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Mumford-Tate Groups

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MUMFORD-TATE GROUPS

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I. Definitions and basic properties

Mumford-Tate groups are the basic symmetry groups of Hodge theory. They lie at the interface of Hodge theory with

- algebraic geometry (variations of Hodge structure)
- representation theory of real, non-compact semi-simple Lie groups (discrete series)
- arithmetic algebraic geometry (theory of Shimura varieties)

Much of the use of Mumford-Tate groups has been in the classical case of weight one polarized Hodge structures (theory of Shimura varieties). Since this topic will be covered elsewhere in the summer school, this set of lectures will emphasize the non-classical, higher weight case which is much less explored.

The first four of these lectures will be largely self-contained, making use of the lectures of Lê Dung Tráng, Cattani, Tu,

Lectures based in large part on joint work with Mark Green and Matt Kerr.

El Zein, Voisin, and Carlson at this summer school. Lectures V and VI will use some of the basic structure theory of semi-simple complex and real Lie groups and Lie algebras, and will be largely without proofs. At the end we indicate, in very general terms, where some potentially interesting directions for further work might lie.

A. **Notations.**

- V is a vector space over $\mathbb{Q}$; $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ etc.;
- $\check{V}$ is the dual vector space;
- $Q : V \otimes V \to \mathbb{Q}$ is a non-degenerate bilinear form with $Q(v, w) = (-1)^n Q(w, v)$ where n will be the weight of the Hodge structure;
- $\mathrm{GL}(V)$, $\mathrm{GL}(V_{\mathbb{R}})$ etc. are the general linear groups;
- $G = \mathrm{Aut}(V, Q)$ is the subgroup of $\mathrm{GL}(V)$ preserving Q.

Definition. A $\mathbb{Q}$-*algebraic group* will be a subgroup of a $\mathrm{GL}(V)$ defined by polynomial equations with $\mathbb{Q}$-coefficients.[1]

Example. G is a $\mathbb{Q}$-algebraic group.

A connected, commutative $\mathbb{Q}$-algebraic group is called an *algebraic torus.*

Examples. $\mathbb{Q}^*$ is an algebraic torus.

$$T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1, \quad a, b \in \mathbb{Q} \right\}$$

is an algebraic torus.

We may also define $\mathbb{R}$-algebraic groups.

[1] The basic reference is Borel, *Linear algebraic groups.*

Example. $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{C}^*$, the restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of $\mathbb{C}^*$, is an $\mathbb{R}$-algebraic group. Throughout these lectures it will be denoted simply by $\mathbb{S}$. It is the semi-direct product of its subgroups $\mathbb{R}^*$ (*scaling*) and $S^1 = \{z \in \mathbb{S} : |z| = 1\}$.

For a $\mathbb{Q}$-algebraic group M, $M(\mathbb{Q})$, $M(\mathbb{R})$, $M(\mathbb{C})$ will denote the set of its $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$-valued points.

- For a subgroup $\Gamma \subset M$, $\overline{\Gamma}^{\mathbb{Q}}$ will denote its $\mathbb{Q}$-Zariski closure.

Definition. The *Lie algebra* $\mathfrak{g}$ of G is the tangent space to G at the identity.

Examples. $\mathrm{gl}(V) = \mathrm{End}(V) \cong V \otimes \check{V}$ with the bracket

$$[X,Y](v) = X(Y(v)) - Y(X(v)), \qquad X, Y \in \mathrm{gl}(V), v \in V.$$

For G as above

$$\mathfrak{g} = \{X \in \mathrm{gl}(V) : Q(Xu, v) + Q(u, Xv) = 0, \quad u, v \in V\}.$$

B. **Polarized Hodge structures.**

Definition (i)**.** A *Hodge structure of weight* n is given by a *Hodge decomposition*

$$\begin{cases} V_{\mathbb{C}} = \bigoplus\limits_{p+q=n} V^{p,q} \\ \overline{V}^{p,q} = V^{q,p}. \end{cases}$$

Definition (ii)**.** A *Hodge structure of weight* n is given by a *Hodge filtration*

$$\begin{cases} F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}} \\ F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}, \qquad p = n, \ldots, 1. \end{cases}$$

Definition (iii)**.** A *Hodge structure of weight* n is given by a non-constant homomorphism of $\mathbb{R}$-algebraic groups

$$\widetilde{\varphi} : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

such that over $\mathbb{C}$ the eigenspace decomposition of $\widetilde{\varphi}:\mathbb{S}\to \mathrm{GL}(V_{\mathbb{C}})$ is

$$\begin{cases} V_{\mathbb{C}} = \underset{p+q=n}{\oplus} V^{p,q} \\ \widetilde{\varphi}(z)v = z^p\bar{z}^q v, \qquad v\in V^{p,q}. \end{cases}$$

Equivalence of definitions.

$$\text{(i)} \Rightarrow \text{(ii)} \qquad F^p = \underset{p'\geqq p}{\oplus} V^{p',q'}$$

$$\text{(ii)} \Rightarrow \text{(i)} \qquad V^{p,q} = F^p\cap \overline{F}^q$$

$$\text{(i)} \Leftrightarrow \text{(iii)} \qquad V^{p,q} = z^p\bar{z}^q \text{ eigenspace for } \widetilde{\varphi}(\mathbb{S}).$$

The *Weil operator* $C\in \mathrm{GL}(V_{\mathbb{R}})$ is defined by

$$C=\widetilde{\varphi}(i) \qquad\qquad (= i^{p-q} \text{ on } V^{p,q})\,.$$

We denote the Hodge structure by $(V,\widetilde{\varphi})$, or sometimes by just $V_{\widetilde{\varphi}}$.

Definition. A *polarized Hodge structure* (V,Q,φ) of weight n is given by

$$\varphi: S^1\to G(\mathbb{R})$$

such that (i) the characters of φ lie in $[-n,n]$, and (ii) the *Hodge-Riemann bilinear relations*

$$\begin{cases} Q(F^p,F^{n-p+1})=0 \\ Q(v,C\overline{v})>0, \qquad\qquad 0\neq v\in V_{\mathbb{C}} \end{cases}$$

are satisfied.

These relations are equivalent to

$$\begin{cases} Q(V^{p,q},V^{p',q'})=0, \qquad p'\neq n-p \\ i^{p-q}Q(v,\overline{v})>0, \qquad 0\neq v\in V^{p,q}. \end{cases}$$

Remark. The action of $\varphi(S^1)$ on $V_{\mathbb{C}}$ decomposes into eigenspaces $V^{p,q}=\overline{V}^{q,p}$ where $\varphi(z)=z^{p-q}$ $(=z^p\bar{z}^q)$ on $V^{p,q}$. The character group

$$X(S^1)\cong \mathbb{Z}$$

where $m\in\mathbb{Z}$ gives the character $\chi_m(z)=z^m$.

Note. The difference between $\widetilde{\varphi}$ and φ is that the scaling action of $\widetilde{\varphi}$ gives the weight

$$\widetilde{\varphi}(r) = r^n \operatorname{id}_V \ .$$

We may define a Hodge structure, without specifying the weight, to be given by

$$\widetilde{\varphi} : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

with the following condition: On $\mathbb{Q}^* \subset \mathbb{S}$ the action of $\widetilde{\varphi}(\mathbb{Q}^*)$ decomposes over $\mathbb{Q}$ into weight spaces

$$V = \bigoplus_n V^{(n)}$$

and $(V^{(n)}, \widetilde{\varphi}\,|_{V^{(n)}})$ is a Hodge structure of weight n. Hodge structures of weight n are sometimes referred to as *pure Hodge structures*. Polarized Hodge structures are always assumed to be pure.

Basic example. $V = \mathbb{Q}^2 =$ column vectors $v = \binom{v_1}{v_2}$,

$$\begin{cases} Q(u,v) = {}^t v Q u, & Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1} & \\ V^{1,0} = \mathbb{C} v_{1,0}, & v_{1,0} = \binom{v_1}{v_2} . \end{cases}$$

The condition

$$iQ(v_{1,0}, \overline{v}_{1,0}) > 0$$

is

$$(-i)(v_1 \overline{v}_2 - \overline{v}_1 v_2) > 0 \, .$$

This implies $v_2 \neq 0$ and when we scale to have

$$v_{1,0} = \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

the above condition becomes $\operatorname{Im} \tau > 0$. We set

$$v_\tau = \frac{1}{\sqrt{2 \operatorname{Im} \tau}} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$
$$\Rightarrow iQ(v_\tau, \overline{v}_\tau) = 1 \, .$$

Then $v_\tau, \overline{v}_\tau$ give a *Hodge basis.*

Denoting by φ_τ the circle giving the Hodge structure, in terms of the Hodge basis φ_τ has the matrix

$$\varphi_\tau(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad z^{-1} = \bar{z}\,.$$

Example. The Tate structure $\mathbb{Q}(n) := 2\pi i\mathbb{Q} \subset \mathbb{C}$ with Hodge structure of pure type $(-n, -n)$.

Hodge structures and polarized Hodge structures admit the usual operations of linear algebra: $\oplus, \mathrm{Hom}, \otimes, \wedge$ and Sym. A subspace $W \subset V$ is a *sub-Hodge structure* of $(V, \widetilde{\varphi})$, not assumed to be pure, if

$$\varphi(\mathbb{S})W_{\mathbb{R}} \subseteq W_{\mathbb{R}}\,.$$

In the polarized case (V, Q, φ) we have that $Q\mid_W$ is non-singular and

$$V = W \oplus W^{\perp}$$

is a direct sum of polarized Hodge structures. Polarized Hodge structures form a semi-smiple abelian category.

Because of

Hodge's Theorem. *The cohomology group $H^n(X, \mathbb{Q})$ of a smooth, projective algebraic variety has a polarized Hodge structure of weight n.*

polarized Hodge structures are the basic objects of Hodge theory. The basic example is the polarized Hodge structure on $H^1(E_\tau, \mathbb{Q})$ where

$$E_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau\,.$$

Note. In practice we will usually be given a lattice $V_{\mathbb{Z}}$ with $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For the theory of Mumford-Tate groups it is better to work over $\mathbb{Q}$ than over $\mathbb{Z}$.

For a Hodge structure $(V, \widetilde{\varphi})$ of even weight $n = 2p$, we define the space of *Hodge vectors* by

$$\mathrm{Hg}_{\widetilde{\varphi}} = V \cap V^{p,q}\,.$$

These are the rational vectors of Hodge type (p,p). The famous Hodge conjecture is that the Hodge vectors in $H^{2p}(X,\mathbb{Q})$ are represented by the fundamental classes of algebraic cycles.

