International Workshop on Cutting-Edge Plasma Physics

5 - 16 July 2010

Conservation Laws in Plasma Dynamics

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Summer College in Plasma Physics
July 1, 2010
Conservation Laws - Conserved Quantities

Certain combinations of dynamical variables that remain "unchanged" during the physical evolution of a system.

The existence of such invariants restricts the dynamics.

This restriction is of great essence.

Very often it is the existence of an invariant (invariants) that facilitates (or guarantees) the solution of a dynamical problem. You all know about the relationship of "integrability" with invariants.

In this lecture we will explore the subject, primarily, in the backdrop of plasma dynamics.
There are two main sources of conservation laws:

1. Symmetries of the underlying Lagrangian
   - Space-time symmetries
   - Gauge invariance

   The existence of such symmetries, for instance, leads to what are known as "Noether currents." These currents are 'Conserved'.

2. Then there are topological invariants

   These invariants do not follow from an explicit invariance symmetry of the Lagrangian.

   The nature and function of this second clan of invariants is very different from the first:
   They are often associated with the stability of solutions.

   (2) also has conserved currents but these are not "Noether currents".
Conserved Currents.

Remember the entire purpose of this exploration is to look for some physical quantity which does not change with time.

There is a simple recipe for achieving this.

I will deliberately deal only with four vector currents (though we could have any tensorial currents).

Let $J^\mu = \{J^0, J^i\}$ be a four current ($\mu = 0 \cdots 3$ and latin indices $i = 1 \cdots 3$).

For it to lead to a conservation law, we must have

$$\partial_\mu J^\mu = 0$$

i.e., the current is divergence less (four).

Let us see how?
Conserved Currents - Invariants

\[ \frac{dt}{dt} J^0 + \nabla \cdot J = 0 \]

\[ \int dt J^0 d^3x = \frac{dQ}{dt} + \int \nabla \cdot J d^3x = 0 \]

Surface term = 0

\[ \Rightarrow \frac{dQ}{dt} = 0, \quad Q = \int J^0 d^3x \text{ in conserved} \]

Some examples:

1. The Continuity equation is called a Conservation law - So what is conserved? What is the current? - The flux \( \Gamma^\mu = (n, v, \mu) \)

   Clearly then, the conserved "charge"

   Corresponding to \( \Gamma^\mu \) is \( \int \Gamma^0 d^3x \),

   \[ N = \int n d^3x = \text{the number of particles} \]

2. So in the equation of motion:

   \[ \partial_\mu T^{\mu\nu} = 0 \]

   \( T^{\mu\nu} \) is the total energy momentum.

   Conserved \( \partial_\mu \text{"charge"} \)

   \[ p^\nu = \int T^{0\nu} d^3x \quad \text{Energy-Momentum} \]
Conservation Laws - An aside

If we have an equation of the form
\[ \partial_\mu J^\mu = \nabla \cdot S \]  \hspace{1cm} (a)

Then again, if \( S_n = 0 = J_n \) at \( \pm \infty \),
\[ Q = \int J^0 \, d^3 \alpha \]
is conserved.

There are situations when an Equation of this type (a) may be allowed by Lorentz invariance.

For instance if \( S^0 \) is time independent and the we could have an
\[ S^\mu = (S^0, S^i) \]
\[ \partial_\mu S^\mu = \nabla \cdot S \]

There is no non-trivial `conserved' `charge' with such on \( S^\mu \).
Finding Topological Invariants in a Dynamics Constitutes fundamental progress

By forbidding certain classes of motion, the topological invariants may guarantee the stability of solutions!
Topological Invariants - A simple Model

The Sine-Gordon Kink

Scalar field in 1+1 dimensions:

\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + a \sinh \phi = 0 \tag{a}
\]

The Lagrangian for S. G

\[ L = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - V(\phi) \tag{b} \]

The Hamiltonian

\[ H = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + V(\phi) \tag{c} \]

\[ V(\phi) = \frac{a}{b} \left( 1 - \cosh b\phi \right) \tag{d} \]

It possesses stationary as well as moving soln.

\[ \phi(x,t) = f(x - ut) = f(s) \tag{e} \]

\[ f(s) = \frac{u}{b} \arctan \frac{r}{s} \tag{f} \]

with \( r = (1-u^2)^{-1/2} \) — A solitary wave

It also has an infinite no. of constant solutions

\[ \phi = \frac{2\pi n}{b} \quad n = 0 \text{ or an integer} \tag{g} \]

