2155-16

International Workshop on Cutting-Edge Plasma Physics

5 - 16 July 2010

Vorticity Creation and Entropy Production

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Lecture Note: Vorticity Creation and Entropy Production

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(Dated: July 12, 2010)

Abstract

The aim of this lecture is to delineate the fundamental relation between the vorticity generation and entropy production. The notion of "vorticity" is naturally (or canonically) generalized to unify fluid vortexes and magnetic fields (EM vorticity). The source of vorticity is a properly defined entropy term TdS (a possibly non-exact differential form breaking the conservation of circulations); the differential is defined in 4D space-time. We will show that the relativistic distortion of space-time can create a seed of cosmic vorticity/magnetic field. Vortexes influence the heat transport and modify the temperature and entropy distributions. The self-organization of convection cells or transport barriers (both are the creations of vorticity) are, then, described as spontaneous mechanisms self-regulating the entropy production. We will show that transport barriers are self-organized with "maximizing" the entropy production rate.

PACS numbers:

I. THERMODYNAMIC LAW AND FLUID MECHANICS

A. Preliminaries

Here we delineate how the thermodynamic laws are built in fluid mechanics equations, or how fluid mechanics can be built around the fundamental thermodynamic relations. In the primitive narrative of thermodynamics, the "energy" does not include the kinetic energy of fluid motion (collective motion); *quasi-static* process is the subject of main interest. To formulate the emergence of infinitely diverse "motion" of fluids, we have to implement the following two formulations into the model:

1. The explicit form of the mechanical energy (consisting of kinetic and potential energies).

2. The geometric formula to calculate variations d in space-time.

Needless to say, it is impossible to describe all possible processes of chaotic dynamics in a non-equilibrium system. Here the notion of "motion" is limited to a class of fluid (or collective) motion (which we will denote by $\boldsymbol{v}$). We consider *elements* of the fluid, and assume that every element obeys thermodynamic laws. Let $\rho = mn$ be the mass density of the fluid; m is the mass of a particle of the fluid constituent and n is the number density. When the volume of a fluid element is V, the number of particles contained in the element is nV. We normalize nV to be unity (in some appropriate unit); in what follows an element is such that $nV = 1$.

The first law associated with each fluid element ($nV = 1$) may be written as

$$dE = \delta W + \delta Q, \tag{1}$$

where E is the total energy of the fluid element, δW is the mechanical work, and δQ is the heat exchange [1]. Here E consists of mechanical (collective) and thermal energies. The mechanical work may not be simply $-PdV$, but we assume that only $-PdV$ works to change E of each fluid element; all other mechanical processes may "rearrange" the partitions among the constituents of E. To represent such "internal work" pertaining to fluid motion, we write

$$dE = d\varepsilon + \delta E_F,$$

where ε denotes the thermal energy (which includes the rest mass energy). The second term is *internal* in the sense that δE_F evaluated along the fluid motion vanishes, while it dictates the motion of the fluid. As to be shown later, we will generalize the energy (enthalpy) to be a tensor, and then, δE_F may be interpreted as the *non-exact* part of the variation of the tensor energy (enthalpy). The first law now reads as

$$d\varepsilon + \delta E_F = -PdV + \delta Q, \tag{2}$$

By the standard Legendre transformation, we introduce the enthalpy $h = \varepsilon + PV$, and rewrite (2) as

$$dh + \delta E_F = n^{-1}dP + \delta Q. \tag{3}$$

The second law is written as $\delta Q = T(dS - \delta S_i)$ with T and S as temperature and entropy, respectively. The quantity $\delta S_i(\geq 0)$ denotes internal entropy production. Combining the first and second laws, we may write

$$dh + \delta E_F = n^{-1}dP + T(dS - \delta S_i). \tag{4}$$

The proper representation of the enthalpy $H = E + VP$ must combine the fluid (collective) kinetic energy ($mv^2/2$), potential energy (ϕ), thermal energy (ε), and $VP = n^{-1}P$. However, in a quasi-static homogeneous fluid, H reduces into the static enthalpy $h = \varepsilon + n^{-1}P$. Then, putting $\delta E_F = 0$ and $\delta S_i = 0$, (4) reduces into the well-known quasi-static thermodynamic relation

$$dh = n^{-1}dP + TdS. \tag{5}$$

The generalization of (5) to (4) requires space-time tensor framework; the generalized H is no longer a scalar state variable, so that dH may include a non-exact part, which has been denoted by δE_F. In the conventional argument of "quasi-static process", the paths of evaluating variations (denoted d or δ) are not explicitly connected to the notion of *time* — they are not dynamical processes, but are caused by *hand*. However, we are now considering a *fluid motion*, in which the variations of parameters are caused by autonomous dynamical processes. Hence, the paths of variations must be explicitly related to the orbits of fluid elements in space-time.

