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MHD plasma acceleration from plasma thrusters to astrophysical jets: a variational approach

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MHD plasma acceleration from plasma thrusters to astrophysical jets: a variational approach

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Abstract

A Hamiltonian formulation of the MHD plasma flow model is discussed. This formulation makes it possible to cast the model equations in a variational form. Applications ranging from plasma thrusters to astrophysical jets are presented

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Electric propulsion

Rocket equation

$$m\dot{\mathbf{v}} = \dot{m}\mathbf{v}_e,$$

where $m(t)$ and $\mathbf{v}(t)$ are the mass and velocity of the rocket-propellant system, $\dot{m}$ is the propellant mass loss rate and $\mathbf{v}_e$ is the velocity of the expelled propellant in the rocket frame.
If $\mathbf{v}_e$ is uniform we obtain $m_0/m_{fin} = \exp(\Delta v/v_e)$.
The required propellant mass decreases by increasing v_e.
For chemical propellants this velocity is limited to $v_e < 4500m/s$. In order to obtain larger values of v_e one can use the acceleration provided by electromagnetic forces.

Electrostatic acceleration dates to a century ago: Goddard (1906)[1] and Tsiolkovsky (1911). In the '60s Hall Effect Thrusters were developed into efficient propulsion devices in the Soviet Union (1962 first flight test, 1969 production). In the same years in the US the main focus was on developing electrostatic ion thrusters.

The power in e.m. thrusters originates from an external (not from the propellant) non-chemical source (solar panels, electrical batteries..).
The power W required for a given thrust T is

$$W = \dot{m} v_e^2/(2\eta) = T v_e/(2\eta)$$

with η an efficiency factor.

[1] Example of scientific newspaper coverage: New York Times. January 13, 1920
..... *Professor Goddard..... does not know the relation of action and reaction, and of the need to have something better than a vacuum against which to react – to say that would be absurd. Of course he only seems to lack the knowledge ladled out daily in high schools.*

Electromagnetic propulsion

Electromagnetic propulsion exploits the interaction in a conductive fluid (quasineutral plasma) between the current density $\mathbf{j}$ and a selfinduced or externally applied magnetic field $\mathbf{B}$.

$$.. = \mathbf{j} \times \mathbf{B}/c$$

in the presence of an applied potential difference.
High thrust densities, wide range of possible geometries, magnetic configurations, steady or pulsed regimes, different types of propellants and ionization mechanisms.

The main lines of development are[2]

Hall effect thrusters (HET)
Magneto-Plasma-Dynamic (MPD) thrusters
Pulsed Plasma Thruster (PPT)

[2] See e.g., R. G. Jahn, Physics of Electric Propulsion, McGraw-Hill, New York, 1968.

In the Hall thrusters electrons are magnetized (in a strong quasiradial magnetic field) and decoupled from the unmagnetized ions that are accelerated by a large axial electric field providing the thrust (at the outside ions are finally neutralized).

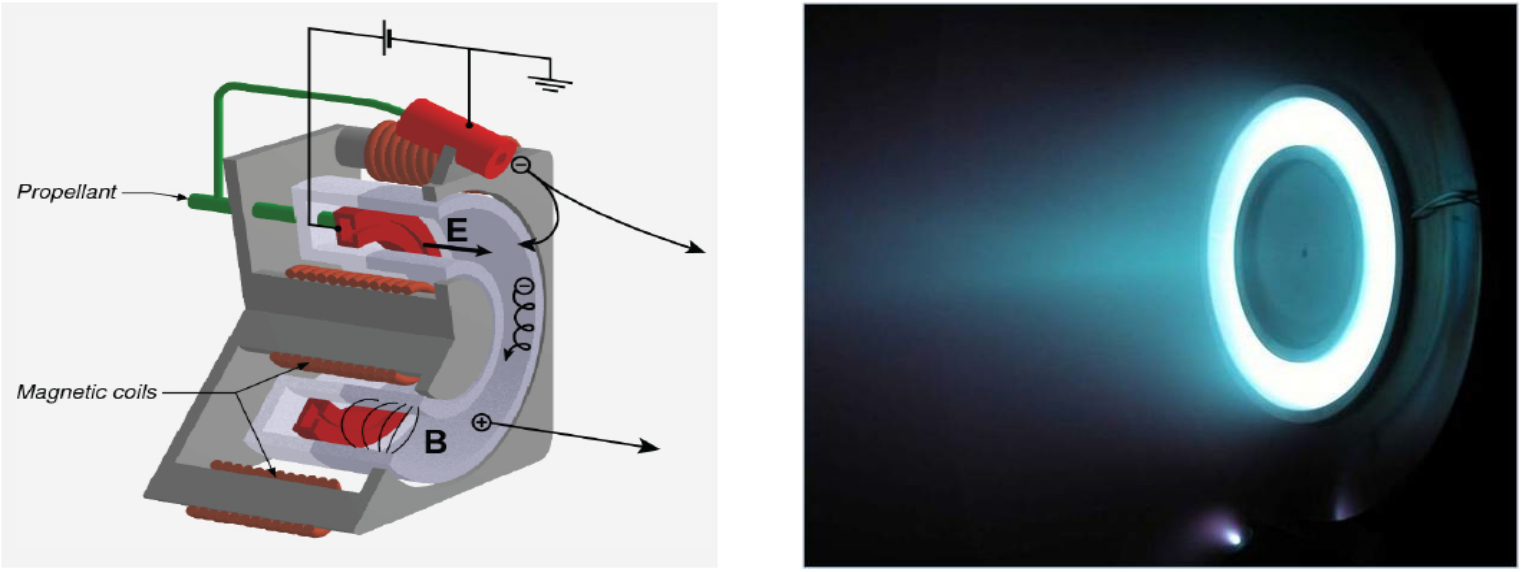
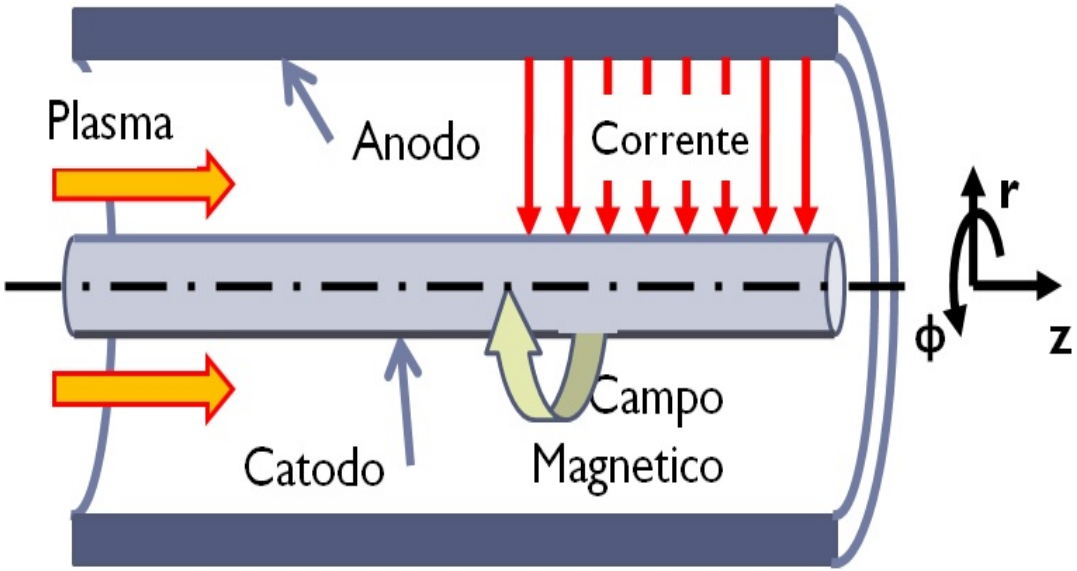


