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Complex Kleinian groups

Jose SEADE

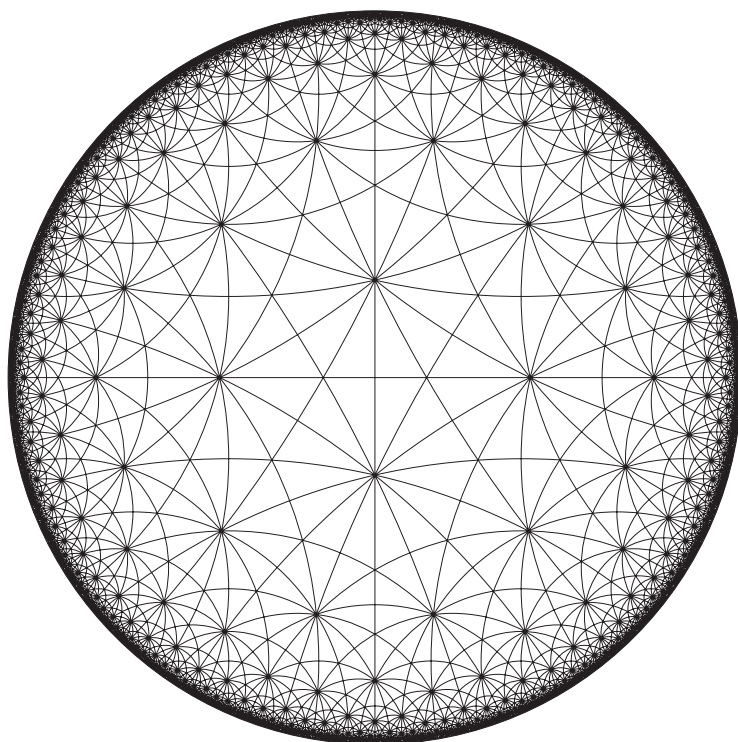*Universidad Nacional Autonoma de Mexico, Cuernavaca, Mexico*

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José Seade

With Angel Cano and Juan-Pablo Navarrete

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INTRODUCTION

The goal of these notes is to have a glimpse of some of the study of the geometry and dynamics of discrete groups of automorphisms of complex projective spaces.

These notes, written jointly by Angel Cano, Juan-Pablo Navarrete and José Seade, have been prepared for the lectures of Seade at the "Advanced School and Workshop on Discrete Groups in Complex Geometry", at the Abdus Salam ICTP, 28th June to 2nd July of 2010. The notes are arranged into four sections, corresponding to the four lectures. These are greatly based on work by the three authors as well as work with Alberto Verjovsky.

Our starting point is the theory of Kleinian groups acting on the Rieman sphere. These were introduced by F. Klein and H. Poincaré as the holonomy groups of certain differential equations, and they can be regarded either as groups of conformal automorphims of the Riemann sphere, as groups of isometries of hyperbolic space, or as groups of holomorphic automorphisms of the complex projective line. Such a diversity of interesting viewpoints obviously brings great richness into the subject, which springs into different theories when going up into higher dimensions. We may thus look, for instance, at complex hyperbolic geometry, or conformal automorphisms of spheres, or holomorphic transformations of complex projective spaces.

The first section is about the classical theory of kleinian groups. This important subject will be dealt with in a much deeper way in the lecture courses by Etienne Ghys and Michael Kapovich. Thus our aim here is to give only a quick introduction to the subject, paving the ground for the following sections.

In Section 2 we introduce the groups, and the concept, that give the title to these lectures: Complex kleinian groups. These are by definition groups of holomorphic automorphisms of complex projective spaces $\mathbb{CP}^n$. For $n = 1$ this is the setting discussed in Section 1. We begin the section by reviewing the definition and construction of the projective spaces. Then we see how this relates to the interesting theory of complex hyperbolic geometry, that will be studied in the lectures by John Parker.

In Section 3 we have a glance on the dynamics of complex kleinian groups. We discuss the concept of limit set for these groups and we see that the quotient of their region of discontinuity by the group action gives rise to orbifolds (often manifolds) with a projective structure. This brings us into the subject of the course that William Goldman will lecture on next week.

Finally, Section 4 is about work with Alberto Verjovsky. Here we see how "twistor theory" can be used to construct complex kleinian groups with rich geometry and dynamics. In fact this shows that the theory of classical kleinian groups embeds into the theory of complex kleinian groups. For simplicity we restrict the discussion to the 4-sphere and its twistor space, which is $\mathbb{CP}^3$. We describe these in detail.

Contents

1   KLEINIAN GROUPS

1.1   Group actions

Let G be a group and M a smooth manifold. An *action* of G on M means a multiplication:

$$\Phi : G \times M \longrightarrow M$$

which preserves the group structure of G. That is, Φ satisfies:
   i) If e denotes the identity in G, then $\Phi(e, x) = x$ for all $x \in M$.
   ii) Let $\cdot$ denote the product in G, then for all $x \in M$ and $g, h \in G$ we have

$$\Phi(g \cdot h, x) = \Phi(g, \Phi(h, x)) \,.$$

Intuitively, an action means a method for "multiplying " elements of the group G by points in M, so that the result is a point in M. Notice that in the definition above we are actually "multiplying " the elements of G by the points in M by the left, so what we have is a *left action*. One has the equivalent notion for right actions. Yet, for the sake of

simplicity, in these we will speak just of "actions", not specifying in general whether they are right or left actions.

We assume in the sequel that G is actually a Lie group. This means that G is itself a smooth manifold and the group operations in G,

$$g \mapsto g^{-1} \qquad \text{and} \qquad (g, h) \mapsto g \cdot h \,,$$

are differentiable. We assume further that the map Φ is differentiable. Notice that in this situation, every subgroup H of G acts on G by multiplication (on the right or on the left). Also, if G acts on a manifold M, then the restriction of the action to H gives an H-action on M.

Observe that for each $g \in G$ we have a smooth map $\phi_g : M \to M$ given by $\phi_g(x) = \Phi(g, x)$. This map has an inverse given by $x \mapsto \Phi(g^{-1}, x)$. Hence each ϕ_g is a diffeomorphism of M. That is, the differentiable group action Φ can be regarded as a family of diffeomorphisms of M parameterized by G.

Similarly, for each $x \in M$ we have a smooth map $\mathcal{O}_x : G \to M$ defined by $g \mapsto \Phi(g, x)$. The image $G(x)$ of $\mathcal{O}_x$ is called *the orbit* of x under the action of G:

$$G(x) = \{ y \in M \mid y = \Phi(g, x) \text{ for some } g \in G \} \,.$$

Given a G-action Φ on M, for each $x \in M$ one has *the stabilizer* of x, also called *the isotropy* subgroup of x, defined by:

$$G_x = \{ g \in G \mid \Phi(g, x) = x \} \,.$$

That is, G_x consists of all the elements in G that leave the point x fixed.

An action is called *free* if all stabilizers are trivial, *i.e.*, if for all $x \in M$ and all $g \in G \backslash e$ we have $\Phi(g, x) \neq x$.

For simplicity, if a group G acts on M we denote the action of $g \in G$ at each point by $g \cdot x$.

Examples 1.1   i. Given fixed real numbers λ_1, λ_2, we may let $\mathbb{R}$ act on $\mathbb{R}^2$ by

$$t \cdot (x_1, x_2) \mapsto (e^{\lambda_1 t} x_1, e^{\lambda_1 t} x_2) \,.$$

An action of the real numbers $\mathbb{R}$ on a manifold M is called *a flow* or also *a one parameter group* of diffeomorphisms of M.

ii. Let $O(2)$ be *the orthogonal group* generated by reflections on all lines through the origin in $\mathbb{R}^2$, and let $Aff(2, \mathbb{R})$ be the group generated by reflections on all lines in $\mathbb{R}^2$. These groups act on $\mathbb{R}^2$ in the obvious way. Now, given integers $p, q, r \geq 2$ such that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \,,$$

let $T := T_{p,q,r}$ be a triangle in $\mathbb{R}^2$ with inner angles $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r}$, [We leave as an exercise to show that the only possible triples are, up to permutation, the triples $(2,3,6)$, $(2,4,4)$ and $(3,3,3)$] and let $\Sigma_{p,q,r}$ be the subgroup of $Aff(2,\mathbb{R})$ generated by the reflections on the edges of T. Then $\Sigma_{p,q,r}$ acts on $\mathbb{R}^2$ in such a way that the various (infinitely many) copies of T cover the plane.

Notice that if ℓ_1 and ℓ_2 are the lines determined by two edges of T that determine, say, the angle π/p, then the reflection on ℓ_1 followed by the reflection on ell_2 is a rotation by an angle $2\pi/p$ around the point where these two lines meet.
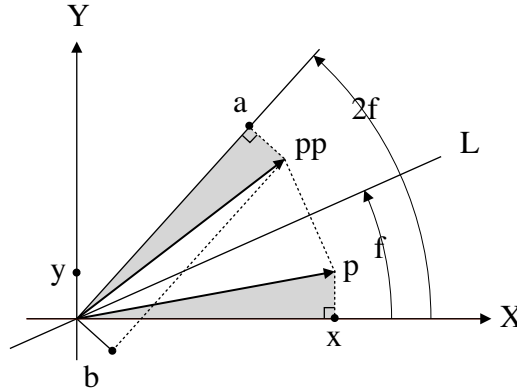

Figure 1: The point PP is the image of P by the reflection on the line $\mathcal{L}$.

Hence the isotropy of this point, which is a vertex of T, is a cyclic group of order $2p$.

iii. Let $O(n)$ be, more generally, the group of linear maps of $\mathbb{R}^n$ generated by the reflections on hyperplanes through the origin, and let $SO(n)$ be the index two subgroup of $O(n)$ consisting of elements that can be expressed by an even number of reflections. This is the group of rotations of $\mathbb{R}^n$. Both of these groups preserve the usual metric in $\mathbb{R}^n$, so they leave invariant each sphere centred at the origin, and we may think of each of them as acting on the unit sphere. It is clear that the origin in $\mathbb{R}^n$ is a fixed point for the corresponding actions, that is $g \cdot \underline{0} = \underline{0}$ for all $g \in O(n)$. We leave it as an exercise to show that all other points have isotropy $O(n-1)$ (and $SO(n-1)$).

iv. Let f be a diffeomorphism of a manifold M. For instance, let M be the 2-sphere $\mathbb{S}^2$ and identify it with the extended plane $\widehat{\mathbb{R}}^2 := \mathbb{R}^2 \cup \infty$ by stereographic projection, so that the origin $(0,0)$ corresponds to the South pole S while ∞ corresponds to the North pole. And let f be the map in $\widehat{\mathbb{R}}^2$ defined by $(x,y) \mapsto (\frac{1}{2}x, 2y)$. Now *iterate* this function. That is, look at the family of maps $f_1 := f$, $f_2 = f \circ f_1$, $f_3 = f \circ f_2$ and so on. Define also $f_0 := Id$ and set $f_{-1} := f^{-1}$; we may thus iterate

f also *backwards* and get a family of maps $\{f_n\}_{n \in \mathbb{Z}}$. Then the assignment $n \mapsto f_n$ determines an action of the integers $\mathbb{Z}$ in $\mathbb{S}^2$.

Notice that in this case we have two points which are fixed by the action, the poles S and N. The points in the axes converge to S and N (either when travelling forwards or backwards), while all other points converge to N.

Observe that in this example we can replace f by any other diffeomorphism and get an action of $\mathbb{Z}$ on $\mathbb{S}^2$. This will be relevant in the sequel. Notice too that if we replace f by some other function which is not a diffeomorphism, then we do not have an inverse f^{-1}. In this case we do not have an action of $\mathbb{Z}$. Yet, we can iterate f forwards and look at the forwards orbits of points. And we can also look at the inverse images of points, and get *the backwards orbit.*

1.2   Inversions and the Möbius group

General references for this and the following sections in this chapter are Beardon's book [3] and the excellent notes of M. Kapovich [13].