B. Energy-Momentum Tensor and Thermodynamic Law

In a dynamic fluid system, the (molar) enthalpy becomes a tensor:

$$\mathcal{H}^{\mu\nu} = hU^\mu U^\nu,$$

where U^μ is the space-time 4-velocity (see Landau-Lifshitz [2] and Appendix A). The specific fluid energy is

$$T^{\mu\nu} = n\mathcal{H}^{\mu\nu} - Pg^{\mu\nu},$$

by which the adiabatic equation of motion is written as

$$\partial_\mu T^{\mu\nu} = 0. \tag{6}$$

To incorporate non-adiabatic effects, we may add a heat term such as $n\Theta^n u = \partial_\mu D^{\mu\nu}$ to the left-hand side (see Appendix A). We may calculate

$$\begin{aligned} \partial_\mu T^{\mu\nu} &\equiv \partial_\mu \left(nhU^\mu U^\nu - Pg^{\mu\nu}\right) \\ &= nU^\mu \partial_\mu \left(hU^\nu\right) - \partial^\nu P \qquad (7) \\ &= n\partial^\nu h + ncU_\mu M^{\mu\nu} - \partial^\nu P, \qquad (8) \end{aligned}$$

where the antisymmetric matter field tensor is defined by

$$cM^{\mu\nu} = \partial^\mu (hU^\nu) - \partial^\nu (hU^\mu). \tag{9}$$

In the first step (7), we calculate

$$\partial_\mu \left(nhU^\mu U^\nu\right) = nU^\mu \partial_\mu \left(hU^\nu\right) + hU^\nu \partial_\mu \left(nU^\mu\right).$$

The second term vanishes because of mass conservation. Contracting both sides of (9) with U_μ, we obtain

$$\begin{aligned} cU_\mu M^{\mu\nu} &= U_\mu \partial^\mu (hU^\nu) - U_\mu \partial^\nu (hU^\mu) \\ &= U^\mu \partial_\mu (hU^\nu) - \partial^\nu h. \end{aligned}$$

Using this relation in (7) yields (8).

Dividing (6) by n, we obtain

$$cU_\mu M^{\mu\nu} = -\partial^\nu h + n^{-1} \partial^\nu P. \tag{10}$$

By the thermodynamic relation (5), the right-hand side of (10) may be written as $-TdS$. A non-adiabatic equation of motion may be written as

$$cU_\mu M^{\mu\nu} = -\partial^\nu h + n^{-1}\partial^\nu P + \Theta^\nu, \tag{11}$$

which is compared with the conceptual equation (3); the term $cU_\mu M^{\mu\nu}$ embodies δE_F, while the heat term Θ^ν corresponds to δQ.

The term $cU_\mu M^{\mu\nu}$ has the following properties:

1. Since $M^{\mu\nu}$ is antisymmetric, contracting with the 4-velocity U_ν yields $U_\nu U_\mu M^{\mu\nu} = 0$, implying that it does not work when evaluated along with each element of the fluid, i.e., it is an "internal" term.

2. In the adiabatic equation of motion (6), the fact that $U_\nu U_\mu M^{\mu\nu} = 0$ implies isentropic flow.

3. While $cU_\mu M^{\mu\nu}$ does not work when evaluated along the streamlines, it can create "vorticity". The geometric implication of this term will be delineated in Sec. II.

C. Plasma Model — matter-EM unification

By generalizing the Hamiltonian and momentum to be *canonical* being amenable to the EM field theory, we obtain the dynamics equations describing matter-EM coupling [3], i.e., model of plasmas. Let $A^\mu = (\varphi, \boldsymbol{A})$ denote the EM 4-potential. The canonical Hamiltonian (enthalpy) and momentum are, respectively, $H = (\wp - q\boldsymbol{A})^2/2m + \phi + h$ and $\wp^\mu = \mathcal{P}^\mu + qA^\mu$. The matter-EM field tensor is, then,

$$M^{\mu\nu} \;\to\; \mathcal{M}^{\mu\nu} = \partial^\mu \wp^n u - \partial^\nu \wp^\mu = M^{\mu\nu} + qF^{\mu\nu}.$$

II. CIRCULATION THEOREM

A. Classical Kelvin's Theorem

The circulation $\oint_L \delta Q$, associated with a physical quantity δQ, calculated along the *loop* L, may be zero or finite depending on whether δQ equals an exact differential $d\varphi$ (φ being a state variable) or not. For example, if $\delta Q = TdS$ (T: temperature, S: entropy), the

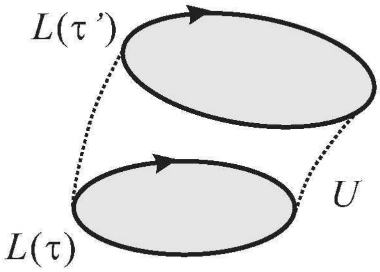

FIG. 1: Transport of a loop and circulation. Given a loop L in space, the circulation of a vector field $\boldsymbol{P}$ is the integral $\oint_L \boldsymbol{P} \cdot d\boldsymbol{x}$. Two loops $L(\tau)$ and $L(\tau')$, connected by the "flow" $d\boldsymbol{x}/d\tau = \boldsymbol{U}$ (the parameter τ may be regarded as time), are shown in the figure. A circulation theorem pertains to a "movement" of loops; the rate of change of circulation is calculated as (12). To generalize the argument to the relativistic regime, we have to immerse the loop in the 4-d space-time and transport it by the 4-velocity $dx_\mu/d\tau = U_\mu$; see Fig. 2. The relativistic space-time circulation conserves in ideal fluids; see (14).

circulation is generally finite and measures the heat gained in a quasi-static thermodynamic cycle.