C. **Mumford-Tate groups.**

Definition. The *Mumford-Tate group* $M_{\widetilde{\varphi}}$ associated to a Hodge structure $(V,\widetilde{\varphi})$ is the smallest $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ such that

$$\widetilde{\varphi}(\mathbb{S}) \subset M_{\widetilde{\varphi}}(\mathbb{R}).$$

It is not required that $V_{\widetilde{\varphi}}$ be of pure weight. If $(V_{\widetilde{\varphi}}) = (V_{1,\widetilde{\varphi}_1}) \oplus (V_{2,\widetilde{\varphi}_2})$ is a direct sum, then $M_{\widetilde{\varphi}} \subseteq M_{\widetilde{\varphi}_1} \times M_{\widetilde{\varphi}_2}$ and the projection onto each factor is surjective but equality may not hold (e.g., if $V_{\widetilde{\varphi}_1} = V_{\widetilde{\varphi}_2}$ then $M_{\widetilde{\varphi}}$ is the diagonal subgroup of the product).

Definition. The *Mumford-Tate group* M_φ associated to a polarized Hodge structure (V,Q,φ) is the smallest $\mathbb{Q}$-algebraic subgroup of G such that

$$\varphi(S^1) \subset M_\varphi(\mathbb{R})\,.$$

Note. In the literature $M_{\widetilde{\varphi}}$ is usually called the Mumford-Tate group and M_φ the special Mumford-Tate group or Hodge group. Because of the centrality in these lectures of polarized Hodge structures, we shall refer to both as Mumford-Tate groups and let the subscripts φ and $\widetilde{\varphi}$ specify to which we are referring. For pure Hodge structures we shall see that $M_{\widetilde{\varphi}}$ and M_φ differ only by the scaling action; $M_{\widetilde{\varphi}}$ will be the semi-direct product of M_φ and $\mathbb{G}_{m,\mathbb{Q}}$.

Basic example (continued). For the above example, we first have $G = \mathrm{SL}_2(\mathbb{Q})$. The change of basis from the Hodge basis to the $\mathbb{Q}$-basis is

$$A_\tau = \frac{1}{\sqrt{2\,\mathrm{Im}\,\tau}}\begin{pmatrix} \tau & \overline{\tau} \\ 1 & -1 \end{pmatrix},$$

and so the matrix of $\varphi_\tau(z)$ in the $\mathbb{Q}$-basis is

$$\psi_\tau(z) := A_\tau \varphi_\tau(z) A_\tau^{-1}.$$

The entries of ψ_τ are not particularly illuminating quadratic expressions in τ and $\overline{\tau}$ with $\mathbb{Q}$-coefficients. To satisfy an additional equation over $\mathbb{Q}$ beyond $\det \psi_\tau = 1$ suggests that τ must be an algebraic number. We shall see later that there are two cases:

(i) $\mathbb{Q}(\tau)$ is a purely imaginary quadratic extension of $\mathbb{Q}$, and M_{φ_τ} are the elements of norm one in $\mathbb{Q}(\tau)^*$;
(ii) $M_{\varphi_\tau} = \mathrm{SL}_2(\mathbb{Q})$.

Example. For $\mathbb{Q}(n)$, the Mumford-Tate group is $\mathbb{G}_{m,\mathbb{Q}}$ for $n \neq 0$ and is trivial for $n = 0$.

We shall now formulate and prove the basic property for M_φ; subsequently, we shall do the same for $M_{\widetilde{\varphi}}$. In each case the basic property is an answer to the question:

What are the defining equations for the $\mathbb{Q}$-algebraic group M_φ, respectively $M_{\widetilde{\varphi}}$?

For this we consider a polarized Hodge structure (V, Q, φ) and let

$$\begin{cases} T^{k,l} = V^{\otimes^k} \otimes \check{V}^{\otimes^l}, \\ T^{\bullet,\bullet} = \underset{k,l \geqq 0}{\oplus} T^{k,l} \end{cases}$$

be the tensor spaces and tensor algebra of V. We then have

$$\begin{cases} \mathrm{Hg}_\varphi^{k,l} \subset T^{k,l} \\ \mathrm{Hg}_\varphi^{\bullet,\bullet} = \oplus \mathrm{Hg}_\varphi^{k,l} \end{cases}$$

consisting of the Hodge tensors in $T^{k,l}$ and the algebra of Hodge tensors.

Basic Property (I). *M_φ is the subgroup of G fixing $\mathrm{Hg}_\varphi^{\bullet,\bullet}$.*

Step one: *If $t \in \mathrm{Hg}_\varphi^{k,l}$, then M_φ fixes t.*

Proof. Since t is rational, fixing it defines a $\mathbb{Q}$-algebraic subgroup $G(t)$ of $\mathrm{GL}(V)$. If t is of Hodge type (p,p) where $n(k-l)=2p$, then $\varphi(z)t = z^{p-p}t = t$. If follows that

$$\varphi(S^1) \subseteq G(t)(\mathbb{R}) \,,$$

and by the minimality of M_φ we conclude that $M_\varphi \subset G(t)$. This gives

$$M_\varphi \subseteq \mathrm{Fix}\ \mathrm{Hg}_\varphi^{\bullet,\bullet} \,.$$

Step two: *If M_φ stabilizes the line $\mathbb{Q}t$ spanned by $t \in T^{k,l}$, then $t \in \mathrm{Hg}_\varphi^{k,l}$ is a Hodge tensor.*

Proof. Since $\varphi(S^1) \subset M_\varphi(\mathbb{C})$, if M_φ stabilizes $\mathbb{Q}t$ then t will be an eigenvector for $\varphi(z)$ for a general $z \in S^1$. Hence t is of pure Hodge type, and since it is rational it must be a Hodge tensor.

Chevalley's theorem (in our context). *Let M be a closed $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$. Then M is the stabilizer of a line in $\bigoplus_{i=1}^{m} T^{k_i,k_i}$.*

Completion of the proof of the basic property (I). There exists

$$\tau = (t_1, \ldots, t_m) \in \bigoplus_{i=1}^{m} T^{k_i,k_i}$$

such that setting $L = \mathbb{Q} \cdot \tau$,

$$M_\varphi = \{g \in \mathrm{GL}(V) : g(L) \subseteq L\} \,.$$

Then $g \in M_\varphi$ fixes each line $\mathbb{Q}t_i \subset T^{k_i,k_i}$, and by step two t_i is a Hodge tensor. Thus

$$\mathrm{Fix}\,\mathrm{Hg}_\varphi^{\bullet,\bullet} \subseteq \bigcap_i \mathrm{Fix}(t_i) = M_\varphi \,. \qquad \square$$

Proof of Chevalley's theorem. From the open embedding $\mathrm{GL}(V) \hookrightarrow \mathrm{End}(V)$ of algebraic varieties comes the injection of coordinate rings

$$\mathbb{Q}[\mathrm{GL}(V)] \hookleftarrow \mathbb{Q}[\mathrm{End}(V)] \cong \bigoplus_{k\geq 0} \mathrm{Sym}^k(\mathrm{End}(V)^\vee) \subseteq \bigoplus_{k \geq 0} T^{k,k}V.$$

The action of $\mathrm{GL}(V)$ on itself by conjugation extends to the adjoint action on $\mathrm{End}(V)$, and these induce compatible actions on the coordinate rings which, moreover, are compatible with the action of $\mathrm{GL}(V)$ on $\oplus T^{k,k}V$. Write $S := \mathbb{Q}[\mathrm{End}(V)]$. If we choose a basis for V, we may think of S as polynomials $P(X^i_j)$ in the matrix entries of $X \in \mathrm{End}(V)$.

The stabilizer of the ideal

$$I(M) \subseteq \mathbb{Q}[\mathrm{GL}(V)]$$

of M, viewed as a subvariety of $\mathrm{GL}(V)$, is the largest algebraic subgroup contained in the Zariski closure of M. But since M is Zariski closed, and a subgroup, the stabilizer is just M. Note that S and

$$I := I(M) \cap S$$

inherit nonnegative $\mathrm{GL}(V)$-invariant gradings from the above injection of coordinate rings, and M is also the subgroup of $\mathrm{GL}(V)$ stabilizing I in S. This follows from the above because $\mathbb{Q}[\mathrm{GL}(V)] = \mathbb{Q}[\mathrm{End}(V)][\frac{1}{\det}]$ and det is nonvanishing on M.

Since M is an algebraic variety in $\mathrm{GL}(V)$, $I(M)$ is finitely generated; the same goes for I, the ideal of the Zariski closure of M in $\mathrm{End}(V)$. Let $P_i \in I^{\leq k}$ be a generating set for I. Since $I^{\leq k}$ generates I and the action is compatible with products, M is the stabilizer of $I^{\leq k}$ in $S^{\leq k}$, and this is, by linear algebra, the same as the stabilizer of

$$L := \wedge^d I^{\leq k} \subseteq \wedge^d S^{\leq k}$$

where $\dim I^{\leq k} = d$ and $\wedge^d S^{\leq k} \subset \bigoplus\limits_{\ell \geq 0} T^{\ell,\ell}V$. $\square$

For a general Hodge structure $(V, \widetilde{\varphi})$, not necessarily of pure weight, using $\mathrm{Gr}_0^W \mathrm{Hg}_{\widetilde{\varphi}}^{\bullet,\bullet}$ to denote the weight zero part $\bigoplus\limits_i \mathrm{Hg}_{\widetilde{\varphi}}^{k_i,k_i}$ of $\mathrm{Hg}_{\widetilde{\varphi}}^{\bullet,\bullet}$, essentially the same argument gives the

Basic Property (II). *$M_{\widetilde{\varphi}}$ is the subgroup of $\mathrm{GL}(V)$ fixing $\mathrm{Gr}_0^W \mathrm{Hg}_{\widetilde{\varphi}}^{\bullet,\bullet}$.*

We next have
If (V, Q, φ) is a polarized Hodge structure, then the following properties of a subspace $W \subset V$ are equivalent:

(i) *W is invariant under M_φ;*
(ii) *W is a sub-Hodge structure.*

Proof of (i) ⇒ (ii). $W_{\mathbb{R}}$ is invariant under $M_\varphi(\mathbb{R})$ which contatins $\varphi(S^1)$; thus the circle acts on $W_{\mathbb{R}}$ to give a sub-Hodge structure, necessarily polarized, of (V, Q, φ).