Figure 1: Hall effect thruster.

In the Magneto-Plasma-Dynamic thrusters both electrons and ions are magnetized and the plasma is accelerated by the interaction between the current density in the plasma and the selfinduced magnetic field



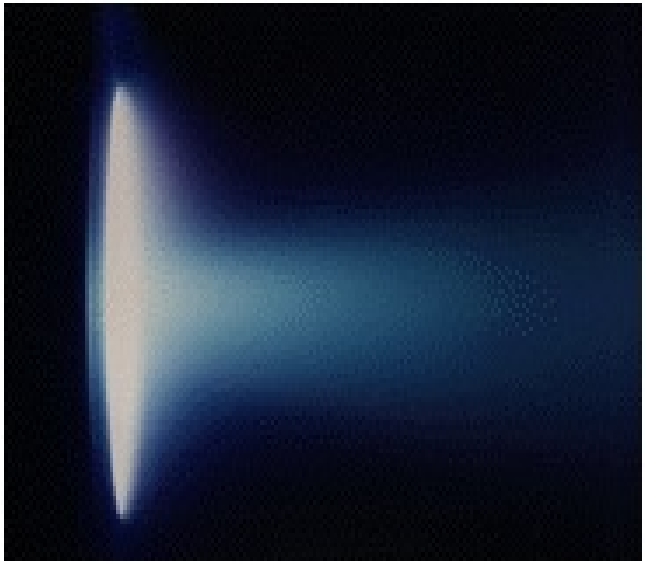


Figure 2: MPD thruster.

In the Applied field Magneto-Plasma-Dynamic thrusters an azimuthal magnetic field is created by an external coil and a magnetic nozzle configuration is adopted to guide the plasma acceleration. This configuration can become kink unstable.

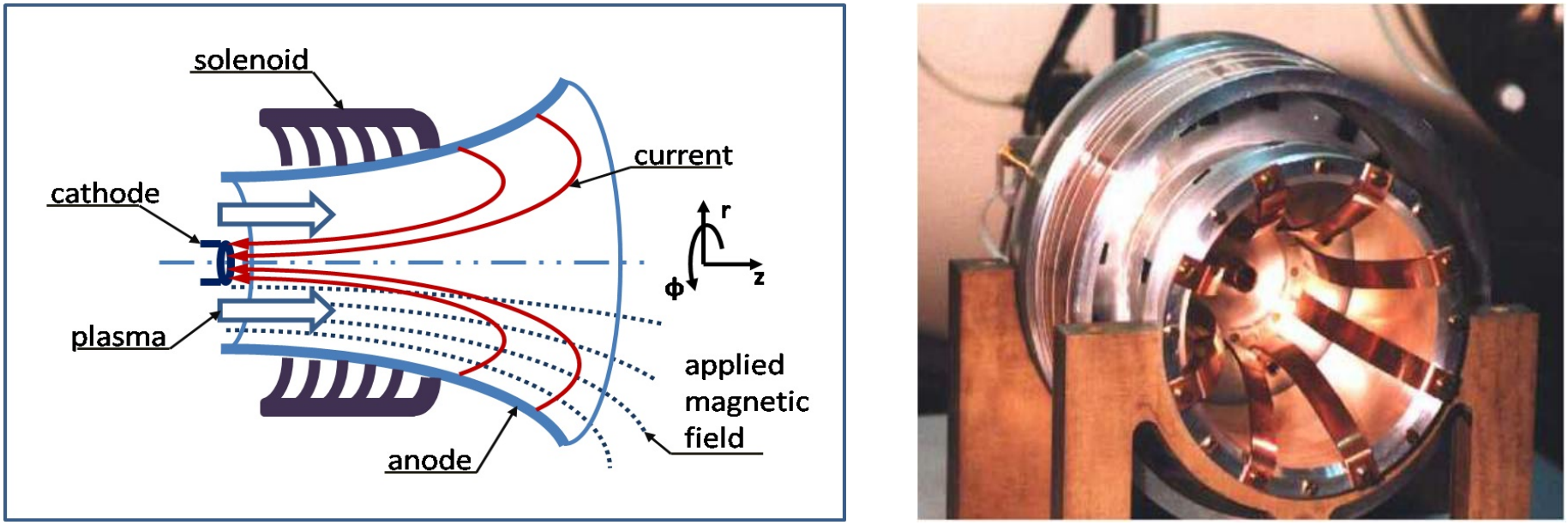


Figure 3: Applied field MPD thruster.

