



**The Abdus Salam
International Centre for Theoretical Physics**



2153-3

**Advanced School and Workshop on Discrete Groups in Complex
Geometry**

28 June - 5 July, 2010

**"The invariants of finite configurations in complex hyperbolic
geometry"**

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The invariants of finite configuration in complex hyperbolic geometry

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Abstract

The main purpose of this course is to construct the invariants of finite configurations in complex hyperbolic geometry and to describe their moduli spaces. We consider the following problems:

- The moduli space of points in complex hyperbolic space.
- The moduli space of points in the boundary of complex hyperbolic space.
- The moduli space of complex geodesics in the complex hyperbolic plane.

Keywords: Complex hyperbolic space; Invariants; Gram matrix.

Introduction: Moduli Spaces

Many of important problems in mathematics concern classification. One has a class of mathematical objects and a notion of when two objects should count as equivalent. It may well be that two equivalent objects look superficially very different, so one wishes to describe them in such a way that equivalent objects have the same description and inequivalent objects have different descriptions. Moduli spaces can be thought of as geometric solutions to geometric classification problems. In general, a moduli problem consists of three ingredients.

- Objects: Which geometric objects would we like to describe?
- Equivalences: When do we identify two of our objects as being isomorphic, or "the same"?
- Families: How do we allow our objects to vary, or modulate?

If one can show that a collection of geometric objects can be given the structure of a geometric space (for instance, the structure of an algebraic or analytic variety, or semi-analytic set, etc.), then one can parametrize such objects by introducing coordinates on the resulting space. In this context, the term "modulus" is used synonymously with "parameter"; moduli spaces were first understood as spaces of parameters rather than as spaces of objects. So, the basic idea for the construction of the moduli space is to give a geometric structure to the totality of the objects we are trying to classify. If we can understand this geometric structure, then we obtain powerful insights into the geometry of the objects themselves. It is natural to require that a moduli space for a family of geometric objects must have the following properties:

- The points of a moduli space correspond bijectively to the points of a family.

- Nearby points of a moduli space represent objects with similar structure.

To explain this better, we consider the following examples. Although of no great interest in itself, it will give us a taste of what a moduli space is.

Example 1: The moduli space of rigid triangles in the euclidian plane.

Let $\Delta = \Delta(p_1, p_2, p_3)$ be a triangle in the euclidian plane \mathbb{R}^2 with vertices p_1, p_2, p_3 . We say that two triangles $\Delta = \Delta(p_1, p_2, p_3)$ and $\Delta^* = \Delta(p_1^*, p_2^*, p_3^*)$ are equivalent or congruent if there exists an isometry f of \mathbb{R}^2 such that $p_i^* = f(p_i)$ for $i = 1, 2, 3$. If we denote the side lengths by $a_1 = d(p_1, p_2)$, $a_2 = d(p_2, p_3)$, and $a_3 = d(p_1, p_3)$, then the equivalence class of a triangle can be uniquely described by the triple (a_1, a_2, a_3) . However, not all triples of positive real numbers give rise to a triangle. Indeed, a necessary and sufficient condition is that the three numbers must satisfy the triangle inequalities. So, we can describe the moduli space of rigid triangles in the euclidian plane as follows:

$$\mathbb{M}_\Delta = \{(a_1, a_2, a_3) \in \mathbb{R}_+^3 : a_1 + a_2 > a_3, a_2 + a_3 > a_1, a_1 + a_3 > a_2\}.$$

Standing at a point in the moduli space corresponds to thinking about the congruence class of a particular triangle. On the other hand, moving through the moduli space corresponds to continuously deforming the triangle.

Example 2: The moduli space of similar triangles in the euclidian plane.

Let again $\Delta = \Delta(p_1, p_2, p_3)$ be a triangle in the euclidian plane \mathbb{R}^2 with vertices p_1, p_2, p_3 . But now we say that two triangles $\Delta = \Delta(p_1, p_2, p_3)$ and $\Delta^* = \Delta(p_1^*, p_2^*, p_3^*)$ are equivalent if there exists a similarity f of \mathbb{R}^2 such that $p_i^* = f(p_i)$ for $i = 1, 2, 3$. Let α_i denote the angle of $\Delta(p_1, p_2, p_3)$ at p_i , $0 < \alpha_i < \pi$. Then the equivalence class of a triangle can be uniquely described by the triple $(\alpha_1, \alpha_2, \alpha_3)$. However, not all triples of such positive real numbers give rise to a triangle: a necessary and sufficient condition is that the three numbers must satisfy the equality $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. So, in this case, we can describe the moduli space of triangles as follows:

$$\mathbb{M}_\Delta = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^3 : \alpha_1 + \alpha_2 + \alpha_3 = \pi\}.$$

Example 3: The moduli space of triangles in the hyperbolic plane.

Let $\Delta = \Delta(p_1, p_2, p_3)$ be a triangle in the hyperbolic plane \mathbb{H}^2 with vertices p_1, p_2, p_3 . We say that two triangles $\Delta = \Delta(p_1, p_2, p_3)$ and $\Delta^* = \Delta(p_1^*, p_2^*, p_3^*)$ are equivalent or congruent if there exists an isometry f of \mathbb{H}^2 such that $p_i^* = f(p_i)$ for $i = 1, 2, 3$. Let α_i denote the angle of $\Delta(p_1, p_2, p_3)$ at p_i , $0 < \alpha_i < \pi$. It is well known that the congruence class of a triangle in the hyperbolic plane can be uniquely described by the triple $(\alpha_1, \alpha_2, \alpha_3)$. However, not all triples of such positive real numbers give rise to a triangle: a necessary and sufficient condition is that the three numbers must satisfy the inequality $\alpha_1 + \alpha_2 + \alpha_3 < \pi$. So, we can describe the moduli space of triangles in the hyperbolic plane as follows:

$$\mathbb{M}_\Delta = \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^3 : \alpha_1 + \alpha_2 + \alpha_3 < \pi\}.$$

We remark that one can also describe the moduli space of triangles in the hyperbolic plane in terms of the side lengths similar to that in Example 1.

Example 4: Projective spaces as moduli spaces of lines.

The real projective space $\mathbb{P}\mathbb{R}^n$ is a moduli space. It is the space of lines in \mathbb{R}^{n+1} which pass through the origin. Similarly, complex projective space $\mathbb{P}\mathbb{C}^n$ is the space of complex lines in \mathbb{C}^{n+1} which pass through the origin.

Example 5: The moduli space of m -tuples of points in the Riemann sphere $\overline{\mathbb{C}}$.

Let $p = (z_1, \dots, z_m)$ and $p' = (w_1, \dots, w_m)$ be two ordered m -tuples of distinct points in $\overline{\mathbb{C}} \equiv \mathbb{P}\mathbb{C}^1$, $m \geq 1$. Then we say that p and p' are congruent (with respect to the diagonal action of $\text{PGL}(2, \mathbb{C})$) if there exists $f \in \text{PGL}(2, \mathbb{C})$ such that $w_i = f(z_i)$ for $i = 1, \dots, m$.

It is well known that if $m = 1, 2, 3$, then p and p' are always congruent in $\text{PGL}(2, \mathbb{C})$. So, in this case, the moduli space is trivial, we have no modulus. This is not true for $m \geq 4$. Next we describe the moduli space for the space of ordered m -tuples of distinct points in $\overline{\mathbb{C}}$ for $m \geq 4$.

First, we recall the definition of the classical cross-ratio. Let $p = (z_1, z_2, z_3, z_4)$ be an ordered quadruple of distinct points in \mathbb{C} . Then the *cross-ratio* of p is defined to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}.$$

The definition can be extended to the case when one of the points z_i is ∞ , for instance,

$$[\infty, z_2, z_3, z_4] = \frac{(z_2 - z_4)}{(z_3 - z_4)}.$$

Also, a conventional value of the cross-ratio can be defined when any three of the four points are distinct.

When the points z_1, z_2, z_3, z_4 are all distinct, the cross-ratio is finite and $\neq 0, 1$. Therefore, in this case, the cross-ratio belongs to $\mathbb{C}_* = \mathbb{C} \setminus \{0, 1\}$.

It is a classical result that the cross-ratio is the only invariant of ordered quadruples of points in $\overline{\mathbb{C}}$ with respect to the diagonal action of $\text{PGL}(2, \mathbb{C})$, that is, the two quadruples p and p' are congruent in $\text{PGL}(2, \mathbb{C})$ if and only if the cross-ratios of p and p' are equal.

We remark that the cross-ratio enjoys the following properties:

1. For any $z \in \mathbb{C}$ such that $z \neq 0$ and $z \neq 1$, $[1, 0, \infty, z] = z$,
2. Given distinct points z_1, z_2, z_3 in $\overline{\mathbb{C}}$, the function $f(z) = [z_1, z_2, z_3, z]$ is a unique element from $\text{PGL}(2, \mathbb{C})$ such that $f(z_1) = 1$, $f(z_2) = 0$, and $f(z_3) = \infty$.

Proposition 0.1 *Let $p = (z_1, \dots, z_m)$ and $p' = (w_1, \dots, w_m)$ be two ordered m -tuples of distinct points in $\overline{\mathbb{C}}$, $m \geq 4$. Then p and p' are congruent with respect to the diagonal action of $\text{PGL}(2, \mathbb{C})$ if and only if*

$$[z_1, z_2, z_3, z_j] = [w_1, w_2, w_3, w_j]$$

for any $j = 4, \dots, m$.

PROOF: Let us assume that $[z_1, z_2, z_3, z_j] = [w_1, w_2, w_3, w_j]$ for all $j = 4, \dots, m$. Applying (2), we find $f, g \in \text{PGL}(2, \mathbb{C})$ such that $f(z_1) = 1$, $f(z_2) = 0$, $f(z_3) = \infty$, and $g(w_1) = 1$, $g(w_2) = 0$, $g(w_3) = \infty$. Then applying (1), we have that

$$f(z_j) = [1, 0, \infty, f(z_j)] = [f(z_1), f(z_2), f(z_3), f(z_j)] = [z_1, z_2, z_3, z_j],$$

and

$$g(w_j) = [1, 0, \infty, g(w_j)] = [g(w_1), g(w_2), g(w_3), g(w_j)] = [w_1, w_2, w_3, w_j].$$

Therefore, our assumption implies that $f(z_i) = g(w_i)$ for all $i = 1, \dots, m$. Let now $h = g^{-1}f$. Then $h(z_i) = w_i$ for all $i = 1, \dots, m$. Hence, p and p' are congruent in $\text{PGL}(2, \mathbb{C})$. \blacksquare

Now let $p = (z_1, \dots, z_m)$ be an ordered m -tuple of distinct points in $\overline{\mathbb{C}}$, where $m \geq 4$. We associate to p the following cross-ratios:

$$x_1 = [z_1, z_2, z_3, z_4], x_2 = [z_1, z_2, z_3, z_5], \dots, x_{(m-3)} = [z_1, z_2, z_3, z_m].$$

Then the above implies that the $\mathrm{PGL}(2, \mathbb{C})$ -congruence class of p is defined uniquely by the cross-ratios $x_i, i = 0, 1, \dots, m - 3$.

Theorem 0.1 *The moduli space of ordered m -tuples of distinct points in $\overline{\mathbb{C}}$, where $m \geq 4$, may be identified with $\mathbb{C}_*^{(m-3)}$.*

The main purpose of this course is to construct the invariants of finite configurations in complex hyperbolic geometry and to describe their moduli spaces. We consider the following problems:

- The moduli space of points in complex hyperbolic space.
- The moduli space of points in the boundary of complex hyperbolic space.
- The moduli space of complex geodesics in the complex hyperbolic plane.

To construct the moduli spaces, we follow the strategy indicated in the examples above. First, we find the invariants which define uniquely the congruence class of an object, and then describe the conditions on these invariants. This gives rise to the description of the moduli space. The main technical tool is the use of Gram matrices of configurations.

The course is organized as follows. In Section 1, we review some basic facts in complex hyperbolic geometry. In Section 2, we give a description of the moduli space of ordered m -tuples of points in complex hyperbolic space. In Section 2, we construct the moduli space of points in the boundary of complex hyperbolic space. Finally, in Section 3, we describe the moduli space of configurations of complex geodesics in the complex hyperbolic space of dimension 2. In particular, in this section, we describe the moduli space of polygonal configurations and the moduli space for the space of representations of plane hyperbolic Coxeter groups.

1 Complex hyperbolic space and its boundary

Let $\mathbb{C}^{n,1}$ be a $(n + 1)$ -dimensional \mathbb{C} -vector space equipped with a Hermitian form $\langle -, - \rangle$ of signature $(n, 1)$. We will use the form such that the Hermitian product is given by $\langle v, w \rangle = v^* J_{n+1} w$, where v^* is the Hermitian transpose of v and $J_{n+1} = (a_{ij})$ is the $(n + 1) \times (n + 1)$ -matrix with $a_{ij} = 0$ for all $i \neq j$, $a_{ii} = 1$ for all $i = 1, \dots, n$, and $a_{ii} = -1$ when $i = n + 1$. Let π denote a natural projection from $\mathbb{C}^{n,1} \setminus \{0\}$ to projective space $\mathbb{P}\mathbb{C}^n$. Let V_-, V_0, V_+ be the subsets of $\mathbb{C}^{n,1} \setminus \{0\}$ consisting of vectors where $\langle v, v \rangle$ is negative, zero, or positive respectively. Vectors in V_0 are called *null* or *isotropic*, vectors in V_- are called *negative*, and vectors in V_+ are called *positive*. Their projections to $\mathbb{P}\mathbb{C}^n$ are called *isotropic*, *negative*, and *positive* points respectively.

The projective model of *complex hyperbolic space* $\mathbf{H}_{\mathbb{C}}^n$ is the set of negative points in $\mathbb{P}\mathbb{C}^n$, that is, $\mathbf{H}_{\mathbb{C}}^n = \pi(V_-)$. It is well known that $\mathbf{H}_{\mathbb{C}}^n$ can be identified with the unit open ball in \mathbb{C}^n . We will consider $\mathbf{H}_{\mathbb{C}}^n$ equipped with the Bergman metric, see [15]. Then $\mathbf{H}_{\mathbb{C}}^n$ is a complete Kähler manifold of constant holomorphic sectional curvature -1 . The boundary $\partial\mathbf{H}_{\mathbb{C}}^n = \pi(V_0)$ of $\mathbf{H}_{\mathbb{C}}^n$ is the $(2n - 1)$ -sphere formed by all isotropic points. Let $U(n, 1)$ be the unitary group corresponding to this Hermitian form. The holomorphic isometry group of $\mathbf{H}_{\mathbb{C}}^n$ is the projective unitary group $\mathrm{PU}(n, 1)$, and the full isometry group $\mathrm{Isom}(\mathbf{H}_{\mathbb{C}}^n)$ is generated by $\mathrm{PU}(n, 1)$ and complex conjugation.

There are two types of totally geodesic submanifolds of $\mathbf{H}_{\mathbb{C}}^n$ of real dimension two:

- *Complex geodesics* (copies of $\mathbf{H}_{\mathbb{C}}^1$) have constant sectional curvature -1 .
- *Totally real geodesic 2-planes* (copies of $\mathbf{H}_{\mathbb{R}}^2$) have constant sectional curvature $-1/4$.

Any complex geodesic is the intersection of a complex projective line in $\mathbb{P}\mathbb{C}^n$ with $\mathbf{H}_{\mathbb{C}}^n$. Complex geodesics c_1 and c_2 are called *ultra-parallel* if the complex projective lines l_1 and l_2 which define c_1 and c_2 intersect at a positive point, *asymptotic* if l_1 and l_2 intersect at an isotropic point, and *concurrent* if l_1 and l_2 intersect at a negative point.

The following fundamental result, the *Witt theorem*, is the basic instrument we will use in our construction of the moduli spaces.

Theorem 1.1 *Any linear injective isometry $\sigma : V \rightarrow W$, where V and W are linear subspaces of $\mathbb{C}^{n,1}$, can be extended to an isometry of $\mathbb{C}^{n,1}$.*

2 The moduli space of points in complex hyperbolic space

In this section, we describe the moduli space of ordered m -tuples of points in complex hyperbolic space of any dimension.

2.1 A characterization of Gram matrices

In this section, we give a characterization of Gram matrices of ordered m -tuples of negative points in complex projective space.