Proof of (ii) ⇒ (i). As previously noted, since W is a sub-Hodge structure, we have the Q-orthogonal direct sum decomposition
$$V = W \oplus W^{\perp}\,.$$
If $G(W) \subset G$ is the $\mathbb{Q}$-algebraic group of those $g \in G$ with $g(W) \subseteq W$, then we have $\varphi(S^1) \subset G(W)(\mathbb{R})$. By the minimality of M_φ, we then have $M_\varphi \subseteq G(W)$. □

Corollary. *M_φ is a reductive algebraic group.*

The above properties of M_φ are also valid for $M_{\widetilde{\varphi}}$. An additional property is: Let $\rho : \mathrm{GL}(V) \to \mathrm{GL}(V_\rho)$ be a representation. Then setting $\widetilde{\varphi}_\rho = \rho \circ \widetilde{\varphi}$, $(V_\rho, \widetilde{\varphi}_\rho)$ is a Hodge structure and
$$M_{\widetilde{\varphi}(\rho)} = \rho(M_{\widetilde{\varphi}})\,.$$
The proof is an exercise.

For properties of algebraic groups see Borel, loc. cit. A consequence of the corollary is that M_φ is an almost direct product $M_1 \times \cdots \times M_k \times A$ of simple $\mathbb{Q}$-algebraic groups M_i and an algebraic torus A.

Below we will see that A is anisotropic; i.e., $A(\mathbb{R})$ is compact.

Exercise. Any non-trivial reductive subgroup of SL_2 is a torus A. If A is anisotropic, it is conjugate to the standard torus given earlier.

II. CM Hodge structures

For reasons of exposition, in this lecture we shall assume that our Hodge structures (V, φ) are of pure weight n, *simple*, and *polarizable*. Simple means that there are no non-trivial, proper sub-Hodge structures (and thus simple $\Rightarrow$ pure weight). Polarizable means that there exists a polarization — we do not specify what it is.[2] We also set $\widetilde{G} = \mathrm{GL}(V)$.

The internal symmetries of (V, φ) are encoded in its *endomorphism algebra*

$$\mathcal{E}_\varphi = \{X \in \mathrm{End}_\varphi(V, V)\} \ .$$

The notation $\mathrm{End}_\varphi(V, V)$ means that as maps of $V_\mathbb{R}$ to itself, for all $z \in S^1$ we have

$$[X, \varphi(z)] = 0 \, .$$

Identifying $\mathrm{End}(V, V)$ with $V \otimes \check{V}$, using the basic property we have

$$\mathcal{E}_\varphi = \mathrm{Hg}^{1,1} \, ,$$

or equivalently

$$\mathcal{E}_\varphi = [\mathrm{End}(V, V)]^{M_\varphi}$$

are the M_φ-invariants in $\mathrm{End}(V, V)$.

Using the assumption that (V, φ) is simple, it follows that $\mathcal{E}_\varphi$ is a division algebra and $\mathcal{E}_\varphi^* = \mathcal{E}_\varphi \backslash \{0\}$ is a subgroup of $\widetilde{G}(\mathbb{Q})$. In fact,

$$\mathcal{E}_\varphi^* = Z_{\widetilde{G}}(M_\varphi)$$

is the centralizer of M_φ in $\widetilde{G}$.

The existence of a polarization Q implies that $\mathcal{E}_\varphi$ is a special type of division algebra; namely, it has a positive (anti-) involution ι: For $X \in \mathcal{E}_\varphi$ and $u, v \in V$, $\iota(X) = X^\dagger$ is defined by

$$Q(Xu, v) = Q(u, X^\dagger v) \, .$$

[2] By the basic property I, φ will preserve any polarization.

The involution property $(X^\dagger)^\dagger = (-1)^n X$ is immediate; the positivity is a consequence of the 2^{nd} Hodge-Riemann bilinear relation (we do not need its explicit form). Division algebras with this property have been classified into four types. For our purposes the ones of type IV are of interest. Denoting by L the center of $\mathcal{E}_\varphi$, division algebras of this type are described as follows:

(i) L is a *CM-field*. That is, it is a totally imaginary extension of $\mathbb{Q}$ having a totally real subfield L_0 with $[L : L_0] = 2$;
(ii) $\mathcal{E}_\varphi$ is a division algebra of rank d^2 over L, where $d^2 = [\mathcal{E}_\varphi : L]$, and
(iii) $\mathcal{E}_\varphi \otimes_{\mathbb{Q}} \mathbb{R} \cong \underbrace{M_d(\mathbb{C}) \times \cdots \times M_d(\mathbb{C})}_{e}$ where $e = [L_0 : \mathbb{Q}]$ and where
$$\iota(A_1, \ldots, A_e) = ({}^t\bar{A}_1, \ldots, {}^t\bar{A}_e)\,.$$

Definition. (V, φ) is a *CM-Hodge structure* if M_φ is a torus.

Proposition. *If (V, φ) is a CM-Hodge structure, then $\mathcal{E}_\varphi$ is a CM-field.*

Step one: $\mathcal{E}_\varphi^*$ is a $\mathbb{Q}$-algebraic group and contains a maximal torus T defined over $\mathbb{Q}$ (cf. Borel). Since it is maximal and centralizes M_φ, we have $T \supset M_\varphi$.

Any sum of non-zero elements of $T(\mathbb{Q}) \subset \mathcal{E}_\varphi^*$ belongs to $\mathcal{E}_\varphi^* \subset \widetilde{G}$ and commutes with T, hence by maximality belongs to $T(\mathbb{Q})$. Thus there is a field L with $\eta : L \hookrightarrow \mathcal{E}_\varphi$ and $\eta(L^*) = T(\mathbb{Q})$. For any $w \in V$, the subspace $W = \eta(L)w$ of V is stabilized by M_φ; hence is a sub-Hodge structure. Thus $W = V$ and V is a 1-dimensional vector space over L. Also $[L : \mathbb{Q}] = \dim V := r$, and we have
$$\mathcal{E}_\varphi^* = T(\mathbb{Q}) = Z_{\widetilde{G}}(M_\varphi)\,.$$

Step two: A *Hodge frame* is given by a basis $\omega = \{\omega_j \in V^{p_j,q_j}, \omega_{r-j} = \overline{\omega}_j\}$ for $V_{\mathbb{C}}$ such that

$$i^{p_j - q_j} Q(\omega_j, \omega_{r-k}) = \delta_{jk}$$

(think of v_τ and $\bar{v}_\tau$ in the standard example). We may choose ω to diagonalize the action of M_φ and hence of $\varphi(\mathbb{S})$. This basis defines and diagonalizes a maximal torus of $\widetilde{G}$ centralizing M_φ and *a priori* defined over $\mathbb{R}$. But since $Z_{\widetilde{G}}(M_\varphi) = T(\mathbb{Q})$, this torus is necessarily T and consequently ω diagonalizes $\eta(L)$.

Step three: Let $\gamma \in L$ be a primitive element for the extension $L/\mathbb{Q}$ and

$$P(\lambda) = \prod_{j=1}^{r} (\lambda - \eta_j(\gamma))$$

its minimal polynomial over $\mathbb{Q}$, where $\eta_i : L \hookrightarrow \mathbb{C}$ are embeddings. This is also a minimal polynomial for $\eta(\gamma)$, and so up to reordering in the Hodge basis we have

$$\eta(l) = \operatorname{diag}\{\eta_1(l), \ldots, \eta_r(l)\}$$

for all $l \in L$. Since $\eta(L) \subset \widetilde{G}(\mathbb{R})$ the eigenvalues of $\eta(\gamma)$ on ω_j and $\omega_{r-j} = \overline{\omega}_j$ must be conjugate, which implies that $\eta_{r-j} = \overline{\eta}_j$. Except when n is even and r is odd, a case that can be ruled out although we shall not do so here, this shows that L is totally imaginary.

Step four: The Rosati involution gives

$$\sigma := \eta^{-1} \circ \dagger \circ \eta \in \operatorname{Gal}(L/\mathbb{Q}),$$

and the defining properties of $\dagger$ and of the Hodge frame give

$$\eta_j \circ \sigma = \overline{\eta}_j$$

for all j. (This computation is a nice exercise.) The existence of such an involution is the defining property for a CM field. $\square$

Standard example (continued): The above argument simplifies in this case. If M_τ is a torus, then it follows from $\mathcal{E}_\varphi = [\mathrm{End}_\varphi(V,V)]$ that V_τ has an extra endomorphism other than $\pm\,\mathrm{id}_V$. If $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ is the lattice with $E_\tau = \mathbb{C}/\Lambda_\tau$, then there is a complex number ξ such that

$$\xi(\Lambda_\tau) \subseteq m\Lambda_\tau$$

for some $m \in \mathbb{Z}$. Then we have

$$\begin{cases} \xi = a + b\tau \\ \xi\tau = c + d\tau \end{cases}$$

where $a, b, c, d \in \mathbb{Z}$. This gives an equation

$$\tau^2 + \alpha\tau + \beta = 0$$

where $\alpha, \beta \in \mathbb{Q}$.

Conversely, suppose that τ satisfies such an equation. Then multiplication by τ gives an isogeny of E_τ to itself, which we may think of as a Hodge class $[\tau]$ in $H^2(E_\tau \times E_\tau, \mathbb{Q})$. The induced action of $[\tau]$ as an element in $\mathrm{End}_\tau(H^1(E_\tau, \mathbb{Q})) = \mathrm{End}_\tau(V)$ diagonalizes with distinct eigenvalues with respect to some Hodge frame $v_\tau, \bar{v}_\tau$. Since the eigenvalues are distinct, the only matrices that commute with it are diagonal. It follows that M_τ is a torus with

$$\begin{cases} M_\tau(\mathbb{C}) &= \mathrm{diag}(z, 1/z) \\ M_\tau(\mathbb{R}) &= \mathrm{diag}(\exp(ix), \exp(-ix))\,. \end{cases}$$

Writing out the elements of this form which are rational with respect to a rational basis of V we have $z = \exp(ix)$ in $\mathbb{Q}(\tau)$. (This is another nice exercise.) From this it follows that M_τ is isomorphic to the elements of norm one in $\mathbb{Q}(\tau)^*$.