An *ideal fluid* can be viewed as a realization of an infinite number of ideal isolated (adiabatic) cycles covering space. Along the time dependent loop $L(t)$, convected by the fluid motion, the rate of change of circulation associated with the canonical momentum $\oint_{L(t)} \boldsymbol{P} \cdot d\boldsymbol{x}$ is identically zero. In fact, if two loops $L(t)$ and $L(t')$ are connected by the "flow" $d\boldsymbol{x}/dt = \boldsymbol{U}$, the rate of change of circulation is calculated as

$$\frac{d}{dt} \oint_{L(t)} \boldsymbol{P} \cdot d\boldsymbol{x} = \oint_{L(\tau)} [\partial_\tau \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{U}] \cdot d\boldsymbol{x}. \tag{12}$$

The fluid equation may be written as (Appendix A)

$$\begin{aligned} \partial_t \boldsymbol{P} + (\nabla \times \boldsymbol{P}) \times \boldsymbol{v} &= -\nabla \left(\frac{mv^2}{2} + \phi \right) - \frac{\nabla p}{n} \\ &= -\nabla H + T \nabla S, \end{aligned} \tag{13}$$

where $H = mv^2/2 + \phi + h$ (ϕ: potential energy, h: static enthalpy; $\nabla(\varepsilon + n^{-1}p) = \nabla h = n^{-1}\nabla p + T\nabla S$). If $\nabla S = 0$ (for example, barotropic relation holds), the rate of change of circulation equals the circulation of an exact fluid-dynamic force derived from the energy density, i.e., $\oint_{L(t)} \nabla H \cdot d\boldsymbol{x} = \oint_{L(t)} dH = 0$. In the standard non-relativistic description of an ideal fluid, therefore, if the initial state has no circulation (vorticity), the later state will also

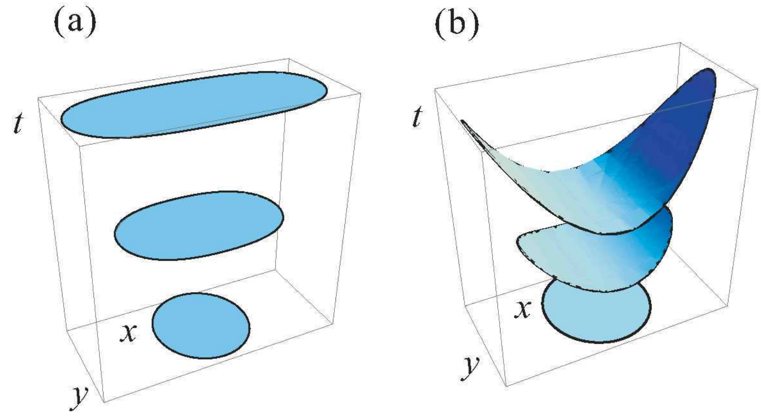

FIG. 2: Transport of a surface (and its boundary) in space-time. Two figures compare the evolution of a surface and its boundary (loop) moved, respectively, by (a) the non-relativistic velocity ($dx_j/dt = V_j$: 3-vector) and (b) the relativistic 4-velocity ($dx_\mu/ds = U_\mu$). The figures are drawn in the space-time x-y-t with $\boldsymbol{v}/c = (\tanh x, 0, 0)$ (thus $\gamma = \operatorname{sech}^{-1} x$). In the Lorentz-covariant theory, the circulation theorem applies to a loop $L(s)$ that is moved by the 4-velocity U_μ in the 4-dimensional space-time.

be vorticity-free (Kelvin's circulation theorem). For the vorticity to be created, the "force" on the fluid must not be an *exact differential.*

B. Generalized 4D Circulation theorem

In the relativistic space-time, the loop $L(t)$ pertaining to a "synchronic space" ($t =$ constant cross section of space-time in a reference frame) ceases to be the appropriate geometric object along which the circulation must be evaluated (see Fig. 1). The loop moves in space-time with a 4-velocity $U^\mu = (\gamma, \gamma V^j/c)$ (V^j: the reference-frame velocity) and the *relativistic circulation* must be described as a function of the proper time s. In Fig. 2, the respective evolutions of the "synchronic loop" $L(t)$ and the "relativistic loop" $L(s)$ are compared. The synchronicity of the loop $L(s)$ is broken by the nonuniformity of the proper time. The circulation of a 4-vector $\wp^\mu$ along the relativistic loop $L(s)$ obeys

$$\frac{d}{ds}\left(\oint_{L(s)} \wp^\mu dx_\mu\right) = \oint_{L(s)} \left(\partial^\mu \wp^\nu - \partial^\nu \wp^\mu\right) U_\nu dx_\mu. \tag{14}$$

If $\wp^\mu$ is an appropriate momentum, the relativistic equation of motion relates the integrand $\left(\partial^\mu \wp^\nu - \partial^\nu \wp^\mu\right) U_\nu$ with an effective force (Appendix A). If the force is exact, the relativistic circulation will be conserved; the ideal fluid does, indeed, obey an appropriate *relativistic Kelvin circulation theorem.* However, vorticity (or magnetic field) is defined on synchronic space (hence, it is reference-dependent); its *circulation* still pertains to the synchronic loop $L(t)$. The field must be mapped from the naturally distorted $L(s)$ back to $L(t)$—this reciprocal distortion, represented by a *Jacobian* γ^{-1}, imparts a *shear* to the thermodynamic force (i.e, changes dH to $\gamma^{-1}dH$) destroying its exactness.