How to model a Magneto-Plasma-Dynamic thruster?

A Magnetohydrodynamic (MHD) description can be used to get the main features of the acceleration mechanism.
In steady regimes we can assume the flow to be stationary and to be essentially axisymmetric.
Magnetic Reynolds numbers can be sufficiently large to allow, at least inside the acceleration channel, for a perfectly conducting fluid description where the magnetic field is frozen in the plasma flow.

Indeed outside the acceleration chamber at least the ions must become detached from the magnetic field as the latter is expected to remain attached to the thruster system while the (neutralized) ions must leave the thruster.
In the following only the acceleration mechanism will be discussed and the effect of the detachment in the external region will be simply accounted for by a boundary condition at the thruster outlet.

Magnetized plasma flows

The behaviour of a plasma flow depends on the interaction between the electromagnetic fields and the conducting medium.

This interaction, which in most cases involves non-stationary fields and dissipation through viscous and resistive effects, leads to a highly complex model.

By neglecting dissipative effects and assuming plasma quasi-neutrality, a general, although simplified, description of the main acceleration processes is possible within the framework of the ideal magnetohydrodynamics (MHD) model.

If in addition the flow configuration is assumed to be time independent and axisymmetric, the MHD equations reduce to a generalized form of the Grad-Shafranov[3] equation, that is used to describe static plasma equilibria in magnetic confinement experiments, and of the Bragg-Hawthorne[4] equation for a hydrodynamic flow with swirl.

[3] H. Grad and H. Rubin, in Proceedings of Second United Nations International Conference on the Peaceful Uses of Atomic Energy (United Nations, Geneva, 1958), Vol. 31, pp. 190–197.
V. D. Shafranov, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2, pp. 103–151.
R. Lüst and A. Schlüter, *Z. Astrophys.* **34**, 263 (1954).

[4] S. L. Bragg and W. R. Hawthorne, *J. Aeronautical. Sci.* **17**, 243 (1950),
R. R. Long, *J. Met.* **10**, 197 (1953),
H. B. Squire, in *Surveys in Mechanics*, edited by G. K. Batchelor and R. M. Davies (Cambridge University Press, Cambridge, 1956) pp. 139-161,
A. R. Elcrat and K. G. Miller, *Diff. Integral Eqs.* **16**, 949 (2003),
H. A. Scott and R. V. E. Lovelace, *Astrophys. J.* **252**, 765 (1982).
R. V. E. Lovelace, C. Mehanian, C. M. Mobarry and M. E. Sulkanen, *Astrophys. J. Suppl. Ser.* **62**, 1 (1986).

Governing equations

The MHD stationary plasma flows are described by the equations

$$\nabla\cdot(\rho\mathbf{v}) = 0, \qquad \rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p + \mathbf{j}\times\mathbf{B}/c,$$

where ρ and S are the plasma density and entropy, $\mathbf{v}$ represents the flow velocity and c is the speed of light.
The pressure p can be expressed in terms of the plasma internal energy per unit mass $U = U(\rho, S)$ as $p = \rho^2\,(\partial U/\partial\rho)$ and, from the entropy equation, we can deduce

$$\rho^{-1}\nabla p = \nabla I - \Theta\nabla S,$$

where $I(p, S) = U + p/\rho$ is the plasma enthalpy per unit mass and $\Theta = \partial U/\partial S$ the plasma temperature.

The current density $\mathbf{j}$ and the magnetic field $\mathbf{B}$, that define the Lorentz force in the momentum equation, satisfy Ampere's law, Faraday's law and the perfect conductivity equation

$$\nabla \times \mathbf{B} = 4\pi \mathbf{j}/c, \qquad \nabla \times \mathbf{E} = 0, \qquad \mathbf{E} + \mathbf{v} \times \mathbf{B}/c = 0,$$

where $\mathbf{E}$ is the electric field.

In order to treat axisymmetric equilibria, we introduce a cylindrical coordinates system (r, ϕ, z), where the azimuthal angle ϕ is the ignorable coordinate and $\phi = const$ defines a poloidal plane. Moreover, in the following $\hat{\phi}$ represents the unit vector in the azimuthal direction and the subscript ϕ indicates the azimuthal component of the magnetic field, B_ϕ, and of the plasma velocity, v_ϕ.

Being divergence free, the poloidal component of the magnetic field can be expressed as

$$\mathbf{B}_p = \nabla\psi \times \nabla\phi, \tag{1}$$

where the scalar function ψ, which labels magnetic flux surfaces, is called the magnetic flux function. The mass conservation equation implies

$$\rho\mathbf{v}_p = \nabla\chi \times \nabla\phi, \tag{2}$$

where χ is the flux function of the mass flow. From the azimuthal component of Ohm's law it follows that

$$\nabla\psi \times \nabla\chi = 0, \tag{3}$$

so the two flux functions, ψ and χ, are functionally dependent.
Eq.(3) is rewritten as

$$\chi = \chi\left(\psi\right). \tag{4}$$

In fact this implies a monotonicity constraint on ψ and thus restricts the possible solutions. For a full discussion see [5] .

