The Abdus Salam
International Centre for Theoretical Physics

United Nations
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2223-2

Winter College on Optics in Imaging Science

31 January - 11 February, 2011

Fourier and Fresnel transforms with some sampling theory

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Feb 2011

Fourier transform

In this section we would like to introduce the Fourier transform before moving onto to its use in describing optical systems. To use Fourier theory to analyze optical systems, we will need to make use of lenses and the by-now familiar Fresnel transform. Fourier theory also provides a very elegant and compact means for understanding (quantifying) the effect of sampling a signal and the resulting loss of information present in the digitized signal. Many text books cover this topic, see for example [1, 2], in far more detail than we can in this lecture course. So we here restrict ourselves to the important aspects of Fourier theory that are relevant to our discussion. Let's begin with the 1-D definition of the Fourier transform and its inverse, which we define as

$$\begin{aligned} U(v) &= FT\{u(x)\}(v), \\ &= \int_{-\infty}^{\infty} u(x) \exp(-j2\pi xv)\, dx. \end{aligned} \tag{1}$$

$$\begin{aligned} u(x) &= IFT\{U(v)\}(x), \\ &= \int_{-\infty}^{\infty} U(v) \exp(j2\pi xv)\, dv. \end{aligned} \tag{2}$$

There are a series of conditions on whether the Fourier transform of a signal actually exists. I have taken the following more or less verbatim from Bracewell [1]

1. *The integral of* $|u(x)|$ *from* $-\infty$ *to* ∞ *exists* - this is equivalent to saying that the function $u(x)$ should have a finite amount of power or energy.

2. *The discontinuities in* $|u(x)|$ *are finite*,

Other important functions such as the Dirac delta function $\delta(x)$ or the $unitstep(x)$ function strictly speaking do not have well-defined Fourier transforms. Similarly with periodic functions since they violate condition 1 above. However since these are so useful for analysis and can be related to physical processes, we bend rules slightly defining special transforms for these important functions. Again we will not examine these interesting fundamental issues referring the reader to Chap 1 and 5 Ref. [1] as well as [3] for a more thorough discussion of these points.

Some examples

Here we wish to provide some examples of Fourier transforms to provide the reader with some insight into the characteristics of the transform. We start be considering the aperture function

$$p_D(x) = \begin{cases} 1, & \text{when } |x| < \text{Ł} \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

Setting $u(x) = p_D(x)$, and subbing into Eq. (1), we arrive at

$$\begin{aligned} P(v) &= \int_{-\infty}^{\infty} p_D(x) \exp(-j2\pi xv)\, dx, \\ &= \int_{-L}^{L} 1 \exp(-j2\pi xv)\, dx, \end{aligned}$$

$$= \frac{\sin(2\pi L v)}{\pi v}. \tag{4}$$

Often this last relationship is written as

$$P(v) \;=\; 2L\text{sinc}(2Lv) \tag{5}$$

where the function $\text{sinc}(x) = \sin(\pi x)/x$. Lets look at another example, this time setting $u(x) = \cos(2\pi f_{\mathrm{x}} x)$. Making the appropriate substitution we arrive at

$$\begin{aligned}
U(v) &= \int_{-\infty}^{\infty} \cos(2\pi f_{\mathrm{x}} x) \exp(-j2\pi x v)\, dx, \\
&= \int_{-\infty}^{\infty} \left[\frac{\exp(j2\pi f_{\mathrm{x}} x) + \exp(-j2\pi f_{\mathrm{x}} x))}{2} \right] \exp(-j2\pi x v)\, dx, \\
&= \frac{1}{2} \left[\delta(v - f_{\mathrm{x}}) + \delta(v + f_{\mathrm{x}}) \right], \qquad (6)
\end{aligned}$$