There is a converse to the above:

Suppose that $\mathcal{E}_\varphi$ has an embedded field $\eta : L \hookrightarrow \mathcal{E}_\varphi$ of degree $r = \dim V$. Then M_φ is abelian.

Proof. Since $\eta(L) \subset \mathrm{Hg}^{1,1}_\varphi$, by the basic property M_φ must centralize $\eta(L)$. Furthermore, $\eta(L^*)$ gives the $\mathbb{Q}$-points of a torus of dimension $[L:\mathbb{Q}]$. Since by assumption $[L:\mathbb{Q}] = \dim V$, this torus is maximal in $\widetilde{G}$, and hence it contains M_φ. □

Anticipating the discussion in the next lecture of the period domain $D = \{$set of polarized Hodge structures on (V,Q) with given Hodge numbers$\}$ as a homogeneous complex manifold

$$D = G(\mathbb{R})/H_\varphi$$

where $H_\varphi = \{g \in G(\mathbb{R}) : [g,\varphi] = 0\}$ is the compact isotropy group preserving a reference polarized Hodge structure (V,Q,φ), we have:

If $M_\varphi(\mathbb{R})$ is contained in H_φ, then M_φ is a torus and φ is a CM Hodge structure.

Proof. First, the Mumford-Tate group M_φ commutes with $\mathcal{E}_\varphi$. Since the action of $M_\varphi(\mathbb{R})$ preserves the Hodge structure it follows that $M_\varphi(\mathbb{Q}) \subseteq \mathcal{E}_\varphi$. Thus $M_\varphi(\mathbb{Q})$ is commutative, and by the Zariski density of the $\mathbb{Q}$-points in a connected, linear $\mathbb{Q}$-algebraic group, M_φ is commutative. Hence it is a torus and the previous result applies. □

It may be shown that the CM polarized Hodge structures are dense in the period domain. They play a central role in the theory of Shimura varieties. Their role in the study of higher weight Hodge structures is in its early stages.

III. Mumford-Tate domains

Given (V,Q) and a set of *Hodge numbers* $h^{p,q} = h^{q,p}$, $p+q=n$ and $\sum_{p+q=n} h^{p,q} = \dim V$, the *period domain* D is the set of polarized Hodge structures (V,Q,φ) with $\dim V^{p,q} = h^{p,q}$. Given $\varphi \in D$, we recall that a Hodge frame ω is $\{\omega_j \in V^{p_j,q_j},\ \omega_{r-j} = \overline{\omega}_j \text{ and } i^{p_j-q_j}Q(\omega_j,\overline{\omega}_k) = \delta_{jk}\}$. Picking a reference point φ, the set of Hodge frames is identified with $G(\mathbb{R})$, and the subgroup H_φ of $G(\mathbb{R})$ fixing φ is compact. Hence,

$$D = G(\mathbb{R})/H_\varphi$$

is a homogeneous manifold.

In fact, D is a *homogeneous complex manifold*. One way to see this is to consider the *compact dual* $\check{D}$, consisting of all filtrations $F^\bullet = \{F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}\}$ satisfying the first Hodge-Riemann bilinear relations

$$Q(F^p, F^{n-p+1}) = 0\,.$$

It may be shown that $\check{D}$ is acted on transitively by $G(\mathbb{C})$ with stability group of $F^\bullet \in \check{D}$ a parabolic subgroup P. Thus

$$\check{D} = G(\mathbb{C})/P$$

is a compact, complex manifold. It is in fact a rational, projective variety defined over $\mathbb{Q}$, as may be seen from the $G(\mathbb{C})$-equivariant embeddings

$$\check{D} \subset \prod_{p=n}^{[\eta/2]} \mathrm{Grass}(f^p, V_{\mathbb{C}}) \subset \prod_{p=n}^{[\eta/2]} \mathbb{P}(\wedge^{f_p} V_{\mathbb{C}})$$

where $f^p = h^{n,0} + \cdots + h^{p,n-p}$, and where the second inclusion is the *Plücker embedding*. Then

$$D \subset \check{D}$$

is an open $G(\mathbb{R})$-orbit of a fixed point $\varphi \in D$, and as such has an induced complex structure.

Basic example (continued). In the case $n = 1$, $h^{1,0} = 1$ we have

$$\mathcal{H} \subset \check{\mathcal{H}} = \mathbb{P}^1 .$$

Another way to see the complex structure on D is to observe that $\mathrm{End}(V, V) \cong V \otimes \check{V}$ has a weight-zero polarized Hodge structure and $\mathfrak{g} \subset \mathrm{End}(V, V)$ is a sub-Hodge structure. In fact

$$\mathfrak{g}_{\mathbb{C}} = \underset{i}{\oplus}\, \mathfrak{g}^{-i,i}$$

where

$$\mathfrak{g}^{-i,i} = \left\{ X \in \mathfrak{g}_{\mathbb{C}} : X(V^{p,q}) \subset V^{p-i,q+i} \right\}$$

shifts the Hodge decomposition i-places to the right.[3] We have

$$\begin{cases} \overline{\mathfrak{g}}^{i,-i} = \mathfrak{g}^{-i,i} \\ \mathfrak{g}^{0,0} = \mathfrak{h}_{\varphi,\mathbb{C}} = \text{the complexified Lie algebra of } H_\varphi \\ [\mathfrak{g}^{-i,i}, \mathfrak{g}^{-j,j}] \subseteq \mathfrak{g}^{-(i+j),i+j}. \end{cases}$$

Setting

$$\mathfrak{g}^{-} = \underset{i>0}{\oplus}\, \mathfrak{g}^{-i,i}, \quad \mathfrak{g}^{+} = \overline{\mathfrak{g}}^{-}$$

the complexified tangent space

$$\begin{aligned} T_\varphi D &\cong \mathfrak{g}_{\mathbb{C}} / \mathfrak{h}_{\varphi,\mathbb{C}} \\ &\cong \mathfrak{g}^{-} \oplus \overline{\mathfrak{g}^{-}} . \end{aligned}$$

Thus, D has a $G(\mathbb{R})$-invariant almost complex structure whose $(1, 0)$ part at the identity coset is $\mathfrak{g}^{-}$. By the bracket relation above this almost complex structure is integrable. Note that if $\varphi \in D \subset \check{D}$ is chosen as a reference point, then

$$\mathfrak{p} = \underset{i \leqq 0}{\oplus}\, \mathfrak{g}^{-i,i} = \mathfrak{h}_{\varphi,\mathbb{C}} \oplus \mathfrak{g}^{+} .$$

Note also that

$$\mathfrak{w} := \mathfrak{g}^{-1,1} \subset \mathfrak{g}^{-}$$

[3] We shall drop the subscript "φ" on the $\mathfrak{g}^{-i,i}$ and on the $\mathfrak{m}^{-i,i}$ below.

gives a $\mathbb{G}(\mathbb{C})$-invariant distribution

$$W \subset T\check{D}\,.$$

A holomorphic map

$$\Phi : S \to \check{D}\,,$$

where S is a complex manifold and $\Phi(s) = F_s^\bullet$ is a holomorphically varying filtration, is said to satisfy the *infinitesimal period relation* (IPR) if

$$\Phi_* : TS \to W\,.$$

We may think of this as

$$dF_s^p \subset F_s^{p-q} \otimes \Omega^1_{S,s}\,.$$

A *local variation of Hodge structure* is given by $\Phi : S \to D$ as above that satisfies the IPR. If we set $W^\perp = I \subset T^*D$, then for local holomorphic sections θ of I we have

$$\begin{cases} \Phi^*(\theta) = 0\,, \\ \Phi^*(d\theta) = d\Phi^*(\theta) = 0\,. \end{cases}$$

The second relation gives the *integrability conditions* associated to the differential constraint $dF_s^p \subset F_s^{p-1} \otimes \Omega^1_{S,s}$. They are equivalent to saying that

$$\Phi_*(T_sS) := E \subset \mathfrak{m}^{-1,1}$$

is an abelian subalgebra.

Definition. Let $\varphi \in D$ have a Mumford-Tate group M_φ. Then the associated *Mumford-Tate domain* is the $M_\varphi(\mathbb{R})$-orbit

$$D_{M_\varphi} = M_\varphi(\mathbb{R}) \cdot \varphi \subset D$$

of φ.

Proposition. *D_{M_φ} is a homogeneous complex submanifold of the period domain D.*

Proof. The basic observation is that the Lie algebra of M_φ

$$\mathfrak{m}_\varphi \subset \mathfrak{g}$$

is a sub-Hodge structure of $\mathfrak{g}$. Then, as above

$$\mathfrak{m}_{\varphi,\mathbb{C}} = \bigoplus_i \mathfrak{m}^{-i,i}$$

where $\mathfrak{m}^{-i,i} = \mathfrak{m}_{\varphi,\mathbb{C}} \cap \mathfrak{g}^{-i,i}$, and where $\mathfrak{m}^{0,0} = \mathfrak{m}_{\varphi,\mathbb{C}} \cap \mathfrak{h}_{\varphi,\mathbb{C}}$ is the complexified Lie algebra of the isotropy group $H_{M_\varphi} = H_\varphi \cap M_\varphi(\mathbb{R})$: thus

$$D_{M_\varphi} = M_\varphi(\mathbb{R})/H_{M_\varphi}\,.$$

We have that $\mathfrak{m}_\varphi^- = \mathfrak{m}_\mathbb{C} \cap \mathfrak{g}^-$ gives the $(1,0)$ part of an $M_\varphi(\mathbb{R})$-invariant almost complex structure, which is integrable for the same reason as in the period domain case. □

A basic fact is given by the

Proposition. *$M_\varphi(\mathbb{R})$ contains a compact maximal torus.*

Proof. We have observed above that $M_\varphi(\mathbb{R})$ is reductive, and thus it is an almost direct product $M_1 \times \cdots \times M_l \times A$ of simple, real Lie groups M_i and an abelian part A.[4] We shall assume that $M_\varphi(\mathbb{R})$ is simple, and comment that the argument may be extended to the general case. For notational simplicity, we shall drop the subscript φ — thus $M_\varphi = M$, etc.