From the equations of the model, a set of five flux functions follows that represent quantities conserved along streamlines

$$
\begin{aligned}
F(\psi) &= 4\pi\chi', \\
G(\psi) &= (v_\phi - v_p B_\phi / B_p)/r, \\
H(\psi) &= rB_\phi - rFv_\phi, \\
J(\psi) &= v^2/2 + I - rv_\phi G, \\
S(\psi) &= S,
\end{aligned}
$$

where J is a generalization of the Bernoulli function and a prime indicates derivation with respect to ψ.

[5] T. Andreussi, P. J. Morrison and F. Pegoraro, *Plasma Phys. Contr. Fus.* **52**, 055001 (2010).

Thus, the equation for the generalized equilibria reads

$$\nabla\cdot\left[\left(\frac{F^2}{4\pi\rho}-1\right)\frac{\nabla\psi}{r^2}\right]-\frac{FF^{'}}{4\pi\rho}\left(\frac{|\nabla\psi|}{r}\right)^2= \tag{5}$$

$$4\pi\rho\left(J^{'}+rv_\phi G^{'}-\Theta S^{'}\right)+\frac{B_\phi}{r}\left(H^{'}+rv_\phi F^{'}\right),$$

where the plasma density is related to ψ and its derivatives through the implicit Bernoulli equation.
The dimensionless parameter $M^2 = F^2/4\pi\rho$ is the square of the Alfvén Mach number, which characterize the second order partial differential equation (5).

The equation for the generalized equilibria is hyperbolic[6] for $M_c^2 \leq M^2 \leq M_s^2$ and for $M^2 \geq M_f^2$, where $M_c^2 \equiv \gamma p / \left(\gamma p + B^2/4\pi\right)$ is the square Alfvén Mach number corresponding to the "cusp velocity" and

$$M_{f,s}^2 \equiv \frac{4\pi\gamma p + B^2}{2B_p^2} \left\{ 1 \pm \left[1 - \frac{16\pi\gamma p B_p^2}{(4\pi\gamma p + B^2)^2} \right]^{1/2} \right\} \tag{6}$$

are the square Alfvén Mach numbers relative to the slow and fast magnetosonic velocities, respectively M_s^2 and M_f^2.

[6] J. P. Goedbloed, *Phys. Plasmas* **11**, L81 (2004)
R. V. E. Lovelace, C. Mehanian, C. M. Mobarry and M. E. Sulkanen, *Astrophys. J. Suppl. Ser.* **62**, 1 (1986).

Hamiltonian formulation[7]

Due to the non dissipative nature of this model, a noncanonical Hamiltonian formulation[8] can be adopted as it allows us to overcome some of the difficulties inherent in the differential formulation by using the Hamiltonian as a variational functional constrained by four "flow invariants" (called Casimirs).

[7] T. Andreussi, F. Pegoraro, Magnetized plasma flows and Magnetoplasmadynamic thrusters, Phys. Plasmas, **17**, 063507 (2010)

[8] P. J. Morrison and J. M. Greene, *Phys. Rev. Lett.* **45**, 790 (1980); **48**, 569 (1982).
P. J. Morrison, *AIP Conf. Proc.*, **88**, 13 (1982).
R. Salmon, *Ann. Rev. Fluid Mech.* **20**, 225 (1988).
P. J. Morrison, *Rev. Mod. Phys.* **70**, 467 (1998). P. J. Morrison, *Phys. Plasmas* **12**, 058102 (2005).

Variational principles associated with generalized equilibria were reported[9] in the literature.

The Casimirs and the Hamiltonian dynamics of axisymmetric plasma flows will be presented[10] in terms of noncanonical Poisson bracket and the equations describing generalized MHD equilibria and the corresponding variational principles will be explicitly deduced.

This variational approach can be extended[11], so as to include open-boundary conditions and discontinuous solutions.

[9] M. Heinemann and S. Olbert, *J. Geophys. Res.* **83**, 2457 (1978).
E. Hameiri, *Phys. Plasmas* **5**, 3270 (1998). E. Hameiri, *Phys. Plasmas* **11**, 3423 (2004).
J. P. Goedbloed, *Phys. Plasmas* **11**, L81 (2004).

[10] T. Andreussi, P. J. Morrison and F. Pegoraro, *Plasma Phys. Contr. Fus.* **52**, 055001 (2010).

[11] T. Andreussi and F. Pegoraro, *Phys. Plasmas* **15**, 092108 (2008).

Variational formulation

The Hamiltonian formulation of the hydrodynamic and MHD equations is possible in terms of a non-canonical Poisson bracket

$$\{\mathcal{F},\mathcal{G}\} = \int_V \left(\frac{\delta \mathcal{F}}{\delta \eta_i} \mathbf{J}_{ij} \frac{\delta \mathcal{G}}{\delta \eta_j} \right) d^3 r, \tag{7}$$

where $\mathcal{F}$ and $\mathcal{G}$ are two functionals and $\delta\mathcal{F}/\delta\eta_i$ represents a functional derivative. The noncanonical Poisson bracket $\{\cdot,\cdot\}$ satisfies the antisymmetry condition and the Jacobi identity and $\mathbf{J}$ is a bilinear skew-symmetric operator, which is called the *co-symplectic operator*

The dynamics of the Eulerian variables $\eta = \{\rho, \mathbf{v}_p, v_\phi, \psi, B_\phi, S\}$ can be written as

$$\frac{\partial \eta}{\partial t} = \{\eta, \mathcal{H}\}\,, \tag{8}$$

where the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_V \left(\rho v^2 + \frac{B^2}{4\pi} + 2\rho U\right) d^3r, \tag{9}$$

is the total energy of the system.