One may note that to get the result in Eq. (6), I made use of the a fundamental Fourier transform pair - $FT\left\{\exp(j2\pi f_{\mathrm{x}} x\right\}(v) = \delta(v - f_{\mathrm{x}})$. In the Appendix to this lecture I have attached a list of some useful Fourier transform pairs and important Fourier transform properties. I would like to look at specific property in more detail namely one of the shifting property relationships for Fourier transforms. We know from Eq. (1) that $U(v) = FT\left\{u(x)\right\}(v)$, let us now calculate

$$\begin{aligned}
FT\left\{u(x)\exp(j2\pi\alpha x)\right\}(v) &= \int_{-\infty}^{\infty} u(x) \exp(j2\pi\alpha x) \exp(-j2\pi x v)\, dx, \\
&= \int_{-\infty}^{\infty} u(x) \exp\left[-j2\pi x (v - \alpha)\right] dx.
\end{aligned} \tag{7}$$

Making the following substitution , i.e. $v' \to v - \alpha$ means that Eq. (7) can be re-written as

$$FT\{u(x)\}(v') = \int_{-\infty}^{\infty} u(x) \exp[-j2\pi x(v')]\, dx, \tag{8}$$

hence we have the following result

$$FT\{u(x)\exp(j2\pi\alpha x)\}(v) = U(v - \alpha). \tag{9}$$

Therefore the FT of some function multiplied by a linear phase term, $\exp(j2\pi\alpha x)$ results in an identical, albeit shifted, Fourier transform. We will make use of this result later when we discuss sampling theory in more detail.

Convolution

Lets look at another important operation: convolution. The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$\begin{aligned} conv(x) &= f(x) * g(x) \\ &= \int_{-\infty}^{\infty} f(u)g(x-u)du. \end{aligned} \tag{10}$$

Examining Eq. (10) we see that to calculate a particular value of conv(x'), we set $x = x'$ in Eq. (10) and integrate (sum) the product of $f(u)g(x' - u)$. As we vary x' we see that we are sliding g relative f, multiplying and integrating. Again I refer the interested reader to Chap 3 of Ref [1] for a thorough discussion of this topic together with helpful graphical depictions of the convolution process. However to provide some insight into the process we now examine what happens when we convolve the following two functions:

$$f(x) = \cos(2\pi f_{x1}x) + \cos(2\pi f_{x2}x),$$

$$g(x) = \mathrm{sinc}(2Lf1x), \tag{11}$$

using the Mathematica Reader file entitled: 'convolution example.nbp'. Examining this distribution we see that we can change the output by varying the different parameters: $f_{x2}, L, alpha$. For certain combinations of these variables, e.g. when $L = 1, f_{x2} = 2$, we see that the presence of the f_{x2} component has been removed. How can we explain this? At this point it is handy to make use of the following property of the Fourier transform (property is taken from the Appendix)

$$\begin{aligned} FT\left\{f(x) * g(x)\right\}(v) &= F(v)G(v), \\ f(x) * g(x) &= IFT\left\{F(v)G(v)\right\}(x) \end{aligned} \tag{12}$$

Making use of the Fourier transform pairs in the Appendix, we note that $f(x)$ maps to $\frac{1}{2}\left[\delta(v - f_{x1}) + \delta(v + f_{x1} + \delta(v - f_{x2}) + \delta(v + f_{x2})\right]$, while $g(x)$ maps to $p_{f_{x1}L}(v)$. Anything outside this aperture will be set to zero and hence the convolution operation can be viewed as a filtering operation. We shall return to this later in this lecture when we consider optical systems that perform such a filtering operation.

Sampling

Here we examine the effect of sampling a signal. What does sampling actually do? The first thing to note is that a continuos signal, $f(x)$ returns a value for every input value of x. If we wish to represent this signal digitally, we must represent the signal with a finite number of values, usually we take a finite number of samples of the signal at uniformly spaced distances. We also wish to ensure that we still have a reasonably accurate representation of our signal, how can we be sure that we have done so?

