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Classical Perfect Diamagnetism

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A Conventional Superconductor-1

A conventional superconductor (SC)=perfect conductor with no magnetic flux in its interior (Meissner–Ochsenfeld effect).

Quantum phenomena-Cooper pairs-Super electrons of charge $(-2e)$ and mass $(2m_e)$. Electrodynamics defined by Steady-state Maxwell

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{J}, \tag{1}$$

and the constitutive relationship

$$\nabla \times \mathbf{J} = -\frac{c}{4\pi}\frac{\mathbf{B}}{\lambda_s^2}, \tag{2}$$

where $\lambda_s = c/\omega_{ps}$ is the skin depth ($\omega_{ps} = (4\pi\, n_s\, e_s^2/m_s)^{1/2}$ is the corresponding plasma frequency). The current $\mathbf{J} = -n_s\, e_s \mathbf{v}_s$ is entirely due to super electrons stipulated to have zero canonical momentum $m_s\, \mathbf{v}_s - (e_s/c)\mathbf{A} = 0$.

A Conventional Superconductor-2

The standard form of the London equation follows

$$\nabla^2 \mathbf{B} = \frac{\mathbf{B}}{\lambda_s^2}, \tag{3}$$

For a system with dimension $\ell \gg \lambda_s$, the magnetic flux is confined to a distance λ_s near the edge; in this narrow skin, the field literally jumps from a value 0 to its external value.

Magnetic field **B** and current **J** are restricted to the same skin depth. Within the London framework, then, the current and flux expulsion are indistinguishable and equivalent; either expulsion could define the canonical superconductivity.

Perfect Diamagnetism- Classical Systems

Can perfect diamagnetism be found in classical systems?

Warning- This is not an an attempt at a classical derivation of standard superconductivity- conventional superconductivity is well known to have quantum origin!

We are attempting here to explore if there are classical systems that mimic the electrodynamics of superconductors- Is something akin to a London state classically accessible?

And if so could the theoretical underpinnings of such systems be experimentally tested?

Survey-A well-known example invoking magnetic helicity

The Woltejer-Taylor state (real α)

$$\nabla \times \mathbf{B} = \alpha^{-1}\mathbf{B} \Rightarrow \nabla^2 \mathbf{B} = -\frac{\mathbf{B}}{\alpha^2} \tag{4}$$

is exactly an antithesis of (3) and implies that the magnetic flux occupies the whole region ($\mathbf{B}$ is oscillatory).

Equation (4), pertaining to a perfectly conducting fluid, was derived in ideal magnetohydrodynamics (MHD) by minimizing the magnetic energy ($<>= \int d^3x$)

$$E_m =< \mathbf{B^2}/8\pi >, \tag{5}$$

subject to the constraint of an invariant magnetic helicity

$$h_m = \frac{1}{8\pi} \langle \mathbf{A} \cdot \mathbf{B} \rangle \tag{6}$$

Systems ”richer” than MHD

Notice that unconstrained minimization would lead to the trivial solution $\mathbf{B}=0$

The recognition of magnetic helicity as an invariant was a major factor in our understanding of the structure of the magnetic fields and the subsequent discovery and development of self-organized states accessible in ideal MHD

The helicity, a measure of the structural- topological complexity of a solenoidal vector field, is easily generalizable to systems more complicated than MHD; the constancy of the so called ”generalized” helicities can be harnessed to generate new and interesting field configurations.

Electrodynamics of perfect conductors

A perfectly conducting super electron gas (charge $-e^*$ and mass m^*) obeys

$$\frac{\partial \mathbf{P}}{\partial t} \equiv \frac{\partial}{\partial t}\left(\mathbf{A} - \frac{cm^*}{e^*}\,\mathbf{u}\right) = \mathbf{u} \times \mathbf{\Omega} + \nabla\left(\frac{u^2}{2} + g\right), \tag{7}$$

where the generalized vorticity (GV),

$$\mathbf{\Omega} = \nabla \times \mathbf{P} = \mathbf{B} - \frac{cm^*}{e^*}\nabla \times \mathbf{u}, \tag{8}$$

$\mathbf{u}$ is the fluid mechanical velocity, $\mathbf{P}$ is proportional to canonical momentum, and the last term represents gradient forces (pressure–). Curl of Eq.7 converts it to the vortex dynamical form:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{\Omega}). \tag{9}$$

A very special solution

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{\Omega}). \tag{10}$$

has a special and unique solution $\mathbf{\Omega} = \mathbf{0}$

If GV is zero, it is also a constant of motion.