Since the Eulerian representation is non-canonical, it is possible to find a set of dynamic invariants, which are called Casimir invariants, that satisfy

$$\{\mathcal{C}, \mathcal{F}\} = 0 \tag{10}$$

for all functionals $\mathcal{F}$.

The functionals $\mathcal{C}$ are determined by the noncanonical Poisson bracket alone and, if we assume the entropy equation $S = S(\psi)$, for the axisymmetric MHD we obtain

$$\mathcal{C}_1 = \int_V \rho J(\psi)\, d^3r,$$
$$\mathcal{C}_2 = \int_V \frac{B_\phi}{r} H(\psi)\, d^3r,$$
$$\mathcal{C}_3 = \int_V r\rho v_\phi G(\psi)\, d^3r,$$
$$\mathcal{C}_4 = \int_V \mathbf{v}\cdot\mathbf{B}\, F(\psi)\, d^3r,$$

where F, G, H and J are generic functions of ψ.

From the condition (10), it follows that the constrained Hamiltonian

$$\mathcal{E} = \mathcal{H} - \int \rho J(\psi) - \int \frac{1}{4\pi r} B_\phi H(\psi) - \int r \rho v_\phi G(\psi) - \int (\mathbf{v} \cdot \mathbf{B}) F(\psi), \quad (11)$$

generates the same dynamics as Eq. (8) and thus equilibria are obtained for

$$\frac{\delta \mathcal{E}}{\delta \eta_j} = 0. \quad (12)$$

In fact, by taking the variations of $\mathcal{E}$ with respect to the variables η_j we see that the generalized equilibria equations correspond to the extrema of the constrained Hamiltonian and that the dynamic invariants generate the set of flux functions of the generalized equilibria model (except for the entropy equation, which is an assumption of the Hamiltonian formulation).

The variational principle (11) depends on the whole set of variables $\eta = \{\rho, \mathbf{v}_p, v_\phi, \psi, B_\phi\}$. It can be rewritten a functional of three independent variables in the form

$$\mathcal{L}(v_\phi, \rho, \psi) = -\int_V [\left(\frac{F^2}{4\pi\rho} - 1\right)\frac{1}{8\pi}\left(\frac{\nabla\psi}{r}\right)^2]d^3r \tag{13}$$

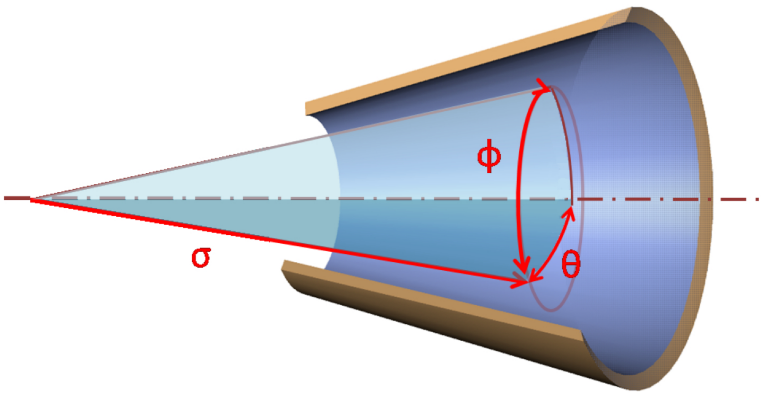
$$+\int_V [\frac{1}{8\pi}\left(\frac{H + 4\pi r v_\phi F}{r}\right)^2 - \frac{1}{2}\rho v_\phi^2 + \rho\left(J + r v_\phi G\right) - \rho U]d^3r.$$

Boundary conditions. For closed geometries, which are typical of magnetic confinement configurations, the boundary is defined by a magnetic flux surface and the corresponding condition is $\psi = const$ on the boundary. However, in order to study open-boundaries problem, we need a description of the configuration that includes the inlet and the outlet regions.

Magnetoplasmadynamic thruster modelling

The approach developed in the previous sections can be used in order to construct explicit solutions of the magnetohydrodynamic equations that describe the acceleration channel of an applied-field MPD thruster and to formulate the appropriate boundary conditions at the exit of the acceleration channel.

Geometry and choice of boundary conditions



The conductive wall and the axis of the (axisymmetric stationary) thruster are two flux surfaces:

$$\psi\left(\sigma, 0\right) = \psi_0 \qquad \text{and} \qquad \psi\left(\sigma, \theta_w\right) = \psi_1,$$

where θ_w is the angle between the symmetry axis and the wall. The difference between these two values can be written in terms of the overall "magnetic flow" $\dot{b}$ as

$$\dot{b} \equiv \int_{S_i} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 2\pi\left(\psi_1 - \psi_0\right)$$

where S_i is the inlet surface. Since the magnetic flux function is defined modulo a constant, we can set the two Dirichlet conditions $\psi|_{\theta=0} = 0 \qquad \text{and}\, \psi|_{\theta=\theta_w} = \dot{b}/(2\pi)$.

Boundary conditions for the open surfaces of the acceleration channel

The boundary condition at the inlet and at the outlet depend on the features of the plasma injection and, more crucially problem, on the external flow which is unknown.

We assume that the plasma attains an extremum of the constrained energy (13) that depends on the injection features only through the shape of the inlet surface.

This determines a *natural boundary condition* for the problem. By taking an arbitrary variation $\delta\psi$ at the boundary of the functional (13), we obtain

$$\frac{1}{r^2}\left(M^2-1\right)\nabla\psi\cdot\mathbf{n}=0, \tag{14}$$

which implies that either the term in the bracket is equal to zero, i.e. the open surface is an Alfvén surface, or that the poloidal magnetic field and mass flow are orthogonal to the open surface.

