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ICTP Latin-American Basic Course on FPGA Design for Scientific Instrumentation

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Introduction to Digital Design

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Introduction to Digital Design

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Outline

- Digital CMOS design
- Arithmetic operators
- Sequential functions

Outline

- Digital CMOS design
  - Boolean algebra
  - Basic digital CMOS gates
  - Combinational and sequential circuits
  - Coding - Representation of numbers

Boolean Algebra

- English mathematician 1815 - 1864

  1854 : *Introduction to the Laws of Thought*

Boolean Algebra

- Let $B = \{0 , 1\}$ — B is called the Boolean set; $0, 1$ are the Boolean constants

- Let $x \in B$ — x is a Boolean variable

Boolean Algebra

- Unary functions : $B \longrightarrow B$

Unary function *0* : $\forall x \in B, \quad x \longmapsto 0$

Unary function *1* : $\forall x \in B, \quad x \longmapsto 1$

Unary function *Identity* : $\forall x \in B, \quad x \longmapsto x$

Unary function *Not* :

$$0 \longmapsto 1$$
$$1 \longmapsto 0$$

Not (x) is denoted $\overline{x}$

Boolean Algebra

- Binary functions : $B^2 \longrightarrow B$

function *And* :

$\forall$ x, y $\in$ B, *And* (x, y) = 1 if and only if x = 1 and y = 1

And (x, y) is also called *Min* is denoted x.y

function *Or* :

$\forall$ x, y $\in$ B, *Or* (x, y) = 0 if and only if x = 0 and y = 0

Or (x, y) is also called *Max* is denoted x+y

Boolean Algebra

- Other binary functions can be defined using *And*, *Or* and *Not*

function *Nand* : $Nand(x, y) = Not(And(x, y))$

function *Nor* : $Nor(x, y) = Not(Or(x, y))$

function *Xor* : $Xor(x, y) = x.\overline{y} + \overline{x}.y$

Xor (x, y) is denoted $x \oplus y$

Boolean Algebra

- Noticeable properties

Not (*Not* (x)) = x $\quad \overline{\overline{x}} = x$

$x.x = x$	$x+x = x$	$x \oplus x = 0$
$x.\overline{x} = 0$	$x+\overline{x} = 1$	$x \oplus \overline{x} = 1$
$x.0 = 0$	$x+1 = 1$	$x \oplus 1 = \overline{x}$
$x.1 = x$	$x+0 = x$	$x \oplus 0 = x$

Boolean Algebra

- Noticeable properties

Commutative	$x.y$	$= y.x$
	$x+y$	$= y+x$
	$x \oplus y$	$= y \oplus x$
Associative	$x.(y.z)$	$= (x.y).z$
	$x+(y+z)$	$= (x+y)+z$
	$x \oplus (y \oplus z)$	$= (x \oplus y) \oplus z$

Boolean Algebra

- Noticeable properties

Distributive	$x \cdot (y+z)$	$= x.y + x.z$
	$x \cdot (y \oplus z)$	$= x.y \oplus x.z$
	$x + (y.z)$	$= (x+y) \cdot (x+z)$
De Morgan	$\overline{x.y}$	$= \overline{x} + \overline{y}$
	$\overline{x+y}$	$= \overline{x} \cdot \overline{y}$
Absorption	$\overline{x}.y + x$	$= y + x$

Boolean Algebra

- Let $B = \{0 , 1\}$ — B is called the Boolean set; $0, 1$ are the Boolean constants
- Let $x \in B$ — x is a Boolean variable
- Let $v \in B^n$ — v is a Boolean vector

Boolean Algebra

- $v \in B^n$, $v = (x_1, \ldots, x_i, \ldots, x_n)$
  $u \in B^n$, $u = (y_1, \ldots, y_i, \ldots, y_n)$

The number of Boolean variables that are different between v and u is called the Hamming distance (v, u)

$$Hd\ (\ (0,0,0,1)\ ,\ (1,0,1,0)\ ) = 3$$

Boolean Algebra

To vectors are said adjacent when their Hamming distance = 1

Hd ((0,0,0,1) , (1,0,0,1)) = 1

Boolean Algebra

- Let $B = \{0, 1\}$ — B is called the Boolean set; $0, 1$ are the Boolean constants
- Let $x \in B$ — x is a Boolean variable
- Let $v \in B^n$ — v is a Boolean vector
- Let $f : B^n \longrightarrow B$ — f is a Boolean function
- $\mathbf{B}_n$ is the set of Boolean Functions

$$card(\mathbf{B}_n) = 2^{(2^n)}$$

Boolean Algebra

- *Card* (B^n) is finite

A Boolean function f may be defined by giving the value $f(v)$ of each Boolean vector v (Truth table)

