possibly be is in its very earliest stage. In this section we shall briefly summarize the role Hodge-theoretic data plays.

II.A. **Algebro-geometric.** Given a Hodge group (M, φ) with associated Hodge domain $D_{\mathfrak{m},\varphi}$, then $D_{\mathfrak{m},\varphi}$ has an invariant complex structure with an invariant exterior differential system and associated integral manifolds

$$\text{(II.A.1)} \qquad\qquad \Phi : S \to \Gamma\backslash D_{\mathfrak{m},\varphi}$$

as explained above. Given a Hodge representation (V, ρ, φ), there is over $\Gamma\backslash D_{\mathfrak{m},\varphi}$ an associated local system $\mathbb{V}$ and Hodge bundles $\mathcal{F}^p \subset \mathbb{V} \otimes \mathcal{O}_{D_{\mathfrak{m},\varphi}}$. From this one may construct cohomology both of the local system and the Hodge bundles (coherent cohomology). This cohomology may be pulled back under (II.A.1) where now the natural target space is the Mumford-Tate domain $D_{M_{\varphi_\rho}}$. The natural objects for this are the characteristic cohomology of $\mathbb{V}$ and the Penrose-Radon transforms of the cohomology of the $\mathcal{F}^p$ (cf. section IV).

On the arithmetic side, in $\Gamma\backslash D_{M_{\varphi_\rho}}$ there are defined the dense set of CM polarized Hodge structures and, more generally, the Noether-Lefschetz loci (cf. [GGK] and the lecture notes).

Assuming that $\Gamma = \Gamma_{\mathbb{Z}}$, there is also an "adelification" of (II.A.1), obtained on the RHS by taking the inverse system over the congruence subgroups of finite index in Γ. On the LHS one takes the corresponding family of finite covering spaces of S and induced global variations of Hodge structure.

II.B. **Representation-theoretic.** The basic reference here is [Schm]. Because of the notations used in the lecture notes we will not be able to use Schmid's notation here. For our purposes the main point is that

> *The real, non-compact semi-simple Lie groups that have*
> *non-trivial discrete series representations are exactly*
> *the real Lie groups $M(\mathbb{R})$ associated to Hodge groups (M, φ).*

Schmid proved that these discrete series representations may be realized as L^2-cohomology of homogeneous line bundles $\mathcal{L}_{\rho_\lambda} \to D_{\mathfrak{m},\varphi}$ over Hodge domains of the form $M(\mathbb{R})/T$ where T is a compact maximal torus in $M(\mathbb{R})$. The notation means this: Associated to a weight λ there is Harish-Chandra's character Θ_λ. A suitable choice of Weyl chamber D_λ then defines a complex structure on $M(\mathbb{R})/T$, together with a character ρ_λ of T that defines the above holomorphic line bundle over $D_{\mathfrak{m},\varphi}$. The discrete series with character Θ_λ is then realized as L^2-cohomology

$$H^d_{(2)}(D_{\mathfrak{m},\varphi}, \mathcal{L}_{\rho_\lambda})$$

where $d = \dim_{\mathbb{C}} K/T$. The other L^2-cohomology groups vanish.

We shall say that φ is *compatible with* λ if the invariant complex structure associated to φ coincides with that given by D_λ as described above. We note that in order to have that $T = Z_{M(\mathbb{R})}(\varphi(S^1))$ we must assume that $\varphi(S^1)$ acts non-trivially on all the root spaces. For each λ there are many φ's that are compatible with it; different φ's will give different Hodge theoretic data. However, for each φ, a power of $\mathcal{L}_{\rho_\lambda}$ will be a Hodge bundle but which Hodge bundle it is depends on the φ. A possibly important point is that *associated to λ there is a natural choice of a φ_λ that is compatible with λ.*

II.C. **Arithmetic.** The assumption that $M(\mathbb{R})$ is the group of real points of a $\mathbb{Q}$-algebraic group M plays no role in the discussion of the discrete series in the section immediately preceding this. This assumption enters in that with it one may define the notion of a *cuspidal automorphic representation* in $L^2(M(\mathbb{Q})\backslash \mathbb{M}(\mathbb{A}))$, where $\mathbb{A}$ are the adeles. We refer to [CK], especially the introduction, and to the references cited therein for the important role that these may play in the arithmetic theory of automorphic forms. Another useful reference, especially to the cohomological aspects of local systems, is [Schm]. In the rich classical theory (Shimura varieties) the choice of a φ is canonical and very special. This is to insure that there is a Hodge representation (V, ρ, φ) *of weight one*, which is in fact essentially unique.