C. Creation of Vorticity/Magnetic Field

The generalized vorticity $\hat{\boldsymbol{\Omega}}$ (or the generalized magnetic field $\hat{\boldsymbol{B}}$) is defined by $\nabla \times \boldsymbol{\wp}$ (or $(c/q)\nabla \times \boldsymbol{\wp}$), where $\boldsymbol{\wp}$ is the vector part of $\wp^{\mu}$. The equation of motion is written succinctly as [3]

$$cU_{\mu}\mathcal{M}^{\mu\nu} = -T\partial^{\nu}S. \tag{15}$$

Substituting (15) into (14) shows that the rate of change of circulation of $\wp^{\mu}$ is balanced by the integral along $L(s)$ of $(T/c)\partial^{\mu}S$. It is the 3-vector part of (15)

$$q\left[\hat{\boldsymbol{E}} + \left(\frac{\boldsymbol{v}}{c}\right) \times \hat{\boldsymbol{B}}\right] = -\frac{cT}{\gamma}\nabla S \tag{16}$$

that explicitly shows the relativistic modification of the force $T\nabla S$ by the factor γ^{-1}. Here, the generalized electric field $\hat{E}^{j} = E^{j} + (mc/q)\mathcal{M}^{0j}$ satisfies Faraday's law $\partial_t\hat{\boldsymbol{B}} = -\nabla \times \hat{\boldsymbol{E}}$. The appearance of γ^{-1} on the right-hand side is due to the mapping back of the relativistic space-time onto the synchronic space in which the conventional circulation and the vorticity are to be calculated. To evaluate the rate of change of $\hat{\boldsymbol{B}}$ (with respect to the reference time t), we must go back to (16) whose curl reveals the source for magnetic field generation:

$$\mathcal{S} = -\nabla \times \left(\frac{cT}{q\gamma}\nabla S\right) = -\nabla\left(\frac{cT}{q\gamma}\right) \times \nabla S, \tag{17}$$

which may be broken into the familiar baroclinic term $\mathcal{S}_B = -(c/q\gamma)\nabla T \times \nabla S$ and the relativistically induced new term

$$\mathcal{S}_R = -\left(\frac{cT}{q}\right)\nabla\gamma^{-1} \times \nabla S = -\left(\frac{c\gamma}{2qn}\right)\nabla\left(\frac{V}{c}\right)^2 \times \nabla p. \tag{18}$$

III. THERMODYNAMICS OF SELF-ORGANIZATION

A. Creation of Vortex and Heat Transport

Here we study the role of vorticity generation term TdS in the self-organization of transport barrier [5]. The following discussion is non-relativistic.

We start by a classical example of fluid-mechanical nonlinear process creating non-trivial (bifurcated) mode of non-equilibrium structure, "vortex". As described in Sec. II, non-exact TdS is the causal of vorticity (see [4] for the geometric and analytic properties of the this *Clebsch form*). Here we study an example of thermal convection (so-called Bénard convection), and delineate how an inhomogeneous entropy —due to an ambient inhomogeneity caused by gravity— can drive circulation. We invoke a simple model:

$$m[\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}] = -n^{-1}\nabla P + \boldsymbol{g}, \tag{19}$$

where $\boldsymbol{g} = -mG\nabla x^1$. We may rewrite (19) as

$$m[\partial_t \boldsymbol{v} - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})] = -\nabla H + T\nabla S, \tag{20}$$

where $H = mv^2/2 + mGx^1 + h$. We consider a small perturbation (but nonlinear with respect to $\boldsymbol{v}$), caused by heat transfer through the fluid, around a constant-temperature ($T = T_0$) mechanical equilibrium (we mark equilibrium quantities by subscript "$_e$"):

$$\boldsymbol{v}_e = 0, \quad h_e + mGx^1 - T_0 S_e = \text{const.} \tag{21}$$

The perturbed (heated) fluid motion obeys

$$m[\partial_t \boldsymbol{v} - \boldsymbol{v}\times(\nabla\times\boldsymbol{v})] = -\nabla(\tilde{h} - T_0\tilde{S}) + \tilde{T}\nabla S_e. \tag{22}$$

The first term on the right-hand side is exact, while the second term is non-exact (see Appendix B for the comparison with the standard Boussinesq's model of thermal convection).

This "baroclinic effect" is caused by the collaboration of the ambient inhomogeneity of S_e (imposed by gravity) and the inhomogeneity of heating producing heat transport. The equilibrium S_e is a function of x^1 (hight). If the heating of the fluid is inhomogeneous only in x^1, the static $\tilde{T}$ may be a function of only x^1, and, then, the driving term $\tilde{T}\nabla S_e$ reduces into an exact term. A flow, however, may break symmetry, and $\tilde{T}$ will become inhomogeneous in other directions.

B. Thermodynamic Balance

From the preceding example of creation of vorticity, it is now clear that the non-exact entropy term TdS is the origin of a dynamic structure=vortex in a fluid. Our question is, then, how efficiently the heat $TdS = \delta Q$ can be converted to a collective motion. The efficiency depends on the "mechanism" of conversion (which involves some instability causing initial motion and some nonlinear process converting the instability to a quasi-static ordered motion of organized vortex), however, there must be an abstract thermodynamic balance relation.

Let us rewrite the thermodynamic relation (4) as

$$dh - T_0 dS + \delta E_F = \delta W + \left(1 - \frac{T_0}{T}\right)\delta Q - T_0 \delta S_i, \tag{23}$$

where T_0 is a positive constant (reference temperature) and $\delta W = n^{-1}dP$. As noted before, δE_F vanishes when evaluated along the streamline of each fluid element. However, if we consider a mean flow velocity averaging out the fluctuating (turbulent) small-scale fluid motion, δE_F may have a finite value.