For (g), \( V = 0 \), and \( \therefore H = 0 \)

These are zero energy solutions ⇒ Degenerate Vacua
\[ \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial V}{\partial \phi} \Rightarrow \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 = V(\phi) \]

For the stationary kink \((\partial t = 0)\)

Let us calculate the energy of this stationary kink.
Sine-Gordon Kink

\[ E = \int H \, dx = \int \left[ \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + v(w) \right] \, dx \]

\[ = \int 2v(w) \, dw = \int_0^{2\pi/b} \left[ \frac{\partial}{\partial q} v(q) \right] \, dq \]

Where we have chosen to integrate between \( n=0 \) and \( n=1 \) of the degenerate vacua (we could have chosen any two)

\[ E = 8 \left( \frac{\alpha}{b^3} \right)^{1/2} \equiv \text{finite} \]

This is a stable soliton solution with finite energy.

Why is it 'mathematically' stable?

The answer lies in the boundary conditions.
Sine-Gordon Kink ↔ Stability

space, here, is an infinite line

\[ n=0 \quad 1 - \text{Kink} \quad n=1 \]

Any such system \((n=0, n=1)\) cannot be continuously deformed, for instance, to the ground state \(n=0, n=0\).

The stability, therefore, depends on the topological properties of the space.

The boundary points of this space is a discrete set.

Just like this Kink, the stability of soliton solutions in nonlinear field theories is a consequence of topology.

But where is the Conservation Law – The Conserved Current \(\ldots\ldots\ldots\)}
Sine-Gordon Kinks - Conerved Current (11)

It is obvious that, in this example, the conserved 'charge' \( Q \) must be an integer \( N \).

Let

\[
J^\mu = \frac{b}{a \pi} \epsilon^{\mu \nu} \partial_\nu \varphi
\]

Remember \( \mu = (0,1) \). \( \epsilon^{\mu \nu} \) is an antisymmetric tensor with \( \epsilon^{01} = 1 \).

So we have

\[
J^0 = \frac{b}{a \pi} \epsilon^{0 \nu} \partial_\nu \varphi = \frac{b}{a \pi} \frac{d\varphi}{dx}
\]

\[
Q = \int_{-\infty}^{+\infty} J^0 dx = \frac{b}{a \pi} \int_{-\infty}^{+\infty} \frac{d\varphi}{dx} dx
\]

\[
= \frac{b}{a \pi} \left[ \varphi(\infty) - \varphi(-\infty) \right] = N
\]

\( N = n_1 - n_2 \).

\( N \) is a topological label for a Kink - since it cannot be continuously deformed to any other number, the Kink is stable.
Plasmas

Hot charged fluids.

Topological invariants of a hot charged fluid interacting with an electromagnetic field.

You have all heard of helicity in classical as well as well as in quantum mechanics.

Definition:

The helicity of a vector field \( \mathbf{B} \)

\[
h = \int \mathbf{B} \cdot (\nabla \times \mathbf{B}) \, d^3\mathbf{x}
\]

(a)

And if \( \mathbf{B} \) is the standard magnetic field, then \( \nabla \times \mathbf{B} = \mathbf{A} \), the vector potential

\[
h = \int \mathbf{A} \cdot \mathbf{B} \, d^3\mathbf{x}
\]

(b)

the familiar expression.

Expression (b) has no reference to any fluid attributes!
Question: One can show that in classical electromagnetism the magnetic helicity $h_m$ is conserved. Is there a corresponding generalized helicity $G$ for a charged fluid-electromagnetic field system?

Naturally for this $G$ to be interesting, it must be a constant of the motion.