III. Cycle spaces and their enlargements

III.A. **General definitions and some properties.** The general reference for this section is [FHW]. We let $D_{\mathfrak{m},\varphi} = M(\mathbb{R})/H_{M_\varphi}$ be a Hodge domain. For the time being, it is only the invariant complex structure, and not the particular φ that gives rise to it, that will be of interest. In a slight departure from the terminology in [GGK], we will refer to the *non-classical case* as meaning that $D_{\mathfrak{m},\varphi}$ does *not* fibre holomorphically over an Hermitian symmetric domain. It may happen that there is a different circle $\varphi' : S^1 \to M(\mathbb{R})$ such that $H_{M_\varphi} = H_{M_{\varphi'}}$ and $D_{\mathfrak{m},\varphi'}$ does fibre holomorphically over an Hermitian symmetric domain (see example 2 below).

Since the φ will not play a role in this discussion, for notational simplicity we set $D = D_{\mathfrak{m},\varphi}$.

Cycle spaces arose from the following considerations. First, in the non-classical case one cannot expect to have automorphic forms in the classical sense. Instead, one expects to have "automorphic cohomology" in degree $d = \dim_{\mathbb{C}} K/H$ (see below). In particular, one cannot say that "an automorphic cohomology class vanishes at a point of D", or that it is defined over a number field. Secondly a cohomology class can be evaluated on a d-dimensional compact, complex analytic submanifold of $\Gamma \backslash D$. This suggests considering automorphic cohomology classes as sections of a bundle over the space of d-dimensional compact, complex analytic subvarieties of $\Gamma \backslash D$. For this we let $Y = K/H_{M_\varphi}$, which is a smooth, projective, algebraic subvariety in D.

Definition. We define the *cycle space U* to be the set of translates gY by those $g \in M(\mathbb{C})$ such that gY remains in D.

Remarks. The compact dual $\check{D}$ of D is a rational, homogeneous projective variety defined over $\mathbb{Q}$

$$\check{D} = M(\mathbb{C})/P$$

where P is a parabolic subgroup of $M(\mathbb{C})$. Then

$$Y = K(\mathbb{C})/P \cap K(\mathbb{C})$$

is also a rational homogeneous variety. It may be shown that the set $\check{U}$ of $M(\mathbb{C})$-translates of Y in $\check{D}$ is given by the affine algebraic variety

$$\check{U} = M(\mathbb{C})/K(\mathbb{C})$$

and is a Zariski open set in the Hilbert scheme of Y in $\check{D}$. For $u \in \check{U}$, we denote by $Y_u \subset \check{D}$ the corresponding subvariety of $\check{D}$.

There is a standard incidence diagram

(III.A.1)
$$
\begin{array}{ccc}
 & I & \subset U \times D \\
\pi_U \swarrow & & \searrow \pi_D \\
U & & D
\end{array}
$$

with fibres
$$
\begin{cases}
\pi_U^{-1}(u) = \{(u, \varphi) : \varphi \in Y_u\} \cong Y \\
\pi_D^{-1}(\varphi) = \{(u, \varphi) : Y_u \text{ passes through } \varphi\}.
\end{cases}
$$

We denote by Z a typical fibre $\pi_D^{-1}(\varphi)$. The main results concerning (III.A.1) are

Theorem ([FHW]). (i) *U is a Stein manifold.*
   (ii) *Z is a contractible Stein manifold.*

See the remark below for a group-theoretic description of U.
   We will verify these results in two examples.

III.B. **Two examples.**

Example 1:[2] Here D is the period domain for polarized Hodge structures of weight two and Hodge numbers $h^{2,0} = 2$, $h^{1,1} = 1$. Then
$$
D \cong \mathrm{SO}(4, 1)/U(2).
$$

Here, $\mathrm{SO}(4, 1)$ will denote the real Lie group acting on $V_{\mathbb{R}} = \mathbb{R}^5$ preserving the quadratic form $x_1^2 + \cdots + x_4^2 - x_5^2$. We denote by $\mathbf{Q} \subset \mathbb{P}V_{\mathbb{C}}$ the corresponding complex quadric; then

   (i) *$\check{D}$ is the set of lines in $\mathbf{Q}$.*

Proof. For $F \in \mathrm{Gr}(2, V_{\mathbb{C}})$ a 2-plane in $V_{\mathbb{C}}$, we denote by $[F] \in \mathbb{P}V_{\mathbb{Q}}$ the corresponding line. The first Hodge-Riemann bilinear relation $Q(F, F) = 0$ is equivalent to $[F] \subset \mathbf{Q}$.
   The condition that $F \in D$, i.e., it satisfies the second bilinear relation, will be written as $Q(F, \overline{F}) > 0$.

   (ii) *D has two components corresponding to the two orientations of $V_{\mathbb{R}}$ given by a Hodge frame associated to $F \in D$.*

A Hodge frame is $e_1, e_2, f, \bar{e}_1, \bar{e}_2$ where $f = \bar{f}$ and $i^2 Q(e_\alpha, \bar{e}_\beta) = \delta_{\alpha\beta}$ and $Q(f, f) = -1$. The orientation is given by $(\frac{i}{2})^2 e_1 \wedge \bar{e}_1 \wedge e_2 \wedge \bar{e}_2 \wedge f$.