Dividing (23) by an infinitesimal time dt, and denoting $\delta Y/dt = \dot{Y}$, we obtain a rate equation:

$$\frac{d}{dt}(h - T_0 S) + \dot{E}_F - \dot{W} = \left(1 - \frac{T_0}{T}\right)\dot{Q} - T_0 \dot{S}_i. \tag{24}$$

Integrating (24) over the fluid, we obtain a macroscopic energy balance relation. In a *quasi-stationary state* (could be far from thermal equilibrium), a sufficiently long-term average of a state variable must be constant. Hence, we may assume that the volume integral of the state variables (h and S) are constant (we neglect the mass flow across both boundaries, so that every fluid element is confined in the fixed domain; see Appendix C). We assume that the system does not absorb or emit the energy mechanically, so $\int \dot{W} dM = 0$. Integrating (24) over all fluid elements (we denote by $dM = \rho d^3x$ the mass element of the fluid; $\rho = mn$ is the mass density), then, yields

$$\int \dot{E}_F dM = \int \left(1 - \frac{T_0}{T}\right)\dot{Q} dM - T_0 \int \dot{S}_i dM. \tag{25}$$

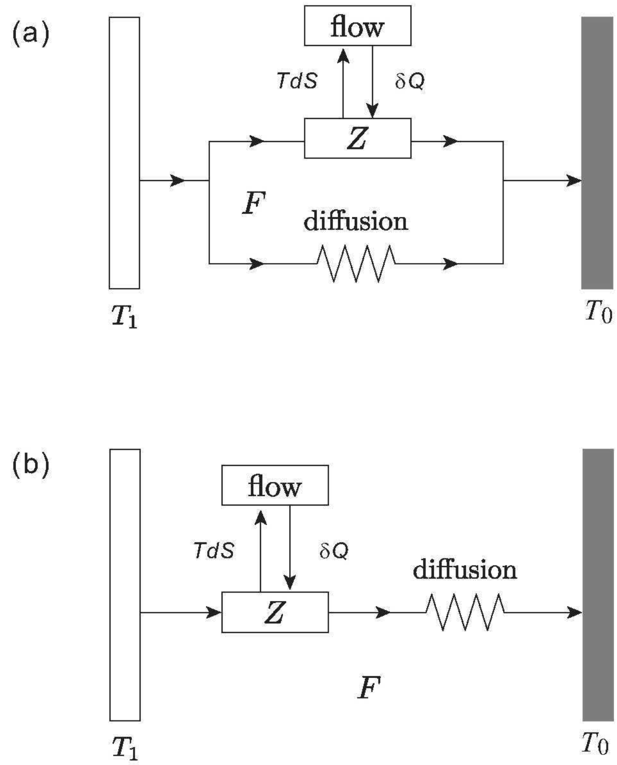

FIG. 3: Equivalent diagram of heat engine in a boundary layer. Creation of vorticity may either (a) open a channel of convective heat transport (for example, Bénard convection), or (b) block the heat transport (for example, zonal flow creating a heat-transport barrier).

C. Quasi-static Layer Model

We will now study the thermodynamics of a "layer" bounded from the inside by an internal heat source, and from the outside by a cold heat bath (see Fig. 3). We will specify the total heat flux F_1 entering the layer through the inner boundary Γ_1 in contact with the core plasma. The temperature of the outer boundary Γ_0 is fixed by the temperature T_0 of the heat bath (hereafter, we set the reference temperature T_0 to be the heat bath temperature). The inner-boundary temperature T_1 (whose value measures the layer temperature gradients), however, is the essential parameter that needs to be determined. The outer-boundary heat flux F_0 must balance F_1 in a quasi-steady state (then, we write $F_1 = F_0 = F$).

In a quasi-stationary state, heat does not accumulate or disperse in any fixed volume element for a sufficiently long term average, so we may put $\dot{Q} = 0$ inside the domain. However, the system may exchange the heat with the exterior at the boundaries. In terms of the heat flux F, we may write

$$\int_{\Omega} \left(\frac{1}{T_0} - \frac{1}{T} \right) \dot{Q} dM = \left(\frac{1}{T_0} - \frac{1}{T_1} \right) F. \tag{26}$$

Using (26) transforms (25) to

$$\int \dot{E}_F dM = \left(1 - \frac{T_0}{T_1} \right) F - T_0 \int \dot{S}_i dM. \tag{27}$$

If the heat were to transport only by diffusion in a stationary medium, (27) holds with $\dot{E}_F = 0$, and the internal entropy production is given by $\dot{S}_i = \boldsymbol{f} \cdot \nabla(1/T)$ with the heat flux $\boldsymbol{f}$. However, a finite temperature difference $T_0^{-1} - T_1^{-1}$ can create *vorticity* —note that the factor $(1 - T_0/T_1)$ is Carnot's efficiency measuring the fraction of the heat flux that may be converted to mechanical energy. If such a mechanism of vorticity creation works, the vorticity term $\dot{E}_F$ becomes finite, which will make a new balance with an enhanced internal entropy production $\dot{S}_i$. The influence, however, is not only that; the creation of vortex may change the heat flux, and thus, T_1 will also change.

The vortex can influence the heat transport in two different ways. In a stationary fluid, the creation of vortex opens a channel of heat transport by convective motion, reducing the effective impedance of the heat conduction (see Fig. 3 (a)). On the other hand, in a turbulent fluid, the creation of an ordered shear flow (or zonal flow) brings about stretching effect and suppresses the turbulent heat transport. In this case, the effective impedance of the heat conduction increases (see Fig. 3 (b)). In the next subsection, we will analyze the latter case.