In the rest of this lecture we will develop an extremely general method for constructing $G$ and then end it by discussing some of the consequences of $G$.
A Hot Relativistic Charged Fluid


A unified Magneto-Fluid Formalism

Equation of Motion

\[ T \partial^\sigma = q \, U_\mu \, M^{\nu \mu} \]  \hspace{1cm} (a)

\( q \) is the charge, \( T \) is the temperature, \( s \) is the entropy

\[ U^\mu = \{ r, v \} \]  \hspace{1cm} (b)

is the four velocity and

\[ M^{\nu \mu} = F^{\nu \mu} + \frac{m}{q} S^{\nu \mu} \]  \hspace{1cm} (c)

is the unified magneto-fluid tensor,

\[ S^{\mu \nu} = \partial^\nu U^\mu - \partial^\mu U^\nu \]  \hspace{1cm} (d)

is the fluid tensor (fully antisymmetric) constructed in analogy to the Faraday tensor

\[ F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \]  \hspace{1cm} (e)

\( f U^\mu \) is the fluid equivalent of the e.m. four potential \( A^\mu \). \( f = f(T) \) is related to the enthalpy density.
Magneto - Fluid Equations

Just to get a better feel, let us examine the unified tensor $M^{uv}$:

We know that, ordinarily, $(F^{23}, F^{31}, F^{12})$ define the magnetic field $\mathbf{B}$.

$(M^{23}, M^{31}, M^{12})$ define

$$\mathbf{B} = \mathbf{B} + \frac{m}{q} \mathbf{v} \times (fr\mathbf{u})$$

Generalized Magnetic field (GM)

Or

$$\mathbf{B} = \frac{q}{m} \mathbf{B} + \nabla \mathbf{v} \times (fr\mathbf{u})$$

Generalized Vorticity (GV)

The role of the two terms on determining the dynamics is exactly the same [The vorticity term could cause Zeeem splitting just as $\mathbf{B}$ does].

Generalized Electric fields

Remarkable thing is that the creation $\Rightarrow$
Magnetofluids

of a unified four potential for a hot fluid

$$\hat{A}^\mu = A^\mu + \frac{\mu}{v} U^\mu$$  \hfill (a)

follows exactly the prescription of the 'minimal coupling' in particle dynamics.

We will exploit this analogy to determine a conserved four vector for this magnetofluid.

Without any fluid, for the pure field

$$h^m = \int A_\mu B^\mu d^3x = \int P^0 d^3x$$  \hfill (b)

the only $P^\mu$ made entirely from the field variable, $A_\mu$, $F^{\mu\nu}$ (or its $F_{\mu\nu}$). That has $A_\mu B^\mu$ as $P^0$ is

$$P^0 = A_\mu F^{\mu\nu}$$  \hfill (c)

For the magnetic fluid, then, in analogy with (a), we must explore the four vector

$$K^\mu = \hat{A}_\nu M^{\nu\mu} = (A_\mu + \frac{\mu}{v} f_{\mu\nu}) H^{\nu\mu}$$  \hfill (d)

Will this be conserved?
In either case

As long as $K_n = 0 = S_n$ at the surface of the domain

$$G = \int K^0 \, d^3 \chi$$

$$= \int d^3 \chi \left[ \hat{A} \cdot \hat{B} \right]$$

$$= \int d^3 \chi \left[ A + \frac{m}{q} \gamma \nu \right] \cdot \left[ \beta + \frac{m}{q} \nu \times \nu \right]$$

is conserved.

In the helicity density $K^0$, $f$ represents temperature and $\gamma$ the kinematic relativistic factor. It holds for 'arbitrary' speeds and temperatures.
Conserved Four Vector - Conserved GH(GV) (16)

First Step: Calculate $\partial \mu K^\mu$

* From SMM-03, one learns

$$\partial \mu H^{\mu \nu} = 0$$

This is because the Homogeneous Maxwell equation, one contained in $\partial \mu f^{\mu \nu} = 0$, and $H^{\mu \nu}$ was constructed totally in analogy.

* 

$$\partial \mu K^\mu = (\partial \mu A^\nu) H^{\mu \nu}$$

$$= \frac{1}{2} M_{\mu \nu} H^{\mu \nu}$$

$$= 2 \hat{E} \cdot \hat{B}$$

Again in analogy with simple fluid-free case

$$F_{\mu \nu} f^{\mu \nu} = -4 \hat{E} \cdot \hat{B}$$

* What is $\hat{E} \cdot \hat{B}$?