   (iii) *The boundary ∂D is smooth (it is a single $\mathrm{SO}(4, 1)$-orbit), and is given by*
$$
\{F \in \check{D} : Q(F, \overline{F}) \text{ has rank one}\}.
$$

---

[2]Based on correspondence with Mark Green.

(iv) *The Hilbert scheme of Y is a 2-sheeted covering of $\mathbb{P}\check{V}_{\mathbb{C}}$ branched along the dual quadric $\check{\mathbf{Q}}$ to $\mathbf{Q}$.*

Proof. For a hyperplane $E \subset V_{\mathbb{C}}$ we denote by $[E] \in \mathbb{P}\check{V}_{\mathbb{C}}$ the corresponding point. If we set $Q_E = Q\big|_E$, then for $[E] \in \mathbb{P}\check{V}_{\mathbb{C}}\backslash\check{\mathbf{Q}}$ the corresponding quadric $\mathbf{Q}_E \subset \mathbb{P}E$ is non-singular and has two rulings. The $SO(4,1)(\mathbb{C})$-translates of a $Y \subset D$ are the sets of lines in one of the two rulings of $\mathbf{Q}_E$. Thus $\check{U}$ *is identified with the sets of rulings on non-singular quadrics* $\mathbf{Q}_E$ *for* $[E] \in \mathbb{P}\check{V}\backslash\check{\mathbf{Q}}$.

In case Q_E is singluar with one double point, the two sets of rulings coincide and give the remaining points in the Hilbert scheme of Y.

(v) $U = \{E \in \mathbb{P}\check{V}_{\mathbb{C}} : Q(E, \overline{E}) > 0\}$.

Proof. Using the above description of $\check{U}$, the condition that $E \in U$ is that $Q(F, \overline{F}) > 0$ for all 2-planes $F \subset E$. Note that

$$U \text{ is unit ball in } \mathbb{P}\check{V}_{\mathbb{C}} \cong \mathbb{P}^4.$$

In particular, it is Stein.

(vi) *In* (III.A.1), *the fibre* $\pi_D^{-1}(F_0)$ *is biholomorphic to a unit ball in* $\mathbb{C}^2$.

Proof. We have

$$\pi_D^{-1}(F_0) = \left\{ \begin{array}{c} (E, F_0) : F_0 \subset E \\ \text{and } Q(E, \overline{E}) > 0 \end{array} \right\}.$$

There is an inclusion

$$\pi_D^{-1}(F_0) \hookrightarrow \mathbb{P}(V_{\mathbb{C}}/F_0)^\vee \cong \mathbb{P}^2,$$

and the constraint $Q(E, \overline{E}) > 0$ defines a unit ball in $\mathbb{P}^2$.

Remark. Fixing a reference maximal compact subvariety $Y_0 \subset D$, we recall that in general U is defined to be the set of translates gY_0, $g \in M(\mathbb{C})$, that remain in U. A natural question is whether there is an explicit group-theoretic description of what these translates are. A very nice answer is given in [FHW]. To state this we let $M(\mathbb{R}) = KAN$ be the Iwasawa decomposition of the real Lie group $M(\mathbb{R})$ where A is a maximal, $\mathbb{R}$-split Cartan subgroup. In the Lie algebra $\mathfrak{a}$ of A denote by ω the interior of intersection of the half-spaces $\{X \in \mathfrak{a} : \langle \alpha, X \rangle \leqq \pi/2\}$ where α runs over the roots of $\mathfrak{a}$. Then

$$U = M(\mathbb{R}) \exp(i\omega) \cdot Y_0 \,.$$

Example 1 (continued). We choose a reference point $F_0 \in D$ where

$$F_0 = \text{span} \left\{ v_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \\ 0 \end{pmatrix} \right\}$$

and set $E_0 = F_0 + \overline{F}_0$. (This is our Y_0 for the example at hand.) Thus, $E_0 = \mathbb{R}^4 \otimes \mathbb{C}$ for the standard $\mathbb{R}^4 \subset \mathbb{R}^5$. Then

$$K = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(4) \right\}$$

and any A is one-dimensional with Lie algebra

$$\mathfrak{a} = \mathbb{R} \begin{pmatrix} 0 & v \\ {}^t v & 0 \end{pmatrix}, \qquad 0 \neq v \in \mathbb{R}^4.$$

A convenient choice is $v_0 = {}^t(0\ 0\ 0\ 1\ 0)$. Then

$$\exp(i\omega) = \left\{ \left(\begin{array}{c|c} I_3 & 0 \\ \hline 0 & \begin{matrix} \cos t & i\sin t \\ i\sin t & \cos t \end{matrix} \end{array} \right) : |t| < \pi/2 \right\}.$$

We let $E_t \in U$ be the corresponding element of $\exp(i\omega) \cdot E_0$.

Now in general, points $E \in U$ are classified into two types:

Type I: $E = \overline{E} \iff \dim(E \cap \overline{E}) = 4 \iff E + \overline{E} \neq \mathbb{C}^5$.
Type II: $E \neq \overline{E} \iff \dim(E \cap \overline{E}) = 3 \iff E + \overline{E} = \mathbb{C}^5$.