In this way we transfer the indetermination due to the processes that occur before the acceleration channel into the geometry of the domain.

The inlet and outlet surfaces are chosen for simplicity as spherical caps of radius σ_i and σ_o. The boundary conditions on the two open surfaces result

$$\partial\psi/\partial\sigma|_{\sigma=\sigma_i}=0 \quad \text{and} \quad (M_o^2-1)\partial\psi/\partial\sigma|_{\sigma=\sigma_o}=0.$$

In the variational formulation of the problem these equations are implicitly satisfied and do not need to be considered in the numerical implementation of the model.

Flux functions

A certain degree of freedom is indeed associated with the choice of these functions which characterize different flow regimes. An approximate knowledge of the main fields in the inlet of the acceleration channel is sufficient to obtain a good guess of these dependencies but some constraints follow from physical considerations.
As an example from the expression of the electric field we obtain $\int_0^{\psi} G\left(\psi^{'}\right) d\psi^{'}/c = \Delta V$ with ΔV the potential difference between the two electrodes.
If we close all the current within the nozzle, and for simplicity consider the case $G = const$ and $F = const$, we can derive the following relationship[12]

$$I_{tot} = \frac{4\pi^2 c^2 \dot{m}}{\dot{b}^2} \frac{r_i^2 - r_o^2}{1 - M_o^2} \Delta V, \tag{15}$$

that can be used to characterize the performance of the thruster.

[12] T. Andreussi, F. Pegoraro, Phys. Plasmas, **17**, 063507 (2010)

Numerical results

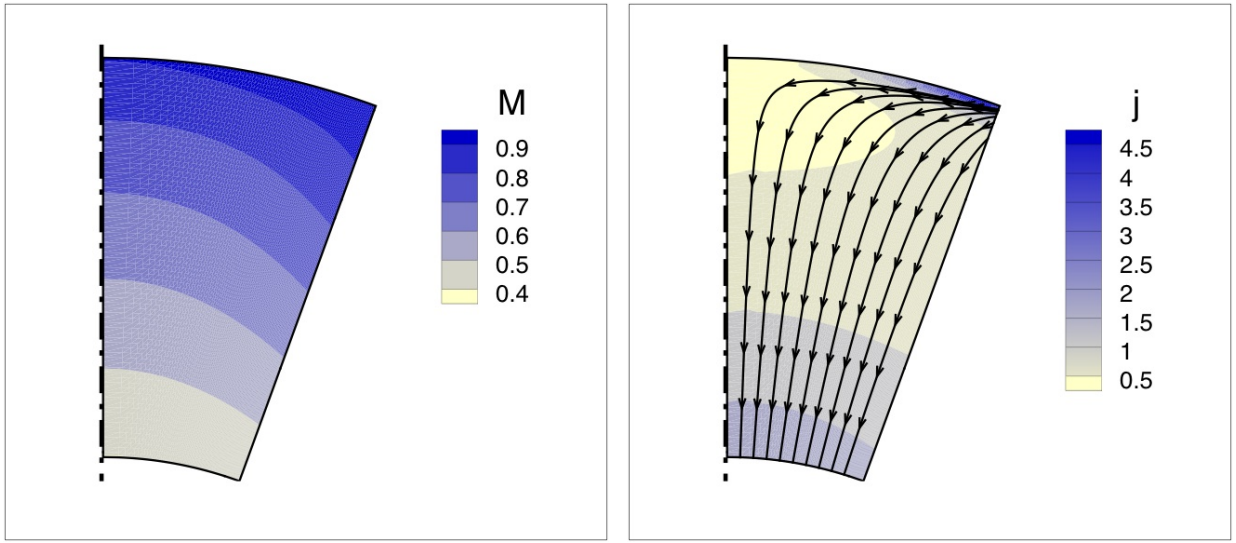


Figure 4: Mach number and current distribution for the optimum acceleration regime