Let us consider now another type of transformations, which are analogous to reflections, the inversions. Given a circle $C = C(a, r)$ in the plane $\mathbb{R}^2$ with centre at a point $a \in \mathbb{R}^2$ and radius r, the inversion in C is the map $\iota = \iota(a, r)$ of the 2-sphere $\mathbb{S}^2 \cong \widehat{\mathbb{R}}^2 := \mathbb{R}^2 \cup \infty$ defined for each $z = (x, y) \neq a, \infty$ by:

$$\iota_{a,r}(x, y) = (a_1, a_2) + \frac{r^2}{|(x, y) - (a_1, a_2)|^2} (x - a_1, y - a_2) ;$$

define $\iota(a) = \infty$ and $\iota(\infty) = a$. Notice that each $z = (x, y) \neq a, \infty$ is carried into the unique point $z' = (x', y')$ in the line determined by z and a which satisfies:

$$d(z, a) \cdot d(z', a) = r^2 ,$$

where $d(\ , a)$ is the usual distance to a. We remark that for circles of maximal length (*i.e.*, radius 1 in the 2-sphere) this map is just a reflection in the corresponding line in $\mathbb{R}^2$.

Notice this formula is easily adapted to describing inversions in $(n - 1)$-spheres in $\mathbb{S}^n \cong \mathbb{R}^n \cup \infty$.

It is an exercise to show that inversions are conformal maps, *i.e.*, they preserve angles. That is, if two curves in $\mathbb{S}^2$ meet with an angle θ, then their images under an inversion also meet with an angle θ. Moreover, one has that if C_1 , C_2 are circles in $\mathbb{S}^2$ and ι_1 is the inversion with respect to C_1, then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally. We leave the prove as an exercise (Show first that two circles C_1, C_2 in $\mathbb{S}^2$ meet orthogonally at the points P_1 y P_2 if and only if the centre z_2 of C_2 is the meeting point of the lines $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$, which are tangent to C_1 at P_1 and P_2, and conversely, the centre z_1 of C_1 is the meeting point of the lines $\mathcal{L}_\in$ and $\mathcal{L}'_\in$, tangent to C_2 at these points.)

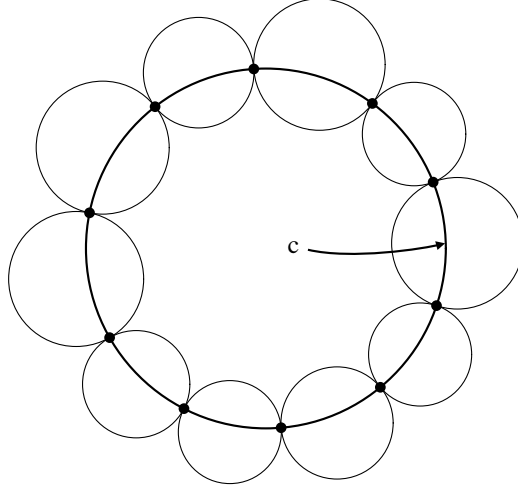In fact the same statement holds in all dimensions (with essentially the same proof):

Figure 2: A collar of circles having C as a common orthogonal circle.

Theorem 1.2 *Let C_1^{n-1}, C_2^{n-1} be spheres of dimension $n-1$ in $\mathbb{S}^n$ and ι_1 the inversion with respect to C_1. Then $\iota_1(C_2) = C_2$ if and only if C_1 and C_2 meet orthogonally.*

We now let $\mathrm{M\ddot{o}b}(\mathbb{S}^n)$ be the group of diffeomorphisms of $\mathbb{S}^n \cong \widehat{\mathbb{R}} = \mathbb{R}^n \cup \{\infty\}$ generated by inversions on all $(n-1)$-spheres in $\mathbb{S}^n$, and let $\mathrm{M\ddot{o}b}(\mathbb{B}^n)$ be the subgroup of $\mathrm{M\ddot{o}b}(\mathbb{S}^n)$ consisting of maps that preserve the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$.

Notice that if the $(n-1)$-sphere $\mathcal{S}_1$ meets $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ orthogonally then $\mathcal{C} := \mathcal{S}_1 \cap \mathbb{S}^{n-1}$ is an $(n-2)$-sphere in $\mathbb{S}^{n-1}$ and the restriction to $\mathbb{S}^{n-1}$ of the inversion $\iota_{\mathcal{S}_1}$ coincides with the inversion on $\mathbb{S}^{n-1}$ defined by the $(n-2)$-sphere $\mathcal{C}$. In other words one has a canonical group homomorphism $\mathrm{M\ddot{o}b}(\mathbb{B}^n) \to \mathrm{M\ddot{o}b}(\mathbb{S}^{n-1})$.

Conversely, given an $(n-2)$-sphere $\mathcal{C}$ in $\mathbb{S}^{n-1}$ there is a unique $(n-1)$-sphere $\mathcal{S}$ in $\mathbb{S}^n$ that meets $\mathbb{S}^{n-1}$ orthogonally at $\mathcal{C}$. The inversion

$$\iota_{\mathcal{C}} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$$

extends canonically to the inversion:

$$\iota_{\mathcal{S}} : \mathbb{B}^n \to \mathbb{B}^n,$$

thus giving a canonical group homomorphism $\mathrm{M\ddot{o}b}(\mathbb{S}^{n-1}) \to \mathrm{M\ddot{o}b}(\mathbb{B}^n)$, which is obviously the inverse morphism of the previous one. Thus one has:

Lemma 1.3 *There is a canonical group isomorphism $\mathrm{M\ddot{o}b}(\mathbb{B}^n) \cong \mathrm{M\ddot{o}b}(\mathbb{S}^{n-1})$.*

Definition 1.4 We call $\mathrm{M\ddot{o}b}(\mathbb{B}^n)$ (and also $\mathrm{M\ddot{o}b}(\mathbb{S}^n)$) *the general Möbius group* of the ball (or of the sphere).

The subgroup $\text{Möb}_+(\mathbb{B}^n)$ of $\text{Möb}(\mathbb{B}^n)$ of words of even length consists of the elements in $\text{Möb}(\mathbb{B}^n)$ that preserve the orientation. This is an index two subgroup of $\text{Möb}(\mathbb{B}^n)$. Similar considerations apply to $\text{Möb}(\mathbb{S}^n)$. We call $\text{Möb}_+(\mathbb{B}^n)$ and $\text{Möb}_+(\mathbb{S}^n)$ *Möbius groups* (of the ball and of the sphere, respectively).

It is easy to see that $\text{Möb}(\mathbb{S}^n)$ includes:

- Euclidean translations: $t(x) = x+a$, where $a \in \mathbb{R}^n$. These are obtained by reflections on parallel hyperplanes.

- Rotations: $t(x) = Ox$, where $O \in SO(n)$; obtained by reflections on hyperplanes through the origin.

- Homotecies, obtained by inversions on spheres with same centre and different radius.

In fact one has:

Theorem 1.5 *The group $Möb(\mathbb{S}^n)$ of Möbius transformations is generated by the previous transformations: Translations, rotations and homotecies, together with the inversion: $t(x) = x/\|x\|^2$.*

It is clear that the rotations are actually contained in $\text{Möb}_+(\mathbb{B}^n)$, since hyperplanes through the origin meet transversally the unit sphere in R^n. In fact one has that $\text{Möb}_+(\mathbb{B}^n)$ contains the orthogonal group $SO(n)$ as the stabilizer (or isotropy) subgroup at the origin 0 of its action on the open ball $\mathbb{B}^n$. The stabilizer of 0 under the action of the full group $\text{Möb}(\mathbb{B}^n)$ is $O(n)$. This implies that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on the space of lines through the origin in $\mathbb{B}^n$. Moreover, $\text{Möb}_+(\mathbb{B}^n)$ clearly acts also transitively on the intersection with $\mathbb{B}^n$ of each ray through the origin. Thus it follows that $\text{Möb}_+(\mathbb{B}^n)$ acts transitively on $\mathbb{B}^n$. In other words we have:

Theorem 1.6 *The group $Möb_+(\mathbb{B}^n)$ acts transitively on the unit open ball $\mathbb{B}^n$ with isotropy $SO(n)$. Furthermore, this action extends to the boundary $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$ and defines a canonical isomorphism between this group and the Möbius group $Möb_+(\mathbb{S}^{n-1})$.*

We remark that for $n > 2$, $\text{Möb}_+(\mathbb{S}^{n-1})$ is the group of (orientation preserving) conformal automorphisms of the sphere (see for instance Apanasov's book). That is, we have:

Theorem 1.7 *For all $n > 2$ we have group isomorphisms*

$$Möb_+(\mathbb{B}^n) \cong Möb_+(\mathbb{S}^{n-1}) \cong \text{Conf}_+(\mathbb{S}^{n-1}) .$$

In fact the previous constructions show that every element in $\text{Möb}_+(\mathbb{B}^n)$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_\infty := \overline{\mathbb{H}}^n_\mathbb{R} \setminus \mathbb{H}^n_\mathbb{R}$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_\infty$ extends to an element in $\text{Möb}_+(\mathbb{B}^n)$.

1.3   Hyperbolic space

We now use Theorem 1.6 to construct a model for hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$. We recall that a riemannian metric g on a smooth manifold M means a choice of a positive definite quadratic form on each tangent space $T_x M$, varying smoothly over the points in M. Such a metric determines lengths of curves as usual, and so defines a metric on M in the usual way, by declaring the distance between two points to be the infimum of the lengths of curves connecting them.

Now consider the open unit ball $\mathbb{B}^n$, its tangent space $T_0 \mathbb{B}^n$ at the origin, and fix the usual riemannian metric on it, which is invariant under the action of $O(n)$. Given a point $x \in \mathbb{B}^n$, consider an element $\gamma \in \text{Möb}(\mathbb{B}^n)$ with $\gamma(0) = x$. Let $D\gamma_0$ denote the derivative at 0 of the automorphism $\gamma : \mathbb{B}^n \to \mathbb{B}^n$. This defines an isomorphism of vector spaces $D\gamma_0 : T_0 \mathbb{B}^n \to T_x \mathbb{B}^n$ and allows us to define a riemannian metric on $T_x \mathbb{B}^n$. In this way we get a riemannian metric at each tangent space of $\mathbb{B}^n$.

We claim that the above construction of a metric on the open ball is well defined, *i.e.*, that the metric one gets on $T_x \mathbb{B}^n$ does not depend on the choice of the element $\gamma \in \text{Möb}(\mathbb{B}^n)$ taking 0 into x. In fact, if $\eta \in \text{Möb}(\mathbb{B}^n)$ is another element taking 0 into x, then $\eta^{-1} \circ \gamma$ leaves 0 invariant and is therefore an element in $O(n)$. Since the orthogonal group $O(n)$ preserves the metric at $T_0 \mathbb{B}^n$, it follows that both maps, γ and η, induce the same metric on $T_x \mathbb{B}^n$. Hence this construction yields to a well-defined riemannian metric on $\mathbb{B}^n$.

It is easy to see that this metric is complete and homogeneous with respect to points, directions and 2-planes, so it has constant (negative) sectional curvature.

Definition 1.8 The open unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ equipped with the above metric serves as a model for the *hyperbolic n-space* $\mathbb{H}^n_{\mathbb{R}}$. The group $\text{Möb}(\mathbb{B}^n)$ is its *group of isometries*, also denoted $\text{Iso}(\mathbb{H}^n_{\mathbb{R}})$, and its index two subgroup $\text{Möb}_+(\mathbb{B}^n)$ is the *group of orientation preserving isometries* of $\mathbb{H}^n_{\mathbb{R}}$, $\text{Iso}_+(\mathbb{H}^n_{\mathbb{R}})$.

In the sequel we denote the real hyperbolic space by $\mathbb{H}^n_{\mathbb{R}}$, to distinguish it from the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ (of real dimension $2n$) that we will consider later. Also, we denote by $\mathbb{S}^{n-1}_\infty$ the sphere at infinity, that is, the boundary of $\mathbb{H}^n_{\mathbb{R}}$ in $\mathbb{S}^n$. We set $\overline{\mathbb{H}}^n_{\mathbb{R}} := \mathbb{H}^n_{\mathbb{R}} \cup \mathbb{S}^{n-1}_\infty$.