The action of $M(\mathbb{R})$ on U preserves type. Above, E_0 is of type I and for $t \neq 0$, E_t is of type II. The orbits of $M(\mathbb{R})$ acting on U are parametrized by $-\pi/2 < t < \pi/2$ and have stability group $\mathrm{SO}(4)$ for $t = 0$ and $\mathrm{SO}(3) \times \mathrm{SO}(1,1)$ for $t \neq 0$.

Example 2 (cf. [EGW] and [C]). In this case

$$D = \mathrm{SU}(2,1)/T,$$

where as above $\mathrm{SU}(2,1) = \mathrm{SU}(2,1)(\mathbb{R})$ for the standard diagonal Hermitian form of signature $(2,1)$ on $\mathbb{C}^3$. The compact dual is the flag manifold

$$\check{D} = \{(P, L) : P \in L\} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2.$$

Up to conjugation, there are three open orbits of $\mathrm{SU}(2,1)$ acting on $\check{D}$. Here we take the one non-classical case where D has the picture
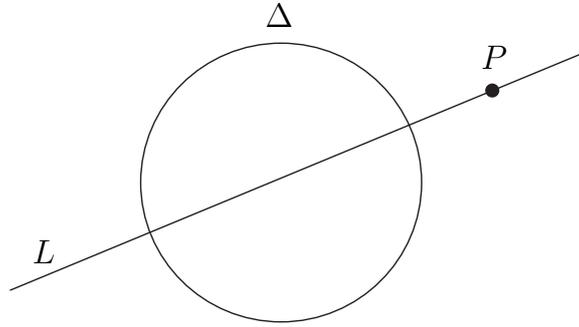
Figure 1

where Δ is the unit ball in $\mathbb{C}^2 \subset \mathbb{P}^2$. Then

(i) *U may be identified with the product of unit balls B and $\check{B}$ in $\mathbb{P}^2$ and $\check{\mathbb{P}}^2$.*
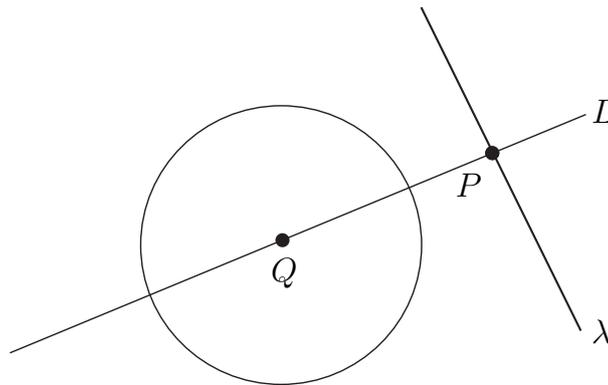
Proof. In the picture

Figure 2

for fixed $Q \in B$ and $\lambda \in \check{B}$ varying (P, L) gives a $\mathbb{P}^1$ in D.

(ii) *In (III.A.1) the fibres Z of π_D are biholomorphic to a disc bundle over a disc.*

Proof. In Figure 2 we have a map

$$Z \to L \backslash L \cap \overline{\Delta}$$

given by $(Q, \lambda; P, L) \in I$ mapping to Q. The fibre of this map consists of all lines λ in Figure 2 that pass through P.

III.C. **Enlargements of cycle spaces.** For Hodge-theoretic purposes it seems desirable to have a space $\mathcal{W}$ that fits in a holomorphic diagram

(III.C.1)

$$
\begin{array}{ccc}
& \mathcal{W} & \\
\pi \swarrow & & \searrow \pi' \\
D & & D
\end{array}
$$

where the fibres are contractible Stein manifolds, and where there also is a fibering

(III.C.2) $\mathcal{W} \to U$

with Stein fibres. We will give the construction in the two above examples; we do not know in what generality such a construction is possible.

  Example 1 (continued). We define $\mathcal{W} \subset D \times D$ to be

$$
\mathcal{W} = \left\{ (F, F') : F \cap F' = (0) \text{ and } Q(F + F', F + \overline{F}' > 0 \right\} .
$$

  (i) *the fibres in* (III.C.1) *are contractible Stein manifolds.*

Proof. We map

(III.C.3) $\pi^{-1}(F) \xrightarrow{\hspace{2cm}} \mathbb{P}(V_{\mathbb{C}}/F)^{\vee} \cong \mathbb{P}^2$

$$
\cup \qquad\qquad\qquad\qquad \cup
$$

$(F, F') \longrightarrow [F'] :=$ projection of F' in $V_{\mathbb{C}}/F$.