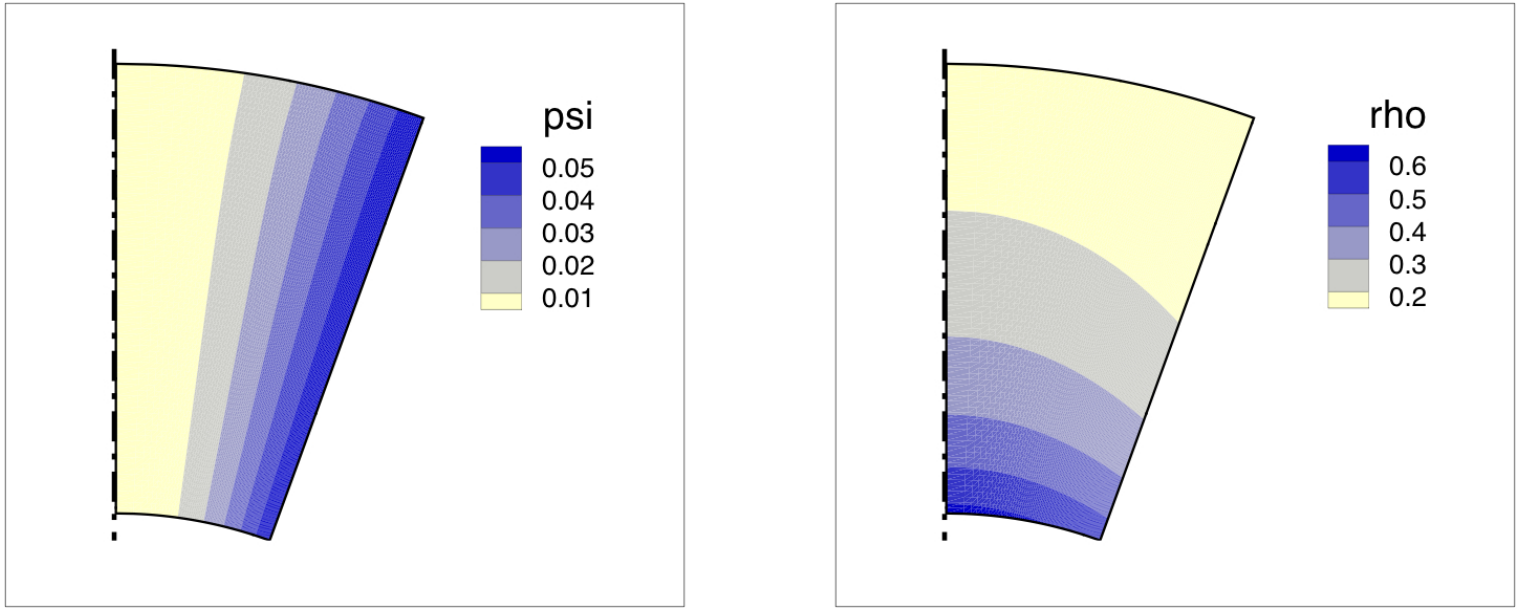
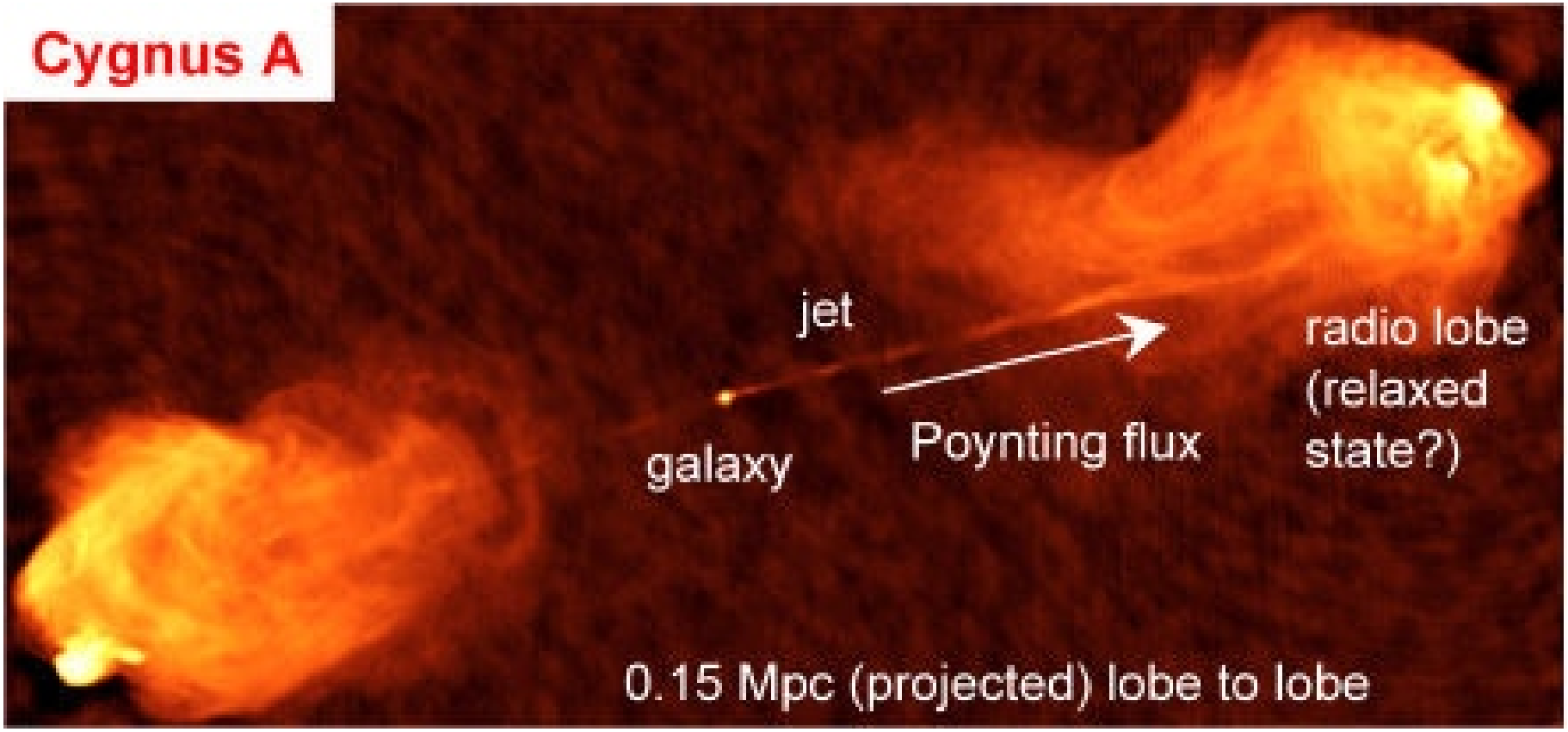


Figure 5: Flux function and density for the optimum acceleration regime corresponding to Mach number = 1 at the outlet. Note the decrease of the density at the anode surface.

