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Thermal Spreading in Charged Particle Beam Transport: Diffusion or Diffraction-Like Process?

FEDELE Renato

Dipartimento di Scienze Fisiche, Università Federico II and INFN
Napoli
Italy

THERMAL SPREADING IN CHARGED PARTICLE BEAM TRANSPORT: DIFFUSION OR DIFFRACTION-LIKE PROCESS?

R. Fedele

Dipartimento di Scienze Fisiche, Università Federico II and INFN, Napoli, Italy

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Onoring Professor Padma Kant SHUKLA in the occasion of his 60th birthday

- **The conventional description of the collective behavior of charged-particle beams is usually given in terms of the Vlasov equation.**

- **An alternative non-conventional description is based on a mathematical formalism fully similar to the ones used for the *propagation of e.m. radiation beams in nonlinear media* a well as the *nonlinear dynamics of the Bose-Einsten condensates*.**

- **In this talk, a tutorial presentation of such a non-conventional description is given.**

Let us consider a *paraxial charged-particle beam* travelling in vacuo along the z-axis with speed βc (β ≈1)

In the comoving frame, the transverse (x-y plane) motion of a single particle of a paraxial beam is nonrelativistic.

In particular, if the beam has a finite transverse temperature (transverse thermal spreading), paraxial approximation means:

$$\boxed{\textit{transverse thermal velocity} \ll c}$$

<u>Adopting the electron optics language we can say</u>: ***the slopes of the electron rays (single particle trajectories) with respect to z-axis are very small:***

$$|p_x| = |dx/dz| \ll 1, \quad |p_y| = |dy/dz| \ll 1$$

(x, p_x) **and** (y, p_x) **are pairs of canonical conjugate variables and the beam transverse motion affected by the thermal spreading should be in principle described statistically by the first- and the second-order moments of the classical phase-space distribution function** $\rho(x, p_x, y, p_x, z)$ **for the electronic rays.**

$\rho(x, p_x, y, p_x, z)$ = ***probability density of finding an electronic ray at the transverse phase space location*** (x, p_x, y, p_x) ***and time*** $t = z/(\beta c)$.

One is naturally lead to consider the transverse thermal spreading as a <u>diffusion process in the real space</u>. In particular, once $\rho(x, p_x, y, p_x, z)$ **is found, its second-order moments can be introduced:**

$$\sigma_j(z) = <(j-<j>)^2>^{1/2}$$

rms dispersion of the j-th electronic ray positions

$$\sigma_{pj}(z) = <(p_j-<p_j>)^2>^{1/2}$$

rms dispersion of the j-th electronic ray slops

$$\sigma_{jpj}(z) = <(j-<j>)(p_j-<p_j>)>^{1/2}$$

correlation term for the j-th transv. direction

$$j = x,y$$

Diffusion coefficient associated with the j-th transverse direction or *rms emittance associated with the j-th transverse direction*:

$$\varepsilon_j(z) = 2[\sigma_j^2(z)\,\sigma_{pj}^2(z) - \sigma_{jpj}^2(z)]^{1/2}$$

Let us assume, for the sake of simplicity, that

$$\varepsilon_x = \varepsilon_y = \varepsilon$$

For free motion or in the presence of linear (elastic-like) forces <u>ε is constant of motion (independent of z)</u>

It is possible to show that, during the free beam motion, σ_j satisfies to the following *envelope equation* (j=x,y):

$$\frac{d^2\sigma_j}{dz^2} - \frac{\varepsilon^2}{4\sigma_j^3} = 0$$

(Pinney equation)

<u>which is in full agreement with the experimental observations</u> (see, f.i., the final focusing stage of linear colliders).
The quantity $\sigma = (\sigma_x^2 + \sigma_y^2)^{1/2}$ (the effective transverse beam width) obeys to a similar differential equation.