We first note that

$$H_M = Z_M(\varphi(S^1))$$

is the centralizer in $M(\mathbb{R})$ of the circle $\varphi(S^1)$. In fact, writing $z \in S^1$ as $z = e^{2\pi i\xi}$, we have that $\varphi(z) = e^{i(2p-n)\xi}$ on $V^{p,n-p}$. It follows that for $g \in M(\mathbb{R})$

$$g\varphi(z) = \varphi(z)g \Leftrightarrow g(V_\varphi^{p,q}) \subseteq V_\varphi^{p,q}\,.$$

Let B be a maximal connected abelian subgroup of $M(\mathbb{R})$ containing $\varphi(S^1)$. Then

$$B \subseteq Z_M(\varphi(S^1)) = H_M\,,$$

and since H_M is compact, so is B. Because B is maximal abelian, rank $M(\mathbb{R}) = \dim B$. □

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[4] M_φ may be simple as a $\mathbb{Q}$-algebraic group but $M_\varphi(\mathbb{R})$ may only be semi-simple as a real Lie group.

Corollary. SL_n *is not a Mumford-Tate group for* $n \geqq 3$.

The other extreme to $M(\mathbb{R})$ being simple is when $M = T$ is an algebraic torus. The non-trivial characters of T acting on $V_{\mathbb{C}}$ occur in conjugate pairs, so that T is anisotropic. In this case the component of the Mumford-Tate domain containing φ is just the point φ, which is a CM-polarized Hodge structure.

Basic example (continued): $\varphi = \sqrt{-d} \in \mathcal{H}$. Then

$$M_\varphi = \left\{ \begin{pmatrix} \alpha & \sqrt{d}\beta \\ -\sqrt{d}\beta & \alpha \end{pmatrix} : \alpha, \beta \in \mathbb{Q} \text{ and } \alpha^2 + d\beta^2 = 1 \right\},$$

and

$$D_{M_\varphi} = \sqrt{-d}$$

is a point in $\mathcal{H}$.

Now we come to Noether-Lefschetz loci.

Definition. Let $\varphi \in D$. Then the *Noether-Lefschetz locus*

$$\mathrm{NL}_\varphi = \left\{ \varphi' \in D : \mathrm{Hg}_{\varphi'}^{\bullet,\bullet} \supseteq \mathrm{Hg}_{\varphi}^{\bullet,\bullet} \right\}.$$

In other words, the Noether-Lefschetz locus is the set of polarized Hodge structures whose algebra of Hodge tensors contains the given algebra $\mathrm{Hg}_\varphi^{\bullet,\bullet}$. Classically, the Noether-Lefschetz loci were defined in the parameter space of a family of smooth projective varieties to be the subvariety where a Hodge class remains a Hodge class. For geometric reasons, the above would seem to be a more natural concept. Denoting by Z° the component containing φ of a subvariety Z of D with $\varphi \in Z$, we have the

Proposition. ${D_{M_\varphi}}^\circ = {\mathrm{NL}_\varphi}^\circ$.

Proof. We clearly have ${D_{M_\varphi}}^\circ \subseteq {\mathrm{NL}_\varphi}^\circ$. Since ${D_{M_\varphi}}^\circ$ is smooth, it will suffice to show equality of the Zariski tangent spaces at φ. Identifying $T_\varphi D$ with $\mathfrak{g}^-$ and $T_\varphi D_{M_\varphi}$ with $\mathfrak{m}^-$, we have

$$T_\varphi \mathrm{NL}_\varphi = \left\{ X \in \mathfrak{g}^- : X(\zeta) = 0 \text{ for all } \zeta \in \mathrm{Hg}_\varphi^{\bullet,\bullet} \right\}.$$

Here, for $\zeta \in \mathrm{Hg}_\varphi^{k,l}$ of Hodge type (p,p) where $n(k-l) = 2p$ and for $X \in \mathfrak{g}^{-i,i}$, $X(\zeta)$ is of Hodge type $(p-i, p+i)$. Thus the above statement in brackets expresses the condition that, to first order in the direction X, ζ remains a Hodge class.

On the other hand, by the basic property of Mumford-Tate groups, the $\mathfrak{m}^-$-part of the complexification of the tangent space $T_e M_\varphi \subset T_e G$ is

$$\{X \in \mathfrak{g}^- : X(\zeta) = 0 \text{ for all } \zeta \in \mathrm{Hg}_\varphi^{\bullet,\bullet}\},$$

which by the above gives

$$T_\varphi \mathrm{NL}_\varphi = T_\varphi D_{M_\varphi}. \qquad \square$$

For applications to algebraic geometry we consider a local variation of Hodge structure

$$\Phi : S \to D,$$

and for a point $s_0 \in S$ we set

$$\mathrm{NL}_{s_0}(S) = \Phi^{-1}(\mathrm{NL}_{\Phi(s_0)}).$$

A classical question is to estimate the codimension of $\mathrm{NL}_{s_0}(S)$ in S. More precisely, if $n = 2p$ and $\zeta \in V^{p,p}_{\varphi(s_0)} \cap V$ is a Hodge class, classically one has been interested in the locus $\mathrm{NL}_{s_0}(\zeta) \subset S$ where ζ remains a Hodge class. An obvious estimate is

$$\operatorname{codim}_S \mathrm{NL}_{s_0}(\zeta) \leqq h^{2p,0} + \cdots + h^{p+1,p-1}$$

since the RHS is the number of conditions to be a Hodge class. This estimate can be significantly sharpened and applied to the algebra of Hodge tensors as follows: Set

$$E = \Phi_*(T_{s_0} S) \subseteq \mathfrak{g}^{-1,1}.$$

There are maps

$$\mathrm{Sym}^{p-2} E \otimes (E \cap \mathfrak{m}^{-1,1}) : V^{2p,0}_{\Phi(s_0)} \to V^{p+1,p-1}_{\Phi(s_0)}. \text{ [5]}$$

Denote by σ_m the rank of this map.

"Proposition": *We have*

$$\operatorname{codim}_S \mathrm{NL}_{s_0}(S) \leqq \dim\left(\mathfrak{g}^{-1,1}/\mathfrak{m}^{-1,1}\right) - \sum_m \sigma_m \,.$$

The quotation marks mean that there are some mild technical assumptions needed that do not effect the essential geometric content of the result. The term $\mathfrak{g}^{-1,1}/\mathfrak{m}^{-1,1}$ represents the part of the normal space to D_{M_φ} in D that satisfies the IPR. The terms σ_m reflect the integrability conditions in the IPR. It may be shown by examples that the above estimate is sharp. In practice it says that for a given Hodge class ζ and when $p \geqq 2$, the codimension of $\mathrm{NL}_{s_0}(\zeta)$ is usually *much less* than the naive estimate $h^{2p,0} + \cdots + h^{p+1,p-1}$. If the Hodge conjecture is true it means that there are more algebraic cycles than predicted by naive dimension counts.

[5] Here $p \geqq 2$; the classical case $p = 1$ is less interesting.

IV. Variation of Hodge structure and Mumford-Tate domains

A *global variation of Hodge structure*

$$\Phi : S \to \Gamma\backslash D$$

is given by the following data:

- S is a smooth, quasi-projective variety;
- $\Gamma \subset G(\mathbb{R})$ is a discrete group;[6]
- Φ is a locally liftable holomorphic mapping whose local lifts satisfy the IPR.

Since Γ is discrete, the intersection of Γ with the compact isotropy group H_φ is a finite group. Being locally liftable means that around each point $s_0 \in S$ there is a neighborhood U and a diagram

$$\begin{array}{ccc} & & D \\ & \overset{\widetilde{\Phi}}{\nearrow} & \downarrow \\ U & \xrightarrow{\Phi} & \Gamma\backslash D. \end{array}$$

Because of this there is an induced mapping

$$\Phi_* : \pi_1(S, s_0) \to \Gamma$$

and the image is called the *monodromy group*. It is *the* fundamental invariant of a global variation of Hodge structure.

Using the local liftings we may obtain a diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\Phi}} & D \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ S & \xrightarrow{\Phi} & \Gamma\backslash D \end{array}$$

where $\widetilde{S} \to S$ is the universal covering. It is sometimes convenient to think of the properties of Φ as given by the

[6] In practice, Γ will usually be a subgroup of an arithmetic group, an arithmetic group being one that is commensurable with $G_{\mathbb{Z}}$.

Γ-invariant properties of $\widetilde{\Phi}$. For example, the Mumford-Tate group M_Φ of Φ may be defined as follows: Outside of a countable union Z of proper subvarieties of $\widetilde{S}$, the Mumford-Tate groups $M_{\widetilde{\Phi}(\tilde{s})} \subset G$ are constant and serve to define M_Φ. More precisely, choosing $s_0 \in S$ to be the image of $\tilde{s}_0 \in \widetilde{S}\backslash Z$, we may think of $V_\mathbb{C}$ as the fibre over $s_0 \in S$ of the flat vector bundle $\mathbf{V}_\mathbb{C} = \widetilde{S} \times_\Gamma V_\mathbb{C} \to S$, and then $\mathrm{Hg}^{\bullet,\bullet}_{s_0}$ gives a subspace of the tensor algebra of $V \subset V_\mathbb{C} = \mathbf{V}_{\mathbb{C},s_0}$ that, as a subspace, is invariant under parallel translation around all paths $\gamma \in \pi_1(S, s_0)$. Then $M_\Phi = \mathrm{Fix}\,\mathrm{Hg}^{\bullet,\bullet}_{s_0}$.

A basic property of the monodromy group is the following

Theorem (Schmid)**.** *If $v \in V^\Gamma_\mathbb{C}$ is a monodromy invariant vector, then the Hodge (p,q) components $v^{p,q}_s$ are constant and Γ-invariant.*

For applications of this theorem, we assume that there is a lattice $V_\mathbb{Z} \subset V$ with $\Gamma \subset G_\mathbb{Z}$. Then the intersection of Γ with any compact subgroup of $\mathrm{GL}(V_\mathbb{R})$ is a finite group. We shall also work up to isogeny, i.e. up to finite coverings $S' \to S$ with the induced variation of Hodge structure $\Phi' : S' \to \Gamma'\backslash D$ where Γ' is a subgroup of finite index in Γ. In particular, referring to the above, we observe that Γ acts on the space $\mathrm{Hg}^{\bullet,\bullet}_{s_0}$ by a finite group. This is because the polarizing form is definite on the spaces $\mathrm{Hg}^{k,l}_{s_0}$. Passing to a a finite covering and relabeling we may assume that $\mathrm{Hg}^{\bullet,\bullet}_{s_0}$ is *pointwise* fixed by Γ. This implies that

$$(*) \qquad \Gamma \subset M_\Phi\,.$$

A consequence of Schmid's theorem is the

Theorem of the fixed part: *If $U = V^\Gamma$ is the subspace of Γ-invariant vectors in V, then $Q\big|_U$ is non-singular, $V = U \oplus U^\perp$ and the variation of Hodge structure splits into a direct sum whose U-part is trivial.*

Proof. If $u \in U_{\mathbb{C}}$, then by Schmid's theorem the Hodge (p,q) components $u(\widetilde{\Phi}(\tilde{s}))^{(p,q)}$ are constant and Γ-invariant. Thus the U-part of $\widetilde{\Phi}(\tilde{s})$ is a constant sub-Hodge structure of $\widetilde{\Phi}(\tilde{s})$. □

Another consequence is the

Rigidity theorem: *If we have two global variations of Hodge structure*

$$\begin{cases} \Phi : S \to \Gamma\backslash D \\ \Phi' : S \to \Gamma\backslash D \end{cases}$$

where $\Phi(s_0) = \Phi'(s_0)$ *and the induced monodromy representations coincide, then* $\Phi = \Phi'$.