2.1.1 Gram matrix

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$ of dimension $n \geq 2$. Then we consider a Hermitian $m \times m$ -matrix

$$G = G(p) = (g_{ij}) = (\langle v_i, v_j \rangle),$$

where $v_i \in \mathbb{C}^{n,1}$ is a lift of p_i . We call G a *Gram matrix* associated to a m -tuple p . Of course, G depends on the chosen lifts v_i . When replacing v_i by $\lambda_i v_i$, $\lambda_i \neq 0$, we get $\tilde{G} = D^* G D$, where D is a diagonal matrix.

We say that two Hermitian $m \times m$ -matrices H and \tilde{H} are *equivalent* if there exists a non-singular diagonal matrix D such that $\tilde{H} = D^* H D$.

Thus, to each ordered m -tuple p of distinct negative points in $\mathbb{P}\mathbb{C}^n$ is associated an equivalence class of Hermitian $m \times m$ -matrices. We remark that for any two Gram matrices G and \tilde{G} associated to an m -tuple p the equality $\det \tilde{G} = \lambda \det G$ holds, where $\lambda > 0$. This implies that the sign of $\det G$ does not depend on the chosen lifts v_i . Also, we remark that $g_{ij} = \langle v_i, v_j \rangle \neq 0$ for negative v_i, v_j .

Proposition 2.1 *Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$. Then the equivalence class of Gram matrices associated to p contains a matrix $G = (g_{ij})$ such that $g_{ii} = -1$ and $g_{1j} = r_{1j}$ are real positive numbers for $j = 2, \dots, m$.*

PROOF: Let $v = (v_1, \dots, v_m)$ be a lift of p . Since the vectors v_i are negative, then by appropriate re-scaling, we may assume that $g_{ii} = \langle v_i, v_i \rangle = -1$. Then we get the result we need by replacing the vectors v_i , if necessarily, by $\lambda_i v_i$, where λ_i is an appropriate unitary complex number. \blacksquare

Remark 2.1 *It is easy to see that such a matrix $G = (g_{ij})$ is unique. We call a unique matrix $G = (g_{ij})$ defined by Proposition 2.1 a normal form of the associated Gram matrix. Also, we call G the normalized Gram matrix.*

2.1.2 Characterization of Gram matrices associated to m -tuples of negative points

Let W be a $(k+1)$ -dimensional subspace of $\mathbb{C}^{n,1}$, $1 \leq k \leq n$. The restriction of the Hermitian product on $\mathbb{C}^{n,1}$ to W has signature $(k, 1)$, $(k, 0)$, or $(k+1, 0)$. We call the subspace W *hyperbolic* if W has signature $(k, 1)$, *parabolic* if W has signature $(k, 0)$, and *elliptic* if W has signature $(k+1, 0)$.

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple p of distinct points in $\mathbb{P}\mathbb{C}^n$ and $v = (v_1, \dots, v_m)$ be a lift of p . Let $V \subset \mathbb{C}^{n,1}$ denote the space spanned by the vectors v_1, \dots, v_m , and let $\dim V = k+1$. Then the following situations are possible:

1. V is hyperbolic of signature $(k, 1)$, where $k \leq n$.
2. V is parabolic of signature $(k, 0)$, where $k \leq n-1$.
3. V is elliptic of signature $(k+1, 0)$, where $k \leq n-1$.

It is easy to see, that this exhausts all possibilities.

For a Hermitian matrix H we denote by $s(H) = (n_-, n_+, n_0)$ the signature (the inertia) of H , where n_- is the number of negative eigenvalues of H , n_+ is the number of positive eigenvalues of H , and n_0 is the number of zero eigenvalues of H .

Now we write down all possible signatures of the Gram matrices $H(p) = (h_{ij}) = (\langle v_i, v_j \rangle)$ associated to an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct points in $\mathbb{P}\mathbb{C}^n$.

Let $p, H(p), V, k$ be as above. Then we have that $H = H(p)$ is a Hermitian $m \times m$ -matrix such that:

1. In the hyperbolic case, $s(H) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.
2. In the parabolic case, $s(H) = (n_-, n_+, n_0)$, where $n_- = 0$, $n_+ = k \leq n-1$, and $n_+ + n_0 = m$.
3. In the elliptic case, $s(H) = (n_-, n_+, n_0)$, where $n_- = 0$, $n_+ = k+1$, $k \leq n-1$, and $n_+ + n_0 = m$.

Therefore, if $H(p)$ is a Gram matrix associated to an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct points in $\mathbb{P}\mathbb{C}^n$, then $\text{rank}(H) \leq n+1$, and the signature of H satisfies one of conditions (1) – (3) above. We call these conditions the *signature conditions*.

Now let $p = (p_1, \dots, p_m)$ be an ordered m -tuple p of distinct negative points in $\mathbb{P}\mathbb{C}^n$ and $v = (v_1, \dots, v_m)$ be a lift of p . Since the space V spanned by v_1, \dots, v_m contains a negative vector, V can be only hyperbolic, and, hence, the Gram matrix G associated to $v = (v_1, \dots, v_m)$ in this case necessarily has signature $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.

Theorem 2.1 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix, $m > 1$, such that $g_{ii} = -1$ and $g_{ij} \neq 0$ for $i \neq j$. Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G) \leq n+1$ and G has signature $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.*

PROOF: Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix with $g_{ii} = -1$ which satisfies the conditions of the theorem. It follows from Sylvester's Law of Inertia that there exists a matrix $S \in GL(m, \mathbb{C})$ such that $S^*GS = B$, where $B = (b_{ij})$ is the diagonal $m \times m$ - matrix such that $b_{11} = -1$, $b_{ii} = 1$, for $1 < i \leq n_+$, and $b_{ij} = 0$ for all other indexes. Now let $A = (a_{ij})$ be the $(n+1) \times m$ -matrix such that $a_{11} = -1$, $a_{ii} = 1$ for $1 < i \leq n_+$, and $a_{ij} = 0$ for all other indexes. It is easy to check that $B = A^*J_{n+1}A$. Then we define v_i to be the i^{th} column vector of the matrix S^*A . One verifies that $\langle v_i, v_j \rangle = g_{ij}$. So, letting $p_i = \pi(v_i)$, we get the result we want. \blacksquare

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct points in $\mathbb{P}\mathbb{C}^n$. Then p is defined to be *hyperbolic* (*parabolic*, *elliptic*) if the subspace $V \subset \mathbb{C}^{n,1}$ spanned by vectors that represent the points p_1, \dots, p_m is hyperbolic (parabolic, elliptic).

We recall that a subspace $W \subset \mathbb{C}^{n,1}$ is defined to be *degenerate* or *singular*, if there exists a vector $w \in W$ such that $\langle w, v \rangle = 0$ for all $v \in W$, and w is not null vector in the euclidian space \mathbb{C}^{n+1} . Of course, such a vector w must be isotropic as vector in the Hermitian space $\mathbb{C}^{n,1}$. It is easy to see that if p is either hyperbolic or elliptic, then the subspace $V \subset \mathbb{C}^{n,1}$ associated to p is regular (non-singular), and if p is parabolic, then V is degenerate (singular).

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$. It follows from the above that the space V spanned by vectors that represent p_1, \dots, p_m is regular. Therefore, applying the Witt theorem, we get the followig.

Proposition 2.2 *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$. Then p and p' are congruent in $\text{PU}(n, 1)$ if and only if their associated Gram matrices are equivalent.*

Corollary 2.1 *Let p and p' be two ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$, and let G and G' be their normalized Gram matrices. Then p and p' are congruent in $\text{PU}(n, 1)$ if and only if $G = G'$.*

In order to describe the moduli space of ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$, we need a characterization of the associated Gram matrices in terms of their minors. In general, there is no simple way to get a characterization of indefinite Hermitian matrices in terms of their minors. Using the fact the normalized Gram matrix of an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$ has a specific structure, we can overcome this difficulty by applying the following trick.

Let now $G = (g_{ij})$ be a Hermitian matrix such that $g_{ii} = -1$, and $g_{1j} \neq 0$ for $j = 2, \dots, m$. In particular, the normalized Gram matrix of an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$ is of this form.

Proposition 2.3 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix, $m > 1$, satisfying the conditions above. Then there exists a sequence of elementary operations which transforms G into the matrix*

$$\begin{bmatrix} -1 & 0 \\ 0 & G^* \end{bmatrix},$$

where $G^* = (g_{ij}^*)$ is the Hermitian $(m-1) \times (m-1)$ -matrix given below

$$G^* = \begin{bmatrix} -1 + g_{12}^2 & g_{23} + g_{13}g_{12} & \cdots & g_{2m} + g_{1m}g_{12} \\ \vdots & & & \vdots \\ g_{m2} + g_{12}g_{1m} & g_{m3} + g_{13}g_{1m} & \cdots & -1 + g_{1m}^2 \end{bmatrix}.$$

PROOF: Let R_i denote the i^{th} row and C_i the i^{th} column of G . Then it is easy to verify that the sequence of elementary operations, row and column additions, given by

- $g_{1j}R_1 + R_j \rightarrow R_j, 2 \leq j \leq m;$
- $g_{1j}C_1 + C_j \rightarrow C_j, 2 \leq j \leq m;$

proves the claim of the proposition. ▀

We call the matrix G^* the *associated matrix* to G .

Corollary 2.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix, $m > 1$, satisfying the conditions of Proposition 2.3 and G^* be its associated matrix. Then*

$$\det G = -\det G^*, \quad \text{rank}(G) = \text{rank}(G^*) + 1.$$

Applying Theorem 2.1, we get the following.

Theorem 2.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix, $m > 1$, such that $g_{ii} = -1$ and $g_{ij} \neq 0$ for all other indexes. Let G^* be the associated matrix to G . Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G^*) \leq n$ and G^* is positive semi-definite.*

As a consequence, we have the following theorem which is the crucial result for our construction of the moduli space of configurations of negative points.

Theorem 2.3 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix, $m > 1$, such that $g_{ii} = -1$ and $g_{ij} \neq 0$ for all other indexes. Let G^* be the associated matrix to G . Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G^*) \leq n$ and all principal minors of G^* are non-negative.*

PROOF: The proof follows from Theorem 2.2 and the characterization of positive semi-indefinite matrices in terms of minors, see, for instance, [22]. ■

We will call the conditions in Theorem 2.3 the *determinant conditions*.

2.2 The moduli space

In this section, we construct the moduli space of ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$.

Let \mathcal{M} be the configuration space of ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$, that is, the quotient of the set of ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$ with respect to the diagonal action of $\text{PU}(n, 1)$ equipped with the quotient topology.

Next, we introduce the invariants we need to construct our moduli space.

Let p_1 and p_2 be distinct negative points in $\mathbb{P}\mathbb{C}^n$ and v_1 and v_2 be their lifts in $\mathbb{C}^{n,1}$. Then we define

$$d(p_1, p_2) = d(v_1, v_2) = \frac{\langle v_1, v_2 \rangle \langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle \langle v_2, v_2 \rangle}.$$

It is easy to see that $d(p_1, p_2)$ is independent of the chosen lifts, and that $d(p_1, p_2)$ is invariant with respect to the diagonal action of $\text{PU}(n, 1)$.

Remark 2.2 *There is no accepted name for this invariant in the literature. Since the distance ρ between p_1 and p_2 is given in terms of $d(p_1, p_2)$, namely, $\cosh^2(\rho/2) = d(p_1, p_2)$, we will call this invariant $d(p_1, p_2)$ the distance invariant or, simply, d -invariant.*

Now let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct negative points in $\mathbb{P}\mathbb{C}^n$ and $v = (v_1, v_2, v_3)$ be a lift of p . Then the *angular invariant* of p is defined to be

$$\mathbb{A} = \mathbb{A}(p_1, p_2, p_3) = \arg(\langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle).$$

One verifies that \mathbb{A} is well defined, \mathbb{A} is independent of the chosen vectors, and that \mathbb{A} is invariant with respect to the diagonal action of $\text{PU}(n, 1)$.

It is convenient (for the use in geometric applications) to consider \mathbb{A} in the interval $(-\pi, \pi]$.

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$. Let $G(p) = (g_{ij})$ be the normalized Gram matrix associated to p . Since $g_{ij} \neq 0$, we write g_{ij} in the following form: $g_{ij} = |g_{ij}|e^{i\alpha_{ij}} = r_{ij}e^{i\alpha_{ij}}$, where $\alpha_{ij} = \arg(g_{ij}) \in (-\pi, \pi]$. In particular, we have that $g_{1j} = r_{1j} > 0$.

Now, given an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$, we define: $d_{ij} = d(p_i, p_j)$, where $i \neq j$, and $\mathbb{A}_{ij} = \mathbb{A}(p_1, p_i, p_j)$, where $i \neq j$; $i, j > 1$.

Then we associate to p the following invariants: d_{ij} , $1 \leq i < j \leq m$, and \mathbb{A}_{ij} , $2 \leq i < j \leq m$. It readily seen that the number of d_{ij} is equal to $m(m-1)/2$, and that the number of \mathbb{A}_{ij} is equal to $(m-1)(m-2)/2$.

Proposition 2.4 *The invariants d_{ij} , $1 \leq i < j \leq m$, and \mathbb{A}_{ij} , $2 \leq i < j \leq m$, define uniquely the $\text{PU}(n, 1)$ -congruence class of an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$.*

PROOF: It follows from the definition of the Gram matrix that

$$d_{ij} = d(c_i, c_j) = g_{ij}g_{ji} = |g_{ij}|^2,$$

and that

$$\mathbb{A}_{ij} = \mathbb{A}(c_1, c_i, c_j) = \arg(g_{1i} g_{ij} g_{j1}) = \arg(r_{1i} g_{ij} r_{j1}).$$

The first equality implies that $|g_{ij}| = \sqrt{d_{ij}}$. Since p is formed by negative points, we have that $r_{1j} > 0$ for all $j > 1$, and that $g_{ij} \neq 0$ for all other indexes. Therefore, the second equality implies that $\mathbb{A}_{ij} = \arg(g_{ij})$. Thus, all the entries of the normalized Gram matrix $G(p)$ of p are recovered uniquely in terms of the invariants d_{ij} and \mathbb{A}_{ij} above. Now the proposition follows from Corollary 2.1. \blacksquare

Corollary 2.3 *The $\text{PU}(n, 1)$ -congruence class of an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$ is defined uniquely by $(m-1)^2$ real numbers.*

Now we are ready to construct the moduli space.

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$ and $G = (g_{ij})$ be the normalized Gram matrix associated to p . By applying the formulae from the proof of Proposition 2.4, we can recover all of the entries of G in terms of the invariants d_{ij} and \mathbb{A}_{ij} . In what follows, we will assume that G has the entries expressed in this way. So, we can write that $G = G(d_{ij}, \mathbb{A}_{ij})$.

Let G^* be the associated matrix to G . We denote by G_{i_1, \dots, i_k}^* , where $1 \leq i_1 < i_2 < \dots < i_k \leq m-1$, the submatrix of G^* obtained by taking the rows i_1, \dots, i_k and the columns i_1, \dots, i_k . Also, we denote by D_{i_1, \dots, i_k}^* the determinant of G_{i_1, \dots, i_k}^* . We will also consider D_{i_1, \dots, i_k}^* as a function of d_{ij} and \mathbb{A}_{ij} .

Let \mathcal{M} denote the configuration space of ordered m -tuples of distinct negative points in $\mathbb{P}\mathbb{C}^n$ and let $[p] \in \mathcal{M}$ be the point represented by p . We define the map

$$\tau : \mathcal{M} \longrightarrow \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d,$$

where $d = d_1 + d_2 = (m-1)^2$, and $d_1 = m(m-1)/2$, $d_2 = (m-1)(m-2)/2$, by associating to $[p]$ the invariants d_{ij} and \mathbb{A}_{ij} above. Given $w \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, we will write $w = (u, v)$, where $u = (u_1, \dots, u_{d_1}) \in \mathbb{R}^{d_1}$ and $v = (v_1, \dots, v_{d_2}) \in \mathbb{R}^{d_2}$. Therefore, the functions $D_{i_1, \dots, i_k}^* = D_{i_1, \dots, i_k}^*(d_{ij}, \mathbb{A}_{ij})$ define the functions $D_{i_1, \dots, i_k}^*(u, v)$ of (u, v) if we use the lexicographic order for d_{ij} and \mathbb{A}_{ij} .