Since we are trying to describe the thermal spreading of the beam as a diffusion process, it is "natural" (but not obvious!) to suppose the existence of a real function f(x,y,z), **defined in the** (x,y) **configuration space (**z **plays the role of timelike coordinate) satisfying the following diffusion equation:**

$$\frac{\partial f}{\partial z} = \frac{\varepsilon}{2} \nabla_{\perp}^{\;2} f$$

$$f(x,y,z) = \int \rho(x,y,p_x,p_y,z)\,dp_x dp_y$$

f(x,y,z) = **classical probability density of finding an electronic ray at the transverse configurational position** (x,y) **and time** t=z/β c

Assuming that initially ($z=z_0$) the beam has a Gaussian transverse shape, a Gaussian non-stationary normalized solution of the diffusion equation can be found:

$$f(x,y,z) = \frac{\exp[-x^2/2\sigma_{fx}{}^2(z) - y^2/2\sigma_{fy}{}^2(z)]}{\sqrt{2\pi}\sigma_{fx}(z)\sigma_{fy}(z)}$$

where

$$\frac{d^2\sigma_{fj}}{dz^2} + \frac{\varepsilon^2}{4\sigma_{fj}^3} = 0$$

$$\frac{\partial f}{\partial z} = \frac{\varepsilon}{2} \nabla_{\perp}{}^{2} f$$ $$f(x,y,z) = \frac{\exp[-x^2/2\sigma_{fx}{}^2(z) - x^2/2\sigma_{fy}{}^2(z)]}{\sqrt{2\pi}\sigma_{fx}(z)\sigma_{fy}(z)}$$	
$$\frac{d\sigma_{fj}{}^2}{dz} = \varepsilon \quad \Rightarrow \quad \frac{d^2\sigma_{fj}}{dz^2} + \frac{\varepsilon^2}{4\sigma_{fj}^3} = 0$$	$$\frac{d^2\sigma_j}{dz^2} - \frac{\varepsilon^2}{4\sigma_j^3} = 0$$
$$\sigma_{fj}(z_0) = \sigma_{j0} \qquad \left(\frac{d\sigma_{fj}}{dz}\right)_{z_0} = \frac{\varepsilon}{2\sigma_{j0}}$$ $$\sigma_{fj}(z) = \sqrt{\sigma_{j0}{}^2 + \varepsilon(z - z_0)}$$	$$\sigma_j(z_0) = \sigma_{j0} \qquad \left(\frac{d\sigma_{j0}}{dz}\right)_{z_0} = \sigma'_{j0}$$ $$\sigma_{j0}(z) = \sqrt{\sigma_{j0}{}^2 + 2E_j(z - z_0)^2}$$ $$E_j = \frac{\varepsilon^2}{4\sigma_{j0}{}^2} + (\sigma'_{j0})^2$$

NOT IN AGREEMENT WITH THE EXPERIMENTAL OBSERVATIONS	**IN AGREEMENT WITH THE EXPERIMENTAL OBSERVATIONS**

$\varepsilon \quad \rightarrow \quad i\varepsilon$	
$$\frac{d^2\sigma_{fj}}{dz^2}+\frac{\varepsilon^2}{4\sigma_{fj}^3}=0 \quad \Rightarrow \quad \frac{d^2\sigma_j}{dz^2}-\frac{\varepsilon^2}{4\sigma_j^3}=0$$	
$$\frac{\partial f}{\partial z}=\frac{\varepsilon}{2}\nabla_\perp{}^2 f \quad \Rightarrow \quad i\frac{\partial \Psi}{\partial z}=-\frac{\varepsilon}{2}\nabla_\perp{}^2\Psi$$	

$$i\varepsilon\frac{\partial\Psi}{\partial z} = -\frac{\varepsilon^2}{2}\nabla_\perp{}^2\Psi$$

$\left|\Psi(x,y,z)\right|^2 \propto$ ***transverse density profile of the beam particles***

$$\Psi(x,y,z) = \sqrt{n(x,y,z)}\exp[\frac{i}{\varepsilon}\theta(x,y,z)]$$

$$\int\left|\Psi(x,y,z)\right|^2 dxdy = \int n(x,y,z)dxdy = 1$$

Fluid interpretation:

