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Quantum plasma fluid model for coherent stimulated emission of radiation by a dense relativistic electron beam

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Table of contents

1 Motivation
   • Interaction scheme

2 Quantum FEL Scheme - Schrodinger-like equation
   • Total electron energy
   • Change of frame

3 Quantum electron fluid
   • Quasi-Classical approximations
   • Hamilton-Jacob
   • 3-D Fluid
   • Ponderomotive potential/Matching conditions
   • Steady-state FEL equations: Universal normalization

4 Conclusions
Motivation

The essence of FEL

- Operates near resonant condition $\lambda_s \approx \lambda_w / 4\gamma_e^2$;
- $e^-$ bunching inside potential well.

Description

- The beam electrons in the ponderomotive potential well are modelled by a discrete ensemble of initially uniform macro-electrons.

HOWEVER…
Motivation

For X-Ray

- Very high frequency such that \( \frac{\hbar \omega_s}{\gamma_e m_e c^2} \geq 1 \)
- For very high frequency, this picture of macro-electrons in the potential well has to be reconsidered, since the longitudinal dimension of the potential is so small that, for any beam density, a potential well would be occupied at the maximum by one single electron.
Interaction scheme

Motivation
Quantum FEL Scheme - Schrödinger-like equation
Quantum electron fluid
Conclusions

Interaction scheme

electron beam $v_e$

wiggler pulse $v_w$

scattered radiation $v_s$

$x$

$y$

$z$
Model

New model → Quantum backscattering

- FEL dynamics as a simple case of backscattering of intense optical wigglers by the energetic, but not ultra energetic, beam electrons, taking into account the photon recoil \((\hbar k_s/\gamma_e m_e c \geq 1)\)

Previous models

- Mclever and Fedorov - Derived a set of general quantum equations starting from K. G. equation.;
- Smetanin - Also from K. G. Eq., showed that Q. FEL behaves as a two-level quantum oscilator;
- Preparata - By using the QFT.
Quantum FEL Scheme - Schrödinger-like equation

Energy conservation

\[ E = \sqrt{p_e^2 c^2 + m_e^2 c^4 + V(\vec{r}, t)} \]

- \( E = \hbar \omega_e \rightarrow e^- \) total relativistic energy;
- \( \vec{p}_e = \hbar \vec{k}_e \rightarrow e^- \) momentum;
- \( V \rightarrow \) ponderomotive potential
- For \( V = 0 \), \( E \) and \( \vec{p}_e \) are constants. But, if \( V \neq 0 \) they will vary slowly in \( \vec{r}, t \)
- Eikonal approximation

\[ E \rightarrow \hbar \left( \omega_e + i \frac{\partial}{\partial t} \right), \quad \vec{p}_e \rightarrow \hbar (\vec{k}_e - i \nabla) \]
Quantum FEL Scheme - Schrodinger-like equation

If $\psi$ is the $e^-$ wave function, then

$$\left| \frac{\partial}{\partial t} \psi \right| << w_e |\psi|, \quad |\nabla \psi| << |k_e|\psi$$

From $E$ expression, the evolution of the $e^-$ wave packet is

$$i\hbar \left( \frac{\partial}{\partial t} + \mathbf{v}_e \cdot \nabla \right) \psi + \frac{c^2}{2w_e} \left[ \nabla^2 - \left( 1 - \frac{m_e^2 c^4}{\hbar^2 w_e^2} \right) (\hat{e}_\parallel \cdot \nabla)^2 \right] \psi =$$

$$[V(\mathbf{r}, t) - e\phi(\mathbf{r}, t)] \psi$$

$\phi$ is the plasma potential; $\mathbf{v}_e$ is the $e^-$ velocity; $\hat{e}_\parallel = \mathbf{k}_e/k_e$. 