This is because the identity map $\mathrm{id}_V \in \mathrm{Hom}(V,V) = V \otimes \check{V}$ is Γ-invariant and of Hodge type $(0,0)$ at $\tilde{s}_0$, hence it is everywhere of type $(0,0)$.

For a third application, we recall that $\overline{\Gamma}^{\mathbb{Q}}$ denotes the $\mathbb{Q}$-Zariski closure of $\Gamma \subset G$. It is the smallest $\mathbb{Q}$-algebraic subgroup of G containing Γ. By $(*)$ above we have

$$\overline{\Gamma}^{\mathbb{Q}} \subseteq M_{\Phi}\,.$$

Semi-simplicity of monodromy theorem: $\overline{\Gamma}^{\mathbb{Q}}$ *is a semi-simple* $\mathbb{Q}$*-algebraic subgroup of* $\mathrm{Aut}(V)$.

In fact, the argument given in the proof of the structure theorem below will show that $\overline{\Gamma}^{\mathbb{Q}}$ is a normal subgroup of the derived group DM_{Φ}.

When the global variation of Hodge structure *arises from geometry*, meaning that we have a family

$$\pi : \mathcal{X} \to S$$

of smooth projective varieties $X_s = \pi^{-1}(s)$ and

$$\Phi(s) = H^n(X_s, \mathbb{Q})_{\mathrm{prim}}\,,$$

the above results are due to Deligne as a consequence of his mixed Hodge theory.

Let now

$$M_\Phi = M_1 \times \cdots \times M_l \times A$$

be the almost product decomposition of the reductive $\mathbb{Q}$-algebraic group M_Φ into simple factors M_i and an abelian part A. Denote by $D_i \subset D$ the $M_i(\mathbb{R})$-orbit of $\widetilde{\Phi}(\tilde{s}_0)$. Then we have the

Structure theorem: (i) *The D_i are homogeneous complex submanifolds of D.* (ii) *After passing to a finite covering of S, the monodromy group splits as a direct product*

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_k, \qquad k \leqq l$$

where $\overline{\Gamma}_i^{\mathbb{Q}} = M_i$.

The proof will give the following:

(i) Setting $P = M_1 \times \cdots \times M_k$ and $R = M_{k+1} \times \cdots \times M_l \times A$, the Mumford-Tate group

$$M_\Phi = P \times R\,,$$

and we denote by D_P and D_R the corresponding subdomains of D;

(ii) the global variation of Hodge structure is given by

$$\Phi : S \to \Gamma\backslash D_P \times D_R$$

and is constant in the D_R-factor;

(iii) the monodromy group Γ and the $\mathbb{Q}$-algebraic group P have the same tensor invariants.

In particular, although it seems not to be known whether or not Γ is a subgroup of finite index in the arithmetic group $P_{\mathbb{Z}}$, they at least have the same tensor invariants.

The main step in the proof of the structure theorem is this:

$\overline{\Gamma}^{\mathbb{Q}}$ *is a normal subgroup of the derived group DM_Φ.*

Proof. For any tensor representation $\rho : V \to V_\rho$ we have a corresponding variation of Hodge structure Φ_ρ and a monodromy group $\Gamma_\rho \subset \mathrm{Aut}(V_\rho)$. By the theorem of the fixed part,

$$U_\rho := \ker(\overline{\Gamma}_\rho^{\mathbb{Q}} - I) \subset V_\rho$$

gives a constant sub-variation of Hodge structure. Recalling that the Mumford-Tate group

$$M_{\Phi_\rho} = \rho(M_\Phi)\,,$$

we infer that $\rho(M_\Phi)$ stabilizes $U_\rho \subset V_\rho$. Given $g \in M_\Phi(\mathbb{Q})$ and $\gamma \in \overline{\Gamma}_\rho^{\mathbb{Q}}$,

$$(**) \qquad \rho(g)\gamma\rho(g)^{-1} \text{ fixes } U_\rho \text{ pointwise.}$$

We now let $\Gamma' \subset \mathrm{Aut}(V)$ be the largest subgroup such that Γ' fixes all tensors of weight zero that are fixed by Γ. Then $\overline{\Gamma}^{\mathbb{Q}} \subseteq \Gamma'$, and by the argument used in the proof of the basic property (Chevalley's theorem where ρ are the respresentations on $T^{k,k}$), it follows that $\overline{\Gamma}^{\mathbb{Q}} = \Gamma'$ and hence by $(**)$, $\overline{\Gamma}^{\mathbb{Q}}$ is a normal subgroup of M_Φ. Since the orbit of the abelian part $A(\mathbb{R})$ of $M(\mathbb{R})$ is a point, $A(\mathbb{R})$ is compact and $\Gamma \cap A$ is a finite group. After passing to a finite index subgroup, we may then assume that $\Gamma \cap A = \{e\}$. It then follows that $\overline{\Gamma}^{\mathbb{Q}}$ is a semi-simple subgroup of DM_Φ. □

We may now infer that Γ splits as in the statement of the structure theorem. Moreover, by the argument just given we have that

$$\overline{\Gamma}_i^{\mathbb{Q}} = M_i\,. \qquad \square$$

Because of the structure theorem we see that Mumford-Tate domains are the natural target spaces for period maps. In the next lectures we shall see that different Mumford-Tate domains may be manifestations of a more basic object, namely a Hodge domain.

V. Hodge representations

In this lecture we will discuss the questions: Given a semi simple $\mathbb{Q}$-algebraic group M

(a) When is M the Mumford-Tate group of a polarized Hodge structure?
(b) In how many ways can such an M be realized as a Mumford-Tate group?

Essentially complete answers are known to these question and we shall describe and illustrate them, largely without proofs. This will require some facts from the structure theory of Lie groups and representation theory, which we will briefly review as needed.

The basic concept is the following:

Definition. A *Hodge representation* (M, ρ, φ) is given by a representation defined over $\mathbb{Q}$

$$\rho : M \to \mathrm{Aut}(V, Q)$$

together with a circle

$$\varphi : S^1 \to M(\mathbb{R})$$

such that, setting $\varphi_\rho = \rho \circ \varphi$, (V, Q, φ_ρ) is a polarized Hodge structure.

Without essential loss of generality we can and do assume that $M(\mathbb{R})$ is simple and that $\rho : M(\mathbb{R}) \to \mathrm{Aut}(V_\mathbb{R}, Q)$ is irreducible. These ρ are then classified using the root and weight structures on the Lie algebra $\mathfrak{m}_\mathbb{R}$. There are two issues:

(i) For which $\rho : M(\mathbb{R}) \to \mathrm{GL}(V_\mathbb{R})$ is there an invariant bilinear form Q?
(ii) Given $\rho : M(\mathbb{R}) \to \mathrm{Aut}(V_\mathbb{R}, Q)$, for which circles $\varphi_\rho = \rho \circ \varphi : S^1 \to M(\mathbb{R})$ is (V, Q, φ_ρ) a polarized Hodge structure?

We have seen in lecture two that if a Hodge representation exists, $M(\mathbb{R})$ contains a compact maximal torus T with

$\varphi(S^1) \subset T$. The action of T on $V_{\mathbb{C}}$ decomposes

$$V_{\mathbb{C}} = \oplus V_\omega$$

where the weight spaces V_ω on which $T = \mathfrak{t}/\Lambda$ acts by a character $\exp i \langle \omega, \bullet \rangle$ where $\omega \in \mathrm{Hom}(\Lambda, \mathbb{Z})$ occur in conjugate pairs: $V_{-\omega} = \overline{V}_\omega$. Restricting the action to $\varphi(S^1) \subset T$ gives a Hodge structure

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}_{\varphi_\rho}$$

where $V^{p,q}_{\varphi_\rho}$ is a sum of weight spaces.[7] Because the form Q is preserved, the first Hodge-Riemann bilinear relation

$$Q(V^{p,q}_{\varphi_\rho}, V^{p',q'}_{\varphi_\rho}) = 0, \qquad p' \neq n - p$$

will be satisfied. The difficult one is the second bilinear relation

$$i^{p-q} Q(V^{p,q}_{\varphi_\rho}, \overline{V}^{p,q}_{\varphi_\rho}) > 0\,.$$

The answer will be expressed in terms of congruences "mod 2" reflecting on the parity of $Q \bmod 2$, and "mod 4" reflecting the fact that the second Hodge-Riemann bilinear relation depends on $p - q \bmod 4$.

The easiest case to analyze is the adjoint representation. Here we recall that if (M, ρ, φ) is a Hodge representation, then so is $(M, \mathrm{Ad}, \varphi)$. Specifically, we have

$$\mathfrak{m} \subset \mathrm{End}(V) \cong V \otimes \check{V}$$

and the bilinear form Q on V induces the Cartan-Killing form $B : \mathfrak{m} \otimes \mathfrak{m} \to \mathbb{Q}$. Then $(\mathfrak{m}, B, \mathrm{Ad}\,\varphi)$ gives a polarized Hodge structure of even weight.