We must go back to the equation of motion 13(a).
Divergence of $K^m$

Remembering that $U^m = (1, \mathbf{r}, \mathbf{v})$, the vector part of 13(a) yields

$$ T \nabla \mathbf{r} = q \mathbf{r} (\hat{\mathbf{E}} + \mathbf{v} \times \hat{\mathbf{B}}) \quad (a) $$

Well let us dot both sides with

$$ T \hat{\mathbf{B}} \cdot \nabla \mathbf{r} = q \mathbf{r} \hat{\mathbf{E}} \cdot \hat{\mathbf{B}} $$

$$ \therefore \quad \partial T \mu T = 2 \hat{\mathbf{E}} \cdot \hat{\mathbf{B}} = \frac{2T}{q \mathbf{r}} \hat{\mathbf{B}} \cdot \nabla \mathbf{r} \quad (b) $$

is the fundamental equation at the heart of Helicity Conservation.

Notice that 17(b) is a remarkably general equation

it holds for a relativistic (kinematically as well as statistical) charged fluid.
Helicities - In different approximations

Non relativistic Limit:

In this limit \( r \to 1, \ f \to 1 \)

\[
\hat{B} = B + \frac{m}{q} \mathbf{v} \times \mathbf{B},
\]

\[
\partial_t K^R = \frac{2T}{q} \hat{B} \cdot \mathbf{v} - \quad (a)
\]

\[
K^0 = (\mathbf{A} + \frac{2m}{q} \mathbf{u}) \cdot (\mathbf{B} + \frac{m}{q} \mathbf{v} \times \mathbf{B})
\]

* If the fluid is homentropic (const. entropy)

\[
\partial_t K^R = 0
\]

G is then conserved.

* If entropy is not uniform, but there exists an equation of state

\[
\mathcal{V} = \mathcal{V} (T) \Rightarrow
\]

\[
T \mathcal{V} \mathcal{R} = \mathcal{V} \mathcal{S}
\]

\[
\partial_t K^R = \nabla \cdot \mathcal{S} \quad (b)
\]

with \( \mathcal{S} = \frac{2}{q} \hat{B} \cdot \mathcal{S} \) (\( \mathcal{S} \) is the solution of \( \frac{dS}{dt} = T \frac{d\mathcal{R}}{dT} \))

Then Again G is conserved

Only baroclinic fluids do not conserve helicity in the non-relativistic limit.
As we go from the non-rel to the relativistic fluids, the situation changes dramatically.

Even when we have an equation of state \( \sigma = \sigma(T) \), Eq. 176 becomes

\[
\frac{\partial \mu}{\partial T} K^\mu = \frac{1}{T} \nabla \cdot S
\]

\[
S = \left( \frac{2}{\alpha} \right) S \hat{\mathbf{e}}_3
\]

For any dynamical relativistic fluid with non-uniform velocity field, \( \gamma \) cannot be absorbed within the divergence and

\[
\frac{dG}{dt} = \oint d^3 \mathbf{x} K^0 K^3 = 0
\]

The fact that special relativity breaks the invariance of a general equilibrium ideal fluid, opens a channel for vorticity or magnetic field generation.
Helicity Density - Evolving Dynamics

Let us see the evolution of helicity density as the dynamics becomes more and more general

\[ K^0 = (A + \frac{m}{q} V \times \mathbf{u}) \cdot (B + \frac{m}{q} V \times \mathbf{u}) \]

Non-relativistic:

* Pure electromagnetic field \( K^0 = A \cdot B \)

* Fluid with mass \( m \) \( K^0 = (A + \frac{m}{q} \mathbf{u}) \cdot (B + \frac{m}{q} V \times \mathbf{u}) \)

The fact that \( K^0 = A \cdot B \) is taken to hold for a plasma because it is the electron helicity with \( m \to 0 \).

*** Each fluid species of a plasma contributes a helicity invariant (under appropriate circumstances)

* Electron-Ion plasma

\[ G_e = \langle (A - \frac{m_e}{e} V_e) \cdot (B - \frac{m_e}{e} V_x V_e) \rangle \]

\[ G_e = \langle (A + \frac{m_e}{e} V_e) \cdot (B + \frac{m_e}{e} V_x V_e) \rangle \]

Relativistic:

For an electron-positron plasma

\[ G_{\pm} = \langle (A \pm f \pm \frac{m}{e} V \pm \frac{m}{e} V \times \mathbf{u}_{\pm}) \cdot (B \pm \frac{m}{e} V \times \mathbf{u}_{\pm} V_e) \rangle \]

Measuring the knottedness of the field (flow) lines, helicity is a fundamental concept of plasma/Fluid Dynamics and generally limits the states accessible.