$n(x,y,z) =$ ***transverse probability density of the beam particles***

$\mathbf{V}(x,y,z) = \nabla_\perp\theta =$ ***transverse current velocity***

Gaussian solution for the BWF

$$\Psi(x,y,z)=\frac{\exp[-x^2/4\sigma_x(z)-y^2/4\sigma_y(z)]}{\sqrt{2\pi\sigma_x(z)\sigma_y(z)}}\exp[\frac{i}{\varepsilon}\theta(x,y,z)]$$

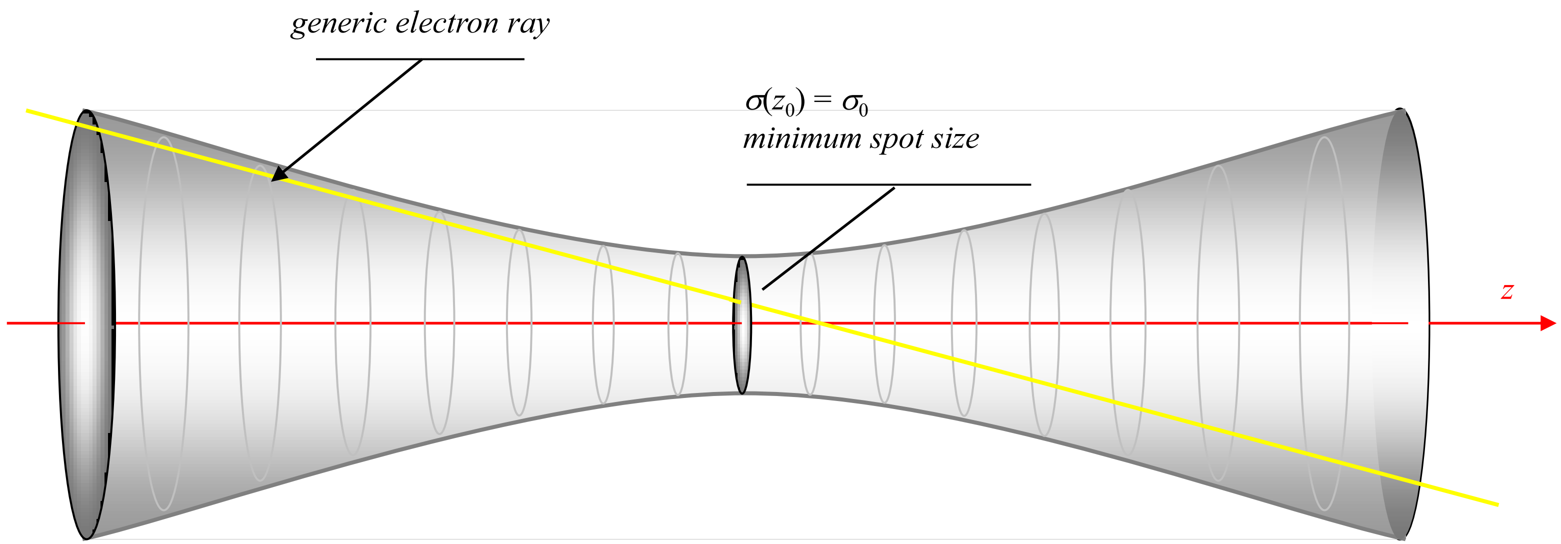
$$\theta(x,y,z)=\frac{x^2}{2R_x(z)}+\frac{y^2}{2R_y(z)}+\phi_x(z)+\phi_y(z)$$