The conditions on φ that $(\mathfrak{m}, B, \mathrm{Ad}\,\varphi)$ give a polarized Hodge structure will now be explained. Under the action of the Cartan sub-algebra $\mathfrak{t}_{\mathbb{C}}$ on $\mathfrak{m}_{\mathbb{C}}$, one has the root space

[7] There is a subtlety here in that since $\varphi(S^1)$ does not have a scaling action, the weight n of the Hodge structure is not uniquely defined. We shall let n be the integer such that $V^{n,0}_{\varphi_\rho} \neq 0$.

decomposition

$$\mathfrak{m}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \left(\underset{\alpha \in \mathfrak{r}}{\oplus} \mathfrak{m}_\alpha \right) .$$

Here, the set of *roots*

$$\mathfrak{r} \subset i\check{\mathfrak{t}}$$

are purely imaginary linear functions on the Lie algebra $\mathfrak{t}$ of T. The root spaces $\mathfrak{m}_\alpha$ are 1-dimensional and occur in conjugate pairs. For a maximal compact subgroup $K \subset M(\mathbb{R})$ we have $T \subset K$ and a Cartan decomposition

$$\mathfrak{m}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$$

where

$$\begin{cases} [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \, . \end{cases}$$

We denote by $\alpha_1, \dots, \alpha_d$ the *compact roots* which belong to $\mathfrak{k}$, meaning that the root space

$$\mathfrak{m}_{\alpha_j} \subset \mathfrak{k}_{\mathbb{C}} \, ,$$

and by $\beta_1, \dots, \beta_e$ the *non-compact roots* belonging to $\mathfrak{p}$.

Writing $T = \mathfrak{t}/\Lambda$, the circle $\varphi : S^1 \to T$ is given by the exponential of the line through a lattice point $\mathfrak{l}_\varphi \in \Lambda$. We then have:

> $(\mathfrak{m}, B, \mathrm{Ad}\,\varphi)$ *is a polarized Hodge structure if, and only if,*
>
> $$\begin{cases} \langle \alpha_j, \mathfrak{l}_\varphi \rangle \equiv 0 \pmod 4 \\ \langle \beta_k, \mathfrak{l}_\varphi \rangle \equiv 2 \pmod 4 \, . \end{cases}$$

Notational caveat. The roots are purely imaginary linear functions on $\Lambda \subset \mathfrak{t}$; we have omitted the "$i$" in the above congruences.

The proof of the above is based on the relation

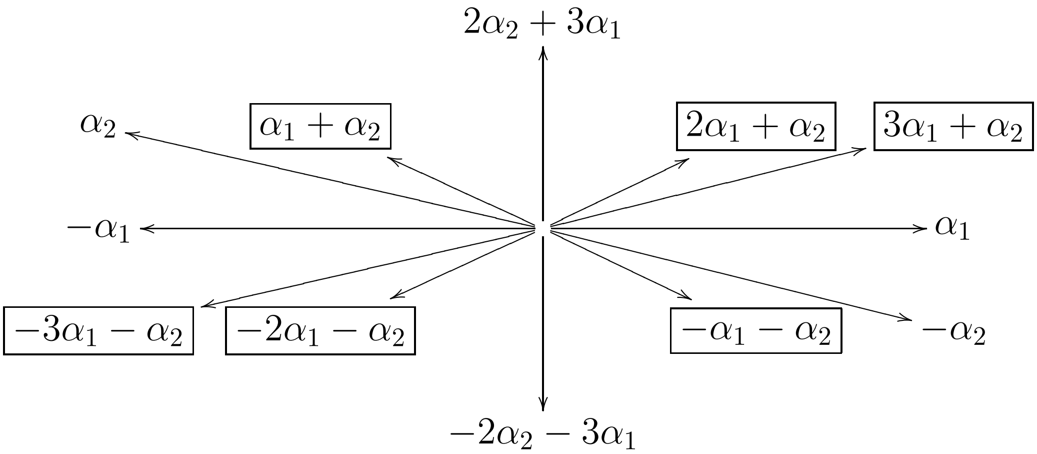
$$\mathfrak{m}^{-k,k} = \bigoplus_{\begin{cases} \alpha \in \mathfrak{r} \\ \langle \alpha, \mathfrak{l}_\varphi \rangle = -2ki \end{cases}} \mathfrak{m}_\alpha \, ,$$

and the fact that

$$\begin{cases} B < 0 & \text{on } (\mathfrak{m}_{\alpha_j} \oplus \mathfrak{m}_{-\alpha_j})_{\mathbb{R}} \\ B > 0 & \text{on } (\mathfrak{m}_{\beta_k} \oplus \mathfrak{m}_{-\beta_k})_{\mathbb{R}} \,. \end{cases}$$

In practice the possible polarized Hodge structures can be read off from the above and standard root tables. If there is one such, then it turns out there are many.

Example. For G_2 with the standard root diagram



where the non-compact roots are framed, the minimal weight of a polarized Hodge structure is six with

$$\begin{aligned} \mathfrak{g}_2^{0,0} &= \operatorname{span}\{e_{\alpha_2}\} \\ \mathfrak{g}_2^{1,5} &= \operatorname{span}\{e_{-3\alpha_1-\alpha_2}, e_{-\alpha_1}\} \\ \mathfrak{g}_2^{2,4} &= \operatorname{span}\{e_{\alpha_1+\alpha_2}, e_{-2\alpha_1-\alpha_2}\} \\ \mathfrak{g}_2^{3,3} &= \operatorname{span}\{\mathfrak{t}_{\mathbb{C}}, e_{2\alpha_2+3\alpha_1}, e_{-2\alpha_2-3\alpha_1}\} \,. \end{aligned}$$

Note. In the classical case of weight one polarized Hodge structures, no exceptional group may occur as a Mumford-Tate group.

Among the main results that come out of the analysis are

(A) *The adjoint group of a simple $\mathbb{Q}$-algebraic group M is a Mumford-Tate group if, and only if, M contains an anisotropic maximal torus.*

To explain the next result, we define a map

$$\psi : \mathfrak{r} \to \mathbb{Z}/2\mathbb{Z}$$

by

$$\psi(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a compact root} \\ 1 & \text{if } \alpha \text{ is a non-compact root.} \end{cases}$$

Since the Cartan involution $\theta : \mathfrak{m}_{\mathbb{R}} \to \mathfrak{m}_{\mathbb{R}}$ given by $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$ is a Lie algebra homomorphism, this map extends linearly to a homomorphism

$$\psi : R \to \mathbb{Z}/2\mathbb{Z}$$

where $R \subset i\check{\mathfrak{t}}$ is the $\mathbb{Z}$-span of the roots, and we define $\Psi : R \to \mathbb{Z}/4\mathbb{Z}$ by

$$\Psi(\alpha) = \begin{cases} 0 & \text{if } \psi(\alpha) = 0 \\ 2 & \text{if } \psi(\alpha) = 1\,. \end{cases}$$

Thinking of $\mathfrak{l}_\varphi \in \Lambda$ as defining an element of $\operatorname{Hom}(R, \mathbb{Z})$, the above condition that $(\mathfrak{m}, B, \operatorname{Ad}\varphi)$ give a polarized Hodge structure may be expressed as

$$\mathfrak{l}_\alpha \equiv \Psi \pmod 4\,;$$

i.e., $\mathfrak{l}_\alpha$ should give a lifting of Ψ in the diagram

$$\begin{array}{ccc} & & \mathbb{Z} \\ & \overset{\mathfrak{l}_\varphi}{\nearrow} & \downarrow \\ R & \xrightarrow{\Psi} & \mathbb{Z}/4\mathbb{Z}\,. \end{array}$$

The list of the non-compact real forms of the simple Lie algebras for which this condition can be satisfied is the following:

A_r su(p,q), $p+q=r+1$, $0 \leq p,q \leq r+1$; sl$_2$

B_r so$(2p,2q+1)$, $p+q=r$, $0 \leq p,q \leq r$

C_r sp(p,q), $p+q=r$, $0 \leq p,q \leq r$; sp(r)

D_r so$(2p,2q)$, $p+q=r$, $0 \leq p,q \leq r$; so$^*(2r)$

E_6 EII, EIII

E_7 EV, EVI, EVII

E_8 EVIII, EIX

F_4 FI, FII

G_2 G

Those that do not have solutions to the above conditions are

$$\text{sl}(n,\mathbb{R}) \text{ for } n \geq 3$$

$$\text{sl}(n,\mathbb{H}) \text{ for } n \geq 2$$

$$\text{EI}$$

$$\text{EIV}$$

As noted, it is more difficult to have a Hodge representation of odd weight. The following is the list of those that do.

$$\begin{aligned}
&\mathrm{su}(4k), \mathrm{so}(4k+2) \quad \text{(compact case)}\\
&\mathrm{su}(2p, 4k-2p), \mathrm{su}(2p+1, 2q+1)\\
&\mathrm{so}(4p+2, 2q+1), \mathrm{so}(2p, 2q) \ \text{ for } \ p+q \text{ odd}\\
&\mathrm{so}^*(4k)\\
&\mathrm{sp}(r)\\
&\mathrm{EV}, \mathrm{EVII}
\end{aligned}$$

Finally, without getting into the details we will let $P \subset i\check{\mathfrak{t}}$ denote the weight lattice and let $\lambda \in P$ be a highest weight. Define

$$P' = \mathbb{Z}\lambda + R\,,$$

so that we have

$$P \supseteq P' \supseteq R\,.$$

We let $\delta \in \mathbb{Z}^+$ be the minimal positive integer such that $\delta\lambda \in R$. Then for M with $M(\mathbb{R})$ having the maximal torus

$$T = \mathfrak{t}/\Lambda$$

where $\Lambda = \mathrm{Hom}(P', \mathbb{Z})$, we have

(B) *There exists* $\mathfrak{l}_\varphi \in \Lambda$ *and an invariant form* Q *such that the representation* ρ_λ *with highest weight* λ *gives a Hodge representation* $(M, \rho_\lambda, \varphi_{\rho_\lambda})$ *if, and only if,*

$$\Psi(\delta\lambda) \equiv \delta m \pmod 4$$

for some integer m.

This result provides a practical test for determining the Hodge representations.