Theorem 2.4 *The configuration space \mathcal{M} is homeomorphic to the set \mathbb{M} of points in $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ defined by the following conditions:*

$$D_{i_1, \dots, i_k}^*(u, v) = 0, \quad \forall 1 \leq i_1 < i_2 < \dots < i_k \leq m, \quad \forall k > n,$$

and

$$D_{i_1, \dots, i_k}^*(u, v) \geq 0, \quad \forall 1 \leq i_1 < i_2 < \dots < i_k \leq m-1.$$

provided that $u_i > 0$ for all $i = 1, \dots, d_1$, and $v_i \in (-\pi, \pi]$ for all $i = 1, \dots, d_2$.

PROOF: It follows from the "if" part of Theorem 2.3 and from the formulae in the proof of Proposition 2.4 that the map τ above defines a map $\tau : \mathcal{M} \rightarrow \mathbb{M}$. First, we show that the map $\tau : \mathcal{M} \rightarrow \mathbb{M}$ is surjective. Given $w = (u, v) \in \mathbb{M}$, we construct a Hermitian $m \times m$ -matrix $G = (g_{ij})$ as follows. Using again the formulae in the proof of Proposition 2.4, we define g_{ij} , $i \neq j$, in terms of (u, v) identifying u_1, \dots, u_{d_1} with d_{ij} and v_1, \dots, v_{d_2} with \mathbb{A}_{ij} . Also, we put $g_{ii} = -1$. This defines G completely. Then it is readily seen that G satisfies all of the conditions in the "only if" part of Theorem 2.3 under the conditions of the theorem. Hence, this implies that G is the normalized Gram matrix for some ordered m -tuple of distinct negative points in $\mathbb{P}\mathbb{C}^n$. This proves that τ is surjective. On the other hand, it follows from Proposition 2.4 that τ is injective. It is clear that $\tau : \mathcal{M} \rightarrow \mathbb{M}$ is a homeomorphism provided that \mathbb{M} is equipped with the topology induced from \mathbb{R}^d . This completes the proof of the theorem. \blacksquare

Remark 2.3 We call \mathbb{M} the moduli space for \mathcal{M} .

Corollary 2.4 The moduli space \mathbb{M} is a semi-analytic set in the euclidian space of dimension $d = (m-1)^2$.

Next, as a corollary of Theorem 2.4, we give an explicit description of the moduli space of ordered triples of distinct negative points in $\mathbb{P}\mathbb{C}^n$. First of all, it follows from Proposition 2.4 that $\text{PU}(n, 1)$ -congruence class of an ordered triple $p = (p_1, p_2, p_3)$ of distinct negative points in $\mathbb{P}\mathbb{C}^n$ is described uniquely by three d-invariants d_{12}, d_{13}, d_{23} and the angular invariant $\alpha = \mathbb{A}(2, 3) = \mathbb{A}(p_1, p_2, p_3)$. Now, let $G = (g_{ij})$ be the normalized Gram matrix of p . Then $g_{ii} = -1$, $g_{1j} = r_{1j} > 0$, and $\arg(g_{23}) = \mathbb{A}(2, 3)$. We have that $d_{1j} = r_{1j}^2$ and $d_{23} = r_{23}^2$. Also, $g_{23} = r_{23} e^{i\alpha} = r_{23}(\cos \alpha + i \sin \alpha)$. A straightforward computation shows that

$$\det G = -1 + (r_{12}^2 + r_{13}^2 + r_{23}^2) + 2r_{12}r_{13}r_{23} \cos \alpha.$$

Using the lexicographic order, we define $r_1 = \sqrt{d_{12}}$, $r_2 = \sqrt{d_{13}}$, $r_3 = \sqrt{d_{23}}$. Then by applying Theorem 2.4, we get the following result.

Corollary 2.5 The configuration space $\mathcal{M}(3)$ of ordered triples of distinct negative points in $\mathbb{P}\mathbb{C}^n$ is homeomorphic to the set

$$\mathbb{M}(3) = \{(r_1, r_2, r_3, \alpha) \in \mathbb{R}^4 : r_i > 0, \alpha \in (-\pi, \pi], -1 + (r_1^2 + r_2^2 + r_3^2) + 2r_1r_2r_3 \cos \alpha \leq 0\}.$$

The equality in the last inequality happens if and only if the points p_1, p_2, p_3 are in a complex geodesic.

3 The moduli space of points in the boundary of complex hyperbolic space

In this section, we construct the moduli space of ordered m -tuples of distinct points in the boundary of complex hyperbolic space of any dimension $n \geq 1$.

3.1 A characterization of Gram matrices

In this section, we give a characterization of Gram matrices of ordered m -tuples of distinct isotropic points in complex projective space of any dimension.

3.1.1 Gram matrix

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ of dimension $n \geq 1$. Then we consider a Hermitian $m \times m$ -matrix

$$G = G(p) = (g_{ij}) = (\langle v_i, v_j \rangle),$$

where $v_i \in \mathbb{C}^{n,1}$ is a lift of p_i . We call G a *Gram matrix* associated to a m -tuple p . Of course, G depends on the chosen lifts v_i . When replacing v_i by $\lambda_i v_i$, $\lambda_i \neq 0$, we get $\tilde{G} = D^* G D$, where D is a diagonal matrix. Since v_i is isotropic, we have that $g_{ii} = 0$. Also, it is easy to show that $g_{ij} \neq 0$ for all $i \neq j$.

We say that two Hermitian $m \times m$ - matrices H and \tilde{H} are *equivalent* if there exists a non-singular diagonal matrix D such that $\tilde{H} = D^* H D$.

Thus, to each ordered m -tuple p of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ is associated an equivalence class of Hermitian $m \times m$ - matrices with 0's on the diagonal. We remark that for any two Gram matrices G and \tilde{G} associated to an m -tuple p the equality $\det \tilde{G} = \lambda \det G$ holds, where $\lambda > 0$. This implies that the sign of $\det G$ does not depend on the chosen lifts v_i . We also remark that $g_{ij} \neq 0$ for $i \neq j$.

Proposition 3.1 *Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$. Then the equivalence class of Gram matrices associated to p contains a matrix $G = (g_{ij})$ such that $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \dots, m$, and $|g_{23}| = 1$.*

PROOF: Let $v = (v_1, \dots, v_m)$ be a lift of $p = (p_1, \dots, p_m)$. Since p_i are distinct and isotropic, we have that $g_{ij} = \langle v_i, v_j \rangle \neq 0$ for $i \neq j$. Then it is easy to see that by appropriate re-scaling we may assume that $g_{12} = g_{13} = 1$. Let now $a = \sqrt{|g_{23}|}$. Taking $v'_1 = av_1$, $v'_2 = (1/a)v_2$, $v'_3 = (1/a)v_3$, we obtain that $g'_{12} = g'_{13} = 1$ and $|g'_{23}| = 1$. Then we get the result we need replacing the vectors v_i , $i = 4, \dots, m$, if necessarily, by $\lambda_i v_i$, where $\lambda_i = 1/\langle v_i, v_1 \rangle$. \blacksquare

Remark 3.1 *It is clear that a matrix $G = (g_{ij})$ defined in this proposition is unique. We call such a matrix G a normal form of the associated Gram matrix. Also, we call G the normalized Gram matrix.*

3.1.2 Characterization of Gram matrices associated to m -tuples of isotropic points

Now let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ and $v = (v_1, \dots, v_m)$ be a lift of p . It is elementary to verify that if $m > 1$, then the space $V \subset \mathbb{C}^{n,1}$ spanned by the vectors v_1, \dots, v_m always contains a negative vector. This implies that V can be only hyperbolic, and, hence, any Gram matrix associated to a m -tuple p of isotropic points necessarily has signature $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.

Proposition 3.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 0$ and $g_{ij} \neq 0$ for $i \neq j$, $m > 1$. Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ if $\text{rank}(G) \leq n+1$ and G has signature $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.*

PROOF: Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix with $g_{ii} = 0$. It follows from Sylvester's Law of Inertia that there exists a matrix $S \in GL(m, \mathbb{C})$ such that $S^* G S = B$, where $B = (b_{ij})$ is the diagonal $m \times m$ - matrix such that $b_{11} = -1$, $b_{ii} = 1$, for $1 < i \leq n_+$, and $b_{ij} = 0$ for all other indexes. Now let $A = (a_{ij})$ be the $(n+1) \times m$ -matrix such that $a_{11} = -1$, $a_{ii} = 1$ for $1 < i \leq n_+$, and $a_{ij} = 0$ for all other indexes. It is easy to check that $B = A^* J_{n+1} A$. Then we define v_i to be the i^{th} column vector of the matrix $S^* A$. One verifies that $\langle v_i, v_j \rangle = g_{ij}$. So, letting $p_i = \pi(v_i)$, we get the result we want. \blacksquare

As a corollary of this proposition, we get the following theorem.

Theorem 3.1 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 0$ and $g_{ij} \neq 0$ for $i \neq j$, $m > 1$. Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G) \leq n+1$ and G is indefinite with the signature $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ \leq n$, and $1 + n_+ + n_0 = m$.*

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$, $m > 1$. It follows from the above that the space $V \subset \mathbb{C}^{n,1}$ spanned by vectors that represent p_1, \dots, p_m is regular. Therefore, applying the Witt theorem [27], we get the following.

Proposition 3.3 Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered m -tuples of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$. Then p and p' are congruent in $\text{PU}(n, 1)$ if and only if their associated Gram matrices are equivalent.

Corollary 3.1 Let p and p' be two ordered m -tuples of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$, and let G and G' be their normalized Gram matrices. Then p and p' are congruent in $\text{PU}(n, 1)$ if and only if $G = G'$.

In order to describe the moduli space for the configuration space \mathcal{M} of ordered m -tuples of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$, we will need a characterization of the associated Gram matrices in terms of their minors. We remark that, in general, there is no simple way to get a characterization of indefinite matrices in terms of their minors. Using the fact that the normalized Gram matrix of an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ has a specific structure, we can overcome this difficulty by applying the following trick.

Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix, $m > 2$, such that $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \dots, m$, and $g_{ij} \neq 0$ for $i \neq j$, that is, G has the following form:

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & g_{23} & g_{24} & \cdots & g_{2m} \\ 1 & \bar{g}_{23} & 0 & g_{34} & \cdots & g_{3m} \\ 1 & \bar{g}_{24} & \bar{g}_{34} & 0 & \cdots & g_{4m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \bar{g}_{2m} & \bar{g}_{3m} & \bar{g}_{4m} & \cdots & 0 \end{bmatrix},$$

where $g_{ij} \neq 0$ for $i \neq j$.

In particular, the normalized Gram matrix of an ordered m -tuple of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ is of this form.

Proposition 3.4 Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix, $m > 2$, such that $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \dots, m$, and $g_{ij} \neq 0$ for $i \neq j$. Then there exists a sequence of elementary operations which transforms G into the matrix

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & G^* \end{array} \right],$$

where $G^* = (g_{ij}^*)$ is the Hermitian $(m-2) \times (m-2)$ - matrix given below

$$G^* = \begin{bmatrix} -(g_{23} + \bar{g}_{23}) & -\bar{g}_{23} - g_{24} + g_{34} & \cdots & -\bar{g}_{23} - g_{2m} + g_{3m} \\ -g_{23} - \bar{g}_{24} + \bar{g}_{34} & -(g_{24} + \bar{g}_{24}) & \cdots & -\bar{g}_{24} - g_{2m} + g_{4m} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{23} - \bar{g}_{2m} + \bar{g}_{3m} & -g_{24} - \bar{g}_{2m} + \bar{g}_{4m} & \cdots & -(g_{2m} + \bar{g}_{2m}) \end{bmatrix}.$$

PROOF: Let R_i denote the i^{th} row and C_i the i^{th} column of G . Then it is easy to verify that the sequence of elementary operations, row and column additions, given by

- $R_i - R_2 \rightarrow R_i$, $3 \leq i \leq m$;
- $C_i - C_2 \rightarrow C_i$, $3 \leq i \leq m$;
- $R_i - \bar{g}_{2i}R_1 \rightarrow R_i$, $3 \leq i \leq m$;
- $C_i - g_{2i}C_1 \rightarrow C_i$, $3 \leq i \leq m$;

proves the claim of the proposition. ■

We call the matrix G^* the associated matrix to G .

Corollary 3.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix satisfying the conditions of Proposition 3.4 and G^* be the matrix associated to G . Then*

$$\det G = -\det G^*, \quad \text{rank}(G) = \text{rank}(G^*) + 2.$$

Applying Theorem 3.1, we get the following.

Theorem 3.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix, $m > 2$, such that $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \dots, m$, and $g_{ij} \neq 0$ for $i \neq j$. Let G^* be the associated matrix to G . Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G^*) \leq n - 1$ and G^* is positive semi-definite.*

As a consequence, we have the following theorem which is the crucial result for our construction of the moduli space for \mathcal{M} .

Theorem 3.3 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix, $m > 2$, such that $g_{ii} = 0$, $g_{1j} = 1$ for $j = 2, \dots, m$, and $g_{ij} \neq 0$ for $i \neq j$. Let G^* be the associated matrix to G . Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G^*) \leq n - 1$ and all principal minors of G^* are non-negative.*

PROOF: The proof follows from Theorem 3.2 and the characterization of positive semi-definite matrices in terms of minors, see, for instance, [22]. ■

We will call the conditions in Theorem 3.3 the *determinant conditions*.

3.2 The moduli space

Let $\mathcal{M}(n, m)$ be the configuration space of ordered m -tuples of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$, that is, the quotient of the set of ordered m -tuples of distinct isotropic points in $\mathbb{P}\mathbb{C}^n$ with respect to the diagonal action of $\text{PU}(n, 1)$ equipped with the quotient topology. In this section, we construct the moduli space for $\mathcal{M}(n, m)$ for any $n \geq 1$ and $m > 3$.

3.2.1 Cartan's angular invariant and the complex cross-ratio

Let $p = (p_1, p_2, p_3)$ be an ordered triple of distinct points in the boundary $\partial\mathbf{H}_{\mathbb{C}}^n$ of complex hyperbolic n -space. Then *Cartan's angular invariant* $\mathbb{A}(p)$ of p is defined to be

$$\mathbb{A}(p) = \arg(-\langle v_1, v_2, v_3 \rangle),$$

where $v_i \in \mathbb{C}^{n,1}$ are corresponding lifts of p_i , and

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle \in \mathbb{C}$$

is the *Hermitian triple product*. It is verified that $\mathbb{A}(p)$ is independent of the chosen lifts and satisfies

$$-\pi/2 \leq \mathbb{A}(p) \leq \pi/2.$$

The Cartan invariant is the only invariant of a triple of points: p and p' are congruent in $\text{PU}(n, 1)$ if and only if $\mathbb{A}(p) = \mathbb{A}(p')$. Basic properties of the Cartan invariant can be found in Goldman [15] and Cartan [5].

In [23], Korányi and Reimann defined a complex-valued invariant associated to an ordered quadruple of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^n$. This invariant generalizes the usual cross-ratio of a quadruple of complex numbers. Let $p = (p_1, p_2, p_3, p_4)$ be an ordered quadruple of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^n$. Following Goldman, we define the Korányi-Reimann complex cross-ratio, or simply the complex cross-ratio, of p , as follows

$$X = X(p) = \frac{\langle v_3, v_1 \rangle \langle v_4, v_2 \rangle}{\langle v_4, v_1 \rangle \langle v_3, v_2 \rangle},$$

where $v_i \in \mathbb{C}^{n,1}$ are corresponding lifts of p_i . It is verified that the complex cross-ratio is independent of the chosen lifts v_i and is invariant with respect to the diagonal action of $\mathrm{PU}(n, 1)$. Since the points p_i are distinct, X is finite and non-zero. More properties of the complex cross-ratio may be found in Goldman [15].

3.2.2 Invariants

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^n$. We associate to p the following cross-ratios:

$$X_{2j} = X(p_1, p_2, p_3, p_j), \quad X_{3j} = X(p_1, p_3, p_2, p_j), \quad X_{kj} = X(p_1, p_k, p_2, p_j),$$

where $m \geq 4$, $4 \leq j \leq m$, $4 \leq k \leq m-1$, $k < j$.

Remark 3.2 *When $m = 4$, we exclude the last cross-ratio which is not defined in this case. Thus, for $m = 4$ we have only two cross-ratios in our list, compare this with [6]. It is easy to show that the number of cross-ratios in the list is equal to $d = m(m-3)/2$.*

Straightforward computations give the following.