It may be shown that the image is a ball, and we claim that the fibre of this map in $\mathbb{C} = \mathbb{P}^1 \backslash \{\text{point}\}$. To see this, we set $E = F + F' \in \mathbb{P}\check{V}_{\mathbb{C}}$ and observe that $[E] = [F'] \in \mathbb{P}(V_{\mathbb{C}}/F)^{\vee}$. The fibre of (III.C.3) over $[E]$ consists of all lines in the quadric $\mathbf{Q}_E$ that do not meet the ruling given by F. Since $F \cap F' = (0)$ the ruling given by F' must be in the same family as that given by F and distinct from F.
  There is a mapping

$$
\widetilde{\omega} : \mathcal{W} \xrightarrow{\hspace{2cm}} U
$$

$$
\cup \qquad\qquad\qquad \cup
$$

$$
(F, F') \longrightarrow E = F + F'.
$$

  (ii) *The fibres of $\widetilde{\omega}$ are bi-holomorphic to two copies of* $\mathbb{P}^1 \times \mathbb{P}^1 \backslash \{\text{diagonal}\}$.

Proof. The lines $[F], [F']$ must be distinct and in the same family of rulings of $\mathbf{Q}_E$.

Example 2 (continued). Here we define $\mathcal{W} \subset D \times D$ to be given by the picture
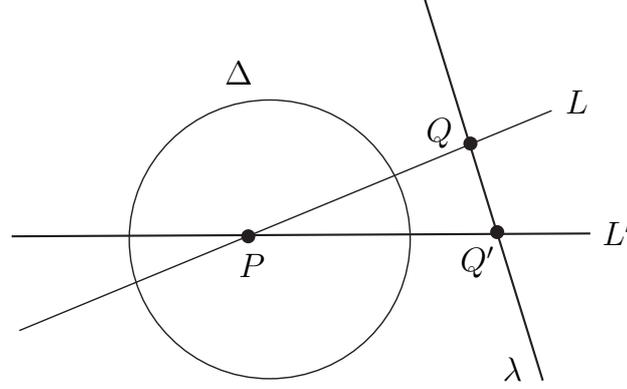

Figure 3

That is

$$\mathcal{W} = \left\{ (Q, L; Q', L') : L \neq L'; L \cap L' \in \Delta; \ \text{and} \ \overline{QQ'} \cap \overline{\Delta} = \emptyset \right\}.$$

(i) *The fibres in* (III.C.1) *are contractible Stein manifolds.*

Proof. There is a surjective submersion

$$\pi^{-1}(Q, L) \longrightarrow \Delta$$

$$\cup \hspace{4cm} \cup$$

$$(Q, L; Q', L') \longrightarrow L \cap L'.$$

The fibre of this map consists of all lines $\overline{QQ'}$ through Q that do not meet $\overline{\Delta}$, which is a disc in $\mathbb{C}$.

There is a surjective map

(III.C.4) $$\widetilde{\omega} : \mathcal{W} \to U$$

given by

$$(P, L; P', L') \to \overline{QQ'}$$

in Figure 3.

(ii) *The fibres of* (III.C.4) *are Stein.*

Proof. There is a surjective submersion

$$\widetilde{\omega}^{-1}(\lambda) \to \Delta$$

sending Figure 3 to $P \in \Delta$. FIxing P, Q and Q' are arbitrary distinct points of $\lambda \cong \mathbb{P}^1$.

One potential advantage of (III.C.1) over the cycle space diagram (III.A.1) is the following: In the next sections we shall see that in

the diagram (III.A.1) cohomology on D, and on quotients $\Gamma \backslash D$, will map to holomorphic data on I and on $\Gamma \backslash I$ and then by a push-down to holomorphic data on U and $\Gamma \backslash U$.[3] The same is also true for the diagram (III.C.1). The latter has the advantage that, given a global variation of Hodge structure

$$\Phi : S \to \Gamma \backslash D \,,$$

there is an associated map

$$\Phi_{\mathcal{W}} : S_{\mathcal{W}} \to \Gamma \backslash \mathcal{W}$$

where $S_{\mathcal{W}}$ is constructed as follows. There is a diagram

$$
\begin{array}{ccc}
\widetilde{S} & \xrightarrow{\ \widetilde{\Phi}\ } & D \\
{\scriptstyle \pi_1}\downarrow & & \downarrow \\
S & \xrightarrow{\ \Phi\ } & \Gamma \backslash D
\end{array}
$$

where $\widetilde{S}$ is the universal covering of $S = \pi_1 \backslash \widetilde{S}$. We then have the

$$\widetilde{S} \times \widetilde{S} \xrightarrow{\ \widetilde{\Phi} \times \widetilde{\Phi}\ } D \times D \,,$$

and we set

$$\widetilde{S}_{\mathcal{W}} = (\widetilde{\Phi} \times \widetilde{\Phi})^{-1} \mathcal{W}$$

and

$$S_{\mathcal{W}} = \pi_1 \backslash \widetilde{S}_{\mathcal{W}} \,.[4]$$

We note that there is a holomorphic submersion

$$S_{\mathcal{W}} \to S$$

whose fibres are of the form $\widetilde{S} \backslash Z$ where Z is a proper analytic sub-variety of $\widetilde{S}$. As will be explained below, automorphic cohomology in $H^d(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda})$ will be mapped to holomorphic data over $\Gamma \backslash \mathcal{W}$ and then pulled back to $S_{\mathcal{W}}$. In particular, we may say that a class $\xi \in H^d(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda})$ vanishes at a point of $S_{\mathcal{W}}$.[5]

---

[3]We note that Γ acts equivariantly in both diagrams (III.A.1) and (III.C.1).