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Transformation to the Stationary beam frame.

\[ \zeta = \vec{r} - \vec{v}_e t; \quad t' = t; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{v}_e \cdot \nabla \]

Decomposing \( \vec{\zeta} \) into parallel and perpendicular components to the \( e^- \) velocity \((\vec{\zeta} = \zeta_\parallel \hat{e}_\parallel + \zeta_\perp \hat{e}_\perp)\), using \( \hbar w_e = \gamma_e m_e c^2 \), we get a Schrödinger-like equation

\[ i\hbar \frac{\partial}{\partial t'} \psi + \frac{\hbar^2}{2m_e \gamma_e} \left\{ \frac{1}{\gamma_e^2} \frac{\partial^2}{\partial \zeta_\parallel^2} + \frac{\partial^2}{\partial \zeta_\perp^2} \right\} \psi = (V - e\phi)\psi \]

Longitudinal field

\[ \nabla \cdot \vec{E} = \nabla \cdot \vec{E}_\perp + \nabla \cdot \vec{E}_\parallel = \nabla \cdot \vec{E}_\perp - \nabla^2 \phi = 4\pi e |\psi|^2 \]
Quantum FEL Scheme - Schrodinger-like equation

**Gradient operator**

\[ \nabla = \left( \frac{\partial}{\partial \zeta_\parallel} \mathbf{\hat{e}}_\parallel, \frac{\partial}{\partial \zeta_\perp} \mathbf{\hat{e}}_\perp \right) \]

- \( \psi \) is the electron ensemble wave function (not single electron) in such a way that its density is \( |\psi|^2 = n \).
- This normalization is valid since we are considering that the electrons in the ensemble interact only collectively through an electromagnetic potential, i.e., we are assuming a pure state ensemble of N-electrons (perfectly coherent particle sample) to represent the relativistic electron beam which interacts with an intense optical wiggler.
Quantum electron fluid

Quasi-classical approximation to the $e^-$ wave function: Madelung transformation

$$\psi = A(\vec{\zeta}, t')e^{iS(\zeta,t')/\hbar};$$

L. D. Landau and E. M. Lifshitz, ”Quantum Mechanics”, 3rd Ed.

$S$ is the action of the particle;

$A$ is a slow-varying amplitude.

Using this into the Schrodinger equation, we get

$$\frac{\partial}{\partial t}S + \frac{1}{2}\gamma_e m_e \left\{ \frac{1}{\gamma_e^2} \left( \frac{\partial}{\partial \zeta_{||}} S \right)^2 + \left( \frac{\partial}{\partial \zeta_{\perp}} S \right)^2 \right\}$$

$$= -(V - e\phi) + \frac{\hbar^2}{2\gamma_e m_e} \left\{ \frac{1}{A} \left( \frac{1}{\gamma_e^2} \frac{\partial^2}{\partial \zeta_{||}^2} + \frac{\partial^2}{\partial \zeta_{\perp}^2} \right) A \right\}$$
Quantum electron fluid

- It reduces to the classical **Hamilton-Jacob** Eq. when $\hbar \to 0$.
- The conservation eq. for the probability density, $A^2 = |\psi|^2$ is

$$
\frac{\partial}{\partial t} A^2 + \frac{\partial}{\partial \zeta_\parallel} \left\{ \frac{A^2}{m_e \gamma_e^3} \frac{\partial}{\partial \zeta_\parallel} S \right\} + \frac{\partial}{\partial \zeta_\perp} \left\{ \frac{A^2}{m_e \gamma_e} \frac{\partial}{\partial \zeta_\perp} S \right\} = 0
$$

**Associations**

$A^2 = n$, $\vec{p} = \nabla S$, $\vec{v} = \left( \frac{p_\parallel}{m_e \gamma_e^3} \hat{e}_\parallel, \frac{p_\perp}{m_e \gamma_e} \hat{e}_\perp \right)$;

**Differential Operator** $D^2$

$$
D^2 = \left( \frac{1}{\gamma_e^2} \frac{\partial^2}{\partial \zeta_\parallel^2} + \frac{\partial^2}{\partial \zeta_\perp^2} \right)
$$
Quantum electron fluid

- **Electron momentum fluid equation:**

\[
\frac{\partial \vec{p}}{\partial t'} + \frac{1}{2} \nabla (\vec{v} \cdot \vec{p}) = -\nabla (V - e\phi) + \frac{\hbar^2}{2m_e \gamma_e} \nabla \left[ \frac{1}{\sqrt{n}} D^2 \sqrt{n} \right];
\]

- **Continuity:**

\[
\frac{\partial}{\partial t'} n + \nabla \cdot (n \vec{v}) = 0;
\]