Proposition 3.5 *Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^n$ and $G = (g_{ij})$ be the normalized Gram matrix of p . Then*

- $\mathbb{A} = \mathbb{A}(p_1, p_2, p_3) = \arg(-g_{23})$,
- $X_{2j} = X(p_1, p_2, p_3, p_j) = \bar{g}_{2j}/\bar{g}_{23}$,
- $X_{3j} = X(p_1, p_3, p_2, p_j) = \bar{g}_{3j}/g_{23}$,
- $X_{kj} = X(p_1, p_k, p_2, p_j) = \bar{g}_{kj}/g_{2j}$,

and

- $g_{23} = -e^{i\mathbb{A}}$,
- $g_{2j} = -e^{i\mathbb{A}} \bar{X}(p_1, p_2, p_3, p_j) = -e^{i\mathbb{A}} \bar{X}_{2j}$,
- $g_{3j} = -e^{-i\mathbb{A}} \bar{X}(p_1, p_3, p_2, p_j) = -e^{-i\mathbb{A}} \bar{X}_{3j}$,
- $g_{kj} = -e^{-i\mathbb{A}} X(p_1, p_2, p_3, p_j) \bar{X}(p_1, p_k, p_2, p_j) = -e^{-i\mathbb{A}} X_{2j} \bar{X}_{kj}$.

where all the indexes satisfy the conditions in the definition of the cross-ratios in question.

Using these formulae, we see that all of the entries of the normalized Gram matrix G of p are recovered uniquely in terms of the invariants above. Applying Corollary 3.1, we get the following important result.

Theorem 3.4 *The invariants X_{2j} , X_{3j} , X_{kj} and \mathbb{A} define uniquely the congruence class of p in $\mathrm{PU}(n, 1)$.*

3.2.3 Moduli space

Now we are ready to construct the moduli space for ordered m -tuples of distinct points in $\partial\mathbf{H}_{\mathbb{C}}^n$.

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct points of $\partial\mathbf{H}_{\mathbb{C}}^n$ and $G = (g_{ij})$ be the normalized Gram matrix associated to p . Applying the formulae from Proposition 3.5, we can recover all of the entries of G in terms of the invariants X_{2j}, X_{3j}, X_{kj} and \mathbb{A} . In what follows, we will assume that G has the entries expressed in this way. So, we can write that $G = G(X_{2j}, X_{3j}, X_{kj}, \mathbb{A})$.

Let G^* be the associated matrix to G . We denote by G_{i_1, \dots, i_k}^* , where $1 \leq i_1 < i_2 < \dots < i_k \leq m - 2$, the submatrix of G^* obtained by taking the rows i_1, \dots, i_k and the columns i_1, \dots, i_k . Also, we denote by D_{i_1, \dots, i_k}^* the determinant of G_{i_1, \dots, i_k}^* . We will consider D_{i_1, \dots, i_k}^* as a function of $X_{2j}, X_{3j}, X_{kj}, \mathbb{A}$.

Let $\mathcal{M} = \mathcal{M}(n, m)$ be the configuration space of ordered m -tuples of distinct points in $\partial\mathbf{H}_{\mathbb{C}}^n$, and let $[p] \in \mathcal{M}$ be the point represented by p . We define the map

$$\tau : \mathcal{M} \longrightarrow \mathbb{R}^{2d+1},$$

where $d = m(m - 3)/2$ by associating to $[p]$ the invariants X_{2j}, X_{3j}, X_{kj} and \mathbb{A} above.

Given $w \in \mathbb{R}^{2d+1} = \mathbb{R}^{2d} \times \mathbb{R}$, we will write $w = (u_1, v_1, \dots, u_d, v_d, t)$. Therefore, the functions $D_{i_1, \dots, i_k}^* = D_{i_1, \dots, i_k}^*(X_{2j}, X_{3j}, X_{kj}, \mathbb{A})$ define the functions $D_{i_1, \dots, i_k}^*(w)$ of w if we use the lexicographic order for X_{2j}, X_{3j}, X_{kj} .

Theorem 3.5 *The configuration space $\mathcal{M}(n, m)$ is homeomorphic to the set $\mathbb{M}(n, m)$ of points in \mathbb{R}^{2d+1} defined by the following conditions:*

$$D_{i_1, \dots, i_k}^*(w) \geq 0, \quad \forall 1 \leq i_1 < i_2 < \dots < i_k \leq m - 2$$

and

$$D_{i_1, \dots, i_k}^*(w) = 0, \quad \forall 1 \leq i_1 < i_2 < \dots < i_k \leq m - 2, \quad \forall k > n - 1,$$

provided that $u_i + iv_i \neq 0$ for all $i = 1, \dots, d$ and $t \in [-\pi/2, \pi/2]$.

PROOF: It follows from the "if" part of Theorem 3.3 and from the formulae in Proposition 3.5 that the map τ above defines a map $\tau : \mathcal{M}(n, m) \longrightarrow \mathbb{M}(n, m)$. First, we show that the map $\tau : \mathcal{M}(n, m) \longrightarrow \mathbb{M}(n, m)$ is surjective. Given $w \in \mathbb{M}(n, m)$, we construct a Hermitian $m \times m$ -matrix $G = (g_{ij})$ as follows. We put $g_{ii} = 0$ and $g_{1j} = 1$ for $j = 2, \dots, m$. Then, using again the formulae in Proposition 3.5, we define g_{ij} for all other indexes in terms of $w = (u_1, v_1, \dots, u_d, v_d, t)$ identifying $u_1, v_1, \dots, u_d, v_d$ with X_{2j}, X_{3j}, X_{kj} and t with \mathbb{A} . This defines G completely. Then it is readily seen that G satisfies all of the conditions of the "only if" part of Theorem 3.3 under the conditions of the theorem. Hence, this implies that G is the normalized Gram matrix for some ordered m -tuples of distinct points in $\partial\mathbf{H}_{\mathbb{C}}^n$. This proves that τ is surjective. On the other hand, it follows from Theorem 3.4 that τ is injective. It is clear that $\tau : \mathcal{M}(n, m) \longrightarrow \mathbb{M}(n, m)$ is a homeomorphism provided that $\mathbb{M}(n, m)$ is equipped with the topology induced from \mathbb{R}^{2d+1} . This completes the proof of the theorem. \blacksquare

We call $\mathbb{M}(n, m)$ the moduli space for $\mathcal{M}(n, m)$.

Corollary 3.3 *The moduli space $\mathbb{M}(n, m)$ is a semi-analytic set in the euclidian space of dimension $s = m^2 - 3m + 1$.*

Remark 3.3 *For $m = 4$, the detailed description of the moduli space is given in [6].*

4 The moduli space of complex geodesics in the complex hyperbolic plane

In this section, we construct the moduli space of ordered m -tuples of distinct complex geodesics in the complex hyperbolic plane.

4.1 A characterization of Gram matrices

In this section, we give a characterization of Gram matrices of ordered m -tuples of positive points in complex projective space of any dimension.

4.1.1 Gram matrix

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct positive points in $\mathbb{P}\mathbb{C}^n$ of dimension $n \geq 2$. Then we consider a Hermitian $m \times m$ -matrix

$$G = G(p) = (g_{ij}) = (\langle v_i, v_j \rangle),$$

where $v_i \in \mathbb{C}^{n,1}$ is a lift of p_i . We call G a *Gram matrix* associated to a m -tuple p . Of course, G depends on the chosen lifts v_i . When replacing v_i by $\lambda_i v_i$, $\lambda_i \neq 0$, we get $\tilde{G} = D^* G D$, where D is a diagonal matrix.

We say that two Hermitian $m \times m$ -matrices H and \tilde{H} are *equivalent* if there exists a non-singular diagonal matrix D such that $\tilde{H} = D^* H D$.

Thus, to each ordered m -tuple p of distinct positive points in $\mathbb{P}\mathbb{C}^n$ is associated an equivalence class of Hermitian $m \times m$ -matrices. We remark that for any two Gram matrices G and \tilde{G} associated to an m -tuple p the equality $\det \tilde{G} = \lambda \det G$ holds, where $\lambda > 0$. This implies that the sign of $\det G$ does not depend on the chosen lifts v_i .

Proposition 4.1 *Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct positive points in $\mathbb{P}\mathbb{C}^n$. Then the equivalence class of Gram matrices associated to p contains a matrix $G = (g_{ij})$ such that $g_{ii} = 1$ and $g_{1j} = r_{1j}$ are real non-negative numbers for $j = 2, \dots, m$.*

PROOF: Let $v = (v_1, \dots, v_m)$ be a lift of p . Since the vectors v_i are positive, then by appropriate re-scaling, we may assume that $g_{ii} = \langle v_i, v_i \rangle = 1$. Then we get the result we need by replacing the vectors v_i , if necessarily, by $\lambda_i v_i$, where λ_i is appropriate unitary complex number. \blacksquare

Remark 4.1 *It is easy to see that such a matrix $G = (g_{ij})$ is unique provided that $g_{1j} = \langle v_1, v_j \rangle \neq 0$ for all $j = 2, \dots, m$. If $g_{1j} = 0$ for some $j = 2, \dots, m$, then $G = (g_{ij})$ admits a further normalization, see Section 4.4.*

If $g_{1j} \neq 0$ for all $j = 2, \dots, m$, we call a unique matrix $G = (g_{ij})$ defined by Proposition 4.1 a *normal form* of the associated Gram matrix. Also, we call G the *normalized Gram matrix*.

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct positive points in $\mathbb{P}\mathbb{C}^n$, and let $G = (g_{ij})$ be a Gram matrix associated to p . We call p *generic* if $g_{ij} \neq 0$ for all $i, j = 1, \dots, m$.

Remark 4.2 *The normalized Gram matrix is defined uniquely for any generic p .*

4.1.2 Characterization of Gram matrices associated to m -tuples of positive points

Now we write down all possible signatures of the Gram matrices associated to positive points.

Let p, G, V, k be as in Section 2.1.2. Then we have that G is a Hermitian $m \times m$ -matrix such that:

1. In the hyperbolic case, $s(G) = (n_-, n_+, n_0)$, where $n_- = 1$, $1 + n_+ = k \leq n$, and $1 + n_+ + n_0 = m$.
2. In the parabolic case, $s(G) = (n_-, n_+, n_0)$, where $n_- = 0$, $n_+ = k \leq n - 1$, and $n_+ + n_0 = m$.
3. In the elliptic case, $s(G) = (n_-, n_+, n_0)$, where $n_- = 0$, $n_+ = k + 1$, $k \leq n - 1$, and $n_+ + n_0 = m$.

Therefore, if G is a Gram matrix associated to an ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct positive points in $\mathbb{P}\mathbb{C}^n$, then $\text{rank}(G) \leq n + 1$, and the signature of G satisfies one of conditions (1) – (3) above. We call these conditions the *signature conditions*.

Proposition 4.2 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 1$. Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct positive points in $\mathbb{P}\mathbb{C}^n$ if and only if G satisfies the signature conditions.*

PROOF: Let $G = (g_{ij})$ be a Hermitian $m \times m$ -matrix with $g_{ii} = 1$. First, we consider the case when the signature of G satisfies condition (1) above. It follows from Sylvester's Law of Inertia that there exists a matrix $S \in GL(m, \mathbb{C})$ such that $S^*GS = B$, where $B = (b_{ij})$ is the diagonal $m \times m$ - matrix such that $b_{11} = -1$, $b_{ii} = 1$, for $1 < i \leq n_+$, and $b_{ij} = 0$ for all other indexes. Now let $A = (a_{ij})$ be the $(n + 1) \times m$ -matrix such that $a_{11} = -1$, $a_{ii} = 1$ for $1 < i \leq n_+$, and $a_{ij} = 0$ for all other indexes. It is easy to check that $B = A^*J_{n+1}A$. Then we define v_i to be the i^{th} column vector of the matrix S^*A . One verifies that $\langle v_i, v_j \rangle = g_{ij}$. So, letting $p_i = \pi(v_i)$, we get the result we want in this case.

In the second case, that is, when the signature of G is $s(G) = (0, n_+, n_0)$, where $n_+ \leq n - 1$, and $n_+ + n_0 = m$, using the arguments above, we construct the matrix $B = (b_{ij}) = S^*GS$, $S \in GL(m, \mathbb{C})$, to be the diagonal $m \times m$ - matrix such that $b_{ii} = 1$, $1 \leq i \leq n_+$, and $b_{ij} = 0$ for all other indexes. Now let $A = (a_{ij})$ be the $(n + 1) \times m$ -matrix such that $a_{ii} = 1$, for $1 \leq i \leq n_+$, and $a_{ij} = 0$ for all other indexes. Again it is easy to check that $B = A^*J_{n+1}A$, and if we define v_i to be the i^{th} column vector of the matrix S^*A , then $\langle v_i, v_j \rangle = g_{ij}$. The third case is similar to the second one. \blacksquare

As a corollary of this proposition, we get the following theorem.

Theorem 4.1 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 1$. Then G is a Gram matrix associated to some ordered m -tuple $p = (p_1, \dots, p_m)$ of distinct positive points in $\mathbb{P}\mathbb{C}^n$ if and only if $\text{rank}(G) \leq n + 1$, and G is either indefinite, or positive definite, or positive semi-definite. In the last two cases, $\text{rank}(G) \leq n$.*

Let $p = (p_1, \dots, p_m)$ be an ordered m -tuple of distinct positive points in $\mathbb{P}\mathbb{C}^n$. Then p is defined to be *hyperbolic (parabolic, elliptic)* is the subspace $V \subset \mathbb{C}^{n,1}$ spanned by vectors that represent the points p_1, \dots, p_m is hyperbolic (parabolic, elliptic).

We recall that a subspace $W \subset \mathbb{C}^{n,1}$ is defined to be *degenerate* or *singular*, if there exists a vector $w \in W$ such that $\langle w, v \rangle = 0$ for all $v \in W$, and w is not null vector in the euclidian space \mathbb{C}^{n+1} . Of course, such a vector w must be isotropic as vector in the Hermitian space $\mathbb{C}^{n,1}$. It is easy to see that if p is either hyperbolic or elliptic, then the subspace $V \subset \mathbb{C}^{n,1}$ associated to p is regular (non-singular), and if p is parabolic, then V is degenerate (singular).

By applying the Witt theorem [27], we get the following.

Proposition 4.3 *Let $p = (p_1, \dots, p_m)$ and $p' = (p'_1, \dots, p'_m)$ be two ordered m -tuples of distinct positive points in $\mathbb{P}\mathbb{C}^n$. Let us assume that p and p' are either hyperbolic or elliptic. Then p and p' are congruent in $\text{PU}(n, 1)$ if and only if their associated Gram matrices are equivalent.*

Remark 4.3 In Section 4.5, we show that this is not true when p is parabolic.

Corollary 4.1 Let p and p' be two ordered generic m -tuples of distinct positive points in $\mathbb{P}\mathbb{C}^n$, and let G and G' be their normalized Gram matrices. If p and p' are either hyperbolic or elliptic, then p and p' are congruent in $\text{PU}(n, 1)$ if and only if $G = G'$.

4.2 The moduli space of complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. Regular case

In this section, we describe the moduli space of configurations of complex geodesics in the complex hyperbolic space of dimension 2 when the space spanned by their polar vectors is not degenerate.

4.2.1 The polar sphere of a configuration of complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$

Let $c \subset \mathbf{H}_{\mathbb{C}}^2$ be a complex geodesic. Then c corresponds to a positive vector $v \in \mathbb{C}^{2,1}$, namely, $c = \pi(v^\perp \setminus \{0\}) \cap \mathbf{H}_{\mathbb{C}}^2$, where v^\perp is the complex linear 2-space consisting of vectors $u \in \mathbb{C}^{2,1}$ such that $\langle u, v \rangle = 0$. We call v a *polar vector* to c and the positive complex line it spans the *polar line*. Also, we call the point $c^\perp = \pi(v)$ the *polar point* to c . Conversely, a positive point $p \in \mathbb{P}\mathbb{C}^2$ defines a complex geodesic c whose polar point is p . Thus, the complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ bijectively correspond to the positive points of $\mathbb{P}\mathbb{C}^2$: the space of complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ identifies with the "outside" of $\mathbf{H}_{\mathbb{C}}^2$ in $\mathbb{P}\mathbb{C}^2$. We call such an identification the *polar identification*, or the *polar duality*.

Let $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. In what follows, we always assume that $m > 1$.