[4]Here, π_1 acts diagonally as deck-transformations on $\widetilde{S} \times \widetilde{S}$; the same is meant by $\Gamma \backslash D \times D$.

[5]The space $S_{\mathcal{W}}$ is a strange object. If we assume that Φ is an immersion and that no $T_i = I$ where T_i are the unipotent monodromies around the branches of the normal crossing divisor $\overline{S} \backslash S$, then a folk result is that

$$\widetilde{S} \text{ is a bounded domain of holomorphy in } \mathbb{C}^N.$$

Consequently, the fibres of $S_{\mathcal{W}} \to S$ are of the form "bounded domain of holomorphy minus a closed analytic subvariety". Thus, $S_{\mathcal{W}}$ is a complex manifold that although not an algebraic variety, does have some algebro-geometric aspects.

The nature of this pullback map has yet to be investigated. Whatever it is, it seems that it will be something rather non-standard for the following reason. First, by passing to a subgroup of finite index if necessary we may assume that $\Gamma \cap K = \{e\}$. In particular, Γ will act without fixed points on D, U and W. It follows that no $\gamma \neq e$ maps a fibre to itself in either (III.A.1) or (III.C.1). In particular, although U and $\mathcal{W}$ are Stein, neither quotient $\Gamma\backslash U$ or $\Gamma\backslash\mathcal{W}$ will be an algebraic variety, at least in any natural way and, as noted above, the same will be true of $S_{\mathcal{W}}$

IV. Penrose-Radon transforms

IV.A. **Work of Eastwood-Gindikin-Wong [EGW].** In algebraic geometry diagrams of the sort

$$
\begin{array}{ccc}
 & X & \\
\pi_Y \swarrow & & \searrow \pi_Y \\
Y & & Z
\end{array}
$$

where X, Y, Z are algebraic varieties and π_X, π_Y are proper morphisms have been used since classical times (correspondences), and more recently have been used to relate cohomology on Y to that on Z, and vice versa. The work referred to above is of a somewhat different character. It seems to have at least in part been motivated by representation theory in the circumstances where one wants to realize a representation geometrically on a space of functions (or sections of a vector bundle) rather than on higher cohomology. It deals with the situation

$$
\left\{
\begin{array}{ccc}
\pi : X & \to & Y \\
\mathcal{E} & \to & Y
\end{array}
\right.
$$

where X and Y are complex manifolds, π is a holomorphic submersion and $\mathcal{E} \to Y$ is a holomorphic vector bundle. One of their results is

Theorem. *Suppose that X is Stein and that the fibres of π are Stein. Then there is a canonical isomorphism*

(IV.A.1) $$H^r(Y, \mathcal{E}) \cong H^r_{\mathrm{DR}}\left(\Gamma\left(\Omega^\bullet_{X/Y}(\mathcal{E}_\pi), d_\pi\right)\right).$$

Here the RHS is the de-Rham cohomology of the complex $\Gamma(\Omega^\bullet_{X/Y}(\mathcal{E}_\pi), d_\pi)$ of global, relative holomorphic forms on X with coefficients in $\mathcal{E}_\pi :=$

$\pi^{-1}(\mathcal{E})$.[6] For $\mathcal{E} = \mathcal{O}_Y$ the idea behind (IV.A.1) is to consider the hypercohomology associated to the exact sheaf sequence on X

$$0 \to \pi^{-1}(\mathcal{O}_Y) \xrightarrow{d_\pi} \Omega_{X/Y} \xrightarrow{d_\pi} \Omega^2_{X/Y} \to \ .$$

Using the stated assumptions, the two spectral sequences degenerate to give the result. For $\mathcal{E}$ one tensors the above sequence by $\underset{\pi^{-1}(\mathcal{O}_Y)}{\otimes} \mathcal{O}_X(\pi^{-1}(\mathcal{E}))$.

The above result can be used to realize representation spaces of $M(\mathbb{R})$ by holomorphic data. In [EGW] it is shown that in certain situations in which there are group actions, including the case $\mathcal{W} \to D$ in example 2 above, and for certain homogeneous bundles, including in that example

(IV.A.2) $\mathcal{L}_{\rho_\lambda} = \mathcal{O}_{\mathbb{P}^2}(r) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^2}(t),$ $r + t \leqq -2$

there are canonical representations of the cohomology classes in the RHS of (IV.A.1) (what [EGW] call "holomorphic, harmonic forms" — cf. Theorem 2.13 there). This then gives a geometric realization of the Harish-Chandra module $H^1(D, \mathcal{L}_{\rho_\lambda})$, for D as in example 2 and $\mathcal{L}_{\rho_\lambda}$ as in (IV.A.2), by holomorphic data.