Astrophysical jets and Conclusions



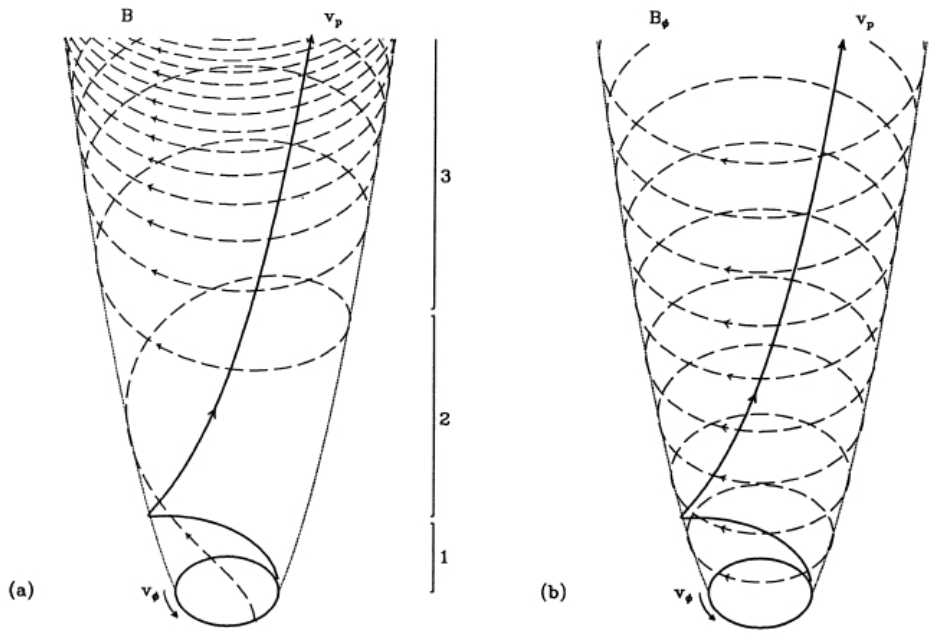


Figure 6: The Crab nebula, Chandra's X-ray image, left and models for an astrophysical jet nozzle, right, taken from: J. Contopoulos, A Simple Type of Magnetically Driven Jets: an Astrophysical Plasma Gun, ApJ., **450** (1995)

There is a clearly wide range of applications of the mathematical approach discussed here. Additional energies (gravitational energy) will be part of the variational principle and appropriate boundary conditions have to be imposed. Starting from an Hamiltonian formulation, if extended to non axisymmetric configurations, can open the way naturally to a stability analysis of the jets.

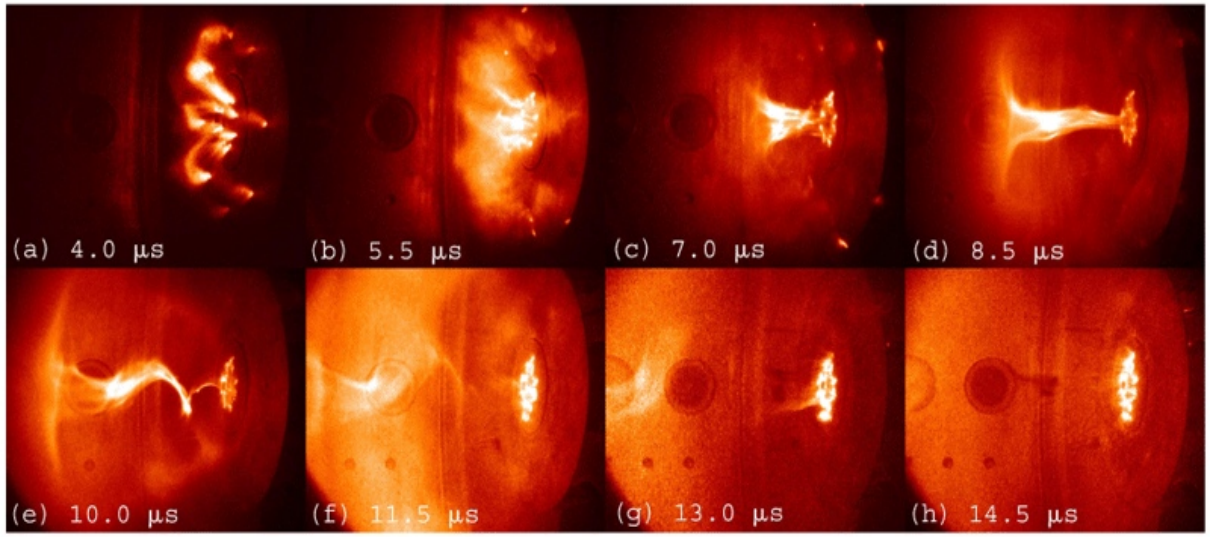


Figure 7: S. C. Hsu, P. M. Bellan, Phys. Rev. Lett. **90**, 215002 (2003).

Appendix[13]

Hall thrusters: nominal operating conditions of a common flight module (e.g., the Russian SPT-100) operating with Xenon are a 2- to 5-mg/sec mass flow rate; a 200- to 300-V applied voltage, yielding a plasma exhaust velocity of 16,000 m/sec; and a thrust of 40-80 mN, at efficiencies of about 50%.

The MPDT has demonstrated its capability of providing specific impulses in the range of 1500-8000 sec with thrust efficiencies exceeding 40$\%$. High efficiency (above 30$\%$) is typically reached only at high power levels (above 100 kW); consequently, the steady-state version of the MPDT is regarded as a high-power propulsion option.

[13]Taken from: Encyclopedia of Physical Science and Technology, Third Edition, Vol. 5, *Electric Propulsion* by Robert G. Jahn and Edgar Y. Choueiri, Princeton University.

When the thruster is operated below 200 kW, the selfinduced magnetic field becomes only marginally sufficient to provide the desired body force, and external fields are frequently added to enhance performance in this range.
However, in its megawatt versions, the self-field MPDT has the unique capability, among all developed electric thrusters, of processing very high power levels in a simple, compact, and robust device that can produce thrust densities as high as 10^5 N/m^2.

These features have rendered the steady-state MPDT particularly attractive for energetic deep-space missions requiring high thrust levels, such as piloted and cargo missions to Mars and the outer planets, as well as for nearer-term orbit raising missions.