Given that we have a metric in $\mathbb{H}^n_{\mathbb{R}}$, we can speak of length of curves, area, volume, and so on. We also have the concept of *geodesics*: curves that minimize (locally) the distance between points. These are the segments of curves in $\mathbb{H}^n_{\mathbb{R}}$ which are contained in circles that meet the boundary $\mathbb{S}^{n-1}_\infty$ orthogonally.

Notice that the constructions above show that every isometry of $\text{Iso}(\mathbb{H}^n_{\mathbb{R}})$ extends canonically to a conformal automorphism of the sphere at infinity $\mathbb{S}^{n-1}_\infty$ and conversely, every conformal automorphism of $\mathbb{S}^{n-1}_\infty$ extends to an isometry of $\mathbb{H}^n_{\mathbb{R}}$.

1.4   Kleinian groups

We now consider a subgroup $\Gamma \subset Iso(\mathbb{H}^n_{\mathbb{R}})$ and look at its action on the hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. We want to study how the orbits of points in $\mathbb{H}^n_{\mathbb{R}}$ (and in $\overline{\mathbb{H}}^n_{\mathbb{R}}$) behave under the action of Γ. Let us begin with an example. Consider, as before, integers $p, q, r \geq 2$, but now we assume that

$$1/p + 1/q + 1/r < 1.$$

Let $T = T_{p,q,r}$ be a triangle in $\mathbb{D}$ bounded by geodesics, with angles π/p, π/q and π/r. Recall these geodesics are segments of circles in $\widehat{\mathbb{R}}^2$ orthogonal to the boundary of $\mathbb{D}$, so we have isometries of $\mathbb{H}^2_{\mathbb{R}}$ defined by the inversions on these three circles, the "sides" of T.

Let Γ^* be the group of isometries of $\mathbb{H}^2_{\mathbb{R}}$ generated by the inversions on the sides of T. Notice that Γ^* has three special orbits of fixed points in $\mathbb{H}^2_{\mathbb{R}}$, which correspond to the three vertices of T. The corresponding isotropy subgroups are cyclic of orders $2p, 2q, 2r$ respectively. The various images of T under the action of Γ cover the whole space $\mathbb{H}^2_{\mathbb{R}}$, and it is intuitively clear that given any point x_0 in the circle $\mathbb{S}^1_\infty = \partial \mathbb{H}^2_{\mathbb{R}}$ and a point x inside $\mathbb{H}^2_{\mathbb{R}}$, there is a sequence $\{\gamma_n\}$ of elements in Γ^* such that the sequence $\{\gamma_n(x)\}$ converges to x_0. This means that in this example the limit set is the whole circle at infinity.

Definition 1.9 Let $\Gamma \subset \mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}})$ be a discrete subgroup. The *limit set* of Γ is the subset $\Lambda = \Lambda(\Gamma)$ of $\mathbb{S}^{n-1}_\infty$ of points which are accumulation points of orbits in $\mathbb{H}^n_{\mathbb{R}}$. That is,

$$\Lambda := \left\{ y \in \mathbb{S}^{n-1}_\infty \mid y = \lim\{g_m(x)\} \;\; \text{for some } x \in \mathbb{H}^n_{\mathbb{R}} \text{ and } \{g_m\} \;\; \text{a sequence in } \mathrm{Iso}(\mathbb{H}^n_{\mathbb{R}}) \right\}.$$

By definition, this is a closed, invariant subset of $\mathbb{S}^{n-1}_\infty$ which is non-empty, unless Γ is finite. This is the set where the dynamics concentrate. It can happen that Λ is the whole sphere at infinity, as for instance in the previous example of the triangle subgroups of isometries of $\mathbb{H}^2_{\mathbb{R}}$.

Definition 1.10 A discrete subgroup of $\mathrm{Iso}(\mathbb{H}^{n+1}) \cong \mathrm{Conf}(\mathbb{S}^n)$ is *kleinian* if its limit set is not the whole sphere at infinity.
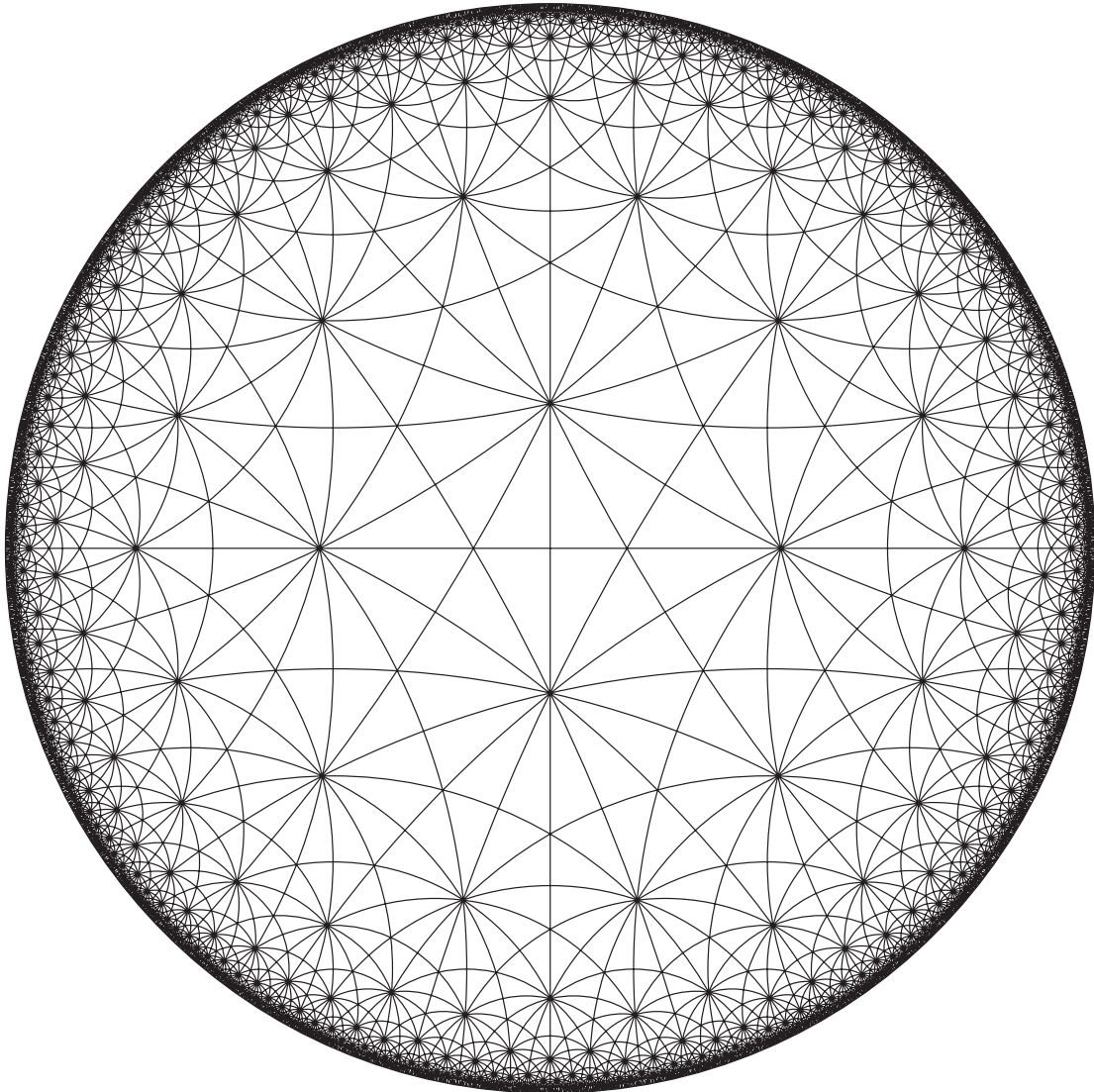
Figure 3: The triangle group $\langle 2, 3, 7 \rangle$

In the sequel we refer to these as *conformal kleinian groups*, to distinguish them from the complex kleinian groups that we shall study later.

We remark that nowadays the term "Kleinian group" is being often used for an arbitrary discrete subgroup of hyperbolic motions, regardless of whether or not the region of discontinuity is empty.

Let us consider for a moment a more general setting. Let G be some group, acting on a smooth manifold M.

Definition 1.11 The action of G is *discontinuous at* $x \in M$ if there is a neighbourhood U of x such that the set

$$\{g \in G \,|\, gU \cap U \neq \varnothing\}$$

is finite. The set of points in M at which G acts discontinuously is called the *region of discontinuity*. The action is *discontinuous on* M if it is discontinuous at every point in M. The action is *properly discontinuous* if for each non empty compact set $K \subset M$ the set

$$\{g \in G \,|\, gK \cap K \neq \varnothing\}\,,$$

is finite.

Example 1.12 (Kulkarni) Consider the map T in $\mathbb{R}^2$ given by $(x, y) \mapsto (\frac{1}{2}x, 2y)$ and all its iterates $\{T_n\}_{n \in \mathbb{Z}}$. This gives an action of $\mathbb{Z}$ in $\mathbb{R}^2$ which is discontinuous away from the origin 0. Notice that if we take a circle C around the origin, then its forward orbit accumulates on the whole $\{y\}$-axe, while the backwards orbit accumulates on the $\{x\}$-axe. So this action is not properly discontinuous on $\mathbb{R}^2 \setminus \{0\}$. Yet we notice that the action is properly discontinuous on $\mathbb{R}^2 \setminus \{x = 0\}$ and also on $\mathbb{R}^2 \setminus \{y = 0\}$.

It is clear that every properly discontinuous action is *a fortiori* discontinuous. The example above shows that the converse statement is false in general, but it is true for conformal kleinian groups. In fact one has (see the literature for a proof):

Theorem 1.13 *Let G be a discrete subgroup of* $\mathrm{Iso}(\mathbb{H}_\mathbb{R}^n)$. *Then G acts properly discontinuously on* $\mathbb{H}_\mathbb{R}^n$ *and its limit set is the complement of the region of discontinuity Ω of its action on* $\mathbb{S}_\infty^{n-1}$. *Furthermore, Ω is the maximal region in* $\mathbb{S}_\infty^{n-1}$ *where the action is properly discontinuous, and it is also the maximal region in the sphere where the action is equicontinuous.*

For instance, consider an arbitrary family of pairwise disjoint closed 2-discs $D_1, ..., D_r$ in the 2-sphere with boundaries the circles $C_1, ..., C_r$. Let $\iota_1, ..., \iota_r$ be the inversions on these r circles, and let Γ be the subgroup of $\mathrm{Iso}(\mathbb{H}_\mathbb{R}^3) \cong \mathrm{M\ddot{o}b}(\mathbb{B}^3) \cong \mathrm{Conf}(\mathbb{S}^2)$ generated by these maps. Then G has nonempty region of discontinuity that contains the complement in $\mathbb{S}^2$ of the union $D_1 \cup ... \cup D_r$ (which is a fundamental domain for Γ). One can show that in this case the limit set is a Cantor set. This is an example of a *Schottky group*, and Schottky groups are all Kleinian groups.

Continuing with this example, move the discs $D_1, ..., D_r$ so that each of them touches tangentially exactly its two neighbors, and there is a common circle C orthogonal to all of them. Then C is the limit set of the corresponding group of inversions.