D. Thermodynamic Model of Self-Organized Transport Barrier

When the impedance η of heat conduction is given, we may write

$$T = T_0 + \eta F, \tag{28}$$

where T_0 (outer-boundary temperature) is a given constant. The heat transport is dominated by turbulence, thus η is a complex function of various parameters and conditions. The power P that is ready to be converted into a coherent (or collective) motion will be the principal parameter dominating η. We put $\eta = \eta_0 + \eta_1(P)$, where η_0 is the minimum (or baseline) impedance corresponding to the non-organized turbulent state, and $\eta_1(P)$ is the increment of the impedance brought about by the self-organization of coherent motion (we assume

$\eta_1(P) \geq 0$ and $\eta_1(0) = 0$). As to be shown later, a larger η yields a larger P. We consider the minimum non-trivial model [5]:

$$T = T_0 + (\eta_0 + \eta_1(P))F, \tag{29}$$

with

$$\eta_1(P) = aP, \tag{30}$$

where a (> 0) is a constant. A positive a yields an increased impedance when the flow produces (by a positive P) an ordered structure such as a zonal flow.

For the convenience of latter calculations, we define $T_D = T_0 + \eta_0 F$, which is the minimum inner-boundary temperature. The power P must be smaller than the Carnot-cycle's power (the ideal conversion of the internal energy to a mechanical energy). Subtracting the minimum entropy production due to the baseline heat diffusion, we may write [see (27)]

$$P = F\left(1 - \frac{T_0}{T}\right) - F\left(1 - \frac{T_0}{T_D}\right) = F\left(\frac{T_0}{T_D} - \frac{T_0}{T}\right). \tag{31}$$

The *non-organized state* is such that $T = T_D$ and $P = 0$; this state may be called a *linear branch* because (29) reduces into a linear relation between T and F.

Let us see how an *organized state* (or *nonlinear branch*) emerges with yielding $T > T_D$ and enhancing the entropy production rate. We solve (29) with (30) and (31) to obtain (see Fig. 4)

$$T = \begin{cases} T_1 \equiv T_D & (F < F_c), \\ T_1 \text{ or } T_2 \equiv aF^2 T_0/T_D & (F \geq F_c), \end{cases} \tag{32}$$

where

$$F_c \equiv \frac{T_0}{\sqrt{T_0 a} - \eta_0} \tag{33}$$

is the threshold of the flux F over which the nonlinear branch bifurcates.

Stability analysis shows that the organized state (nonlinear branch) $T = T_2 \equiv aF^2 T_0/T_D$ ($\geq T_D$) is stable, while the non-organized state (linear branch) $T = T_1 \equiv T_D$ destabilizes over the threshold [5].

Going back to the essentials of the model, we note that the new element that could impart this non-standard behavior to the heat transport is our choice of the inner boundary condition; instead of specifying the temperature T_1 at the inner boundary, we have chosen to specify the amount of heat flux F entering the layer. In fact, we can easily verify that the

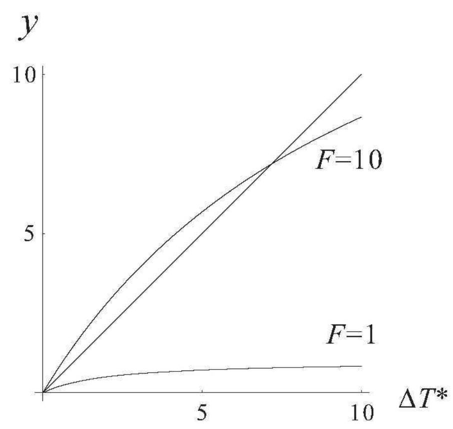

FIG. 4: Bifurcation of solutions (intersections of the graphs $y = g(\Delta T^*)$ and $y = \Delta T^*$). If $g'(0) > 1$, we have the second branch of solutions with $\Delta T^* > 0$. In this graph, parameters are $T_0 = 1$, $a = 2$, $\eta_0 = 1$, and $F = 1,\ 10$.

higher impedance (nonlinear) branch still self-organizes if T_1 is fixed instead of controlling F, but, then, the entropy production rate is reduced from that of the lower impedance (linear) branch. One may invoke the analogy of electric circuit: If a nonlinear impedance (Z) is connected to a constant-current (I) power supply, the entropy production rate is ZI^2, while it is V^2/Z if connected to a constant-voltage (V) power supply.

It is F that brings in the energy that would be eventually channeled into an ordered flow; unless F is large enough (larger than a threshold value), the "heat engine" does not work, i.e, the high temperature-contrast state is not accessible. Since the high temperature-contrast state is the final product of the heat engine, the factor measuring the temperature-contrast, $(1-T_0/T_1)$ scales the strength of the two seemingly contradictory constituent processes –the Carnot efficiency for generating mechanical energy (flow), and the entropy production (or emission). Such a state of affairs could pertain if, for instance, the dissipation mechanisms that create the total entropy were independent of the mechanisms that convert the "free energy" into an ordered flow.

The two opposing mechanisms could, indeed, act independently and simultaneously if the "domains" of their efficient operation were non-overlapping. We propose that a recourse to scale-separation does precisely what is needed : (1) the total entropy production is dominated by small scale perturbations with a large damping rate ($\propto L^{-2}$; L: eddy size) keeping the eddy amplitudes (sacrifice for the dissipation) to be very small, (2) The flow, being a coherent macroscopic structure, is created in the large scale, perhaps, from an instability driven by the entering heat flux F; its creation/characteristics are not affected by the short scale dissipation responsible for entropy production.