Note. In general, for a homogeneous bundle $\mathcal{L}_{\rho_\lambda} \to D$ over a Hodge domain, there is a L^2-cohomology $H^r_{(2)}(D, \mathcal{L}_{\rho_\lambda})$ and ordinary coherent cohomology $H^r(D, \mathcal{L}_{\rho_\lambda})$. There is an obvious map

$$H^r_{(2)}(D, \mathcal{L}_{\rho_\lambda}) \to H^r(D, \mathcal{L}_{\rho_\lambda}),$$

and only in certain cases, including the one above, is this map injective with a dense image (cf. [Schm]).

An alternative method of realizing representation spaces by holomorphic data is based on the diagram (III.A.1). Recognizing that the discrete series representation is realized as $H^d_{(2)}(D, \mathcal{L}_{\rho_\lambda})$ where $d = \dim_\mathbb{C} Y$ for a maximal compact subvariety, and that the condition $|\lambda| \gg 0$ implies that $\mathcal{L}_{\rho_\lambda} \otimes \mathcal{O}_Y$ is "very negative" and hence $H^d(\mathcal{L}_{\rho_\lambda} \otimes \mathcal{O}_Y)$ is "very big", suggests mapping $H^d_{(2)}(D, \mathcal{L}_{\rho_\lambda})$ to holomorphic sections of a vector bundle $\mathcal{E}_\lambda \to U$ over the cycle space discussed above (cf. [G],§11). The basic result here is Corollary 14.5.3 in [FHW], which informally may be stated as follows:

> *Under suitable conditions on λ the Penrose-Radon transform*
>
> $$\mathbb{P} : H^d(D, \mathcal{L}_{\rho_\lambda}) \to H^0(U, \mathcal{E}_\lambda)$$
>
> *is injective.*

---

[6]We are using Γ to denote both a discrete sub-group of M and global sections; we hope that the context will make clear which use is intended.

IV.B. **Work of Carayol** [C]. The preceding section was concerned with transforming cohomology on Hodge domains to holomorphic data on an associated space. In this section we will discuss the situation when we factor by a discrete group $\Gamma \subset M$, which we shall assume to be co-compact in $M(\mathbb{R})$ and to operate without fixed points on D, U and $\mathcal{W}$. Then for a suitable choice of λ with $|\lambda| \gg 0$ we have

$$(\text{IV.B.1}) \quad \dim H^q(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda}) = \begin{cases} 0 & q \neq d \\ \sim C|\lambda|^{\dim D}, & c > 0 \ \text{ when } \ q = d. \end{cases}$$

Thus there is a lot of automorphic cohomology.

Remark. When Γ is an arithmetic group, by analogy with the classical case a natural group to consider is $H^d_{(2)}(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda})$. In [WW] there is constructed a Poincaré series map

$$H^d_{(2;1)}(D, \mathcal{L}_{\rho_\lambda}) \to H^d_{(2)}(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda})$$

where the LHS are the L^2-cohomology classes represented by an L^1-form. It seems not to be known if the vanishing result in (IV.B.1) and finite dimensionality of $H^d_{(2)}(\Gamma \backslash D, \mathcal{L}_{\rho_\lambda})$ are valid.

In [C] a slightly modified situation in example 2 is analyzed. Namely, the action of $\text{SU}(2,1)(\mathbb{R})$ on the compact dual $\check{D}$ realized as the flag manifold

$$\check{D} = \{(P, L) : P \in L\} \subset \mathbb{P}^2 \times \check{\mathbb{P}}^2 \,,$$

has six open orbits corresponding to three distinct complex structures on $\text{SU}(2,1)(\mathbb{R})/T$ and their conjugates. These complex structures labelled as D, D' and D'', may be pictured as
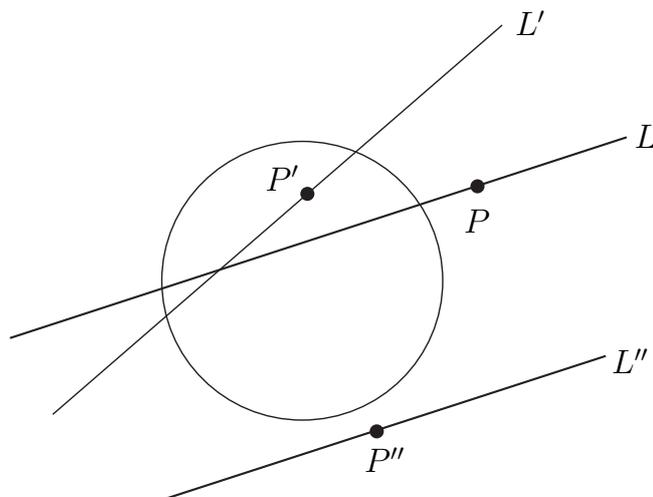