Now move the circles slightly $C_1, ..., C_r$, breaking the condition that they have a common orthogonal circle, keeping the condition that each disc touches with its two neighbors. Then one has (this is not at all obvious) that the limit set becomes a fractal curve of Hausdorff dimension between 1 and 2, and choosing appropriate deformations one can cover
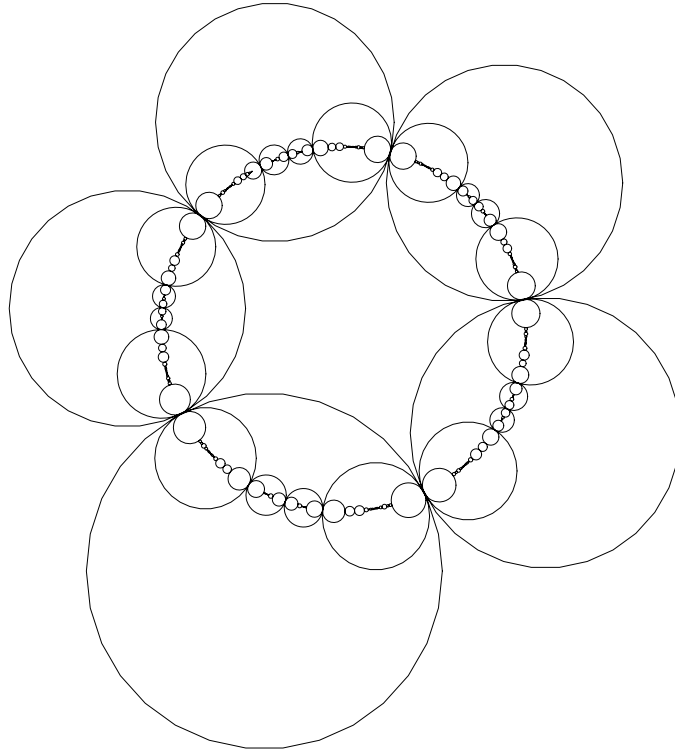
Figure 4: A "kissing Schottky" group with C as limit set.

the whole range of Hausdorff dimension between 1 and 2. This is depicted in the figure below, and this is an example of a more general result by Rufus Bowen.

These are all examples of kleinian groups. So we see that whenever we have a kleinian group, the sphere $\mathbb{S}^{n-1}_{\infty}$ splits in two sets, which are invariant under the group action: the limit set Λ, where the dynamics concentrates, and the region of discontinuity Ω where the dynamics is "mild" and plays an important role in geometry, as we will see later.

Figure 5: Deformation of a fuchsian group: The limit set is a quasi-circle

2   COMPLEX KLEINIAN GROUPS

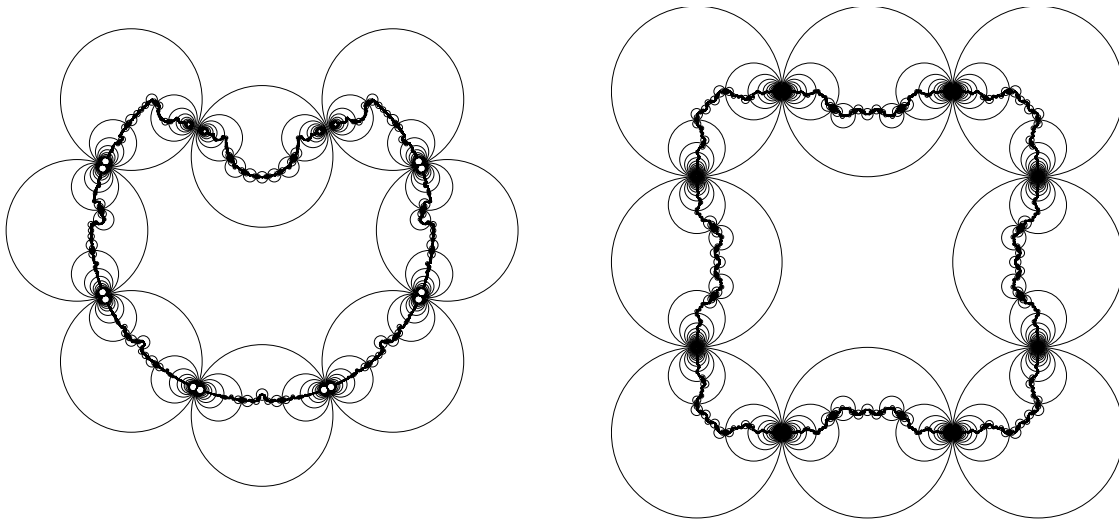In the previous section we studied discrete subgroups of isometries of real hyperbolic spaces $\mathbb{H}_{\mathbb{R}}^n$. We remark that when $n = 3$, the sphere at infinity is 2-dimensional and we can think of it as being the Riemann sphere $\mathbb{S}^2$, which is a complex 1-dimensional manifold, diffeomorphic to the projective line $\mathbb{CP}^1$. Moreover, in this case one has that every (orientation preserving) element in the conformal group $\mathrm{Conf}_+(\mathrm{S}^2)$ is actually a Möbius transformation:

$$z \mapsto \frac{az + b}{cz + d} \, ,$$

where a, b, c, d are complex numbers such that $ad - bz = 1$. The set of all such maps forms a group, which is isomorphic to the group $PSL(2, \mathbb{C})$ of projective automorphisms of $\mathbb{CP}^1$:

$$PSL(2, \mathbb{C}) := SL(2, \mathbb{C})/ \pm \mathrm{Id} \, ,$$

where $SL(2, \mathbb{C})$ is the group of 2×2 matrices with complex coefficients and determinant 1, and Id is the identity matrix. Hence, considering discrete subgroups of $\mathrm{Iso}_+(\mathbb{H}_{\mathbb{R}}^3)$ is the same thing as considering discrete subgroups of $PSL(2, \mathbb{C})$.

What about higher dimensions? That is, what about discrete subgroups of $PSL(n + 1, \mathbb{C})$, the group of automorphisms of the complex projective space $\mathbb{CP}^n$? In the real case, we have the group of isometries of the real hyperbolic space. And in the low dimensions this coincides with groups of automorphisms of $\mathbb{CP}^1$, which is a complex manifold. But in higher dimensions, there is nothing similar.

The topic we shall focus on will be precisely the study of discrete subgroups of automorphisms of complex projective spaces, so we start by describing these spaces.

2.1 Complex projective space

We recall that the complex projective space $\mathbb{CP}^n$ is defined as:

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\sim,$$

where "$\sim$" denotes the equivalence relation given by $x \sim y$ if and only if $x = \alpha y$ for some nonzero complex scalar α. In short, $\mathbb{CP}^n$ is *the space of complex lines through the origin* in $\mathbb{C}^{n+1}$.

Consider for instance $\mathbb{CP}^1$. Every point here represents a complex line through the origin in $\mathbb{C}^2$. Recall that a complex line ℓ through the origin is always determined by a unit vector in it, say v, together with all its complex multiples. In other words, a unit vector v in $\mathbb{C}^2$ determines the complex line

$$\ell = \{\lambda \cdot v \mid \lambda \in \mathbb{C}\}.$$

Notice that the unit vectors in $\mathbb{C}^2$ form the 3-sphere $\mathbb{S}^3$, just as the unit vectors in $\mathbb{C}$ form the circle

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid z = e^{i\theta}, \theta \in [0, 2\pi]\}.$$

Notice that the circle $\mathbb{S}^1$ acts on $\mathbb{C}^2$ in the obvious way: $e^{i\theta} \cdot (z_1, z_2) \mapsto (e^{i\theta}z_1, e^{i\theta}z_2))$. This action preserves distances in $\mathbb{C}^2$, so given a point $v \in \mathbb{S}^3 \subset \mathbb{C}^2$, its orbit under this $\mathbb{S}^1$-action is the set $\{(e^{i\theta} \cdot v)\}$, which is a circle in $\mathbb{S}^3$ contained in the complex line determined by v. That is *the intersection of $\mathbb{S}^3$ with every complex line through the origin in $\mathbb{C}^2$* is a circle, and one has:

$$\mathbb{CP}^1 \cong \mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{S}^2.$$

The projection $\mathbb{S}^3 \to \mathbb{CP}^1 \cong \mathbb{S}^2$ is known as *the Hopf fibration*.

More generally, $\mathbb{CP}^n$ is a compact, connected, complex n-dimensional manifold, diffeomorphic to the orbit space $\mathbb{S}^{2n+1}/U(1)$, where $U(1) \cong \mathbb{S}^1$ is acting coordinate-wise on the unit sphere in $\mathbb{C}^{n+1}$. In fact, we usually represent the points in $\mathbb{CP}^n$ by *homogeneous coordinates* $(z_1 : z_2 : \cdots : z_{n+1})$. This means that we are thinking of a point in $\mathbb{CP}^n$ as being the equivalence class of the point $(z_1, z_2 : \cdots, z_{n+1})$ up to multiplication by non-zero complex numbers. Hence if, for instance, we look at points where the first coordinate z_1 is not zero, then the point $(z_1 : z_2 : \cdots : z_{n+1})$ is the same as $(1 : \frac{z_2}{z_1} : \cdots : \frac{z_{n+1}}{z_1})$. Notice this is just a copy of $\mathbb{C}^n$. That is, every point in $\mathbb{CP}^n$ that can be represented by a point $(z_1 : z_2 : \cdots : z_{n+1})$ with $z_1 \neq 0$, has a neighbourhood diffeomorphic to $\mathbb{C}^n$, consisting of all points with homogeneous coordinates $(1 : w_2 : \cdots : w_{n+1})$. Of course similar remarks apply for points where $z_2 \neq 0$ and so on. This provides the classical way for constructing an atlas for $\mathbb{CP}^n$ with $(n+1)$ coordinate charts.

Notice one has a projection $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$, a *Hopf fibration*, and the usual riemannian metric on $\mathbb{S}^{2n+1}$ is invariant under the action of $U(1)$. Therefore this metric descends to a riemannian metric on $\mathbb{CP}^n$, which is known as the *Fubini-Study* metric.

It is clear that every linear automorphism of $\mathbb{C}^{n+1}$ defines a holomorphic automorphism of $\mathbb{CP}^n$, and it is well-known that every automorphism of $\mathbb{CP}^n$ arises in this way. Thus one has that the group of projective automorphisms is:

$$PSL(n+1, \mathbb{C}) \; := \; GL(n+1, \mathbb{C})/(\mathbb{C}^*)^{n+1} \cong SL(n+1, \mathbb{C})/\mathbb{Z}_{n+1} \,,$$

where $(\mathbb{C}^*)^{n+1}$ is being regarded as the subgroup of diagonal matrices with a single nonzero eigenvalue, and we consider the action of $\mathbb{Z}_{n+1}$ (viewed as the roots of the unity) on $SL(n+1, \mathbb{C})$ given by the usual scalar multiplication. Then $PSL(n+1, \mathbb{C})$ is a Lie group whose elements are called projective transformations.

There is a classical way of decomposing the projective space that paves the way for studying complex hyperbolic geometry. For this we think of $\mathbb{C}^{n+1}$ as being a union $N_- \cup N_0 \cup N_+$, where each of these sets consists of the points $(z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$ satisfying that $|z_0|^2$ is, respectively, larger, equal or smaller than $|z_1|^2 + \cdots + |z_n|^2$. It is clear that each of these sets is a complex cone, that is, union of complex lines through the origin in $\mathbb{C}^{n+1}$, with (deleted) vertex at 0.

Obviously

$$S := \{(z_0, \cdots, z_n) \in N_0 \,|\, z_0 = 1\}\,,$$

is a sphere of dimension $(2n-1)$, and N_0 is the union of all complex lines in $\mathbb{C}^{n+1}$ joining the origin $0 \in \mathbb{C}^{n+1}$ with a point in S; each such line meets S in a single point. Hence the projectivisation $[S] = (N_0 \setminus \{0\})/\mathbb{C}^*$ of N_0 is a $(2n-1)$-sphere in $\mathbb{CP}^n$ that splits this space in two sets, which are the projectivisations of N_- and N_+. The set N_0 is often called *the cone of light*.

Similarly, notice that the projectivisation of N_- is an open $(2n)$-ball $\mathbb{B}$ in $\mathbb{CP}^n$, bounded by the sphere $[S]$. This ball serves as model for complex hyperbolic geometry, as we will see in the following section, where we describe its full group of holomorphic isometries, which is naturally a subgroup of projective transformations. This gives a natural source of discrete subgroups of $PSL(n+1, \mathbb{C})$, those coming from complex hyperbolic geometry.

2.2   Complex Kleinian groups

Our aim in these lectures is to study discrete groups G of $PSL(n+1, \mathbb{C})$ which act on $\mathbb{CP}^n$ with non-empty region of discontinuity. Recall from the previous section that the action of G is *discontinuous at* $x \in \mathbb{CP}^n$ if there is a neighbourhood U of x such that the set

$$\{g \in G \,|\, gU \cap U \neq \varnothing\}$$

is finite. The set of points in $\mathbb{CP}^n$ at which G acts discontinuously is called the *region of discontinuity*.