APPENDIX A: RELATIVISTIC AND NONRELATIVISTIC 4-DIMENSIONAL REPRESENTATIONS

1. Basic Definitions

Following the standard notation, we write

$$x^\mu = (ct, x, y, z), \quad x_\mu = (ct, -x, -y, -z),$$

and

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \nabla\right), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\nabla\right).$$

The non-relativistic (NR) 4-velocity is (normalizing by c)

$$u^\mu = \frac{dx^\mu}{cdt} = (1, \boldsymbol{v}/c), \quad u_\mu = \frac{dx_\mu}{cdt} = (1, -\boldsymbol{v}/c).$$

The relativistic (R) 4-velocity is defined by the proper-time derivative:

$$U^\mu = \frac{dx^\mu}{ds} = (\gamma, \gamma\boldsymbol{v}/c), \quad U_\mu = \frac{dx_\mu}{ds} = (\gamma, -\gamma\boldsymbol{v}/c),$$

where $ds^2 = dx^\mu dx_\mu$ and $\gamma = 1/\sqrt{1 - v^2/c^2}$. Obviously, $U^\mu U_\mu = 1$.

The NR particle energy-momentum 4-vector is

$$\wp^\mu = (mv^2/(2c), m\boldsymbol{v}).$$

The R particle energy-momentum 4-vector is

$$\mathcal{P}^\mu = mc U^\mu = (E/c, \boldsymbol{p}).$$

where $E = \gamma mc^2$ and $\boldsymbol{p} = \gamma m\boldsymbol{v}$. Obviously, $\mathcal{P}^\mu \mathcal{P}_\mu = m^2c^2$. The classical limit of E is $mc^2 + mv^2/2$, thus $\wp^0 \approx \mathcal{P}^0 - mc^2$.

2. Fluid Energy-Momentum

For a *fluid*, we invoke the enthalpy 4-vector; the NR 4-enthalpy is

$$p^\mu = (H/c, m\boldsymbol{v}),$$

where

$$H = \frac{mv^2}{2} + \phi + \varepsilon + n^{-1}P.$$

The R 4-enthalpy is

$$\mathcal{P}^\mu = (h/c)\, U^\mu = (\gamma h/c, \gamma(h/c^2)\boldsymbol{v}).$$

where

$$h = \varepsilon + n^{-1}P$$

is the proper (static) molar enthalpy. The energy ε includes the rest mass energy mc^2 as well as the thermal energy , so that h/c^2 is the *effective rest mass* of the fluid element. In the NR limit, we may approximate

$$\begin{aligned}\mathcal{P}^0 = \frac{\gamma h}{c} &\approx \frac{1}{c}\left(mc^2 + \frac{mv^2}{2} + \varepsilon + \phi + n^{-1}P\right) \\ &= \frac{1}{c}\left(mc^2 + H\right) = mc + p^0, \end{aligned} \tag{A1}$$

and $\mathcal{P}^j = \gamma(h/c^2)\boldsymbol{v} \approx m\boldsymbol{v} = p^j$.

3. Field Tensor and Equation of Motion

The matter field tensor is, for the NR fluid,

$$M^{\mu\nu} = \partial^\mu p^\nu - \partial^\nu p^\mu,$$

and for the R fluid,

$$\mathcal{M}^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu.$$

The adiabatic fluid equation is written as

$$c\,U_\mu \mathcal{M}^{\mu\nu} + \partial^\nu h - n^{-1}\partial^\nu P = 0. \tag{A2}$$

By the thermodynamic law (5), we may write

$$\partial^\nu h - n^{-1}\partial^\nu P = T\partial^\nu S \tag{A3}$$

with the entropy S. Hence, (A2) may be rewritten as

$$cU_\mu \mathcal{M}^{\mu\nu} + T\partial^\nu S = 0. \tag{A4}$$

Contracting both sides of (A4) with U_ν, we obtain $TU_\nu \partial^\nu S = 0$, implying entropy conservation.

Non-adiabatic effects modify (A4) as

$$cU_\mu \mathcal{M}^{\mu\nu} + T\partial^\nu S = \Theta^\nu \tag{A5}$$

with a heat term such that $\Theta^\nu = -n^{-1}\partial_\mu D^{\mu\nu}$. If a heat flux $\boldsymbol{f}$ occurs in the fluid, we should include $\Theta^0 \equiv \delta Q/c = -n^{-1}\partial_j f^j$ (i.e., we set $D^{,0} = f^j$). Contracting both sides of (A4) with U_ν, we now obtain $TU_\nu \partial^\nu S = \delta Q$.

4. Explicit Form of NR Equations

Let us examine how the tensor equation (A2) reproduces the NR fluid equations. First we assume that the space dimension = 1. Then,

$$M^{\mu\nu} = \begin{pmatrix} 0 & [\partial_t(mv) + \partial_x H]/c \\ -[\partial_t(mv) + \partial_x H]/c & 0 \end{pmatrix},$$

and, thus, the equation of motion (A2) reads as

$$\begin{aligned} cu_\nu M^{\mu\nu} + T\partial^\nu S &= \begin{pmatrix} -v[\partial_t(mv) + \partial_x H]/c \\ -[\partial_t(mv) + \partial_x H] \end{pmatrix} \\ &\quad + \begin{pmatrix} T\partial_t S/c \\ -T\partial_x S \end{pmatrix} = 0 \end{aligned}$$

Plugging the second equation to the first equation (i.e., contracting both sides with U_ν) yields

$$T[\partial_t S + (v \cdot \nabla)S] = 0,$$

which implies the entropy conservation. The heat term $\Theta^0 = \delta Q$ must be added if heat transport occurs.