Example (G_2-continued)**.** We consider the standard representation of G_2 in SO(3, 4). We use block matrices

$$\begin{matrix} & \overbrace{\quad}^{3} & \overbrace{\quad}^{4} \\ {}_3\{ & A & B \\ {}_4\{ & {}^tB & C \end{matrix}$$

where $A = -{}^tA$, $C = -{}^tC$ is an element of so(3, 4). Then $\mathfrak{g}_{2,\mathbb{R}} \subset \mathrm{so}(3,4)$ is defined by seven independent linear equations (whose exact form is not necessary for our purposes). Setting $E = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and

$$H_1 = \left(\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad H_2 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{array}\right),$$

H_1 and H_2 give a basis for $\mathfrak{t}_{\mathbb{C}}$; the exponentials of $2\pi i$ times $\mathbb{R}H_1 + \mathbb{R}H_2$ give a maximal torus $T \subset M(\mathbb{R})$, and $T \cong \mathbb{R}^2/\Lambda$ where $\Lambda = \mathbb{Z}H_1 + \mathbb{Z}H_2$. We have

> *The co-character φ whose differential is $\mathfrak{l}_\varphi = l_1H_1 + l_2H_2$ gives a Hodge representation for* **every** *representation of G_2 if, and only if, the conditions*
>
> $$l_1 \equiv 0 \pmod 4$$
> $$l_2 \equiv 2 \pmod 4$$
>
> *are satisfied.*

For the standard representation on $\mathbb{Q}^7$ we obtain the following illustrative cases:

(a) $l_1 = 0$, $l_2 = 2$. Then the weight $n = 2$ and the Hodge numbers $h^{2,0} = 2$, $h^{1,1} = 3$.

(b) $l_1 = 4$, $l_2 = -2$. Then the weight $n = 4$ and $h^{4,0} = 1$, $h^{3,1} = 2$, $h^{2,2} = 1$.

(c) $l_1 = 4$, $l_2 = 6$. Then $n = 6$ and all $h^{p,q} = 1$.

The Mumford-Tate groups in cases (a) and (c) appeared (not as Mumford-Tate groups, of course) in the classic 1905 paper of Elie Cartan in which G_2 was first realized geometrically as a transformation group.

VI. Hodge domains

Given the pair (M,φ) consisting of a semi-simple $\mathbb{Q}$-algebraic group M containing an anisotropic maximal torus and a non-trivial circle $\rho : S^1 \to M(\mathbb{R})$, there may be many different ρ's giving Hodge representations (M,ρ,φ). For each of these there is a Mumford-Tate domain $D_{(M,\rho,\varphi)}$ contained in the corresponding period domain. A natural question is:

> *What is the relation among the $D_{(M,\rho,\varphi)}$'s for a given (M,φ)?*

With the notation $D_{\mathfrak{m},\varphi}$ for the Mumford-Tate domain for $(\mathfrak{m}, B, \operatorname{Ad}\varphi)$ we have the result:

Theorem. *As homogeneous complex manifolds together with the invariant distribution in the tangent bundle given by the IPR,*

$$D_{(M,\rho,\varphi)} \cong D_{\mathfrak{m},\varphi}\,.$$

Proof. Denoting by $H_\varphi = Z_{M(\mathbb{R})}(\varphi(S^1))$ the compact centralizer in $M(\mathbb{R})$ of the circle $\varphi(S^1)$, we have

$$D_{(M,\rho,\varphi)} \cong M(\mathbb{R})/H_\varphi$$

independently of ρ. Remark that the finite center Z_M of $M(\mathbb{R})$ is contained in the compact maximal torus $T \subset H_\varphi$, so that the RHS above may be replaced by the images of $M(\mathbb{R})$ and H_φ in the adjoint group $M(\mathbb{R})_a = M(\mathbb{R})/Z_M$.

If we identify the tangent space to $D_{\mathfrak{m},\varphi}$ at the identity coset with $\mathfrak{m}^-$, then the subbundle W of the tangent bundle has fibre $\mathfrak{w} = \mathfrak{m}^{-1,1}$; thus

$$\mathfrak{w} = \left\{ \begin{array}{c} (-2i)\text{-eigenspace of } \varphi(S^1) \\ \text{acting on } \mathfrak{m}^-\,. \end{array} \right\} \qquad \square$$

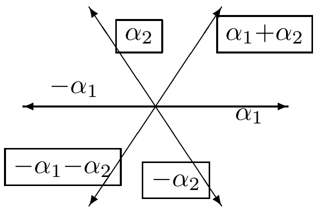
Definition. A *Hodge domain* is given by $D_{\mathfrak{m},\varphi} = M(\mathbb{R})/H_\varphi$ where $H_\varphi = Z_{M(\mathbb{R})}(\varphi(S^1))$ and (M,φ) is as above.

Thus, Hodge domains are the *universal parameter spaces* for families of polarized Hodge structures whose generic Mumford-Tate group is M and whose stability group of a fixed polarized Hodge structure is H_φ.

Example. This first non-classical case is when

$$D_{\mathfrak{m},\varphi} = \mathrm{SU}(2,1)/T\,.$$

The root diagram is (the non-compact roots are framed)



The co-characters that give Hodge domains are those $\mathfrak{l}_\varphi \in \Lambda \subset \mathfrak{t}$ such that

$$\begin{cases} \langle \alpha_1, \mathfrak{l}_\varphi \rangle = 4k_1 & \qquad k_1 \neq 0 \\ \langle \alpha_2, \mathfrak{l}_\varphi \rangle = 4k_2 + 2 & \end{cases}$$

where k_1, k_2 are integers. The root plane is divided into six Weyl chambers corresponding to the inequalities

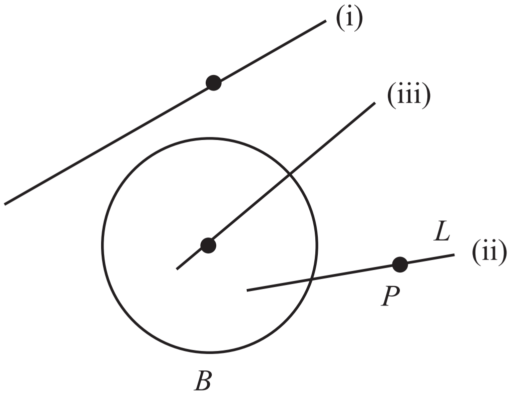
	Inequalities	Basis for $\mathfrak{m}^-$
(i)	$k_1 > 0, k_2 \geq 0$	$e_{\alpha_1}, e_{-\alpha_2}, e_{-\alpha_1-\alpha_2}$
(ii)	$k_1 + k_2 \geq 0, k_2 < 0$	$e_{-\alpha_1}, e_{\alpha_2}, e_{-\alpha_1-\alpha_2}$
(iii)	$k_1 + k_2 \geq 0, k_1 < 0$	$e_{\alpha_1}, e_{-\alpha_2}, e_{\alpha_1-\alpha_2}$
(i)*	$k_1 < 0, k_2 < 0$	$e_{\alpha_1}, e_{\alpha_2}, e_{\alpha_1+\alpha_2}$
(ii)*	$k_1 + k_2 < 0, k_2 \geq 0$	$e_{\alpha_1}, e_{-\alpha_2}, e_{\alpha_1+\alpha_2}$
(iii)*	$k_1 + k_2 < 0, k_1 > 0$	$e_{-\alpha_1}, e_{\alpha_2}, e_{\alpha_1+\alpha_2}$.

Here (i)* is the conjugate complex structure to (i), etc.. Note that φ induces a complex structure on $\mathrm{SU}(2,1)/T$, which is different in each of the six cases.

The complex structures (i)–(iii) may be visualized as follows (cf. [1] in the references): Taking $V = \mathbb{Q}^3$ and for Q the form diag$(-1,1,1)$, we have the unit ball $B = \{[z_0, z_1, z_2] : |z_1|^2 + |z_2|^2 < |z_0|^2\} \subset \mathbb{P}^2$. In $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ there is the standard incidence correspondence

$$I = \{(P, L) : P \in L\},$$

which is the flag variety for the complex group $\mathrm{SL}(3,\mathbb{C}) \cong \mathrm{SU}(2,1)(\mathbb{C})$ acting on $\mathbb{P}^2$. The pictures of the three complex structures are



The complex structure (ii) is the non-classical one; i.e., it does *not* fibre holomorphically or anti-holomorphically over an Hermitian symmetric space. There is a unique choice of φ for which the IPR is non-trivial and non-integrable; it is when $k_2 = -1$, $k_1 = 1$. Then

$$\mathfrak{m}^{-1,1} = \mathrm{span}\{e_{\alpha_2}, e_{-\alpha_1-\alpha_2}\}$$

and

$$[e_{\alpha_2}, e_{-\alpha_1-\alpha_2}] = ce_{-\alpha_2}, \qquad c \neq 0\,,$$

shows that the bracket of the 2-plane field given by the IPR is non-trivial. It is a contact structure; geometrically the 2-plane field is spanned by the two tangent vectors given by the conditions that to 1^{st} order either P or L does not move. The polarized Hodge structures on $\mathrm{su}(2,1)$ have Hodge numbers

$$h^{4,0} = 1,\ h^{3,1} = 2,\ h^{2,2} = 2\,.$$

It is not known if there is a family of smooth, projective fourfolds having a non-trivial period map to a quotient of this $D_{\mathfrak{m},\varphi}$.[8]

We mention this example because it is indicative of the rich geometry of Mumford-Tate domains, and because it has been the subject of some very interesting work in representation theory and automorphic cohomology that goes beyond the classical case of Shimura varieties (cf. loc. cit.). In this regard we have:

- *The real forms $M(\mathbb{R})$ of non-compact Hodge groups are exactly those that have non-trivial discrete series representations in $L^2(M(\mathbb{R}))$;*
- *The $\mathbb{Q}$-algebraic Hodge groups whose real forms are non-compact are those for which one may hope to have cuspidal automorphic representations in $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}))$.*

Aside from the work cited above, the rich confluence of

- algebraic geometry/Hodge theory
- representation theory (discrete series)
- arithmetic (automorphic forms)

present in the case when the Mumford-Tate domains parametrize the complex points of Shimura varieties seems to be largely unexplored in the higher weight case. In particular, one may ask what if any role the enhanced data (M,φ), beyond just the $\mathbb{Q}$-algebraic group M, might play? For example, when there is a "φ" there is a well-defined notion of CM points in quotients $\Gamma\backslash D_{\mathfrak{m},\varphi}$ of Hodge domains by arithmetic groups. In the classical case these CM points are a

[8] Since the IPR is non-integrable, they could not be Calabi-Yau's.

central part of the story. In general, even though in the non-classical case $\Gamma\backslash D_{\mathfrak{m},\varphi}$ is not an algebraic variety, the CM points — and more generally the Noether-Lefschetz loci — have an arithmetic structure arising from the "φ" and that may play an important role.

Selected references[9]

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[8] ———, Variation of Hodge structure: The singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.

[9] These are a small sample; [2]–[5] contain extensive further references to the literature.