We say that C is *regular* if and only if the subspace $V(C) \subset \mathbb{C}^{2,1}$ spanned by polar vectors to c_1, \dots, c_m is regular. Otherwise, C is called to be *degenerate*, or *parabolic*. It follows from Section 4.1 that C is regular if and only if $V(C)$ is either hyperbolic or elliptic, and C is parabolic if and only if $V(C)$ is parabolic. In the hyperbolic case, we have that either $V(C) = \mathbb{C}^{2,1}$ or $V(C)$ is a proper subspace of $\mathbb{C}^{2,1}$. In the elliptic or parabolic case, $V(C)$ is always a proper subspace of $\mathbb{C}^{2,1}$. Since $m > 1$, we have that if $V(C)$ is a proper subspace, then $\dim V(C) = 2$. Therefore, in this case, $S(C) = \pi(V \setminus \{0\})$ is a complex projective line in $\mathbb{P}\mathbb{C}^2$. We call such a complex projective line $S(C)$ the *polar sphere* defined by C . So, when $V(C)$ is a proper subspace, the polar points p_1, \dots, p_m to the complex geodesics c_1, \dots, c_m belong to a polar sphere. One sees, that if C is hyperbolic and $V(C)$ is proper, then $S(C)$ intersects $\mathbf{H}_{\mathbb{C}}^2$. If C is elliptic, then $S(C)$ contains only positive points. Finally, if C is parabolic, then $S(C)$ is tangent to $\partial\mathbf{H}_{\mathbb{C}}^2$ at a single point.

4.2.2 The moduli space of regular configurations. Generic case

Let \mathcal{M} be the configuration space of ordered m -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, that is, the quotient of the set of ordered m -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ with respect to the diagonal action of $\text{PU}(2, 1)$ equipped with the quotient topology. In this section, we construct the moduli space for \mathcal{M} in the regular generic case.

First, we rewrite Theorem 4.1 in the form adapted to dimension 2. Let $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and $p = (p_1, \dots, p_m)$ be the ordered m -tuple of polar points to c_1, \dots, c_m . Let $G = (g_{ij})$ be a Gram matrix associated to p . Then $G = (g_{ij})$ is defined to be a *Gram matrix associated to C* .

Theorem 4.2 Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 1$. Then G is a Gram matrix associated to some ordered m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ if and only if $\text{rank}(G) \leq 3$, and G is either indefinite, or positive definite, or positive semi-definite. In the last two cases, $\text{rank}(G) \leq 2$.

As a consequence, we have the following theorem which is the crucial result for our construction of the moduli space for \mathcal{M} .

Theorem 4.3 *Let $G = (g_{ij})$ be a Hermitian $m \times m$ - matrix such that $g_{ii} = 1$. Then G is a Gram matrix associated to some ordered m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ if and only if all principal minors G of order $k \geq 4$ vanish, and all principal minors of G of order 3 are non-positive.*

PROOF: Since any k vectors in $\mathbb{C}^{2,1}$ are linearly dependent for $k \geq 4$, all principal minors of G of order $k \geq 4$ vanish. The rest follows from Sylvester's criterion and Theorem 3.1. \blacksquare

We will call the conditions in Theorem 4.3 the *determinant conditions*.

Corollary 4.2 *Any Hermitian 2×2 -matrix $G = (g_{ij})$ such that $g_{ii} = 1$ is a Gram matrix associated to some ordered pair of distinct complex geodesics $C = (c_1, c_2)$ in $\mathbf{H}_{\mathbb{C}}^2$. Moreover, G is positive semi-definite (in this case, $\det G = 0$) if and only if c_1 and c_2 are asymptotic, G is positive definite (in this case, $\det G > 0$) if and only if c_1 and c_2 are concurrent, G is indefinite (in this case, $\det G < 0$) if and only if c_1 and c_2 are ultra-parallel.*

Corollary 4.3 *Let $G = (g_{ij})$ be a Hermitian 3×3 - matrix such that $g_{ii} = 1$. Then G is a Gram matrix associated to some ordered triple $C = (c_1, c_2, c_3)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ if and only if $\det G \leq 0$.*

Remark 4.4 *If $m \geq 3$ and all principal minors of G of order 3 vanish, then the subspace $V(C) \subset \mathbb{C}^{2,1}$ spanned by polar vectors to c_1, \dots, c_m has dimension 2, and C is either parabolic (in this case, c_1, \dots, c_m intersect at an isotropic point), or elliptic (in this case, c_1, \dots, c_m intersect at a negative point), or hyperbolic (in this case, there exists a complex geodesic c orthogonal to c_1, \dots, c_m , see below). In all these cases, the polar points to c_1, \dots, c_m belong to a polar sphere, see Section 4.2.1.*

Next, we introduce the invariants we need to construct our moduli space.

Let c_1 and c_2 be distinct complex geodesics corresponding to polar vectors v_1 and v_2 in $\mathbb{C}^{2,1}$. Then we define

$$d(c_1, c_2) = d(v_1, v_2) = \frac{\langle v_1, v_2 \rangle \langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle \langle v_2, v_2 \rangle}.$$

It is easy to see that $d(c_1, c_2)$ is independent of the chosen polar vectors, and that $d(c_1, c_2)$ is invariant with respect to the diagonal action of $\text{PU}(2, 1)$. Also, it is well known, see, for instance, Goldman [15], that: c_1 and c_2 are concurrent if and only if $d(c_1, c_2) < 1$, c_1 and c_2 are asymptotic if and only if $d(c_1, c_2) = 1$, c_1 and c_2 are ultra-parallel if and only if $d(c_1, c_2) > 1$ (compare this with Corollary 4.2). Moreover, $d(c_1, c_2)$ is the only invariant of an ordered pair of complex geodesics. We have also that the angle θ between the complex geodesics c_1 and c_2 (in the case $d(c_1, c_2) < 1$) is given by $\cos^2(\theta) = d(c_1, c_2)$, and the distance ρ between c_1 and c_2 (in the case $d(c_1, c_2) \geq 1$) is given by $\cosh^2(\rho/2) = d(c_1, c_2)$, see [15]. We remark that $0 < \theta_i \leq \pi/2$. We say that c_1 and c_2 are *orthogonal* if $\theta = \pi/2$, this is equivalent to the equality $d(c_1, c_2) = 0$.

Remark 4.5 *There is no accepted name for this invariant in the literature. Since the distance or the angle between c_1 and c_2 is given in terms of $d(c_1, c_2)$, we will call this invariant $d(c_1, c_2)$ the distance-angular invariant or, simply, d -invariant.*

Now let $C = (c_1, c_2, c_3)$ be an ordered triple of distinct complex geodesics corresponding to polar vectors v_1, v_2 , and v_3 . In what follows, we will assume that c_i is not orthogonal to c_j for all $i, j = 1, 2, 3$. Then the *angular invariant* of C is defined to be

$$\mathbb{A} = \mathbb{A}(c_1, c_2, c_3) = \arg(\langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \langle v_3, v_1 \rangle).$$

One verifies that \mathbb{A} is well defined, \mathbb{A} is independent of the chosen polar vectors, and that \mathbb{A} is invariant with respect to the diagonal action of $\text{PU}(2, 1)$.

Remark 4.6 *It is convenient (for the use in geometric applications) to consider \mathbb{A} in the interval $(-\pi, \pi]$.*

Remark 4.7 *The angular invariant \mathbb{A} enjoys a lot of properties of Cartan's angular invariant defined for ordered triples of isotropic points, see, Goldman [15]. We just remark here that \mathbb{A} is not the only invariant of a triple of complex geodesics, and that \mathbb{A} does not enjoy the cocycle property: the equality*

$$\mathbb{A}(c_1, c_2, c_3) + \mathbb{A}(c_1, c_3, c_4) = \mathbb{A}(c_1, c_2, c_4) + \mathbb{A}(c_2, c_3, c_4)$$

does not hold in general.

An ordered m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ is said to be *generic* if the complex geodesics c_i and c_j are not orthogonal for all $i, j = 1 \dots, m$. Otherwise, C is called *special*. It is clear that C is generic if and only if the corresponding m -tuple of polar points is generic, see Section 4.1.

Remark 4.8 *It is easy to see that the subset of the configuration space \mathcal{M} of ordered m -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ corresponding to all generic configurations is open in \mathcal{M} , and that the subset of \mathcal{M} corresponding to all special configurations is nowhere dense in \mathcal{M} .*

Let $C = (c_1, \dots, c_m)$ be an ordered generic m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and let $p = (p_1, \dots, p_m)$ be the corresponding m -tuple of their polar points. Let $G(p) = (g_{ij})$ be the normalized Gram matrix associated to p , see Section 4.1. We call the matrix $G(C) = G(p)$ the *normalized Gram matrix associated to C* . Since $g_{ij} \neq 0$, we write g_{ij} in the following form: $g_{ij} = |g_{ij}|e^{i\alpha_{ij}} = r_{ij}e^{i\alpha_{ij}}$, where $\alpha_{ij} = \arg(g_{ij}) \in (-\pi, \pi]$. In particular, we have that $g_{1j} = r_{1j} > 0$.

Now, given an ordered generic m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, we define: $d_{ij} = d(c_i, c_j)$, where $i \neq j$, and $\mathbb{A}_{ij} = \mathbb{A}(c_1, c_i, c_j)$, where $i \neq j$; $i, j > 1$.

Then we associate to C the following invariants: d_{ij} , $1 \leq i < j \leq m$, and \mathbb{A}_{ij} , $2 \leq i < j \leq m$. It readily seen that the number of d_{ij} is equal to $m(m-1)/2$, and that the number of \mathbb{A}_{ij} is equal to $(m-1)(m-2)/2$.

Proposition 4.4 *The invariants d_{ij} , $1 \leq i < j \leq m$, and \mathbb{A}_{ij} , $2 \leq i < j \leq m$, define uniquely the $\text{PU}(2, 1)$ -congruence class of an ordered regular generic m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$.*

PROOF: It follows from the definition of the Gram matrix that

$$d_{ij} = d(c_i, c_j) = g_{ij}g_{ji} = |g_{ij}|^2,$$

and that

$$\mathbb{A}_{ij} = \mathbb{A}(c_1, c_i, c_j) = \arg(g_{1i} g_{ij} g_{j1}) = \arg(r_{1i} g_{ij} r_{j1}).$$

The first equality implies that $|g_{ij}| = \sqrt{d_{ij}}$. Since C is generic, we have that $r_{1j} > 0$ for all $j > 1$, and that $g_{ij} \neq 0$. Therefore, the second equality implies that $\mathbb{A}_{ij} = \arg(g_{ij})$. Thus, all the entries of the normalized Gram matrix $G(C)$ of C are recovered uniquely in terms of the invariants d_{ij} and \mathbb{A}_{ij} above. Now the proposition follows from Corollary 4.1. \blacksquare

Corollary 4.4 *The $\text{PU}(2, 1)$ -congruence class of an ordered regular generic m -tuple $C = (c_1, \dots, c_m)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ is defined uniquely by $(m-1)^2$ real numbers.*

Now we are ready to construct the moduli space in the regular generic case.

Let $C = (c_1, \dots, c_m)$ be an ordered regular generic m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and $G = (g_{ij})$ be the normalized Gram matrix associated to C . By applying the formulae from the proof of Proposition 3.1, we can recover all of the entries of G in terms of the invariants d_{ij} and \mathbb{A}_{ij} . In what follows, we will assume that G has the entries expressed in this way. So, we can write that $G = G(d_{ij}, \mathbb{A}_{ij})$. We denote by G_{i_1, \dots, i_k} , where $1 \leq i_1 < i_2 < \dots < i_k \leq m$, the submatrix of G obtained by taking the rows i_1, \dots, i_k and the columns i_1, \dots, i_k . Also, we denote by D_{i_1, \dots, i_k} the determinant of G_{i_1, \dots, i_k} . We will also consider D_{i_1, \dots, i_k} as a function of d_{ij} and \mathbb{A}_{ij} .

Let \mathcal{M}_0 denote the configuration space of ordered regular generic m -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and let $[C] \in \mathcal{M}_0$ be the point represented by C . We define the map

$$\tau : \mathcal{M}_0 \longrightarrow \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d,$$

where $d = d_1 + d_2 = (m-1)^2$, and $d_1 = m(m-1)/2$, $d_2 = (m-1)(m-2)/2$, by associating to $[C]$ the invariants d_{ij} and \mathbb{A}_{ij} above. Given $w \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, we will write $w = (u, v)$, where $u = (u_1, \dots, u_{d_1}) \in \mathbb{R}^{d_1}$ and $v = (v_1, \dots, v_{d_2}) \in \mathbb{R}^{d_2}$. Therefore, the functions $D_{i_1, \dots, i_k} = D_{i_1, \dots, i_k}(d_{ij}, \mathbb{A}_{ij})$ define the functions $D_{i_1, \dots, i_k}(u, v)$ of (u, v) if we use the lexicographic order for d_{ij} and \mathbb{A}_{ij} .

Theorem 4.4 *The configuration space \mathcal{M}_0 is homeomorphic to the set \mathbb{M}_0 of points in $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ defined by the following conditions:*

$$D_{i_1, \dots, i_k}(u, v) = 0, \quad \forall 1 \leq i_1 < i_2 < \dots < i_k \leq m, \quad \forall k \geq 4,$$

and

$$D_{i_1, i_2, i_3}(u, v) \leq 0, \quad \forall 1 \leq i_1 < i_2 < i_3 \leq m,$$

provided that $u_i > 0$ for all $i = 1, \dots, d_1$, and $v_i \in (-\pi, \pi]$ for all $i = 1, \dots, d_2$.

PROOF: It follows from the "if" part of Theorem 4.3 and from the formulae in the proof of Proposition 4.4 that the map τ above defines a map $\tau : \mathcal{M}_0 \longrightarrow \mathbb{M}_0$. First, we show that the map $\tau : \mathcal{M}_0 \longrightarrow \mathbb{M}_0$ is surjective. Given $w = (u, v) \in \mathbb{M}_0$, we construct a Hermitian $m \times m$ -matrix $G = (g_{ij})$ as follows. Using again the formulae in the proof of Proposition 4.4, we define g_{ij} , $i \neq j$, in terms of (u, v) identifying u_1, \dots, u_{d_1} with d_{ij} and v_1, \dots, v_{d_2} with \mathbb{A}_{ij} . Also, we put $g_{ii} = 1$. This defines G completely. Then it is readily seen that G satisfies all of the conditions in the "only if" part of Theorem 4.3 under the conditions of the theorem. Hence, this implies that G is the normalized Gram matrix for some ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. This proves that τ is surjective. On the other hand, it follows from Proposition 4.4 that τ is injective. It is clear that $\tau : \mathcal{M}_0 \longrightarrow \mathbb{M}_0$ is a homeomorphism provided that \mathbb{M}_0 is equipped with the topology induced from \mathbb{R}^d . This completes the proof of the theorem. \blacksquare

Remark 4.9 *We call \mathbb{M}_0 the moduli space for \mathcal{M}_0 .*

Corollary 4.5 *The moduli space \mathbb{M}_0 is a semi-analytic set in the euclidian space of dimension $d = (m-1)^2$.*

Next, as a corollary of Theorem 4.4, we give an explicit description of the moduli space of ordered regular generic triples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. First of all, it follows from Proposition 4.4 that PU(2, 1)-congruence class of an ordered regular generic triple $C = (c_1, c_2, c_3)$ of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ is described uniquely by three d-invariants d_{12}, d_{13}, d_{23} and the angular invariant $\alpha = \mathbb{A}(2, 3) = \mathbb{A}(c_1, c_2, c_3)$. Now, let $G = (g_{ij})$ be the normalized Gram matrix of C . Then $g_{ii} = 1$, $g_{1j} = r_{1j} > 0$, and $\arg(g_{23}) = \mathbb{A}(2, 3)$. We have that $d_{1j} = r_{1j}^2$ and $d_{23} = r_{23}^2$. Also, $g_{23} = r_{23} e^{i\alpha} = r_{23}(\cos \alpha + i \sin \alpha)$. A straightforward computation shows that

$$\det G = 1 - (r_{12}^2 + r_{13}^2 + r_{23}^2) + 2r_{12}r_{13}r_{23} \cos \alpha.$$

Using the lexicographic order, we define $r_1 = \sqrt{d_{12}}$, $r_2 = \sqrt{d_{13}}$, $r_3 = \sqrt{d_{23}}$. Then by applying Theorem 4.4, we get the following result which generalizes Lemma 2.2.1 in [18] and Proposition 1 in [25], see also [26].