- **Poisson:**

\[
\nabla^2 \phi = 4\pi e (n - n_0); \quad -\nabla \cdot \vec{E}_\perp = 4\pi en_0;
\]

\(n_0\) → average beam density; \(n\) → density of an \(e^-\) fluid element; \(\vec{v}(\vec{p})\) → velocity(momentum) of this element in the moving frame.

\[
\frac{\hbar^2}{2m_e \gamma_e} \nabla \left[ \frac{1}{\sqrt{n}} D^2 \sqrt{n} \right]
\]
Quantum electron fluid

- FEL is a 1-D phenomenon. **Transverse effects are neglected**
- Electron momentum fluid equation:
  \[ \frac{\partial}{\partial t'} p + v \frac{\partial}{\partial \zeta} p = -\frac{\partial}{\partial \zeta} (V - e\phi) + \frac{\hbar^2}{2m_e \gamma_e^3} \frac{\partial}{\partial \zeta} \left( \frac{1}{\sqrt{n}} \frac{\partial^2}{\partial \zeta^2} \sqrt{n} \right) \]
  
- Continuity
  \[ \frac{\partial}{\partial t'} n + \frac{\partial}{\partial \zeta} (nv) = 0 \]

- Poisson
  \[ \frac{\partial^2}{\partial \zeta^2} \phi = 4\pi e (n - n_0) \]

- \( p, v \) and \( \zeta \) now stand for the longitudinal components of \( \vec{p}, \vec{v} \) and \( \vec{\zeta} \).
Radiation and wiggler fields

- Assuming the circularly-polarized fields, and using the wave equation, we get the Slow-varying amplitude equations:

- **Optical wiggler**

  \[
  \left[ \frac{\partial}{\partial t'} - (v_w + v_e) \frac{\partial}{\partial \zeta} \right] a_w = \frac{w_p^2}{2\gamma_e \omega_w} a_s \delta ne^{-i(k_l \zeta - [w_l - k_l v_e] t')}
  \]

- **Radiation field**

  \[
  \left[ \frac{\partial}{\partial t'} + (v_s - v_e) \frac{\partial}{\partial \zeta} \right] a_s = -\frac{w_p^2}{2\gamma_e \omega_s} a_w \delta ne^{i(k_l \zeta - [w_l - k_l v_e] t')}
  \]

- \( v_s(w) = c^2 k_s(w) / \omega_s(w) \) is the radiation(optical wiggler) group velocity.
Plasma equation

- 1-D $e^{-}$ fluid perturbed equations:

$$
\frac{\partial}{\partial t'} \delta p = -\frac{\partial}{\partial \zeta} V + e \frac{\partial}{\partial \zeta} \phi + \frac{\hbar^2}{4m_e \gamma_e^3} \frac{\partial^3}{\partial \zeta^3} \delta n
$$

$$
\frac{\partial}{\partial t'} \delta n + \frac{1}{m \gamma_e^3} \frac{\partial}{\partial \zeta} \delta p = 0
$$

$$
\frac{\partial^2}{\partial \zeta^2} \phi = (4\pi en_0) \delta n
$$

- A simple manipulation lead us to Eq. which represents the space-chage oscillation excited by a ponderomotive force through $V$

$$
\left[ \frac{\partial^2}{\partial t'^2} + \left( \frac{\hbar}{2m_e \gamma_e^3} \right)^2 \frac{\partial^4}{\partial \zeta^4} + \frac{w_p^2}{\gamma_e^3} \right] \delta n = \frac{1}{m \gamma_e^3} \frac{\partial^2}{\partial \zeta^2} V
$$
From our fields and Ponderomotive Force definition

$$V = \frac{mc^2}{2\gamma_e} \left( i a_w a_s^* e^{i(k_l\zeta - \Omega_l t)} + cc \right)$$

- **Matching conditions:**
  - Energy: \( \hbar w_l = \hbar w_s - \hbar w_w \);
  - Momentum: \( \hbar k_l = \hbar k_s + \hbar k_w \);
  - \( \Omega_l = w_l - k_l v_e \)
Free-electron laser instability