Plugging $H = mv^2/2 + \phi + h$ into the second equation, we obtain

$$\begin{aligned} m\frac{\partial}{\partial t}v &= -\nabla H + T\nabla S \\ &= -\nabla\left(\frac{mv^2}{2} + \phi\right) - n^{-1}\nabla P. \end{aligned} \tag{A6}$$

Extending the space dimension to 3, we obtain

$$
\begin{aligned}
m\left(\frac{\partial}{\partial t}v + (\nabla\times\boldsymbol{v})\times\boldsymbol{v}\right) &= -\nabla H + T\nabla S \\
&= -\nabla\left(\frac{mv^2}{2}+\phi\right) - n^{-1}\nabla P.
\end{aligned}
\tag{A7}
$$

APPENDIX B: BOUSSINESQ MODEL OF CREATION OF VORTICITY

Here we reveal the relation between the Boussinesq model of buoyancy-driven vorticity generation and the general formulation of the vorticity drive in terms of TdS.

The thermal convection may be described by a fluid equation including gravity:

$$
m\left[\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = \frac{-\nabla P}{n} + \boldsymbol{g},
\tag{B1}
$$

where $\boldsymbol{g} = -\nabla(mGx)$. The static force balance is written as $n_0^{-1}\nabla P_0 = \boldsymbol{g}$. For a sufficiently small variation of temperature $\tilde{T}$ (in a constant pressure process), the change of density $\tilde{n}$ around the static density n_0 may be written as

$$
\tilde{n} = \left(\frac{\partial n}{\partial T}\right)_P \tilde{T} \equiv \alpha n_0 \tilde{T}.
$$

Using this relation, we may approximate the right-hand side of (B1) as

$$
\begin{aligned}
\frac{-\nabla P}{n} + \boldsymbol{g} &= \frac{-\nabla(P_0+\tilde{P})}{n_0+\tilde{n}} + \boldsymbol{g} \\
&\approx \frac{-\nabla\tilde{P}}{n_0} + \frac{\tilde{n}\nabla P_0}{n_0^2} \\
&= \frac{-\nabla\tilde{P}}{n_0} + \frac{\alpha\tilde{T}\nabla P_0}{n_0}.
\end{aligned}
\tag{B2}
$$

Plugging (B2) into (B1) yields the well-known Boussinesq model of buoyancy.

The ambient density n_0 is often approximated to be a constant; then the first term of (B2) is exact, while the second term may be non-exact and may create vorticity. We may rewrite it as the entropy term. By Maxwell's relation

$$
\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{n^2}\left(\frac{\partial n}{\partial T}\right)_P = \frac{\alpha}{n},
$$

we may write

$$
\alpha\frac{\nabla P_0}{n_0} = \nabla S_0,
$$

so the non-exact vorticity generation term reads as

$$\frac{\alpha\tilde{T}\nabla P_0}{n_0} = \tilde{T}\nabla S_0.$$

On the other hand, the exact terms are related as (approximating that n_0 and T_0 are constants)

$$\frac{-\nabla\tilde{P}}{n_0} = -\nabla\tilde{h} + T_0\nabla\tilde{S},$$

proving that (B2) is equivalent to the linearization of $-\nabla h + T\nabla S + \boldsymbol{g}$.

APPENDIX C: FLUID-ELEMENT INTEGRAL AND CONSERVATION LAWS

Denoting by dM the mass element, the total amount of some state variable X (evaluated for a unit mass) is given by $\overline{X} = \int X dM$. Note that this representation uses the Lagrangian frame (dM moves with the fluid). To evaluate the time derivative of $\overline{X}$, it is convenient to use the Eulerian frame. With defining the mass density $\rho = mn$, we may write $dM = \rho d^3x$, where d^3x is the volume element (Lebesgue measure) of the laboratory frame. We observe, using the mass conservation law $\partial\rho/\partial t + \nabla\cdot(\boldsymbol{v}\rho) = 0$ ($\boldsymbol{v}$ is the flow velocity),

$$\begin{aligned} \frac{d}{dt}\overline{X} &= \int_\Omega \frac{\partial}{\partial t}(X\rho)d^3x \\ &= \int_\Omega \left[\frac{\partial}{\partial t}(X\rho) + \nabla\cdot(\boldsymbol{v}X\rho)\right] d^3x - \int_\Omega \nabla\cdot(\boldsymbol{v}X\rho)d^3x \\ &= \int_\Omega \left(\frac{\partial}{\partial t}X + \boldsymbol{v}\cdot\nabla X\right)\rho d^3x - \int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{v})\rho X d^2x \\ &= \int_\Omega \left(\frac{d}{dt}X\right) dM - \int_{\partial\Omega}(\boldsymbol{n}\cdot\boldsymbol{v})\rho X d^2x \end{aligned} \tag{C1}$$

where $\boldsymbol{n}$ is the unit normal vector, directed outward, on the boundary $\partial\Omega$, and $dX/dt = \partial X/\partial t + \boldsymbol{v}\cdot\nabla X$ is the convective (Lagrangian) derivative. In a steady state, $dX/dt = 0$. If we assume that the mass flow is confined in the domain, $(\boldsymbol{n}\cdot\boldsymbol{v})\rho X$ must vanish on the boundary. Then, we obtain $d\overline{X}/dt = 0$, the constancy of the specific X for every mass element of the fluid.

---

[1] In this lecture, we denote by dX the variation of a state variable X, which is an exact differential form on the parameter space. General variation is denoted by δY, which may not be a variation of some variable Y, but only variation δY may be describable.

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