$$\frac{1}{R_j(z)}=\frac{1}{\sigma_j(z)}\frac{d\sigma_j(z)}{dt}$$

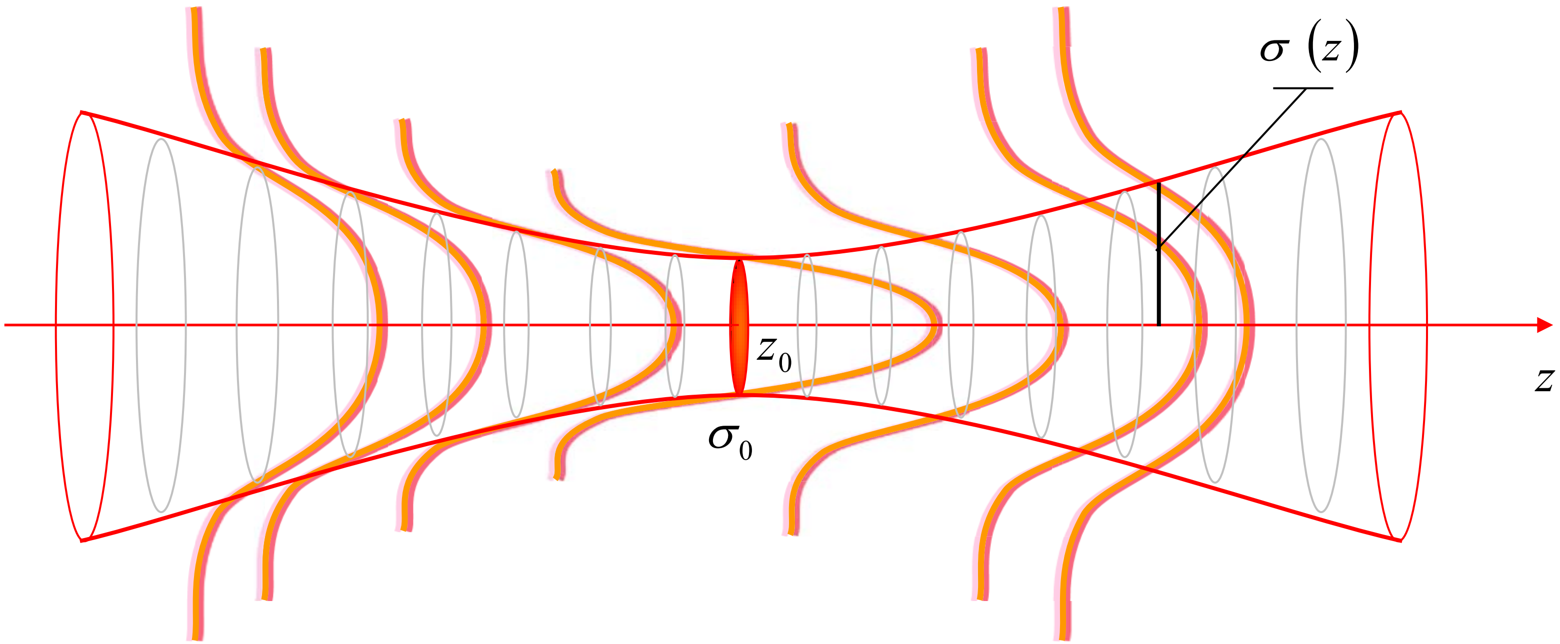
$$\frac{d\phi_{j0}(z)}{dt}=-\frac{\varepsilon}{4\sigma_j^2(z)}$$

$$j=x,y$$

Qualitative rappresentation of the free envelope motion (paraxial approximation) of a cilindrically-symmetric beam travelling in vacuo.



Qualitative envelope evolution of a cilindrically-symmetric Gaussian beam propagating in vacuo.



$$\sigma\,(z)=\sqrt{\sigma_0{}^2+2E(z-z_0)^2}$$

TRANSVERSE NONLINEAR BEAM DYNAMICS IN A COLD PLASMA

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Self-consistent interaction between the plasma wake field and the driving relativistic electron beam

R. Fedele
Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Mostra D'Oltremare pad.20, I-80125 Napoli, Italy

P. K. Shukla
Institut für Theoretische Physik IV, Ruhr-Universität Bochum, W-4630 Bochum 1, Germany
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It is shown that the self-consistent interaction between wake fields and the driving electron bunch in a collisionless, unmagnetized, overdense ($n_p \gg n_b$) plasma is governed by three coupled equations. In the long-beam limit, they reduce to a pair consisting of an appropriate nonlinear Schrödinger equation for the *beam wave function* Ψ, and an equation for the wake-field potential that is driven by the transverse profile of the beam density, which is proportional to $|\Psi|^2$. The pair of equations are suitable for studying the beam self-focusing (self-pinching equilibrium) for the case in which the beam-spot size is larger (smaller) than the wavelength of the wake fields. It is demonstrated that our self-consistent theory, which is based on the recently proposed *thermal wave model for relativistic charged-particle beam propagation,* is capable of reproducing the main results for the beam-filamentation threshold and the self-pinching equilibrium condition that are already known in the conventional theory of the beam self-interaction in collisionless plasmas.