The condition $\mathbf{\Omega} = 0$ (with $\mathbf{J} = -n\, e^* \, \mathbf{u}$) is precisely the constitutive relation (2) that yields the London equation.

Electrodynamically, then, Quantum transitions simply supply an initial condition that assures perfect diamagnetism.

The "singular" $\mathbf{\Omega} = 0$ is electromagnetic 'signature' of the superconducting state. The more general $\mathbf{\Omega} = \mu \mathbf{u}$, permitted as an equilibrium solution of (7) is not

Surely for a classical system, the latter will may be the general solution with $\mathbf{\Omega} = 0$ as a possible limiting case.

Ideal Perfect Fluids- Accessible States-a primer-1

A plasma consisting of several dynamic species (the standard superconductor has only one – the super electrons) is embedded in a strong confining magnetic field $\mathbf{B}_0 = \widehat{e}_z\, B_0$

The total magnetic field $\mathbf{B}_T = \mathbf{B} + B_0\widehat{e}_z$, **B** produced by plasma currents.

Each of these perfectly conducting components, derived for a fluid with constant density and isotropic pressure, obeys

$$\frac{\partial}{\partial t}\mathbf{P}_\alpha = \mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha + B_0\left(\mathbf{v}_\alpha \times \widehat{e}_z\right) - \nabla\psi_\alpha, \qquad (11)$$

where $\mathbf{P}_\alpha = \mathbf{A} + (m_\alpha c/q_\alpha)\mathbf{v}_\alpha$, $\mathbf{\Omega}_\alpha = \nabla \times \mathbf{P}_\alpha = \mathbf{B} + (m_\alpha c/q_\alpha)\nabla \times \mathbf{v}_\alpha$ is the generalized vorticity for the species α with mass (charge) $m_\alpha(q_\alpha)$, and $\psi_\alpha = c/q_\alpha(p_\alpha/n_\alpha + .5m_\alpha v_\alpha^2 + q_\alpha\phi)$ spells out the gradient forces; $p_\alpha(n_\alpha)$ is the pressure(density) and ϕ is the electrostatic potential.

Ideal Perfect Fluids- Accessible States-a primer-2

The uniform static external field $B_0(\partial B_0/\partial t = 0)$ distinguisshed from the dynamic magnetic field $\mathbf{B}$. Tremendously simplification results when: 1) no variation along confining field $(\partial/\partial z = 0)$, and 2) motions are compressible $(\nabla \cdot \mathbf{v}_\alpha = 0)$.

The allowed velocity field

$$\mathbf{v}_\alpha = v_{z\alpha}\,\widehat{e}_z + \widehat{e}_z \times \nabla\chi_\alpha, \tag{12}$$

implying

$$\mathbf{v}_\alpha \times \widehat{e}_z = \nabla\chi_\alpha, \tag{13}$$

converts (11) into $(\widehat{\psi}_\alpha = \psi_\alpha - B_0\chi)$

$$\frac{\partial \mathbf{P}_\alpha}{\partial t} = \mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha - \nabla\widehat{\psi}_\alpha, \tag{14}$$

Dynamics in strong magnetic field reduced to one with no confining field; the confining field has simply gone to modify the gradient force -not pertinent for current investigation

Ideal Perfect Fluids- Accessible States-a primer-3

Dynamics is governed by Eq.(14), and its curl

$$\frac{\partial \mathbf{\Omega}_\alpha}{\partial t} = \nabla \times \left(\mathbf{v}_\alpha \times \mathbf{\Omega}_\alpha \right), \tag{15}$$

and the Ampere's law $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$, $\mathbf{J} = \sum q_\alpha n_\alpha \mathbf{v}_\alpha$.

Following $(n+1)$ bilinear invariants emerge: the total energy $[<>=\int d^3x]$

$$E = \left\langle \frac{B^2}{8\pi} + \frac{1}{2}\sum_\alpha n_\alpha m_\alpha v_\alpha^2 \right\rangle, \tag{16}$$

and a generalized helicity (GH) for each species

$$h_\alpha = \frac{1}{8\pi}\left\langle \mathbf{P}_\alpha \cdot \mathbf{\Omega}_\alpha \right\rangle \tag{17}$$

Unless the electron inertia is neglected (like in MHD) it is h_α, and not the magnetic helicity h_m that is conserved.

Variational principle- Accessible States

Relaxed states are derived via the variational principle

$$\delta\left(E - \mu_\alpha^{-1} h_\alpha\right) = 0, \tag{18}$$

minimizing the energy with the helicity constraints. The constant μ_α are Lagrange multipliers. The Euler–Lagrange equations

$$\mathbf{\Omega}_\alpha = \mathbf{B} + \frac{m_\alpha c}{q_\alpha}\nabla\times\mathbf{v}_\alpha = (4\pi/c)\mu_\alpha q_\alpha n_\alpha \mathbf{v}_\alpha \tag{19}$$

align generalized vorticities of each species along its velocity.