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Boolean Algebra

- Unary functions : $\mathbf{B}_n \longrightarrow \mathbf{B}_n$

  function *Not* : $(Not\ (f))\ (v) = Not\ (f\ (v))$

- Binary functions : $\mathbf{B}_n{}^2 \longrightarrow \mathbf{B}_n$

  function *And* : $(And\ (f, g))\ (v) = And\ (f\ (v), g\ (v))$

  function *Or* : $(Or\ (f, g))\ (v) = Or\ (f\ (v), g\ (v))$

Boolean Algebra

- $\forall v \in B^n$, $v = (x_1, \dots, x_i, \dots, x_n)$

  The Boolean function $f \in \mathbf{B}_n$ /

  $f(v) = x_i$ is denoted x_i

Boolean Algebra

- A Boolean function f may be defined by giving a Boolean expression

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

$$f = \overline{x}.y.z + x.\overline{y}.\overline{z} + x.z$$

$$f = x.\overline{y} + y.z$$

There is not a unique expression

Boolean Algebra

- Let $f \in \mathbf{B}_n$

$$f = \sum (\alpha_j . \prod \tilde{x}_i)$$

$$f = \bar{x}.y.z + x.\bar{y}.\bar{z} + x.\bar{y}.z + x.y.z$$

min-term

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Boolean Algebra

- Let $f \in \mathbf{B}_n$

$$f = \prod (\beta_j + \sum \tilde{x}_i)$$

$$f = (x+y+z) \,.\, (x+y+\overline{z}) \,.\, (x+\overline{y}+z) \,.\, (\overline{x}+\overline{y}+z)$$

max-term

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Boolean Algebra

- Let $f \in \mathbf{B}_n$ 　 f is said independent from the variable x_i

$$\forall v \in B^n, v = (x_1, \ldots, x_i, \ldots, x_n)$$

$$f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, \overline{x_i}, \ldots, x_n)$$

Boolean Algebra

- Let $f \in \mathbf{B}_n$

  $\exists!\ f_{i0}\ ,\ f_{i1}$ independent from the variable x_i

  $f = x_i \,.\, f_{i1} + \overline{x_i} \,.\, f_{i0}$

  Shannon decomposition

Boolean Algebra

- Let $f \in \mathbf{B}_n$

$$f = x_i \cdot f_{i1} + \overline{x_i} \cdot f_{i0}$$

$$f = x.(\overline{y}+z) + \overline{x}.(y.z)$$

x	y	z	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Boolean Algebra

- Let $f = x_i \cdot f_{i1} + \overline{x_i} \cdot f_{i0}$

  if f is independent from the variable x_i $f = f_{i0} = f_{i1}$

  $$f_{i0} \oplus f_{i1} = 0$$

  if $f_{i0} \oplus f_{i1} = 0$ then f is insensitive to x_i

  notion of derivative

Boolean Algebra

- Let $f = x_i \cdot f_{i1} + \overline{x_i} \cdot f_{i0}$

$$\frac{\partial f}{\partial x_i} = f_{i0} \oplus f_{i1}$$

Boolean Algebra

- Let $f = x_i \cdot f_{i1} + \overline{x_i} \cdot f_{i0}$

  f may be sensitive to x_i in two ways

$$\frac{\partial f}{\partial x_i} = f_{i1} \cdot \overline{f_{i0}} + \overline{f_{i1}} \cdot f_{i0}$$

$f_{i1} \cdot \overline{f_{i0}}$ and $\overline{f_{i1}} \cdot f_{i0}$ cannot be 1 for the same vector

Boolean Algebra

- $f = x_i \cdot f_{i1} + \overline{x_i} \cdot f_{i0}$ $\qquad \dfrac{\partial f}{\partial x_i} = f_{i1} \cdot \overline{f_{i0}} + \overline{f_{i1}} \cdot f_{i0}$

if $f_{i1} \cdot \overline{f_{i0}}\,(v) = 1$, f varies in a direct way with x_i
f is a positive function of x_i

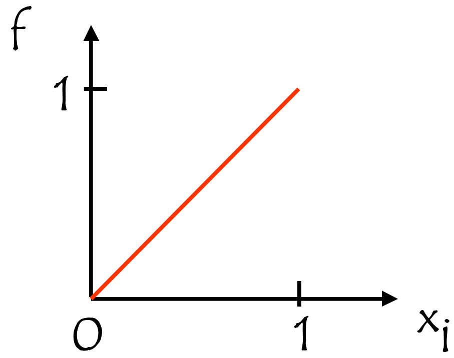
if $\overline{f_{i1}} \cdot f_{i0}\,(v) = 1$, f varies in an opposite way with x_i
f is a negative function of x_i

$$\frac{\partial f^{+}}{\partial x_i} = f_{i1} \cdot \overline{f_{i0}}$$

$$\frac{\partial f^{-}}{\partial x_i} = \overline{f_{i1}} \cdot f_{i0}$$

Boolean Algebra

- $$\frac{\partial f^{+}}{\partial x_i} = f_{i1} . \overline{f_{i0}}$$



- $$\frac{\partial f^{-}}{\partial x_i} = \overline{f_{i1}} . f_{i0}$$