Definition 2.1 A discrete subgroup Γ of $PSL(n+1, \mathbb{C})$ is *complex kleinian* if its region of discontinuity in $\mathbb{CP}^n$ is non-empty.

   This is a concept introduced by Alberto Verjovsky and Jos Seadesome years ago (see SV1,SV2,SV3), which puts together several important areas of current research, as we shall see (we refer to [11] for more on the topic). For $n = 1$, $\mathbb{CP}^1$ is the Riemann sphere, $PSL(2, \mathbb{C})$ can be regarded as being the group of (orientation preserving) isometries of the hyperbolic space $\mathbb{H}^3_{\mathbb{R}}$ and we are in the situation envisaged previously, of classical kleinian groups.

   Notice that in this classical case, there is a particularly interesting class of kleinian subgroups of $PSL(2, \mathbb{C})$: Those which are conjugate to a subgroup of $PSL(2, \mathbb{R})$. This latter group can be regarded as the group of Möbius transformations with real coefficients:

$$z \mapsto \frac{az + b}{cz + d} \quad , \quad ad - bc = 1 \ , \ a, b, c, d \in \mathbb{R} \ .$$

These are the Möbius transformations that preserve the upper half plane in $\mathbb{C}$. And if we identify the Riemann sphere with the extended plane $\mathbb{C} \cup \infty$, via stereographic projection, these are the conformal automorphisms of the sphere that preserve the Southern hemisphere, *i.e.*, they leave invariant a 2-ball in $\mathbb{S}^2$. Equivalently, these are subgroups of Iso $\mathbb{H}^3_{\mathbb{R}}$ which actually are groups of isometries of the hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$. These are called *Fuchsian groups*. In higher dimensions, this role is played by the so-called complex hyperbolic groups. These are, by definition, subgroups of $PSL(n+1, \mathbb{R})$ which act on $\mathbb{CP}^n$ leaving invariant a certain open ball of complex dimension n, which serves as model for complex hyperbolic geometry. In the subsection below we speak a few words about this interesting subject.

2.3   Complex hyperbolic groups

Let us look at the subset $[N_-]$ of $\mathbb{CP}^n$ consisting of points whose homogeneous coordinates satisfy:

$$|z_0|^2 < |z_1|^2 + \cdots |z_n|^2 \ . \tag{2.2}$$

As noticed above, this set is an open ball $\mathbb{B}$ of real dimension $2n$ and its boundary,

$$[N_0] := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n \,|\, |z_0|^2 = |z_1|^2 + \cdots |z_n|^2\} \ ,$$

is a sphere of real dimension $2n - 1$. This set $[N_-]$ is the usual starting point for complex hyperbolic geometry; for this one needs to introduce a metric, which is known as the Bergman metric. We shall do that in a way similar to the one we used for real hyperbolic space.

   Let $U(n+1)$ be the unitary group. By definition, its elements are the $(n+1) \times (n+1)$ matrices which satisfy

$$\langle Uz, Uw \rangle = \langle z, w \rangle \ ,$$

for all complex vectors $z = (z_0, ..., z_n)$ and $w = (w_0, ..., w_n)$, where $\langle \cdot, \cdot \rangle$ is the usual hermitian product on $\mathbb{C}^{n+1}$: $\langle z, w \rangle = \sum_{i=0}^{n} z_i \cdot \bar{w}_i$. This is equivalent to saying that the columns of U form an orthonormal basis of $\mathbb{C}^{n+1}$ with respect to the hermitian product.

We now let $U(1, n)$ be the subgroup of $U(n + 1)$ of transformations that preserve the quadratic form

$$Q(z_0, \cdots, z_n) = |z_0|^2 - |z_1|^2 - \cdots - |z_n|^2. \tag{2.3}$$

In other words, an element $U \in U(n + 1)$ is in $U(1, n)$ if and only if $Q(z) = Q(Uz)$ for all points in $\mathbb{C}^{n+1}$. Let $PU(1, n)$ be its projectivization. Then the action of $PU(1, n)$ on $\mathbb{CP}^n$ leaves invariant the set $[N_-]$. To see this, recall that a point in $\mathbb{CP}^n$ is in $[N_-]$ if and only if its homogeneous coordinates satisfy equation (2.3). If $(z_0 : \cdots : z_n)$ is in $[N_-]$ and γ is in $PU(1, n)$, then the point $\gamma(z_0 : \cdots : z_n)$ is again in $[N_-]$. Therefore the group $PU(1, n)$ *acts on* the ball $[N_-] \cong \mathbb{B}^{2n}$.

Recall that to construct the real hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$ we considered the unit open ball $\mathbb{B}^n$ in $\mathbb{R}^{n+1}$, and we looked at the action of the Möbius group $\text{Möb}_+(\mathbb{B}^n)$ on this ball. This action was transitive with isotropy $O(n, \mathbb{R})$. So we can consider the usual metric at the space $T_0(\mathbb{B}^n)$, tangent to the ball at the origin, and spread it around using that the action is transitive; we get a well-defined metric on the ball using the fact that the isotropy $O(n, \mathbb{R})$ preserves the usual metric.

Let us now do the analogous construction for the ball $[N_-]$ using the action of $PU(1, n)$: It is an exercise to show that this action is transitive, with isotropy $PU(n)$. Let P be the center of this ball, $P := (0 : 0 : \cdots : 0 : 1)$. We equip the tangent space $T_P([N_-]) \cong \mathbb{C}^n$ with the usual hermitian metric, and spread this metric around $[N_-]$ using the action of $PU(1, n)$. Since the isotropy $PU(n)$ preserves the metric in $T_P([N_-])$ we get a well-defined metric on the ball $[N_-] \cong \mathbb{B}^{2n}$. This is the Bergman metric on the ball $[N_-]$, which thus becomes a model for the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$, with $PU(1, n)$ as its group of holomorphic isometries. Its boundary $[N_0]$ is the sphere at infinity $\mathbb{S}_{\infty}^{2n-1}$.

Since the action of $PU(1, n)$ on $\mathbb{H}_{\mathbb{C}}^n$ is by isometries, then one has (by general results of groups of transformations) that every discrete subgroup of $PU(1, n)$ acts discontinuously on $\mathbb{H}_{\mathbb{C}}^n$. Hence, regarded as a subgroup of $PU(n + 1)$, such a group acts on $\mathbb{CP}^n$ with non-empty region of discontinuity. In other words, we have:

> *Every complex hyperbolic discrete group is a complex kleinian group,*

a statement that generalises to higher dimensions the well-known fact that every fuchsian subgroup of $PSL(2, \mathbb{R})$ is kleinian when regarded as a subgroup of $PSL(2, \mathbb{C})$.

2.4   Complex affine groups

There is another classical way of constructing the projective space, and this also plays a significant role for producing discrete subgroups of $PSL(n + 1, \mathbb{C})$. This is by thinking of $\mathbb{CP}^n$ as being the union of $\mathbb{C}^n$ and the "hyperplane at infinity":

$$\mathbb{CP}^n = \mathbb{C}^n \cup \mathbb{CP}^{n-1}.$$

A way for doing so is by writing

$$\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C} = \{(Z, z_n) \,|\, Z = (z_0, ..., z_{n-1}) \in \mathbb{C}^n \text{ and } z_n \in \mathbb{C}\}.$$

Then every point in the hyperplane $\{(Z, 1)\}$ determines a unique line through the origin in $\mathbb{C}^{n+1}$, *i.e.*, a point in $\mathbb{CP}^n$; and every point in $\mathbb{CP}^n$ is obtained in this way except for those corresponding to lines (or "directions") in the hyperplane $\{(Z, 0)\}$, which form the "hyperplane at infinity" $\mathbb{CP}^{n-1}$. It is clear that every affine map of $\mathbb{C}^{n+1}$ leaves invariant the hyperplane at infinity $\mathbb{CP}^{n-1}$. Furthermore, every such map carries lines in $\mathbb{C}^{n+1}$ into lines in $\mathbb{C}^{n+1}$, so the map naturally extends to the hyperplane at infinity. This gives a natural inclusion of the affine group

$$Aff(\mathbb{C}^n) \cong GL(n, \mathbb{C}) \ltimes \mathbb{C}^n,$$

in the projective group $PSL(n+1, \mathbb{C})$. Hence every discrete subgroup of $Aff(\mathbb{C}^n)$ is a discrete subgroup of $PSL(n+1, \mathbb{C})$.

A discrete subgroup of $Aff(\mathbb{C}^n)$ is called a *complex crystallographic group* (sometimes this name is reserved for groups with compact quotient). Crystallographic groups have been studied by various authors.

3   GEOMETRY AND DYNAMICS

3.1   The limit set: an example

In the first section of these notes we defined the limit set of a kleinian group in the classical way: It is the set of accumulation points of the orbits. This is indeed a good definition in that setting in all possible ways: Its complement Ω is the maximal region of discontinuity for the action of the group on the sphere, and Ω is also the region of equicontinuity, *i.e.*, the set of points where the group forms a normal family.

It could be nice to have such a "universal" concept in the setting we envisage in these notes, that of groups of automorphisms of $\mathbb{CP}^n$. Alas this is not possible in general and there is not a concept of limit set which can be said to be " the correct" one. Rather, there are several possible definitions, each with its own interest and characteristics and which coincide under certain conditions. This is illustrated by the example below.

Indeed the question of giving "the definition" of limit set can be rather subtle, as pointed out by R. Kulkarni in the general setting of discrete group actions [14].

Consider the following example from [16]. Let $\gamma \in PSL(3, \mathbb{C})$ be the projectivisation of the linear map $\tilde{\gamma}$ given by:

$$\tilde{\gamma} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

where $\alpha_1\alpha_2\alpha_3 = 1$ and $|\alpha_1| < |\alpha_2| < |\alpha_3|$. We denote by Γ the cyclic subgroup of $PSL(3,\mathbb{C})$ generated by γ; we may choose the α_i so that Γ is conjugate to a subgroup of $PU(1,2)$. Denote by $\{e_1, e_2, e_3\}$ the usual basis of $\mathbb{C}^3$. Each of these vectors represents a complex line in $\mathbb{C}^3$ and therefore determines a point in $\mathbb{CP}^2$, that for simplicity we denote also by $\{e_1, e_2, e_3\}$; these are fixed points of the action, since the corresponding lines are invariant. The conditions on the norm of the eigenvalues imply that the backwards orbit of "almost" every point in $\mathbb{CP}^2$ converges to e_1, while most of the forward orbits converge all to e_3. To be precise, notice that since $\{e_1, e_2, e_3\}$ are fixed points, the lines joining them, $\overleftrightarrow{e_i, e_j}$, are invariant lines and the set of accumulation points of all orbits consists of the points $\{e_1, e_2, e_3\}$. The first of these is an attractor, the second is a saddle point and the latter is a source.

It is not hard to show that:

i. Γ acts discontinuously on $\Omega_0 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_3, e_2})$, and also on $\Omega_1 = \mathbb{CP}^2 - (\overleftrightarrow{e_1, e_2} \cup \{e_3\})$ and $\Omega_2 = \mathbb{CP}^2 - (\overleftrightarrow{e_3, e_2} \cup \{e_1\})$.

ii. Ω_1 and Ω_2 are the maximal open sets where Γ acts properly discontinuously; and Ω_1/Γ and Ω_2/Γ are compact complex manifolds. (In fact they are Hopf manifolds).

iii. Ω_0 is the largest open set where Γ forms a normal family.

It follows that even if the set of accumulation points of the orbits consists of the points $\{e_1, e_2, e_3\}$, in order to actually get a properly discontinuous action we must remove a larger set. Furthermore, in this example we see that there is not a largest region where the action is properly discontinuous, since neither Ω_1 nor Ω_2 is contained in the other.

So one has several candidates to be called as "limit set":

- The points $\{e_1, e_2, e_3\}$ where all orbits accumulate. But the action is not properly discontinuous on all of its complement. Yet, this definition is good if we make this group conjugate to one in $PU(1,2)$ and we restrict the discussion to the "hyperbolic disc" $\mathbb{H}^2_\mathbb{C}$ contained in $\mathbb{CP}^2$. This corresponds to taking the Chen-Greenberg limit set of Γ, that we shall define below.