All variations are incompressible and normal components of fields vanish at the boundaries. Eq.(19) is an equilibrium solution provided the Bernoulli condition $(\nabla\psi) = 0$ is satisfied- Bernoulli conditions are not directly relevant to today's lecture.

The structure of the magnetic and velocity fields for these relaxed states can be obtained by solving (19) in conjunction with Ampere's law.

Search for Perfect Diamagnetism-A Model

The model plasma has two distinct components – a bulk plasma of essentially stationary (non–dynamic) electrons and ions, and a dynamic fast component that could be either electrons or ions.

The dynamic component (to be designated fast with a label f) carries all the current as well as the kinetic energy of the system, i.e., $|q_f n_f \mathbf{v}_f| \gg |q_e n_e \mathbf{v}_e|, |q_i n_i \mathbf{v}_i|$, and $n_f m_f v_f^2 \gg n_e m_e u_e^2,\ n_i m_i v_i^2$.

The fast dynamical component mimics the superelectrons; can be justifiably treated as ideal(infinite conductivity). A single generalized helicity h_f is of essence. Magnetic fields are determined by Eq. 19, and

$$\nabla \times \mathbf{B} = \left(\frac{4\pi}{c}\right) q_f n_f \mathbf{v}_f. \tag{20}$$

Normalization (λ_f is the skin depth, $\lambda_f^2 = c^2/\omega_{pf}^2$, $\omega_{pf}^2 = (4\pi q_f^2 n_f/m_f)$) yields ($\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$):

$$\nabla \times \nabla \times \mathbf{B} + \mathbf{B} = \frac{\mu_f}{\lambda_f} \nabla \times \mathbf{B}, \tag{21}$$

Working out the Model-1

But for the Lagrange multiplier $\mu_f = 0$, Eq.(21) is nothing but the London equation with fields restricted to a skin depth λ_f

Surely, then, electrodynamically $\mu_f = 0$ (which for the relaxed state insures $\mathbf{\Omega}_f = 0$) is just the necessary and sufficient condition for perfect diamagnetism!.

The model system has a general solution ($A_\pm$ are constants)

$$\mathbf{B} = A_+\mathbf{G}_+ + A\mathbf{G}_- \tag{22}$$

where $\mathbf{G}_\pm$, known as Beltrami fields, are, in turn, the solutions of

$$\nabla \times \mathbf{G}_\pm = \lambda_\pm \mathbf{G}_\pm \tag{23}$$

with

$$\lambda_\pm = .5\left[\frac{\mu_f}{\lambda_f} \pm \left[\left(\frac{\mu_f}{\lambda_f}\right)^2 - 4\right]^{1/2}\right]. \tag{24}$$

Working out the Model-2

The roots $\lambda_\pm$ are real for $(\mu_f/\lambda_f)^2 > 4$ - **no diamagnetism**

Form a complex conjugate pair for $(\mu_f/\lambda_f)^2 < 4$-**partial diamagnetism**

In the latter case $\mathbf{G}_+^* = \mathbf{G}_-$ and A_+ must be A_-^* in order for the solution $[\mathbf{B} = 2\,\mathrm{Re}(A_+\mathbf{G}_+]$ to be real.

As $|\mu_f/\lambda_f|$ goes from zero to larger values, the system begins with perfect diamagnetism ($\lambda_\pm = \pm i$), switches to partial diamagnetism ($\pm$ complex), and finally succumbs to the W-T state ($\lambda_\pm$ real)

The transition from complex to real roots happens at the critical value $|\mu_f/\lambda_f| = 2$.

Evidently the amount of 'diamagnetism' displayed by this relaxed state is controlled by the parameter $|\mu_f/\lambda_f|$.

Working out the Model-3

An expression for μ_f for the relaxed equilibrium state (19-20) is readily derived through a the following steps:

$$
\begin{aligned}
h_f &= \frac{1}{8\pi}\langle \mathbf{P}_f \cdot \boldsymbol{\Omega}_f \rangle = \frac{\mu_f}{2c}\left\langle \left(\mathbf{A} + \frac{m_f c}{q_f}\mathbf{v}_f\right) \cdot q_f n_f \mathbf{v}_f \right\rangle \\
&= \mu_f \left\langle \frac{1}{2} m_f n_f v_f^2 + \frac{B^2}{8\pi} \right\rangle
\end{aligned}
$$

yielding the revealing identification

$$\mu_f = \frac{h_f}{E} \tag{25}$$

The Lagrange multiplier (dimensions of a length) measures the generalized helicity as a fraction of the total energy. A ratio of two constants of motion, the control parameter μ_f is an invariant of the system and is fully determined by the initial 'preparation' of the system.