- Normalized set of FEL equations:

\[
\begin{align*}
\left[ \frac{\partial}{\partial \tau} - (\beta_w + \beta_e) \frac{\partial}{\partial \chi} \right] a_w &= \frac{\omega_p}{2\gamma_e \omega_w} \bar{a}_s \bar{n} \\
\left[ \frac{\partial}{\partial \tau} - (\beta_e - \beta_s) \frac{\partial}{\partial \chi} - i \frac{\delta}{\omega_p} \right] \bar{a}_s &= -\frac{\omega_p}{2\gamma_e \omega_s} a_w \bar{n}^* \\
\left[ \frac{\partial}{\partial \tau} + \beta_q \frac{\partial}{\partial \chi} + \frac{\nu}{\omega_p} \right] \bar{n} &= \frac{c^2 k_i^2}{4\gamma_e^4 \Omega_i \omega_p} a_w \bar{a}_s
\end{align*}
\]

where the damping (\(\nu\)) and the mismatching (\(\delta\)) have been introduced. Here, \(\tau = \omega_p t'\), \(\chi = (\omega_p/c)\zeta\), and \(\Omega_i^2 - 1/4(\hbar/\gamma_e^3 m_e)^2 k_i^4 - \omega_p^2/\gamma_e^3 = 0\).
$\beta_q = \frac{\hbar k_l}{\gamma_e m_e c} \geq 1$ is the quantum parameter. $\beta_e$ and $\beta_s(\beta_w)$ are the normalized electron beam and radiation (wiggler) group velocities.
Steady-state FEL equations: Universal normalization

\[
\frac{\partial \hat{a}_w}{\partial \chi} = -\hat{a}_s \hat{n} \cos \phi \\
\frac{\partial \hat{a}_s}{\partial \chi} = S \hat{a}_w \hat{n} \cos \phi \\
\frac{\partial \hat{n}}{\partial \chi} + \hat{\nu} \hat{n} = \hat{a}_w \hat{a}_s \cos \phi \\
\frac{\partial \phi}{\partial \chi} = -\hat{\nu} - \left( \frac{\hat{a}_w \hat{a}_s}{\hat{n}} + S \frac{\hat{a}_w \hat{n}}{\hat{a}_s} - \frac{\hat{a}_s \hat{n}}{\hat{a}_w} \right) \sin \phi
\]

where

\[
\hat{n} = n \sqrt{|C_1| C_2}; \quad \hat{a}_s = a_s \sqrt{C_0 C_2}; \quad \hat{a}_w = a_w \sqrt{C_0 |C_1|}; \quad S = \frac{|C_1|}{C_1}
\]
These equations can be solved by numerical methods or by the scheme of Galeev et al. Reviews of Plasma Physics, (Plenum, New York), Vol.7, Chapter 1 (1979)
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Quantum plasma fluid model for coherent stimulated emission of solitons.

\[ \delta = 0 \]
\[ v = 0 \]
\[ a_s \text{ and } n \]
\[ a_w \]
repeated explosion ($\delta = 2$, $\nu = 0$)
soliton + shock ($\nu=4, \delta=2$)
Physical parameters: $\gamma_e=5.0$, $a_w=1.2 \times 10^{-3}$, $\delta = 0.20699$, $n_0 = 10^{-17} \text{ cm}^{-3}$, $\lambda_s=1.06 \times 10^{-12} \text{ cm}$, $\lambda_w=1.1 \times 10^{-3} \text{ cm}$, $\beta_q=1.85$
We have presented a quantum plasma fluid model to describe the stimulated emission of coherent radiation by a relativistic electron beam interacting with an intense electromagnetic wave as a three-wave interaction process. This stimulated emission is treated as a Compton scattering process, without the usual nonlinear electron bunching in the ponderomotive potential;

S. charge is negligible for dense and relativistic e\(^-\) beam, since the quantum term \(\frac{1}{4}(\hbar/\gamma^3_e m_e)^2 k_f^4 \gg \omega_p^2/\gamma_e^3\);

This fluid model is simpler and more realistic to describe and simulate quantum FEL than previous quantum model based on the Wigner distribution description. (Bonifacio, 2005, Nucl. Instr. and Meth. A; Europhys. Lett.; and Serbeto, 2008, Phys. Plasmas)
That’s all folks.