To represent the effect of the sampling operation we introduce the comb function,

$$\delta_T(x) \quad = \quad \sum_{n=-\infty}^{\infty} \delta(x - nT) \tag{13}$$

where T is the space between adjacent samples. To sample the function $f(x)$, we perform the following operation

$$f(nT) = \delta_T(x) f(x). \tag{14}$$

Another important relationship is that $\delta_T(x)$ can also be expressed as

$$\delta_T(x) \quad = \quad \sum_{n=-\infty}^{\infty} \exp\left(j2\pi\frac{n}{T}x\right). \tag{15}$$

Using Eq. (14) and (15) and subbing into Eq. (1) we calculate the Fourier transform of our sampled distribution

$$F_S(v) \quad = \quad \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} \exp\left(j2\pi\frac{n}{T}x\right) f(x)\right] \exp\left(-j2\pi xv\right) dx. \tag{16}$$

where we have included a subscript S to indicate that we are calculated the continuos distribution $F_S(v)$ from a set of discretely sampled points. Taking the sum outside the integral in Eq. (17) and using the shifting theory that we saw earlier, i.e. Eq. (9), we can rewrite Eq. (FFS.16) as

$$F_S(v) \quad = \quad \sum_{n=-\infty}^{\infty} F\left(v - \frac{n}{T}\right), \tag{17}$$

where $F(v) = FT\left\{f(x)\right\}(v)$ and so we see that the Fourier transform of our sampled function consists of an infinite number of shifted copies of the original Fourier distribution. We now explore some of this implications using the file entitled: 'Sampling+Example.nbp'.

Fresnel transform and sampling

The following paragraph is taken from Ref. [4]. We now turn our attention to the sampling of a diffracted Fresnel field. Let our original field be called $u(X)$ and the field in the Fresnel domain $u_z(x)$.

$$\begin{aligned} u^s(X) &= \frac{1}{T\sqrt{-j\lambda z}} \int u_z(x)\delta_T(x)\exp[\frac{-j\pi}{\lambda z}(X-x)^2]dx \\ &= \frac{1}{T\sqrt{-j\lambda z}} \sum_{n=-\infty}^{\infty} \int u_z(x)\exp(j2\pi nx/T)\exp[\frac{-j\pi}{\lambda z}(X-x)^2]dx \end{aligned} \tag{18}$$

We also note the shifting property [5, 4] of the Fresnel transform (for an arbitrary linear phase ξ), for some analytical signal $f(X)$,

$$\chi_z\left\{f(X)\exp(j2\pi\xi X)\right\}(x) = \exp\left(\frac{-j\pi\xi^2}{\lambda z}\right)\exp\left(j2\pi x\xi\right)\chi_z\left\{f(X)\right\}(x-\xi\lambda z) \tag{19}$$

Combining the results from Eq. (18) and Eq. (19) we arrive at

$$\begin{aligned} u^s(X) &= \frac{1}{T\sqrt{-j\lambda z}} \sum_{n=-\infty}^{\infty} \int u_z(x)\exp(j2\pi nx/T)\exp[\frac{-j\pi}{\lambda z}(X-x)^2]dx \\ u^s(X) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \chi_{-z}\left\{u_z(x)\exp\left[j2\pi\left(\frac{n}{T}\right)x\right]\right\}(X) \\ u^s(X) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp\left[\frac{-j\pi\left(n/T\right)^2}{\lambda z}\right]\exp\left(j2\pi Xn/T\right)\chi_{-z}\left\{u_z(x)\right\}\left(X-\frac{n\lambda z}{T}\right). \end{aligned} \tag{20}$$

Thus from Eq. (20) we can relate $u^s(X)$ to the actual field $u(X)$. The sampling process however causes differences between the actual signal and our approximation to it. We note several points in relation to this: (i) the sampling process creates an infinite number of replicas in the object plane, (ii) the centers of adjacent replicas are separated by a distance $\lambda z/T$, (iii) each of the replicas is also multiplied by a different linear phase as well as some unimportant constant phase factor.

If we impose the constraint that our object (field) has a finite support Δ in the object plane then this field can be imaged without overlapping replicas provided that $T \leq (\lambda z)/\Delta$. This important result is known [**?**, 6, **?**, 7, 8, 9, 10, 11] and means that under certain conditions it is possible to sample the diffracted field at rates below the Nyquist limit and to recover through a generalized interpolation formula super-Nyquist frequencies. In Ref. [12] some implications of this result are explored in more detail using a simple analytical example. As we shall see however the effect of the finite pixel size and camera extent impose resolutions limits in addition to this constraint.