Corollary 4.6 *The configuration space $\mathcal{M}_0(3)$ of ordered regular generic triples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ is homeomorphic to the set*

$$\mathbb{M}_0(3) = \{(r_1, r_2, r_3, \alpha) \in \mathbb{R}^4 : r_i > 0, \alpha \in (-\pi, \pi], 1 - (r_1^2 + r_2^2 + r_3^2) + 2r_1r_2r_3 \cos \alpha \leq 0\}.$$

The equality in the last inequality happens if and only if the complex geodesics c_1, c_2, c_3 intersect in a point $p \in \mathbf{H}_{\mathbb{C}}^2$, or c_1, c_2, c_3 have a common perpendicular.

Remark 4.10 *If $\det G = 0$, then the polar points to the complex geodesics c_1, c_2, c_3 belong to a polar sphere $S(C)$ of C . Since C is regular, we have that C is either elliptic (in this case, c_1, c_2, c_3 intersect in a negative point which is the polar point to $S(C)$), or hyperbolic (in this case, c_1, c_2, c_3 are orthogonal to $S(C)$).*

4.3 Polygonal configurations of complex geodesics

Given a complex geodesic c in $\mathbf{H}_{\mathbb{C}}^2$, let \bar{c} denote its closure in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. An ordered m -tuple $S = (s_1, \dots, s_m)$ of distinct points in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ is said to be in *general position*, or, equivalently, S is of *general type* if the set $\{s_1, \dots, s_m\}$ contains no sub-triples of distinct points lying in \bar{c} for some complex geodesic c . In particular, a triple $S = (s_1, s_2, s_3)$ of distinct points in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ is in general position, if it is not contained in the closure of a complex geodesic.

Let $S = (s_1, \dots, s_m)$ be an ordered m -tuple of distinct points in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$. We define c_i to be a unique complex geodesic determined by s_i and s_{i+1} , $i = 1, \dots, m$, where the indices are taken modulo m . That is, the complex geodesic c_i is defined by a unique complex projective line spanned by s_i and s_{i+1} : c_i is the intersection of this complex projective line with $\mathbf{H}_{\mathbb{C}}^2$.

Now let $S = (s_1, \dots, s_m)$ be an ordered m -tuple of distinct points in $\mathbf{H}_{\mathbb{C}}^2 \cup \partial\mathbf{H}_{\mathbb{C}}^2$ in general position. Since S is of general type, all the complex geodesics c_i defined above are distinct, and, therefore, the above construction associates to S an ordered m -tuple of distinct complex geodesics $P = P(S) = (c_1, \dots, c_m)$. We say that $P = (c_1, \dots, c_m)$ is a *closed polygonal configuration* of complex geodesics, or, simply, that P is a *c-polygon* (a closed polygon of complex geodesics). The points s_1, \dots, s_m are called the *vertices* of P , and the complex geodesics c_1, \dots, c_m the *sides* of P . Vertex s_i is called *proper* if $s_i \in \mathbf{H}_{\mathbb{C}}^2$. Otherwise, it is called *ideal*. We say that a c-polygon P is *proper* if all of its vertices are proper. Also, we say that a c-polygon P is *ideal* if all of its vertices are ideal. We define a c-polygon $P = (c_1, \dots, c_m)$ to be *simple* if $\bar{c}_i \cap \bar{c}_j \neq \emptyset$ implies that either $j = i - 1$, or $j = i$, or $j = i + 1$ (modulo m). Since the complex geodesics c_i are defined uniquely by s_i and s_{i+1} , it follows that $\bar{c}_i \cap \bar{c}_{i+1} = s_{i+1}$ for all $i = 1, \dots, m$ (modulo m).

Let $P = (c_1, \dots, c_m)$ be a c-polygon with vertices $S = (s_1, \dots, s_m)$. If s_i is proper, then the *complex angle* θ_i at s_i is defined to be the angle between the complex geodesics c_{i-1} and c_i (again, the indices are taken modulo m). So, in this case, $\cos^2(\theta_i) = d_{(i-1)i}$. We remark that $0 < \theta_i \leq \pi/2$.

We say that a c-polygon $P = (c_1, \dots, c_m)$ with vertices $S = (s_1, \dots, s_m)$ is *acute* if for any proper vertex s_i of P the complex angle $\theta_i \neq \pi/2$. Since any c-polygon $P = (c_1, \dots, c_m)$ is obviously regular, Proposition 4.4 implies the following.

Theorem 4.5 *The invariants d_{ij} , $1 \leq i < j \leq m$, and \mathbb{A}_{ij} , $2 \leq i < j \leq m$, define uniquely the $\text{PU}(2, 1)$ -congruence class of an acute c-polygon $P = (c_1, \dots, c_m)$.*

Let $P = (c_1, \dots, c_m)$ be a proper c-polygon, and let $\theta_1, \dots, \theta_m$ be its complex angles. In this case, we will use the following notation: $P = P(\theta_1, \dots, \theta_m)$. We denote by $\mathbb{D}(\theta_1, \dots, \theta_m)$ the set of the $\text{PU}(2, 1)$ -congruence classes of c-polygons with fixed complex angles $\theta_1, \dots, \theta_m$. We call $\mathbb{D}(\theta_1, \dots, \theta_m)$ the *deformation space* of $P(\theta_1, \dots, \theta_m)$. By rewriting Theorem 4.4, one can describe the moduli space $\mathbb{M}(\theta_1, \dots, \theta_m)$

for $\mathbb{D}(\theta_1, \dots, \theta_m)$ in terms of the invariants d_{ij} and \mathbb{A}_{ij} in the case of acute proper c-polygons. Also, it is not difficult to describe the moduli space of ideal c-polygons (it is useful to compare our description with the description of ideal triangles in [16] given in terms of Cartan's angular invariant). Simple c-polygons are characterized by the following condition: the d-invariants $d_{ij} > 1$ for any pair (c_i, c_j) of non-adjacent sides of P . We leave the details to the reader.

As an application of the results above, we give a description of the moduli space for the space of representations of plane hyperbolic co-compact Coxeter groups in $\text{PU}(2, 1)$. We recall that a plane hyperbolic co-compact Coxeter group is defined as a group generated by reflections in the sides of a geodesic convex closed compact polygon in the hyperbolic plane \mathbb{H}^2 . Such a group Γ has the following presentation:

$$\Gamma = \langle \gamma_1, \dots, \gamma_m : \gamma_i^2 = (\gamma_i \gamma_{i+1})^{n_i} = 1, i = 1, \dots, m \rangle,$$

where $n_i \geq 2$, $m \geq 3$, and the indices are taken modulo m .

Now let $P = (c_1, \dots, c_m)$ be a proper c-polygon with complex angles $\theta_1, \dots, \theta_m$, where $\theta_i = \pi/n_i$. We denote by Γ^* the subgroup of $\text{PU}(2, 1)$ generated by g_i , where g_i is inversion in c_i , $i = 1, \dots, m$. Since $(g_i g_{i+1})^{n_i} = 1$ (see, for instance, Goldman [15]), it follows that the map $\gamma_i \mapsto g_i$ defines a homomorphism $\Gamma \rightarrow \Gamma^*$, and, therefore, a representation of Γ into $\text{PU}(2, 1)$. Let $\text{Rep}(\Gamma)$ denote the space of such representations (up to the diagonal action of $\text{PU}(2, 1)$) equipped with the topology of convergence on generators. We call $\text{Rep}(\Gamma)$ the *deformation space* of Γ . Applying Theorem 4.4, we get the following.

Theorem 4.6 *The deformation space $\text{Rep}(\Gamma)$ is homeomorphic to the space $\mathbb{M}(\theta_1, \dots, \theta_m)$, where $\theta_i = \pi/n_i$, provided that $n_i > 2$ for all $i = 1, \dots, m$.*

Remark 4.11 *Since the group Γ contains a subgroup of finite index isomorphic to the fundamental group of a closed orientable surface of genus $g \geq 2$, this theorem gives a method for constructing a class of representations of surface groups in $\text{PU}(2, 1)$.*

4.4 The moduli space of regular configurations. Special case

When $C = (c_1, \dots, c_m)$ is special, the construction of the invariants which define the $\text{PU}(2, 1)$ -congruence class of C is more complicated. A reason is that in the special case it is impossible to recover all of the entries of Gram matrices of C in terms of the "natural" invariants d_{ij} and \mathbb{A}_{ij} defined above. So, one needs new invariants. In this section, we show how to construct the invariants which describe uniquely the $\text{PU}(2, 1)$ -congruence class of any special configuration of complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and describe the moduli space of such configurations.

Let $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and $G = (g_{ij})$ be a Gram matrix associated to C . For brevity, we will call the complex geodesics c_1, \dots, c_m the *lines*. A line c_i is defined to be *bad* if the number of lines in C orthogonal to c_i is greater than one. A configuration $C = (c_1, \dots, c_m)$ is called *good* if it contains no bad lines. In particular, for any good configuration C the number of zeros in any line of the matrix G is less than two. It is clear that any generic configuration of complex geodesics is good.

Let c_1 and c_2 be ultra-parallel complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and let v_1 and v_2 be polar vectors to c_1 and c_2 . Then there exists a complex geodesic c orthogonal to c_1 and c_2 . A polar vector v to c can be found as the cross-product $v = v_1 \boxtimes v_2$, see, Goldman [15]. We recall that such a complex geodesic c is called a common perpendicular to c_1 and c_2 . We did not find in the literature any proof of the fact that common perpendicular is unique. For completeness, we provide a proof of this simple fact as an illustration of how our Theorem 4.4 works.

We say that a proper c-polygon $P = (c_1, \dots, c_m)$ is *right-angled* if $\theta_i = \pi/2$ for all $i = 1, \dots, m$.

Proposition 4.5 *Right-angled proper c-polygons do not exist for $m = 4$.*

PROOF: Let us suppose that such a c-polygon $P = (c_1, \dots, c_4)$ exists. Let v_1, \dots, v_4 be polar vectors to c_1, \dots, c_4 , and $G = (g_{ij})$ be the Gram matrix of P defined by these vectors. Then $g_{12} = 0$, $g_{23} = 0$, $g_{34} = 0$, $g_{14} = 0$. Applying Proposition 4.1, we may normalize so that $g_{ii} = 1$, and $g_{13} = r_{13} \geq 0$. In this case, G admits a further normalization: replacing the vector v_2 , if necessarily, by $\lambda_2 v_2$, where λ_2 is appropriate unitary complex number, one may assume that $g_{24} = r_{24} \geq 0$. A straightforward calculation shows that $\det G = (1 - r_{13}^2)(1 - r_{24}^2)$, and that $D_{123} = 1 - r_{13}^2$, $D_{124} = 1 - r_{24}^2$. Applying Theorem 4.4, we conclude that $\det G = 0$, and that $D_{123} \leq 0$, $D_{124} \leq 0$. This is elementary to verify that the vectors v_1, v_2, v_3 as well the vectors v_1, v_2, v_4 are linearly independent, so $D_{123} < 0$, $D_{124} < 0$. All this implies that

$$(1 - r_{13}^2)(1 - r_{24}^2) = 0, \quad 1 - r_{13}^2 < 0, \quad 1 - r_{24}^2 < 0.$$

These conditions are clearly incompatible. This proves the claim. \blacksquare

Corollary 4.7 *If two complex geodesics c_1 and c_2 are ultra-parallel, then they have a common orthogonal complex geodesic c . This complex geodesic c is unique.*

Let now $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. It is not difficult to show that there exists an ordered sub-tuple $C^* = (c_{i_1}, \dots, c_{i_k})$, where $1 \leq c_{i_1} < \dots < c_{i_k} \leq m$, $k \geq 2$, which contains no bad lines. For instance, one can construct such a sub-tuple using the following algorithm: if C contains no bad lines, then we have done; if C contains a bad line, say c_i , then we remove c_i ; if a new configuration (a sub-tuple) contains no bad lines, then we have done; if a new configuration contains a bad line, then we remove it to get a new configuration, and so on. This algorithm defines a configuration having two or more lines because a configuration with only two lines has no bad lines. We call C^* a *reduced* configuration associated to C . We remark that C^* is not defined uniquely. It is clear that C^* is good, and, as it follows from Corollary 4.7, the configuration C is reconstructed uniquely from any its reduced configuration C^* .

Let C^* be a reduced configuration associated to C . Given a line c_i in the complement of C^* , we denote by C_i^* the set of all lines in C^* orthogonal to c_i . We call the lines in C_i^* the lines *associated* to c_i . Then all these lines are obviously ultra-parallel, and, therefore, all the d-invariants defined by pairs of distinct complex geodesics in C_i^* are greater than one. We call these conditions the *orthogonality conditions* defined by c_i . The orthogonality conditions defined by all c_i we call the *inherited conditions*.

Now we are ready to explain how we are going to construct the moduli space of special configurations. First, we construct the invariants which describe uniquely the PU(2,1)-congruence class of any good configuration of complex geodesics (the set of these invariants contains d-invariants) and describe the moduli space \mathbb{M}^* of such configurations. Then it follows from the discussion above that the moduli space of special configurations \mathbb{M}_m^s can be identified with an open subset of the moduli space of any reduced configuration associated to C : this open set is defined by the inherited conditions.

Let $C = (c_1, \dots, c_k)$ be an ordered k -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and $v = (v_1, \dots, v_k)$ be a k -tuple of polar vectors to c_1, \dots, c_k . We define a complex number

$$\delta_k = \delta_k(c_1, \dots, c_k) = \frac{\langle v_1, v_2 \rangle \langle v_2, v_3 \rangle \cdots \langle v_{k-1}, v_k \rangle \langle v_k, v_1 \rangle}{\langle v_1, v_1 \rangle \cdots \langle v_k, v_k \rangle}.$$

One verifies that δ_k is well defined (we recall that v_i is positive, hence $\langle v_i, v_i \rangle \neq 0$), δ_k is independent of the chosen polar vectors, and that δ_k is invariant with respect to the diagonal action of PU(2,1). Note that this invariant is a natural generalization of the d-invariant, since as it is easy to see $\delta_2(c_1, c_2) = d(c_1, c_2)$. We call $\delta_k = \delta_k(c_1, \dots, c_k)$ the *k-delta invariant*, or, simply, the *delta-invariant*. For $k = 3$, the angular invariant $\mathbb{A}(c_1, c_2, c_3)$ is the argument of $\delta_3 = \delta_3(c_1, c_2, c_3)$ provided that a triple (c_1, c_2, c_3) contains no orthogonal complex geodesics. We remark also that for $k = 3$, a similar invariant was considered by Hakim and Sandler [21] in the case of negative points. We will show that the δ -invariant serves well for describing the PU(2,1)-congruence classes of good configurations of complex geodesics.

Next, we describe the moduli space of regular good configurations of complex geodesics.

Let $\mathcal{M}^*(k)$ be the configuration space of ordered regular good k -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, that is, the quotient of the set of ordered regular good k -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ with respect to the diagonal action of $\text{PU}(2, 1)$ equipped with the quotient topology. Then we split $\mathcal{M}^*(k)$ in the following form:

$$\mathcal{M}^*(k) = \mathcal{M}_1^* \cup \mathcal{M}_2^* \cup \dots \cup \mathcal{M}_{(k-1)}^* \cup \mathcal{M}_k^*,$$

where

1. \mathcal{M}_1^* is the subset of $\mathcal{M}^*(k)$ whose points $[C]$ are represented by k -tuples $C = (c_1, \dots, c_k)$ such that c_1 is not orthogonal to c_j for all $j = 2, \dots, k$;
2. \mathcal{M}_n^* , $n = 2, \dots, (k-1)$, is the subset of $\mathcal{M}^*(k)$ whose points $[C]$ are represented by k -tuples $C = (c_1, \dots, c_k)$ such that c_1 is orthogonal to c_n and c_1 is not orthogonal to c_j for all $j \neq n$;
3. \mathcal{M}_k^* is the subset of $\mathcal{M}^*(k)$ whose points $[C]$ are represented by k -tuples $C = (c_1, \dots, c_k)$ such that c_1 is orthogonal to c_k and c_1 is not orthogonal to c_j for all $j = 2, \dots, (k-1)$.

It is readily seen that $\mathcal{M}^*(k)$ is the disjoint union of \mathcal{M}_i^* , $i = 1, \dots, k$.

Let $C = (c_1, \dots, c_k)$ be an ordered regular good k -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. We say that C is of the *first type* if $[C] \in \mathcal{M}_1^*$, C is of the *second type* if $[C] \in \mathcal{M}_n^*$ for some $n = 2, \dots, (k-1)$, and, finally, C is of the *third type* if $[C] \in \mathcal{M}_k^*$.