Helicity is an impediment to perfect diamagnetism!

Summary-Conclusions-1

Similar Electrodynamics for a standard superconductor and that of a magnetically confined ideal plasma - consisting of a fast dynamic component and an essentially stationary ion–electron bulk plasma

The generalized helicity $h = (8\pi)^{-1} \langle \mathbf{P} \cdot \boldsymbol{\Omega} \rangle$, an invariant measure of the 'knottedness' of the field of generalized vorticity $\boldsymbol{\Omega}$, emerges as a fundamental determinant of the class of magnetic field configurations that the system can entertain.

The state of Perfect Diamagnetism corresponds to $h = 0$. For minimum energy relaxed states, this condition implies the constitutive relationship $\boldsymbol{\Omega} = 0$ $[\boldsymbol{\Omega} \propto \mu \mathbf{v}, \mu = h/E]$, the very definition of superconductivity.

The constant of the motion h is determined by the initial conditions =¿ the electrodynamics of a super–conductor is fully reproduced if the quantum correlations provided the correct initial condition, $h = 0$. Quantum correlations do produce the super–electrons precisely in this state.

Summary-Conclusions-2

With the helicity h established as the fundamental determinant of the “ diamagnetic content”, whole series of experiments become possible.

The classical plasma is not bound to be in a helicity–free state; it can, in principle, entertain (or be prepared in) configurations of arbitrary helicity (or helicity/energy). Its magnetic behavior, therefore, can vary over a broad range; from perfect or nearly perfect diamagnetism to no diamagnetism.

Perfect diamagnetism is asymptotically accessible to a classical system with length $\mu = h/E$ (may be termed the decorrelation length) much smaller than the skin depth

A clever experimentalist can play, for example, with a beam- plasma system (by experimenting with how to inject an ion or an electron beam in an ambient plasma) to bring the generalized helicity to any arbitrary value. She could accomplish the same feat for a classical plasma what quantum mechanics does in a conventional superconductor.

Summary-Conclusions-3

Even in the zero or near zero helicity state, classical systems display immense variety in the degree of localization.

With appropriate choices of fast electron and/or ion beams with a range of densities, one can create skin lengths which can vary over several orders of magnitude.

Current channels of arbitrary extent could be experimentally created.

Whenever one finds excessive localization of current in space, astrophysical or laboratory plasmas, one should look, it seems, for a classical "super–conducting" explanation.

Refrences

S.M.Mahajan, Phys. Rev. Lett **100**, 075001-4 (2008)

F. and H. London, Physica **2**, 341(1935); F. London, Physica **3**, 450 (1936); F. London, Nature **137**, 991 (1936)

See for example, J. R. Schreiifer, *Theory of Superconductivity*, (W. A. Benjamin Incorporated, New York, 1964)

There was an attempt to derive the London equation from classical dynamics by W. F. Edwards, Phys. Rev. Lett **47**, 1963 (1981) and was shown to be erroneous by J. B. Taylor, Nature **29**, 681(1982).

L.Woltjer, Proc. Natl. Acad. Sci. U.S.A 44, 489 (1958), S.Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A **42**, 489 (1956), S.Chandrasekhar and L.Woltjer, Proc. Natl. Acad. Sci. U.S.A **44**, 489 (1958)

J. B. Taylor, Phys. Rev. Lett. **33**, 1139 (1974)

For an excellent introduction to magnetic helicity, see, H.K.Moffatt, *Magnetic field generation in electrically conducting fluids*, (Cambridge University Press, Cambridge,1978). For a very general relativistic formulation of generalized helicities, see, S.M.Mahajan, Phys. Rev. Lett **90**, 035001-4 (2003)

A. Hasegawa, Adv. Phys. **34**, 1 (1985). In this review, the use of the helicity invariant as an agent for inverse cascading is thoroughly described.

S. M. Mahajan, and Z. Yoshida, Phys. Plasmas **7**, 635 (2000)

R.N.Sudan, Phys. Rev. Lett **42**, 1277 (1979), J.M.Finn and T.M.Antonsen Jr.,Phys. Fluids **26**, 3540(1983), L.C.Steinhauer and A.Ishida, Phys. Rev. Lett **79**, 3423 (1997), S. M. Mahajan and Z. Yoshida, Phys. Rev. Lett **81**, 4863 (1998)