Figure 4

The same space $\mathcal{W} \subset D \times D$ as pictured in Figure 3 now fibres *holomorphically* in the diagram

$$
\begin{array}{ccc}
 & \mathcal{W} & \\
{}^{\pi_{D'}}\swarrow & & \searrow^{\pi_D} \\
D & & D'
\end{array}
$$

where, in the notation of Figure 3,

$$
\begin{cases}
\pi_D(Q, L; Q', L') = (Q, L) \\
\pi_{D'}(Q, L, Q', L') = (P, L).
\end{cases}
$$

A modification of (IV.A.1), based on explicit calculations special to the situation at hand, enables Carayol to relate an H^1 on D to an H^0 on the Hermitian symmetric space D'. Specifically, setting $\mathcal{L}(r, t) = \mathcal{O}_{\mathbb{P}^2}(r) \boxtimes \mathcal{O}_{\check{\mathbb{P}}^2}(t)$, he constructs an isomorphism

$$
H^0(D', \mathcal{L}(r, t)) \xrightarrow{\sim} H^1(D, \mathcal{L}(-r-2, r+t+1)
$$

with a similar one for D''.

In order to be able to pass to the quotients by Γ, the following result, which seems to be special to example 2, is used:

The space $\Gamma \backslash \mathcal{W}$ is Stein.

Using this the above drops down to give an isomorphism

$$
H^0(\Gamma \backslash D', \mathcal{L}(r, t)) \to H^1(\Gamma \backslash D, \mathcal{L}(r+t+1, -t-2)
$$

for $(r, t) \neq (0, -3)$, explicitly exhibiting — for the first time to my knowledge — automorphic cohomology in a non-classical case.

As noted in lecture VI there is exactly one circle $\varphi : S^1 \to \mathrm{SU}(2, 1)(\mathbb{R})$ for which the infinitesimal period relation is non-trivial and non-integrable. One can imagine the existence of a global variation of Hodge structure

$$
\Phi : S \to \Gamma \backslash D
$$

where S is an algebraic curve. The construction in section III.C leads to a map

$$
\Phi_\mathcal{W} : S_\mathcal{W} \to \Gamma \backslash \mathcal{W}
$$

where $S_\mathcal{W}$ is a surface that fibres over the curve S. The automorphic cohomology above then gives holomorphic sections of a line bundle $\mathcal{L}_\mathcal{W} \to S_\mathcal{W}$ that may be explicitly written out using the formulae in [C]. What its' interpretation — either algebro-geometric or arithmetic — may be is not clear.

Hodge-theoretic reprise: A Hodge group (M, φ) and Hodge domain $D_{\mathfrak{m}, \varphi}$ has finer data than just an invariant complex structure on $M(\mathbb{R})/H$. This is due to

(a) the $\mathbb{Q}$-algebraic group M whose real points are $M(\mathbb{R})$;
(b) the choice of a circle $\varphi : S^1 \to M(\mathbb{R})$.

One may ask what role this refined data might play? As mentioned before, given λ and a compatible φ, a power of $\mathcal{L}_{\rho_\lambda}$ is a Hodge bundle, but which Hodge bundle it is depends on φ. Additionally, we note that

- From (a) and (b), we have $D_{\mathfrak{m}, \varphi} \subset \check{D}_{\mathfrak{m}, \varphi}$ where $\check{D}_{\mathfrak{m}, \varphi}$ is a homogeneous rational, projective variety defined over $\mathbb{Q}$. Hence, points of $D_{\mathfrak{m}, \varphi}$ have a Γ-invariant field of definition.
- There are special Γ-invariant arithmetically defined points in $D_{\mathfrak{m}, \varphi}$; e.g., those for which the polarized Hodge structure on $(\mathfrak{m}, B, \varphi)$ are of CM type. These induce special points on

$$\Gamma \backslash \mathcal{W} \subset \Gamma \backslash D_{\mathfrak{m}, \varphi} \times D_{\mathfrak{m}, \varphi}.$$

The above data is only present if we have (a) *and* (b).

To conclude we will use this to very loosely formulate a specific speculation. First, from sections of bundles one may construct meromorphic functions in the well known way (e.g., ratios of sections of a line bundle, taking determinants of vector bundles to get line bundles, etc.).

Secondly, we consider a space of eigenfunctions for the Hecke algebra $\mathcal{H}$ operating on the right $\mathcal{H}$-module $L^2(M(\mathbb{Q}) \backslash M(\mathbb{A}))$ and whose part over the place $\nu = \infty$ corresponds to $H^d_{(2)}(\Gamma \backslash D_{\mathfrak{m}, \varphi}, \mathcal{L}_{\rho_\lambda})$ as discussed above. Let f be a meromorphic function constructed from the image of that space in $H^0(\Gamma \backslash \mathcal{W}, \mathcal{E}_{\rho_\lambda})$.

Question. *Does the value of a suitably normalized f at a CM point lie in a field constructed from the CM field?*

The point is not so much the question itself, but rather that the introduction of enlarged cycle spaces enables one to formulate it.

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