- The two lines $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_3, e_2}$, which are attractive sets for the iterations of γ (in one case) or γ^{-1} (in the other case). This corresponds to Kulkarni's limit set of Γ, that we define below, and it has the nice property that the action on its complement is properly discontinuous and also, in this case, equicontinuous. And yet, the proposition above says that away from either one of these two lines the action of Γ is discontinuous. So this region is not "maximal".

- Then we may be tempted to taking as limit set the complement of the "maximal region of discontinuity", but there is no such region: there are two of them, the complements of each of the two invariant lines, so which one we choose?

- Similarly we may want to define the limit set as the complement of "the equicontinuity region". In this particular example, that definition may seem appropriate. The problem is that this would rule out important cases, as for instance the Hopf manifolds, which can not be written in the form U/G where G is a discrete subgroup of $PSL(3, \mathbb{C})$ acting equicontinuously on an open set U of $\mathbb{CP}^2$. Moreover, there are examples where Γ is the fundamental group of certain compact complex surfaces (Inoue surfaces) and the action of Γ on $\mathbb{CP}^2$ has no points of equicontinuity.

Thus one has different definitions with nice properties in different settings. For example, we will see that for Schottky groups in higher odd dimensional projective spaces, there is yet another definition of limit set which seems appropriate.

We shall say more about limit sets later.

3.2   The limit set for complex hyperbolic groups

Consider now a discrete subgroup G of $PU(1, n)$. As before, we take as model for complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ the ball $\mathbb{B} \cong \mathbb{B}^{2n}$ in $\mathbb{CP}^n$ consisting of points with homogeneous coordinates satisfying

$$|z_1|^2 + \cdots + |z_n|^2 < |z_0|^2 \,,$$

whose boundary is a sphere $\partial \mathbb{H}^n_{\mathbb{C}} \cong \mathbb{S}^{2n-1}_{\infty}$, and we equip $\mathbb{B}$ with the Bergman metric ρ to get $\mathbb{H}^n_{\mathbb{C}}$.

The following notion was introduced in [12].

Definition 3.1 *If G is a discrete subgroup of $PU(1, n)$, then its Chen-Greenberg limit set , denoted $\Lambda_{CG}(G)$, is the set of accumulation points of the G-orbit of any point in $\mathbb{H}^n_{\mathbb{C}}$.*

As remarked earlier, the fact that the action on $\mathbb{H}^n_{\mathbb{C}}$ is by isometries and G is discrete implies that the orbit of every $x \in \mathbb{H}^n_{\mathbb{C}}$ must accumulate in $\partial \mathbb{H}^n_{\mathbb{C}}$. Hence the limit set $\Lambda_{CG}(G)$ is contained in the sphere at infinity, likewise in the conformal case. Moreover, one also has the following result of Chern-Greenberg:

Proposition 3.2 *If the set $\Lambda_{CG}(G) \subset X$ has more than two points, then every orbit in $\Lambda_{CG}(G)$ is dense in $\Lambda_{CG}(G)$.*

The proof of this result is entirely analogous to the proof in the classical case, for groups of isometries of the real hyperbolic space, and it relies on the fact that the action is by isometries of $\mathbb{H}^n_{\mathbb{C}}$.

It is clear from the definition that $\Lambda_{CG}(G)$ is a closed invariant subset of $\mathbb{S}^{2n-1}_{\infty}$, and the result above says that this set is minimal. In particular $\Lambda_{CG}(G)$ does not depend on the choice of the orbit of the point in $\mathbb{H}^n_{\mathbb{C}}$.

Thus, when considering subgroups of $PU(1, n)$ acting on the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, this definition of limit set is a good definition. Yet, if we consider the action of G on

the whole projective space, it is easy to show that the action is not properly discontinuous away from the ball $[N_-] \subset \mathbb{CP}^n$, which serves as model for $\mathbb{H}_{\mathbb{C}}^n$. So we need to introduce another notion of limit set.

R. Kulkarni introduced in [14] a concept of limit set that he motivated through an example, which actually inspired the example we gave above. This definition of limit set applies in a very general setting of discrete group actions, and it has the important property of assuring that its complement is an open invariant set where the group acts properly discontinuously. For this, recall that given a family $\{A_\beta\}$ of subsets of X, where β runs over some infinite indexing set B, a point $x \in X$ is a *cluster (or accumulation) point* of $\{A_\beta\}$ if every neighbourhood of x intersects A_β for infinitely many $\beta \in B$.

Given a space X and a group G as above, let $L_0(G)$ be the closure of the set of points in X with infinite isotropy group. Let $L_1(G)$ be the closure of the set of cluster points of orbits of points in $X - L_0(G)$, *i.e.*, the cluster points of the family $\{\gamma(x)\}_{\gamma \in G}$, where x runs over $X - L_0(G)$.

Finally, let $L_2(G)$ be the closure of the set of cluster points of $\{\gamma(K)\}_{\gamma \in G}$, where K runs over all the compact subsets of $X - \{L_0(G) \cup L_1(G)\}$. We have:

Definition 3.3   i. Let X be as above and let G be a group of homeomorphisms of X. The *Kulkarni limit set* of G in X is the set

$$\Lambda_{Kul}(G) := L_0(G) \cup L_1(G) \cup L_2(G).$$

ii. The *Kulkarni region of discontinuity* of G is

$$\Omega_{Kul}(G) \subset X := X - \Lambda_{Kul}(G).$$

It is easy to see that the set $\Lambda_{Kul}(G)$ is closed in X and it is G-invariant (it can be empty). The set $\Omega_{Kul}(G)$ (which also can be empty) is open, G-invariant, and G acts properly discontinuously on it.

When G is a Möbius (or conformal) group, the classical definitions of the *limit set* and the *discontinuity set* coincide with the above definitions.

In the example in Section 3.1 one has that the sets $L_0(G)$ and $L_1(G)$ are equal, and they consist of the three points $\{e_1, e_2, e_3\}$, while $L_2(G)$ consists of the lines $\overleftrightarrow{e_1, e_2}$ and $\overleftrightarrow{e_2, e_3}$, passing through the saddle point.

That example also shows that although Kulkarni's limit set has the property of assuring that the action on its complement is discontinuous, this region is not always maximal. Yet, one can show that in the case of actions of complex hyperbolic groups on $\mathbb{CP}^2$, "generically" this region is the largest open set where the action is discontinuous, and it coincides with the region of equicontinuity, by [16]. In fact a similar statement holds "generically" for discrete subgroups of $PSL(3, \mathbb{C})$, by [5, 6, 7, 8].

A first obvious question is to determine the relation between these two notions of limit set in the case of complex hyperbolic groups, as well as the relation of these sets with the corresponding region of equicontinuity.

One has the following theorem due to J. P. Navarrete [16] in the two dimensional case, and to A. Cano and J. Seade [10] in higher dimensions. For this, let $\mathbb{S}_\infty^{2n-1} \subset \mathbb{CP}^n$ be the boundary of $\mathbb{H}_{\mathbb{C}}^n$, so it is the sphere at infinity. Notice that for each point $x \in \mathbb{S}_\infty^{2n-1}$ there is a unique complex projective subspace $\mathcal{H}_x$ of complex dimension $n-1$ which is tangent to $\mathbb{S}_\infty^{2n-1}$ at x. Given a discrete subgroup $G \subset PU(1,n)$, let $\mathcal{H}_G$ be the union of all these projective subspaces for all points in the limit set $\Lambda_{CG}(G) \subset \mathbb{S}_\infty^{2n-1}$. This set is clearly G-invariant, since $\Lambda_{CG}(G)$ is invariant and the G action on $\mathbb{S}_\infty^{2n-1}$ is by holomorphic transformations.

Theorem 3.4 (Navarrete, Cano-Seade) *Let $G \subset PU(1,n)$ be a discrete subgroup and let $Eq(G)$ be its equicontinuity region in $\mathbb{P}_{\mathbb{C}}^n$. Then $\mathbb{P}_{\mathbb{C}}^n \setminus Eq(G)$ is the union of all complex projective hyperplanes tangent to $\partial\mathbb{H}_{\mathbb{C}}^n$ at points in $\Lambda(G)$, and G acts properly discontinuously on $Eq(G)$. Moreover, if $n = 2$ then $Eq(G)$ coincides with the Kulkarni region of discontinuity $\Omega_{Kul}(G)$.*

We do not know yet whether or not in higher dimensions the Kulkarni region of discontinuity of G coincides with the region of equicontinuity.

Let us mention some other interesting examples of complex kleinian groups. We refer to [9] for details.

Example 3.5 (Fundamental groups of complex tori.) *Let $v_1, \ldots, v_4 \in \mathbb{C}^2 - \{0\}$ be $\mathbb{R}$-linearly independent vectors and $g_i : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the projective transformations induced by the translation generated by v_i. Then $\Gamma = <g_1, \ldots, g_4>$ is a group isomorphic to $\mathbb{Z}^4$ and satisfies*

$$Eq(\Gamma) = \Omega_{Kul}(\Gamma) = \mathbb{C}^2.$$

Moreover $\mathbb{C}^2$ is the largest open set where Γ acts properly discontinuously.

Example 3.6 (Fundamental groups of Hopf surfaces.) *Let $\Gamma_g = <g>$ be the cyclic group of $PSL(3, \mathbb{C})$ induced by the affine transformation $g(z, w) = (\lambda z, \lambda w)$, where $|\lambda| < 1$. Then $Eq(\Gamma_g) = \Omega_{Kul}(\Gamma_g) = \mathbb{C}^2 \setminus \{0\}$ is the largest open set where Γ_g acts properly discontinuously. One can show that $(\mathbb{C}^2 \setminus \{0\})/\Gamma_g$ is a compact manifold, and in fact this is a Hopf Manifold.*

Example 3.7 (Suspensions.) *Every element $g \in PSL(2, \mathbb{C})$ has two liftings to $SL(3, \mathbb{C})$, that we may denote $\pm g$. We call the projective transformations induced by the following matrices*

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} -g & 0 \\ 0 & 1 \end{pmatrix}$$

the suspension of g on $PSL(3, \mathbb{C})$. Now given $\Gamma \subset PSL(2, \mathbb{C})$ a Kleinian group and $G \subset \mathbb{C}^*$ a discrete group, we will define the suspension of Γ with respect G, in symbols $Sus(G, \Gamma)$, as the group

$$\left\langle \{g \in PSL(3, \mathbb{C}) | g \text{ is a lift of an element } \tilde{g} \in \Gamma \} \cup \left\{ \begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g^{-2} \end{pmatrix} : g \in G \right\} \right\rangle.$$

Then $Eq(Sus(G, \Gamma)) = \mathbb{P}^2_{\mathbb{C}} \setminus \mathcal{C} = \Omega_{Kul}(G)$ is the largest open set where Γ acts properly discontinuously; here:

$$\mathcal{C} = \begin{cases} \bigcup_{p \in \Lambda(\Gamma)} \overleftrightarrow{p, e_3} & \text{if } G \text{ es finite.} \\ \bigcup_{p \in \Lambda(\Gamma)} \overleftrightarrow{p, e_3} \cup \overleftrightarrow{e_1, e_2} & \text{if } G \text{ es infinite.} \end{cases}$$

Example 3.8 (An Inoue Surface.) *Let $M \in SL(3, \mathbb{Z})$ be a matrix with eigenvalues $\alpha, \beta, \overline{\beta}$ where $\alpha > 1$, $\beta \neq \overline{\beta}$. Let (a_1, a_2, a_3) be a real eigenvector belonging to α and (b_1, b_2, b_3) an eigenvector belonging to β. Now, let G_M be the group induced by the transformations:*

$$\gamma_0(w, z) = (\alpha w, \beta z),$$
$$\gamma_i(w, z) = (w + a_i, z + b_i) \, i = 1, 2, 3.$$

Then $\Omega_{Kul}(G_M) = \mathbb{H}^+ \times \mathbb{C}$ is the largest open set where G_M acts properly discontinuously and $\Omega_{Kul}(G_M)/G_M$ is a compact surface, called Inoue Surface. However in this case $Eq(G_M) = \emptyset$.