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TRANSVERSE NONLINEAR BEAM DYNAMICS IN A COLD PLASMA

- **A cilindrically symmetric Gaussian relativistic charged particle beam, with transverse rms** R_0 **(initial beam radius) and unperturbed number density** n_{b0}**, travelling along the z-axis with velocity** βc ($\beta \approx 1$) **and transverse emittance** ε**.**
- **At** $z=0$ **the beam enters a semi- infinite slab of cold unmagnetized plasma with unperturbed number density** n_{p0} **in "overdense condition"** ($n_{b0} \ll n_{p0}$)**.**

- **The beam length $\sigma_z >> \lambda_p$ (the plasma density perturbation n_1 is produced adiabatically) and therefore**

$$e\, n_1(r,\xi) \approx q\, n_{b0}(r,\xi),$$

r=cilindrical radial coordinate, $\xi = z - \beta ct$

- **According to the theory of plasma wake field excitation:**

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - k_p^2\right] U_\perp^{coll}(r,\xi) = \frac{4\pi q^2 n_b}{m\gamma\beta^2 c^2} \approx \frac{4\pi q^2 n_b}{m\gamma c^2}$$

$$\Downarrow$$

$$\boxed{i\varepsilon\frac{\partial\Psi}{\partial\xi} = -\frac{\varepsilon^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + U_\perp^{coll}\left[|\Psi(r,\xi)|^2\right]\Psi}$$

(a). $k_p R >> 1,\qquad i\varepsilon\frac{\partial\Psi}{\partial\xi} = -\frac{\varepsilon^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) - \frac{n_{b0}}{n_{p0}\gamma}\left|\Psi\right|^2\Psi$

$$\mathcal{A} = \frac{\varepsilon^2}{2}\int_0^\infty\left|\frac{\partial\Psi}{\partial r}\right|^2 rdr - \frac{n_{b0}}{2n_{po}\gamma}\int_0^\infty\left|\Psi\right|^4 rdr = const. \quad\Rightarrow\quad \mathcal{A} = \frac{1}{2}\left[\frac{\varepsilon^2}{R_0^2} - \frac{1}{2}\frac{n_{b0}}{n_{p0}\gamma}\right]$$

$$R^2(\xi) = R_0^2 + 2\mathcal{A}\left(\xi\text{-}\xi_0\right)^2$$

$\mathcal{A}$<0: *self-focusing*

$\mathcal{A}$>0: *self-defocusing*

$\mathcal{A}$=0: *stationary solution; Weibel instability threshold:*

$$\beta_\perp = \frac{v_{th}}{c} \approx 0.7\left(\frac{n_{b0}}{n_{p0}\gamma}\right)^{1/2}$$

(b). $k_p R << 1,$ $$\frac{d^2 R}{d\xi^2} = \frac{\varepsilon^2}{R^3} - \frac{2K}{R}\left\langle \int_0^r \left|\Psi\left(r',\xi\right)\right|^2 r' dr' \right\rangle$$

$$K = \frac{2\pi e^2 n_{b0}}{m\gamma\beta c^2}$$

$$\frac{dR}{d\xi} = 0 \qquad \Rightarrow$$ *Bennett self-pinch equilibrium condition:*

$$\frac{\varepsilon^2}{R_0^2} = \frac{1}{2} K R_0^2 \quad \Rightarrow \quad \frac{I^2}{c^2} = \frac{N}{\sigma_z} T$$ (cgs unts)

- Aberrationless approximate solution of NLSE:

$$\frac{d^2 R}{d\xi^2} + KR - \frac{\varepsilon^2}{R^3} = 0$$

Dear Padma
HAPPY BIRTHAY TO YOU!