We assume that $C = (c_1, \dots, c_k)$ corresponds to a k -tuple $v = (v_1, \dots, v_k)$ of polar vectors. Let $G = (g_{ij})$ be the Gram matrix associated to $v = (v_1, \dots, v_k)$. Applying Proposition 4.1, we may normalize so that $g_{ii} = 1$, and $g_{1j} = r_{1j} \geq 0$, for all $j = 2, \dots, k$. Since C is good, the number of zeros in the first line $L = (1, r_{12}, \dots, r_{1k})$ of $G = (g_{ij})$ is less than two.

If C is of the first type, we have that $r_{1j} > 0$ for all $j = 2, \dots, m$. In this case, the Gram matrix $G = (g_{ij})$ admits no further normalization. If C is of the second or of the third type, then the Gram matrix $G = (g_{ij})$ of C admits a further normalization. First, we remark that if $r_{1n} = 0$ for some $n = 2, \dots, (k-1)$, then necessarily $g_{in} \neq 0$ for all $i = 2, \dots, k$, since if $g_{in} = 0$, then c_n is orthogonal to c_1 and c_i , but this is impossible because C contains no bad lines. By the same reason, if $r_{1k} = 0$, then $g_{jk} \neq 0$ for all $j = 2, \dots, (k-1)$. For configurations of the second type, replacing the vector v_n , if necessarily, by $\lambda_n v_n$, where λ_n is appropriate unitary complex number, we normalize so that $g_{nk} = r_{nk} > 0$. Similarly, for configurations of the third type, we normalize so that $g_{(k-1)k} = r_{(k-1)k} > 0$. We remark that such a normalization defines $G = (g_{ij})$ uniquely. We call a matrix G obtained by this normalization the *completely normalized* Gram matrix of a good configuration C .

Proposition 4.6 *The d -invariants $d_j = d_{1j}$, $1 < j \leq k$, and the 3-delta invariants $\delta(1, i, j) = \delta_3(c_1, c_i, c_j)$, $1 < i < j \leq k$, define uniquely the $\text{PU}(2, 1)$ -congruence class of an ordered regular good k -tuple of distinct complex geodesics of the first type.*

PROOF: Let $C = (c_1, \dots, c_k)$ be an ordered good k -tuple of distinct complex geodesics of the first type and $G = (g_{ij})$ be its completely normalized Gram matrix. Then we have that $g_{ii} = 1$ and $g_{1j} = r_{1j} > 0$ for all $j = 2, \dots, k$. It follows from the definition of the Gram matrix that

$$d_{1j} = d(c_1, c_j) = g_{1j}g_{j1} = |g_{1j}|^2 = r_{1j}^2,$$

and that

$$\delta(1, i, j) = \delta_3(c_1, c_i, c_j) = g_{1i} g_{ij} g_{j1} = r_{1i} g_{ij} r_{j1}.$$

The first equality implies that $r_{1j} = \sqrt{d_{1j}}$. Since $r_{1i} > 0$, $r_{j1} > 0$ for all $i, j > 1$, the second equality implies that $g_{ij} = \delta(1, i, j)/(d_{1i} d_{1j})^{1/2}$, $1 < i < j \leq k$. Thus, all the entries of the completely normalized Gram matrix G of C are recovered uniquely in terms of the invariants d_{1j} and $\delta(1, i, j)$ above. Now the claim of the proposition follows from Proposition 4.3. \blacksquare

Remark 4.12 *It follows from this proposition that if $k = 2$, then we need only one d -invariant.*

Proposition 4.7 *Let $C = (c_1, \dots, c_k)$ be an ordered regular good k -tuple of distinct complex geodesics of the second type such that $[C] \in \mathcal{M}_n^*$ for some fixed $n \in \{2, \dots, (k-1)\}$. Then the $\text{PU}(2,1)$ -congruence class of C is defined uniquely by the d -invariants d_{1j}, d_{nk} , $1 < j \leq k, j \neq n$, the 3-delta invariants $\delta(1, i, j) = \delta_3(c_1, c_i, c_j)$, $1 < i < j \leq k, i, j \neq n$, and the 4-delta invariants $\delta(1, j, n, k) = \delta_4(c_1, c_j, c_n, c_k)$, $1 < j < k, j \neq n$.*

PROOF: Let $G = (g_{ij})$ be the completely normalized Gram matrix of C . Then we have that $g_{ii} = 1$, $g_{1j} = r_{1j} > 0$ for all $j = 2, \dots, k, j \neq n$, $r_{1n} = 0$, and $g_{nk} = r_{nk} > 0$. This immediately implies that $r_{1j} = \sqrt{d_{1j}}$ for all $j = 2, \dots, k, j \neq n$. Therefore, we have already recovered all the entries of the first line of G . Next we write

$$\delta(1, i, j) = \delta_3(c_1, c_i, c_j) = g_{1i} g_{ij} g_{j1} = r_{1i} g_{ij} r_{j1}.$$

Since $r_{1i} > 0$ and $r_{j1} > 0$ for $i, j \neq n$, we get that $g_{ij} = \delta(1, i, j)/(d_{1i}d_{1j})^{1/2}$ for all $1 < i < j \leq k$ and $i, j \neq n$. Finally, let us write

$$\delta(1, j, n, k) = \delta_4(c_1, c_j, c_n, c_k) = r_{1j} g_{jn} r_{nk} r_{k1}.$$

Since $r_{1j} > 0$ for $j \neq n$, and $r_{k1} > 0, r_{nk} > 0$, we get that

$$g_{jn} = \delta(1, j, n, k)/(d_{1j} d_{1k} d_{nk})^{1/2}$$

for $j = 2, \dots, k, j \neq n$. Thus, all the entries of the completely normalized Gram matrix G of C are recovered uniquely in terms of the invariants $d_{1j}, d_{nk}, \delta(1, i, j)$, and $\delta(1, j, n, k)$ above. Again the claim follows from Proposition 4.3. \blacksquare

Proposition 4.8 *Let $C = (c_1, \dots, c_k)$ be an ordered regular good k -tuple of distinct complex geodesics of the third type. Then the $\text{PU}(2,1)$ -congruence class of C is defined uniquely by the d -invariants $d_{1j}, d_{k(k-1)}$, $1 < j \leq k$, the 3-delta invariants $\delta(1, i, j) = \delta_3(c_1, c_i, c_j)$, $1 < i < j \leq (k-1)$, and the 4-delta invariants $\delta(1, i, k, k-1) = \delta_4(c_1, c_i, c_k, c_{(k-1)})$, $1 < i \leq (k-2)$.*

PROOF: Let $G = (g_{ij})$ be the completely normalized Gram matrix of C . Then we have that $g_{ii} = 1$, $g_{1j} = r_{1j} > 0$ for all $j = 2, \dots, (k-1)$, $r_{1k} = 0$, and $g_{(k-1)k} = r_{(k-1)k} > 0$. This immediately implies that $r_{1j} = \sqrt{d_{1j}}$ for all $j = 2, \dots, (k-1)$. Therefore, this recovers all the entries of the first line of G . Next we write

$$\delta(1, i, j) = \delta_3(c_1, c_i, c_j) = g_{1i} g_{ij} g_{j1} = r_{1i} g_{ij} r_{j1}.$$

Since $r_{1i} > 0$ and $r_{j1} > 0$ for all $2 \leq i, j \leq (k-1)$, we get that $g_{ij} = \delta(1, i, j)/(d_{1i} d_{1j})^{1/2}$ for all $1 < i < j \leq (k-1)$. Finally, let us write

$$\delta(1, i, k, k-1) = \delta_4(c_1, c_i, c_k, c_{(k-1)}) = r_{1i} g_{ik} r_{k(k-1)} r_{(k-1)1}.$$

Since $r_{1i} > 0$ for $1 < i \leq (k-1)$, and $r_{(k-1)1} > 0, r_{(k-1)k} > 0$, we get that

$$g_{ik} = \delta(1, i, k, k-1)/(d_{1i} d_{1(k-1)} d_{k(k-1)})^{1/2}$$

for $i = 2, \dots, (k-2)$. Thus, we have recovered uniquely all the entries of the matrix G in terms of the invariants above. As before, the claim follows again from Proposition 4.3. \blacksquare

In order to construct the moduli space \mathbb{M}_k^* for the configuration space $\mathcal{M}^*(k)$ of regular good k -tuples of distinct complex geodesics, we proceed as in Theorem 4.4. First, applying the formulae from Propositions 4.6-4.8, we can express all of the entries of the completely normalized Gram matrix $G = (g_{ij})$ in terms of the invariants constructed in these propositions. Then $\mathcal{M}^*(k)$ is homeomorphic to the set \mathbb{M}_k^* in the

space of invariants \mathbb{R}^d (d depends on a number of the invariants in Propositions 4.6-4.8) defined by the determinant conditions analogous to those in Theorem 4.4. We omit the details.

The set \mathbb{M}_k^* is defined to be the *moduli space* for $\mathcal{M}^*(k)$.

Let now $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and let $C^* = (c_{i_1}, \dots, c_{i_k})$, where $1 \leq c_{i_1} < \dots < c_{i_k} \leq m$, be a reduced configuration associated to C . Then summarizing all the above, we get the following description of the moduli space \mathbb{M}_m^s of special configurations.

Theorem 4.7 *The moduli space \mathbb{M}_m^s is homeomorphic to the open subset of $\mathbb{M}_k^* = \mathbb{M}_k^*(c_{i_1}, \dots, c_{i_k})$ defined by the inherited conditions.*

Next, as an application of Theorem 4.7, we obtain an explicit description of the moduli space of proper right-angled c -polygons $P = (c_1, \dots, c_m)$ for $m = 5$ and $m = 6$. We recall that a proper c -polygon $P = (c_1, \dots, c_m)$ with vertices $S = (s_1, \dots, s_m)$ is right-angled if $\theta_i = \pi/2$ for all $i = 1, \dots, m$. We show that, surprisingly, any right-angled proper c -pentagon is R-plane, that is, all of its vertices are in a totally geodesic real submanifold of $\mathbf{H}_{\mathbb{C}}^2$. On the other hand, proper c -hexagons are interesting, and, it seems, they provide an important class of c -polygons for constructing non-trivial discrete representations of the Coxeter groups in $\text{PU}(2, 1)$, see Section 4.3.

We begin with $m = 5$. Let $P = (c_1, \dots, c_5)$ be a right-angled proper c -pentagon with vertices $S = (s_1, \dots, s_5)$. First, we construct a reduced configuration associated to P . For instance, one may consider $P^* = (c_1, c_3, c_5)$. It is seen that P^* is good, and that P^* is of the third type. The completely normalized Gram matrix $G = (g_{ij})$ of P^* has the following entries: $g_{ii} = 1$, $g_{13} = r_{13} > 0$, $g_{15} = 0$, and $g_{35} = r_{35} > 0$. It follows from the above that P^* is completely determined by two d-invariants: d_{13} and d_{35} . Moreover, $r_{13} = \sqrt{d_{13}}$ and $r_{35} = \sqrt{d_{35}}$. The inherited conditions are defined by c_2 and c_4 . They are $d_{13} > 1$ and $d_{35} > 1$. A straightforward computation shows that $\det G = 1 - (r_{13}^2 + r_{35}^2)$. So, the determinant condition is equivalent to the inequality $\det G = 1 - (r_{13}^2 + r_{35}^2) < 0$. It is seen that the inherited conditions imply the determinant condition. All this implies that the moduli space of right-angled c -pentagons may be identified with $\mathbb{M}_5^s = \{(x, y) : x > 1, y > 1\}$. To prove that P is R-plane, we construct a c -pentagon P' in the totally real totally geodesic hyperbolic plane $\mathbb{H}_{\mathbb{R}}^2 \subset \mathbf{H}_{\mathbb{C}}^2$ having the same invariants $d_{13} > 1$ and $d_{35} > 1$. Then it follows from Theorem 4.6 that P and P' are in the same $\text{PU}(2, 1)$ -congruence class.

Now we consider the case $m = 6$. Let $P = (c_1, \dots, c_6)$ be a proper right-angled c -hexagon with vertices $S = (s_1, \dots, s_6)$. First, we construct a reduced configuration associated to P . For instance, one may consider $P^* = (c_1, c_3, c_5)$. Then we have that P^* is good, and that P^* is of the first type. It follows from Proposition 4.6 that the d-invariants $d_1 = d_{13}$, $d_2 = d_{35}$, and the 3-delta invariant $\delta = \delta_3(c_1, c_3, c_5)$ define uniquely the $\text{PU}(2, 1)$ -congruence class of P^* . The inherited conditions are defined by c_2 , c_4 , and c_6 . They are $d_{13} > 1$, $d_{35} > 1$, and $d_{15} > 1$, where d_{15} is defined from the equality $d_{15} = |\delta|/(d_1 d_2)$. Therefore, applying Theorem 4.6, we get a description of the moduli space of proper right-angled c -hexagons in terms of the d-invariants and the 3-delta invariant. It should be noticed that in this case, P^* is generic under the inherited conditions: its complex sides are all ultra-parallel. So, one may describe the moduli space of P^* , and, hence, the moduli space of P using more "natural" invariants: the d-invariants and the angular invariant. This has been done in Corollary 4.6. Rewriting the statement of this corollary and adding the inherited conditions, we have the following description of the moduli space of proper right-angled c -hexagons.

Theorem 4.8 *The configuration space $\mathcal{M}(6)$ of proper right-angled c -hexagons is homeomorphic to*

$$\mathbb{M}(6) = \{(r_1, r_2, r_3, \alpha) \in \mathbb{R}^4 : r_i > 1, \alpha \in (-\pi, \pi], 1 - (r_1^2 + r_2^2 + r_3^2) + 2r_1r_2r_3 \cos \alpha < 0\}.$$

As a corollary, we get the following.

Theorem 4.9 *The deformation space $\text{Rep}(\Gamma)$, where Γ is a Coxeter group with presentation*

$$\Gamma = \langle \gamma_1, \dots, \gamma_6 : \gamma_i^2 = (\gamma_i \gamma_{i+1})^2 = 1, i = 1, \dots, 6 \rangle$$

is homeomorphic to the space $\mathbb{M}(6)$ above.

4.5 The moduli space of degenerate configurations

Let $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and $v = (v_1, \dots, v_m)$ be a m -tuple of polar vectors to c_1, \dots, c_m . We recall that C degenerate or parabolic iff the subspace $V(C) \subset \mathbb{C}^{2,1}$ spanned by the vectors v_1, \dots, v_m is degenerate. In this section, we construct the moduli space of degenerate configurations. First, we will show that it is impossible to describe the $\text{PU}(2,1)$ -congruence classes of degenerate configurations using invariants defined in terms of hermitian products: a reason is that the Witt theorem does not work in the degenerate case. Then we will introduce new invariants (they have non-hermitian nature) which describe the $\text{PU}(2,1)$ -congruence classes of degenerate configurations.

In this section, it is more convenient to work with coordinates in $\mathbb{C}^{2,1}$ in which the Hermitian product is represented by:

$$\langle Z, W \rangle = z_1 \bar{w}_3 + z_2 \bar{w}_2 + z_3 \bar{w}_1,$$

where

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

These coordinates were used by Burns-Shnider [4], Epstein [10], Parker-Platis [24], and also in [19], [20], [9],[6], [7] among others. In these coordinates, the boundary of complex hyperbolic space is identified with the one point compactification of the boundary of the Siegel domain \mathcal{S}^2 . Following Goldman-Parker [17], we give the Siegel domain *horospherical coordinates*. Recall that the Heisenberg group is $\mathcal{H} = \mathbb{C} \times \mathbb{R}$ with the group law

$$(z, t) * (z', t') = (z + z', t + t' + 2\text{Im}(zz')).$$

Complex hyperbolic space (the Siegel domain) is parametrised in horospherical coordinates by $\mathcal{H} \times \mathbb{R}_+$:

$$\psi : (z, t, u) \mapsto \begin{bmatrix} -|z|^2 - u + it \\ z\sqrt{2} \\ 1 \end{bmatrix} \quad \text{for } (z, t, u) \in \overline{\mathcal{S}^2} - \{q_\infty\}; \quad \psi : q_\infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

where $q_\infty = \infty$ is a distinguished point in the boundary of $\mathbf{H}_{\mathbb{C}}^n$.

We start with the following example. Let $C = (c_1, c_2, c_3)$ be an ordered triple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ corresponding to polar vectors v_1, v_2, v_3 :

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} z \\ 1 \\ 0 \end{bmatrix},$$

where $z \neq 0, 1$.