Example 3.9 (A Group induced by a Toral Automorphism.) *Let M be the matrix*

$$M = \begin{pmatrix} 3 & 5 \\ -5 & 8 \end{pmatrix}.$$

It is not hard to check that the eigenvalues of M are

$$\alpha_{\pm} = \frac{-5 \pm \sqrt{21}}{2}$$

and a choice of the corresponding eigenvectors could be

$$v_+ = (1, \frac{-11 + \sqrt{21}}{10}), \, v_- = (\frac{-11 + \sqrt{21}}{10}, 1).$$

Set $\Gamma_M^{\ltimes}$ be the group induced by the following transformations:

$$\gamma_0(w, z) = (\alpha_+ w, \alpha_- z);$$
$$\gamma_1(w, z) = (w + 1, z + \tfrac{-11 + \sqrt{21}}{10});$$
$$\gamma_2(w, z) = (w + \tfrac{-11 + \sqrt{21}}{10}, z + 1);$$
$$\gamma_3(w, z) = (z, w).$$

Then $\Omega_{Kul}(\Gamma_A^{\ltimes}) = Eq(\Gamma_A^{\ltimes}) = \bigcup_{j, j = 0, 1} (\mathbb{H}^{(-1)^i} \times \mathbb{H}^{(-1)^j})$ is the largest open set where $\Gamma_A^{\ltimes}$ acts properly discontinuously.

3.3 Divisible open sets and proyective structures

As mentioned before, roughly speaking, when we have a discrete group action, we generally have two invariant sets: one of them is where the dynamics concentrates, the limit set (in a suitable definition), and another where the dynamics is in some sense "mild".

For instance, one of the main reasons for studying discrete subgroups of $PU(1, n)$ is because given any such subgroup G, the quotient $\mathbb{H}^n_{\mathbb{C}}/G$ is a complex hyperbolic orbifold. Orbifolds are a generalization of manifolds. If G acts freely on $\mathbb{H}^n_{\mathbb{C}}$ then the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is actually a compact manifold with a complex hyperbolic structure. This is a very rich and interesting geometric structure, and many of the lectures we shall hear about in this meeting will be precisely about complex hyperbolic manifolds.

For instance, when $n = 1$, the complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}$ coincides with the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, and the Riemann-Köebe uniformization theorem says that every oriented, closed surface of genus more than 1 has a hyperbolic structure.

We now recall that we have already shown that complex hyperbolic groups are canonically subgroups of $PSL(n + 1, \mathbb{C})$, the group of holomorphic automorphisms of complex projective space. And the similar statement applies to real hyperbolic geometry, that is, every discrete subgroup of Iso $\mathbb{H}^n_{\mathbb{R}}$ is canonically a subgroup of $PSL(n + 1, \mathbb{R})$, the group of automorphisms of real projective space. As we explain below, this means that every real (or complex) hyperbolic manifold is canonically a real (or complex, respectively) projective manifold.

Let us explain briefly what this means.

A topological n-dimensional manifold means a topological, metric, space X, equipped with a locally finite covering by open sets $U\alpha$, each of these coming with a homeomorphism h_α from U_α into an open subset of $\mathbb{R}^n$. The set $\{(U_\alpha, h_\alpha)\}$ is called an atlas for X. In other words, an atlas tells us how X looks by pieces, just as a usual "atlas" for the earth tells us how it looks by pieces, since we can not make a round picture of the earth fit into a book! Again, as in a usual atlas for the earth, given the local maps (in general called coordinate charts), in order to get the global picture we must indicate how each chart is glued together with its neighbours. In other words, consider two of these open sets U_α and U_β with non-empty intersection, then we have homeomorphisms h_α, h_β from $U_\alpha \cap U_\beta$ into open subset of $\mathbb{R}^n$, and we can consider the homeomorphism:

$$\phi_{\alpha,\beta} := h_\beta \circ h_\alpha^{-1} : h_\alpha(U_\alpha \cap U_\beta) \rightarrow h_\beta(U_\alpha \cap U_\beta).$$

These homeomorphisms are called *the transition functions* for the given atlas, and they tell us how the little pieces, or coordinate charts, must be glued together in order to reconstruct the manifold X.

Notice each $\phi_{\alpha,\beta}$ is a homeomorphism between open subsets of $\mathbb{R}^n$ so it makes sense to ask whether this is also a diffeomorphism, or a diffeomorphism of class C^r, $r \geq 1$. By definition, a topological manifold X is a differentiable manifold of class C^r if we can equip it with an atlas such that all the transition functions are differentiable manifold of class C^r.

For instance, a teardrop is a topological manifold of dimension 2, for it is homeomorphic to a 2-sphere, so we can cover by two open sets (the sphere minus the North pole, and the sphere minus the South pole), each homeomorphic to a 2-disc

Figure 6: A teardrop

It is a deep theorem in low dimensional topology, due to Hirsch, that every closed (= compact with no boundary) topological manifold of dimension ≤ 3 is homeomorphic to a smooth manifold. But in higher dimensions this is no longer true.

In general, we may ask whether the manifold X can be given an atlas where the transition functions satisfy further restrictions, besides being differentiable. For instance, if n is even, then the functions $\phi_{\alpha,\beta}$ can be regarded as homeomorphisms between open subsets of $\mathbb{C}^n$ and we can ask whether they are actually holomorphic functions. If this is the case then we say that X is a complex manifold. That is, a complex n-manifold means a differentiable manifold of dimension $2n$ equipped with an atlas for which all the transition functions are not only differentiable of class $\mathbb{C}^\infty$, but they are actually complex

analytic maps.

A complex manifold of (complex) dimension one is called a Riemann surface. Such a manifold is then a 2-dimensional real surface, equipped with the structure of a complex manifold. And we shall hear a lot about Riemann surfaces in this meeting.

Of course that the same differentiable manifold can be equipped with many different complex structures. For instance the 2-sphere only admits one complex structure (up to equivalence), and this is what we call the Riemann sphere. However, for the torus $\mathbb{S}^1 \times \mathbb{S}^1$, its different (non-equivalent) complex structures are in one-to-one correspondence with the points in $\mathbb{S}^2$. And for closed surfaces of genus $g > 1$, it is know (essentially since Riemann) that its different complex structures form a complex manifold of complex dimension $3g-3$.

Now, given a smooth manifold X we may try to equip it with an atlas such that the coordinate functions satisfy some other type of restriction. For instance, we may ask the transitions functions to be all local isometries of the real hyperbolic space. That is, we may ask the transitions functions to be differentiable maps between open subsets of, say, the upper half space

$$\mathbb{R}^n_+ := \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, x_n > 0 \right\},$$

that we can equip with the hyperbolic metric, and ask the transition functions to preserve this metric. If this is possible, *i.e.*, if we can endow X with such an atlas, then we say we have equipped X with a hyperbolic structure.

Similarly, we can think of $\mathbb{R}^n$ as been an open subset of the real projective space $\mathbb{RP}^n_{\mathbb{R}}$ and ask the transition functions to be restriction of projective automorphisms. If we can do so, then *we have equipped the manifold X with a projective structure.*

In other words, to say that X is a projective manifold means that X is modelled locally by open subsets of $\mathbb{RP}^n_{\mathbb{R}}$ which are glued by elements in the projective group $PSL(n+1,\mathbb{R})$.

Of course that analogous statements hold for complex manifolds: A complex manifold of (complex dimension) n admits a complex projective structure if it can be equipped with an atlas where all the transition functions are elements in the projective group $PSL(n+1,\mathbb{C})$. This is a topic very much related to what we have been discussing so far in these lectures.

For instance, let Γ be a discrete subgroup of $PSL(n+1,\mathbb{C})$ and let Ω_{Kul} be its Kulkarni region of discontinuity. Suppose for simplicity that Γ acts freely on Ω_{Kul}. That is, if $\gamma(x) = x$ for some point $x \in \Omega_{Kul}$ and some $\gamma \in \Gamma$, then γ is the identity. Then one has that:

The quotient $M := \Omega_{Kul}/\Gamma$ is a complex manifold with a projective structure,

and of course it is interesting to study the geometry and topology of the manifolds one gets in this way. For instance, if Γ is a cocompact complex hyperbolic group, then Ω_{Kul} is $\mathbb{H}^n_{\mathbb{C}}$ and what we get is a complex hyperbolic manifold. But there are many other interesting families of manifolds arising in this way. It is also interesting to study complex hyperbolic group which are not cocompact, and consider their action on $\mathbb{CP}^n$, not only on $\mathbb{H}^n_{\mathbb{C}}$. This is

being done in dimension 2 by Waldemar Barrera, Juan-Pablo Navarrete and Angel Cano (see [9]; also [10]).

In dimension two, there are interesting articles by various authors studying the compact complex surfaces with a projective structure, such as Kobayashi, Ochiai, Inoue and Klingler, among others. And if we look at these questions from the viewpoint of the group actions, then this is the work of Angel Cano's thesis.

Following Y. Benoist, let us say that an open set $\Omega \subset \mathbb{CP}^n$ is *divisible* if there exists a discrete subgroup $\Gamma \subset PSL(n+1, \mathbb{C})$ for which Ω is an invariant set and the quotient Ω/Γ is compact. Then, for $n = 2$, Cano show's that there are two families of possible divisible sets, according to whether or not the corresponding groups are elementary (in a sense that we make precise below).

In the first case, the region Ω is *a-fortiori* the Kulkarni region of discontinuity of the group and up to projective equivalence we have the following four possibilities:

i. $\Omega_{Kul}(\Gamma) = \mathbb{CP}^2$. Then the group is finite and the Kulkarni limit set $\Lambda_{Kul}(\Gamma)$ is empty.

ii. $\Omega_{Kul}(\Gamma) = \mathbb{C}^2$. Then Γ is affine, the quotient $\Omega_{Kul}(\Gamma)/\Gamma$ is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus or a Kodaira surface, and the limit set $\Lambda_{Kul}(\Gamma)$ is a projective line.

iii. $\Omega_{Kul}(\Gamma) = \mathbb{C} \times \mathbb{C}^*$. Then $\Omega_{Kul}(\Gamma)/\Gamma$ is a finite covering (possibly ramified) of a surface biholomorphic to a complex torus. The limit set $\Lambda_{Kul}(\Gamma)$ consists of two lines.

iv. $\Omega_{Kul}(\Gamma) = \mathbb{C}^* \times \mathbb{C}^*$. Then $\Omega_{Kul}(\Gamma)/\Gamma$ is also a finite covering (possibly ramified) of a surface biholomorphic to a complex torus. The Kulkarni limit set now consists of three lines.

The second case is when the limit set contains infinitely many projective lines or points away from these lines. In this situation he proves that up to projective equivalence one has the following possibilities:

i. The group is complex hyperbolic and $\Omega_{Kul}(\Gamma)$ is $\mathbb{H}^2_{\mathbb{C}}$.

ii. The group is affine, $\Omega_{Kul}(\Gamma) = \mathbb{C} \times (\mathbb{H}^- \cup \mathbb{H}^+)$, and $\Omega_{Kul}(\Gamma)/\Gamma$ is equal to M or $M \sqcup M$ where $M = (\mathbb{C} \times \mathbb{H})/\Gamma_{\mathbb{C} \times \mathbb{H}}$ is an Inoue Surface ($\sqcup$ denotes disjoint union).

iii. The group is affine and $\Omega_{Kul}(\Gamma) = \mathcal{U} \times \mathbb{C}^*$ where $\mathcal{U} \subset \mathbb{P}^1_{\mathbb{C}}$ is the discontinuity region of a classical kleinian group. In this case $\Omega_{Kul}(\Gamma)/\Gamma = \bigsqcup_{i \in \mathcal{I}} N_i$ where $\mathcal{I}$ is at most countable, and the N_i are orbifolds whose universal covering orbifold is biholomorphic to $\mathbb{H} \times \mathbb{C}$. Every compact connected component is a finite covering (possibly ramified) of an elliptic affine surface.