Then we have that $g_{ij} = \langle v_i, v_j \rangle = 1$ for all $i, j = 1, 2, 3$. So, the Gram matrix $G = (g_{ij})$ associated to C which is defined by the vectors v_1, v_2, v_3 does not depend on $z \in \mathbb{C}$. It is easy to verify that v_1, v_2, v_3 are orthogonal to the isotropic vector $v_0 = (1, 0, 0)^t$. This implies that C is degenerate and that the polar points p_1, p_2, p_3 to c_1, c_2, c_3 belong to the polar sphere $S(C)$ which is tangent to $\partial\mathbf{H}_{\mathbb{C}}^2$ at p_0 , where

$p_0 = \pi(v_0) \in \partial\mathbf{H}_{\mathbb{C}}^2$. Now we show that the triples $C = (c_1, c_2, c_3)$ and $C' = (c_1, c_2, c'_3)$ corresponding to the triples of polar vectors $v = (v_1, v_2, v_3)$ and $v' = (v_1, v_2, v'_3)$, where $v_3 = (z, 1, 0)^t$ and $v'_3 = (z', 1, 0)^t$ are not congruent in $\text{PU}(2, 1)$ provided that $z \neq z'$. It is clear that $p_3 \neq p'_3$ iff $z \neq z'$. Let us assume that there exists $\gamma \in \text{PU}(2, 1)$ such that $\gamma(C) = C'$. Then $\gamma(p_1) = p_1$ and $\gamma(p_2) = p_2$. On the other hand, as it is readily seen, $\gamma(p_0) = p_0$. All this implies that γ is the identity, a contradiction, because $p_3 \neq p'_3$. This example shows that there exist ordered triples of distinct complex geodesics (necessarily degenerate) which are not congruent in $\text{PU}(2, 1)$ but have the same Gram matrix. Therefore, there are no invariants defined in terms of hermitian products which distinguish between the $\text{PU}(2, 1)$ -congruence classes of ordered triples of distinct complex geodesics in the degenerate case. In what follows, we will introduce new invariants to solve this problem.

4.5.1 The polar spheres of degenerate configurations of complex geodesics

In this section, we study the geometry of polar spheres of degenerate configurations in more details.

Proposition 4.9 *For any isotropic point $p \in \partial\mathbf{H}_{\mathbb{C}}^2$ there exists a unique complex projective line $S_p \subset \mathbb{P}\mathbb{C}^2$ tangent to $\partial\mathbf{H}_{\mathbb{C}}^2$ at p .*

PROOF: Let $l_p \subset \mathbb{C}^{2,1}$ be the complex line determined by the point p . Let $V_p \subset \mathbb{C}^{2,1}$ be the Hermitian complement to l_p , that is,

$$V_p = \{v \in \mathbb{C}^{2,1} : \langle v, w \rangle = 0, \forall w \in l_p\}.$$

Then it is easy to see that V_p is a complex linear subspace of dimension 2 and $l_p \subset V_p$. Moreover, the intersection of V_p with the set of isotropic vectors is exactly l_p , and all the vectors in $V_p \setminus l_p$ are positive. All this implies that the complex projective line $S_p = \pi(V_p \setminus \{0\}) \subset \mathbb{P}\mathbb{C}^2$ is tangent to $\partial\mathbf{H}_{\mathbb{C}}^2$ at p . It is easy to see that such S_p is unique. \blacksquare

Corollary 4.8 *The complex 2-space V_p is a degenerate (singular) subspace of the Hermitian space $\mathbb{C}^{2,1}$. Moreover, any degenerate subspace of dimension 2 of the Hermitian space $\mathbb{C}^{2,1}$ is of the form V_p for some isotropic point p .*

Remark 4.13 *In what follows, we will identify S_p with the Riemann sphere $\overline{\mathbb{C}}_p = \mathbb{C}_p \cup \{p\}$, where \mathbb{C}_p is identified with the set of complex numbers. We call S_p the polar sphere based at p , or the parabolic sphere based at p .*

Corollary 4.9 *Let $C = (c_1, \dots, c_m)$ be an ordered m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, and let $P = (p_1, \dots, p_m)$ be the corresponding m -tuple of their polar points. Then C is degenerate if and only if all the complex geodesics c_1, \dots, c_m intersect at a single isotropic point $p \in \partial\mathbf{H}_{\mathbb{C}}^2$. In this case, the points p_1, \dots, p_m belong to the polar sphere S_p based at p .*

Proposition 4.10 *Let S_p and S_q be the polar spheres based at p and q respectively, where $p, q \in \partial\mathbf{H}_{\mathbb{C}}^2$. Let $\gamma \in \text{PU}(2, 1)$ such that $\gamma(p) = q$. Then $\gamma(S_p) = S_q$. Moreover, $\gamma : S_p \rightarrow S_q$ determines a projective isomorphism between S_p and S_q considered as complex projective lines.*

PROOF: The proof of this proposition follows easily from the proof of Proposition 4.9. \blacksquare

Proposition 4.11 *Let S_p and S_q be the polar spheres based at p and q respectively, where $p, q \in \partial\mathbf{H}_{\mathbb{C}}^2$. Then S_p and S_q are projectively equivalent as complex projective lines, where a projective isomorphism can be defined by $\gamma \in \text{PU}(2, 1)$ such that $\gamma(p) = q$.*

PROOF: Since the group $\text{PU}(2, 1)$ acts transitively on $\partial\mathbf{H}_{\mathbb{C}}^2$, the proof follows from Proposition 4.10. \blacksquare

4.5.2 Action of $\text{PU}(2,1)$ on the parabolic spheres

Let $L \equiv \mathbb{P}\mathbb{C}^1 \subset \mathbb{P}\mathbb{C}^2$ be a complex projective line in $\mathbb{P}\mathbb{C}^2$, and let $G(L) \equiv \text{PGL}(2, \mathbb{C})$ be the projective automorphism group of L . In general, not every subgroup of $G(L)$ is the restriction of a subgroup of the stabilizer of L in $\text{PU}(2,1)$. In this section, we show that if $L = S_p$ is a polar sphere based at a point $p \in \partial\mathbf{H}_{\mathbb{C}}^2$, then every element of the stabilizer $\Gamma_p \subset G(L)$ of p in $G(L)$ has a natural extension to an element of the stabilizer Γ_p^* of p in $\text{PU}(2,1)$. In other words, the restriction homomorphism $\Gamma_p^* \rightarrow \Gamma_p$ is surjective.

Proposition 4.12 *Let $p \in \partial\mathbf{H}_{\mathbb{C}}^2$, and let S_p be the polar sphere based at p . Let Γ_p be the stabilizer of p in the projective automorphism group $G(S_p)$ of S_p and Γ_p^* be the stabilizer of p in $\text{PU}(2,1)$. Then the restriction homomorphism $\Gamma_p^* \rightarrow \Gamma_p$ is well defined and surjective.*

Remark 4.14 *It will be shown in the proof of this proposition that the restriction homomorphism $\Gamma_p^* \rightarrow \Gamma_p$ has non-trivial kernel, that is, the action of Γ_p^* on S_p is not faithful.*

PROOF: Let $p \in \partial\mathbf{H}_{\mathbb{C}}^2$, and let S_p be the polar sphere based at p . Then it follows from Proposition 4.9 that $\Gamma_p^*(S_p) = S_p$. Moreover, applying Proposition 4.10, we may assume that $p = \infty$ which is represented by the isotropic vector $v_0 = (1, 0, 0)^t$. This implies that the subspace $V_p \subset \mathbb{C}^{2,1}$ may be identified with $V_\infty = \{v = (z_1, z_2, 0)^t : z_1, z_2 \in \mathbb{C}\}$. Since $z_2 \neq 0$ for all $v \in V_\infty \setminus \{v_0\}$, then projectivising, we may projectively identify S_p with $S_\infty \equiv \{v = (z, 1, 0)^t : z \in \mathbb{C}\} \cup \{\infty\} \equiv \mathbb{C} \cup \{\infty\}$. Therefore, Γ_p is identified with the group of affine automorphisms of $\mathbb{C} : \Gamma_p \equiv \Gamma_\infty = \{\gamma(z) = az + b : a, b \in \mathbb{C}\}$.

It is well known, see, for instance, Falbel-Parker [13], that if $A \in U(2,1)$ fixes the line spanned by v_0 , then it is upper triangular, and the stabilizer of this line in $U(2,1)$ is generated by the following matrices:

$$T = \begin{bmatrix} 1 & -\bar{z}_0 & (-|z_0|^2 + it_0)/2 \\ 0 & 1 & z_0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix},$$

where $z_0 \in \mathbb{C}$, and $t_0, \theta, \lambda \in \mathbb{R}$, $\lambda > 0$.

A straightforward computation shows that

$$T(z, 1, 0)^t = (z - \bar{z}_0, 1, 0)^t, \quad R(z, 1, 0)^t = (z, e^{i\theta}, 0)^t, \quad D(z, 1, 0)^t = (\lambda z, 1, 0)^t.$$

Hence, the projectivisations T^*, R^*, D^* of $T, R,$ and D act on S_∞ as follows

$$T^*(z) = z - \bar{z}_0, \quad R^*(z) = e^{-i\theta}z, \quad D^*(z) = \lambda z,$$

and $T^*(\infty) = \infty, R^*(\infty) = \infty, D^*(\infty) = \infty$.

These formulae imply that the restriction homomorphism $\Gamma_\infty^* \rightarrow \Gamma_\infty$ is well defined and surjective. Moreover, it follows that it has non-trivial kernel which is generated by the elements in Γ_∞^* which correspond to matrices T with $z_0 = 0$ and $t_0 \neq 0$. ■

4.5.3 Parabolic invariant of degenerate configurations

In this section, we introduce the invariants which distinguish between the $\text{PU}(2,1)$ -congruence classes of ordered m -tuples of distinct complex geodesics in the degenerate case.

First, we recall the definition of the classical cross-ratio. Let $p = (z_1, z_2, z_3, z_4)$ be an ordered quadruple of distinct points in \mathbb{C} . Then the *cross-ratio* of p is defined to be

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}.$$

The definition can be extended to the case when one of the points z_i is ∞ , for instance,

$$[\infty, z_2, z_3, z_4] = \frac{(z_2 - z_4)}{(z_3 - z_4)}.$$

Also, a conventional value of the cross-ratio can be defined when any three of the four points are distinct. It is a classical result that the cross-ratio is the only invariant of ordered quadruples of points in $\overline{\mathbb{C}}$ with respect to the diagonal action of $\mathrm{PGL}(2, \mathbb{C})$, see, for instance, Berdon [1].

We remark that the cross-ratio enjoys the following properties:

1. For any $z \in \mathbb{C}$ such that $z \neq 0$ and $z \neq 1$, $[1, 0, \infty, z] = z$,
2. Given distinct points z_1, z_2, z_3 in $\overline{\mathbb{C}}$, the function $f(z) = [z_1, z_2, z_3, z]$ is a unique element from $\mathrm{PGL}(2, \mathbb{C})$ such that $f(z_1) = 1$, $f(z_2) = 0$, and $f(z_3) = \infty$.

Proposition 4.13 *Let $p = (z_1, \dots, z_m)$ and $p' = (w_1, \dots, w_m)$ be two ordered m -tuples of distinct points in $\overline{\mathbb{C}}$, $m \geq 4$. Then p and p' are congruent with respect to the diagonal action of $\mathrm{PGL}(2, \mathbb{C})$ if and only if*

$$[z_1, z_2, z_3, z_j] = [w_1, w_2, w_3, w_j]$$

for any $j = 4, \dots, m$.

PROOF: The proof follows closely, with a slight modification, the proof of the fact that the cross-ratio is the only invariant of ordered quadruples given in [1]. Let us assume that $[z_1, z_2, z_3, z_j] = [w_1, w_2, w_3, w_j]$ for all $j = 4, \dots, m$. Applying (2), we find $f, g \in \mathrm{PGL}(2, \mathbb{C})$ such that $f(z_1) = 1$, $f(z_2) = 0$, $f(z_3) = \infty$, and $g(w_1) = 1$, $g(w_2) = 0$, $g(w_3) = \infty$. Then applying (1), we have that

$$f(z_j) = [1, 0, \infty, f(z_j)] = [f(z_1), f(z_2), f(z_3), f(z_j)] = [z_1, z_2, z_3, z_j],$$

and

$$g(w_j) = [1, 0, \infty, g(w_j)] = [g(w_1), g(w_2), g(w_3), g(w_j)] = [w_1, w_2, w_3, w_j].$$

Therefore, our assumption implies that $f(z_i) = g(w_i)$ for all $i = 1, \dots, m$. Let now $h = g^{-1}f$. Then $h(z_i) = w_i$ for all $i = 1, \dots, m$. Hence, p and p' are congruent in $\mathrm{PGL}(2, \mathbb{C})$. \blacksquare

Now we are ready to define the invariants which describe the $\mathrm{PU}(2, 1)$ -congruence classes of degenerate configurations of complex geodesics.

Let $C = (c_1, c_2, c_3)$ be an ordered parabolic triple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$ and $P = (p_1, p_2, p_3)$ be the triple of polar points to c_1, c_2, c_3 . Let S_p be the polar sphere of C based at $p \in \partial\mathbf{H}_{\mathbb{C}}^2$. Considering S_p as a complex projective line and using the identifications in Section 4.1, we define

$$\chi(C) = \chi(c_1, c_2, c_3) = \chi(p_1, p_2, p_3) = [p, p_1, p_2, p_3].$$

Then it follows from Section 4.5.2 that χ is well defined, and, since the points p, p_1, p_2, p_3 are all distinct, χ is finite and $\chi \neq 0, 1$, so $\chi \in \mathbb{C}_* = \mathbb{C} \setminus \{0, 1\}$. It also follows from Section 4.5.3 that χ is invariant with respect to the diagonal action of $\mathrm{PU}(2, 1)$. We call χ the *parabolic invariant*. Next we show that χ is the only $\mathrm{PU}(2, 1)$ -invariant of degenerate triples of complex geodesics.

Theorem 4.10 *Let $C = (c_1, c_2, c_3)$ and $C' = (c'_1, c'_2, c'_3)$ be two ordered degenerate triples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$. Then C and C' are congruent with respect to the diagonal action of $\mathrm{PU}(2, 1)$ if and only if $\chi(C) = \chi(C')$.*

PROOF: Let us suppose that $\chi(C) = \chi(C')$. Let S_p and $S_{p'}$ be the polar spheres of C and C' . Since $\text{PU}(2, 1)$ acts transitively on $\partial\mathbf{H}_{\mathbb{C}}^2$, one may assume that $p = p' = \infty$. Since $\chi(C) = \chi(C')$, we have that $[\infty, p_1, p_2, p_3] = [\infty, p'_1, p'_2, p'_3]$. Hence there exists an element γ in the projective automorphism group $G(S_{\infty}) \equiv \text{PGL}(2, \mathbb{C})$ of S_{∞} such that $\gamma(\infty) = \infty$, $\gamma(p_1) = p'_1$, $\gamma(p_2) = p'_2$, and $\gamma(p_3) = p'_3$. Since $\gamma(\infty) = \infty$, it follows that γ belongs to the stabilizer Γ_{∞} of ∞ in $G(S_{\infty})$. Applying Proposition 4.12, we get that γ is the restriction of an element γ^* from $\text{PU}(2, 1)$. All this implies that there exists an element $g \in \text{PU}(2, 1)$ such that $g(p_i) = p'_i$, $i = 1, 2, 3$. This proves the theorem. \blacksquare

Now let $C = (c_1, \dots, c_m)$ be an ordered degenerate m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, where $m \geq 3$. We associate to C the following parabolic invariants:

$$\chi_0 = \chi(c_1, c_2, c_3), \chi_1 = \chi(c_1, c_2, c_4), \dots, \chi_{(m-3)} = \chi(c_1, c_2, c_m).$$

Applying Proposition 4.13 and Theorem 4.10, we get the main result of this section.

Theorem 4.11 *Let $C = (c_1, \dots, c_m)$ be an ordered degenerate m -tuple of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, where $m \geq 3$. Then the $\text{PU}(2, 1)$ -congruence class of C is defined uniquely by the parabolic invariants χ_i , $i = 0, 1, \dots, (m - 3)$.*

Corollary 4.10 *The moduli space of ordered degenerate m -tuples of distinct complex geodesics in $\mathbf{H}_{\mathbb{C}}^2$, where $m \geq 3$, may be identified with $\mathbb{C}_*^{(m-2)}$.*

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