4 KLEINIAN GROUPS AND TWISTOR THEORY

We have been talking so far about complex kleinian groups, that is, discrete subgroups of $PSL(n+1, \mathbb{C})$ acting on $\mathbb{CP}^n$ with non-empty region of discontinuity. For $n = 1$ this is the theory of classical kleinian groups, which has been for decades one of the main areas of geometry, low dimensional topology and holomorphic dynamics.

It is natural to ask how rich this theory is, or can be, for $n > 1$. In higher dimensions we know it includes complex hyperbolic groups as well as the complex affine groups. In this section we shall discuss a construcion by Seade and Verjovsky, that gives other interesting families of complex kleinian groups. This gives a way for constructing compact manifolds that arising as quotient of open invariant subsets of $\mathbb{CP}^n$ divided by some kleinian action. There us also a rich dynamics happening.

4.1 The Calabi-Penrose fibration

To start, we need to speak about *twistor theory*. This is an important area of geometry and mathematical physics, developed by various authors, most notably by Roger Penrose, in the late 1970s. There are also important contributions by M. Atiyah, N. Hitchin and several other authors. The idea is that each even-dimensional, oriented riemannian manifold M has its twistor space $\mathfrak{Z}(M)$, a manifold which is a fibre bundle over M, and which under certain differential geometric restrictions on M, has a canonical complex structure. Furthermore, Penrose's twistor program springs from the fact that there is a rich interplay between the conformal geometry of the manifold M and the complex geometry of its twistor space. What Seade and Verjovsky did was showing that this interplay between the conformal geometry of M and the complex geometry of its twistor space can be pushed forward to dynamics. As a consequence we obtaine that every conformal kleinian group, or rather, every group of isometries of a real hyperbolic space (with non-empty region of discontinuity in the sphere at infinity) can be realised as a complex kleinian group, *i.e.*, as a discrete group of holomorphic transformations of some complex projective space, with non-empty region of discontinuity.

This theory is particularly nice when the manifold M is the 4-sphere $\mathbb{S}^4$ endowed with its usual metric, and that is what we shall focus on in this section. The corresponding twistor space turns out to be the complex projective space $\mathbb{CP}^3$. This particular case is also relevant for other interesting problems in differential geometry, studied independently by E. Calabi. Hence in this case the twistor fibration:

$$\pi : \mathbb{CP}^3 \longrightarrow \mathbb{S}^4 ,$$

is also known as the Calabi-Penrose fibration.

To construct this fibration, recall first that the complex projective line $\mathbb{CP}^1$ is the space of lines through the origin in $\mathbb{C}^2$, and so it is diffeomorphic to the sphere $\mathbb{S}^2$. We

claim that, similarly, the sphere $\mathbb{S}^4$ is diffeomorphic to the quaternionic projective line $\mathcal{H}\mathbb{P}^1$. Let us explain this.

Recall that the complex numbers can be regarded as being $\mathbb{R}^2$ with a richer structure, coming from the fact that we have added the symbol i, which corresponds to the point $(0,1)$ in $\mathbb{R}^2$, with $i^2 = -1$. Similarly, we have the space of quaternions $\mathcal{H}$. As a set, this is $\mathbb{R}^4$, a four-dimensional vector space over the real numbers, equipped with a richer structure, obtained by quaternionic multiplication. To define this multiplication we consider the usual basis of $\mathbb{R}^4$ and let $i = (0,1,0,0)$, $j = (0,0,1,0)$ and $k = (0,0,0,1)$; we identify the scalar 1 with the vector $(1,0,0,0)$. Then we define a multiplication by setting:

$$i^2 = j^2 = k^2 = ijk = -1$$

From this we get the well-known relations $ij = k; jk = i; ki = j$, and also $ij = -ji$ and so on. We extend this multiplication to all elements in $\mathcal{H}$ in the obvious way using that $1, i, j, k$ form a basis it as a vector space.

Notice that every quaternion can be expressed as:

$$q = a_0 + a_1 i + a_2 j + a_3 k = (a_0 + a_1 i) + (a_2 + a_3 i)j = z_1 + z_2 j \,.$$

So we see that every quaternion can be regarded as a pair of complex numbers, just as each complex number can be regarded as a pair of real numbers.

We consider now the space $\mathbb{C}^4$ and we identify it with $\mathcal{H} \times \mathcal{H} = \mathcal{H}^2$. Notice that we can multiply vectors in $v \in \mathbb{C}^4$ by complex numbers (scalars) in the usual way:

$$\lambda \cdot (z_1, z_2, z_3, z_4) = (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4) \,.$$

Doing so, each vector $v \in \mathbb{C}^4$ determines a unique complex line ℓ_v in $\mathbb{C}^4$ passing through the origin:

$$\ell_v := \left\{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \,|\, z = \lambda v \quad , \text{ for some } \lambda \in \mathbb{C} \right\}.$$

Similarly, given a vector $v \in \mathcal{H}^2 \cong \mathbb{C}^4$, we can multiply it by quaternions, but one must decide to use either right or left multiplication (now this does matter, since this multiplication is non-commutative). In either case one gets, for each vector v, a quaternionic line $\mathcal{L}_v$, which is a 4-plane:

$$\mathcal{L}_v := \left\{ q = (q_1, q_2) \in \mathcal{H}^2 \,|\, q = \lambda v \quad , \text{ for some } \lambda \in \mathcal{H} \right\}.$$

Notice that each quaternionic line is actually a copy of $\mathbb{C}^2$ embedded in $\mathbb{C}^4$, spanned by the complex lines ℓ_v and ℓ_{jv}. In fact $\mathcal{L}_v$ is filled by complex lines.

Just as $\mathbb{C}\mathbb{P}^3$ is obtained from $\mathbb{C}^4 \setminus 0$ by identifying points in the same complex line, so too we can form the quaternionic projective space:

$$\mathcal{H}\mathbb{P}^1 := \frac{\mathcal{H}^2 \setminus 0}{\mathcal{H}^*} \; \cong \; \mathbb{S}^7/\mathbb{S}^3 \,,$$

the space of left quaternionic lines in $\mathcal{H} \times \mathcal{H}$. In other words, two non-zero quaternions q_1, q_2 are identified if there is another quaternion q such that $q\, q_1 = q_2$.

We leave it as an exercise to show that, just as one has:

$$\mathbb{RP}^1 \cong \mathbb{S}^1 \qquad \text{and} \qquad \mathbb{CP}^1 \cong \mathbb{S}^2 \,,$$

so too one has:

$$\mathcal{H}\mathbb{P}^1 \cong \mathbb{S}^4 \,.$$

Therefore we see that if in $\mathbb{C}^4 \cong \mathcal{H}^2$:

i) We identify each complex line to a point, then we get $\mathbb{CP}^3$;

ii) And if we identify each quaternionic line to a point, we get $\mathcal{H}\mathbb{P}^1 \cong \mathbb{S}^4$.

Since every complex line is contained in a unique quaternionic line, we thus get a projection map:

$$\pi : \mathbb{CP}^3 \longrightarrow \mathcal{H}\mathbb{P}^1 \,,$$

which is easily seen to be a locally trivial fibration, *i.e.*, a fibre bundle. For each point $[q_1 : q_2] \in \mathcal{H}\mathbb{P}^1$ the fiber $\pi^{-1}([q_1 : q_2])$ consists of all the complex lines through the origin in $\mathbb{C}^4 \cong \mathcal{H}^2$ which are contained in the same quaternionic line, which is a copy of $\mathbb{C}^2$. Hence each fibre is diffeomorphic to $\mathbb{CP}^1 \cong \mathbb{S}^2$.

This is the Calabi-Penrose fibration, also known as the twistor fibration of the 4-sphere.

4.2 Conformal dynamics *versus* holomorphic dynamics

We now recall that one has a group isomorphism:

$$\mathrm{Conf}_+(\mathbb{S}^2) \cong \left\{ \frac{az + b}{cz + d} \,\middle|\, a, b, c, d \in \mathbb{C} \right\} \cong \mathrm{PSL}(2, \mathbb{C}) \,.$$

The proof of these facts can be adapted to showing the analogous statements (see Ahlfors' works [1, 2]):

$$\mathrm{Conf}_+(\mathbb{S}^4) \cong \left\{ (az + b)(cz + d)^{-1} \,\middle|\, a, b, c, d \in \mathcal{H} \right\} \cong \mathrm{PSL}(2, \mathcal{H}) \,,$$

where the latter is the projectivisation of the group of 2×2 invertible matrices with coefficients in $\mathcal{H}$ and determinant one. Notice that one such matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts linearly on $\mathcal{H}^2$, and so it also acts on $\mathbb{C}^4 \cong \mathcal{H}^2$, with quaternionic multiplication being regarded as a 2×2 complex matrix. Hence there is a natural embedding

$$\mathrm{Conf}_+(\mathbb{S}^4) \hookrightarrow \mathrm{PSL}(4, \mathbb{C}) \,.$$

Therefore we get:

Proposition 4.1 *Every group of orientation preserving isometries of the real hyperbolic space $\mathbb{H}^5_{\mathbb{R}}$ has a canonical lifting to a group of holomorphic automorphisms of the complex projective space $\mathbb{CP}^3$.*

$$
\begin{array}{ccc}
\mathbb{CP}^3 & \xrightarrow{\ \widetilde{\Gamma}\ } & \mathbb{CP}^3 \\
\pi \downarrow & & \downarrow \pi \\
\mathbb{S}^4 & \xrightarrow{\ \Gamma\ } & \mathbb{S}^4
\end{array}
$$

This result is well-known in full generality (not only for the 4-sphere) for people working in twistor theory; this is also proved in [19] in a different way, using twistor theory. Notice that given an element $\gamma \in \mathrm{Conf}_+(\mathbb{S}^4)$, its lifting to $PSL(4,\mathbb{C})$ is an automorphism of $\mathbb{CP}^3$ that carries fibres of π into fibres of π, and these are copies of $\mathbb{S}^2$. The fibres of π are called *twistor lines*, and it turns out that the action of Γ on $\mathbb{CP}^3$ carries twistor lines into twistor lines isometrically. Using this one can prove (see [19]):

Theorem 4.2 *Let $\Gamma \subset \mathrm{Conf}_+(\mathbb{S}^4)$ be a Kleinian group and let $\Omega(\Gamma) \subset \mathbb{S}^4$ be its region of discontinuity in the sphere. Denote by $\widetilde{\Gamma}$ its lifting to $PSL(4,\mathbb{C})$. Then:*

- *The Kulkarni region of discontinuity $\Omega_{Kul}(\widetilde{\Gamma})$ is $\pi^{-1}(\Omega(\Gamma))$.*

- *The action of $\widetilde{\Gamma}$ on the limit set $\Lambda_{Kul}(\widetilde{\Gamma}) := \mathbb{CP}^3 \setminus \Omega_{Kul}(\widetilde{\Gamma})$ is minimal if and only if Γ is either Zariski dense in $\mathrm{Conf}_+(\mathbb{S}^4)$ or else it is conjugate in $\mathrm{Conf}_+(\mathbb{S}^4)$ to a Zariski dense subgroup of $\mathrm{Conf}_+(\mathbb{S}^3)$.*

- *The quotient $\Omega_{Kul}(\widetilde{\Gamma})/\widetilde{\Gamma}$ is an orbifold with a complex projective structure, and it is a manifold whenever Γ is torsion-free.*

Notice also that $\mathrm{Conf}_+(\mathbb{S}^4)$ is a Lie group of real dimension 15, while $PSL(4,\mathbb{C})$ is a complex Lie group of complex dimension 15. The construction above says that every deformation of a kleinian subgroup of $\mathrm{Conf}_+(\mathbb{S}^4)$ lifts to a deformation of the corresponding kleinian subgroup of $PSL(4,\mathbb{C})$. Yet, in this latter group we have plenty of space to get deformations of the group that do not come form deformations below. We conclude that:

The theory of kleinian subgroups of $PSL(4,\mathbb{C})$ is richer than that of $\mathrm{Iso}_+(\mathbb{H}^5)$!

Similar statements hold in higher dimensions (see [19]; also [20]).

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José Seade
Unidad Cuernavaca del Instituto de Matemáticas,
Universidad Nacional Autónoma de México.
e-mail: jseade@matcuer.unam.mx