Optical Fourier Transform

The analysis presented in this section is taken from Goodman [2]. Consider the optical system depicted in Fig. 1(a). A normally incident unit amplitude plane wave is incident on a transparency, $t(x)$, that is placed directly in front of a converging lens. The field immediately before the lens can be described as

$$U_l(x) \quad = \quad t(x). \tag{21}$$

We can account for the effect of the thin lens using our earlier representation, i.e. we describe the effect of the lens with the following expression:

$$\mathrm{e}^{\frac{-j\pi}{\lambda f}x^2}, \tag{22}$$

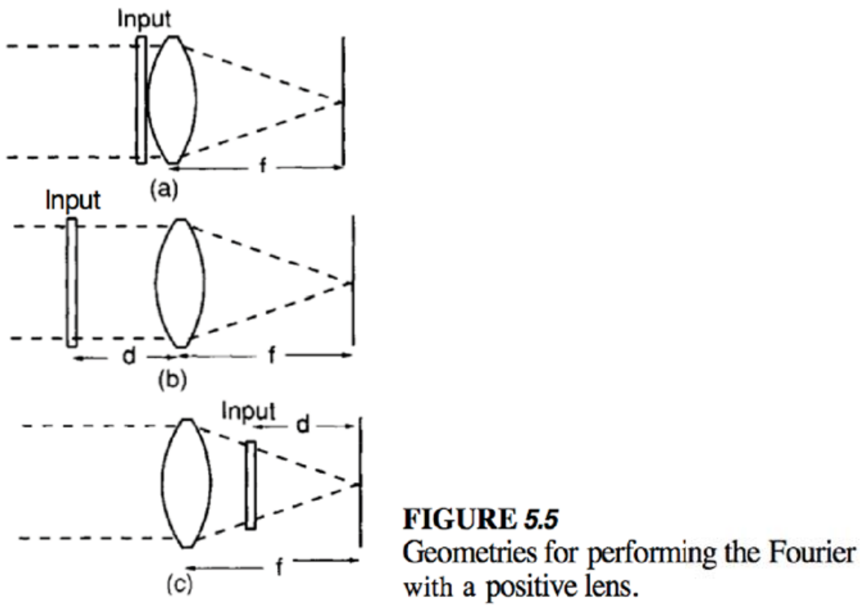

Figure 1: Three OFT systems taken from Ref. [2]

and write the field after the lens as

$$U_l'(x) \quad = \quad t\,(x)\,p(x)\mathrm{e}^{\frac{-j\pi}{\lambda f}x^2}, \tag{23}$$

where we account for the finite extent of the thin lens using the aperture function $p(x)$, such that

$$p(x) = \begin{cases} 1, & \text{inside the aperture} \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

We now calculate the distribution a distance z away from the lens using the Fresnel transform

$$U_f(u) \quad = \quad \frac{1}{\sqrt{j\lambda z}} \int_{\infty}^{\infty} U_l'(x) \exp\left[\frac{j\pi}{\lambda z}(u-x)^2\right] dx. \tag{25}$$

Substituting Eq. (23) into 25 and expanding out the $\exp(\bullet)$ term gives

$$U_f(u) \quad = \quad \frac{\exp\left(\frac{j\pi u^2}{\lambda z}\right)}{\sqrt{j\lambda z}} \int_{\infty}^{\infty} p(x)t(x) \exp\left(\frac{-2j\pi x u}{\lambda z}\right) \exp\left[\frac{j\pi x^2}{\lambda}\left(\frac{1}{z} - \frac{1}{f}\right)\right] dx. \tag{26}$$

Assuming an aperture of infinite extent, we note that in the back focal plane of the lens, i.e. when $z = f$ then Eq. (26) reduces to

$$U_f(u) \quad = \quad \frac{\exp\left(\frac{j\pi u^2}{\lambda f}\right)}{\sqrt{j\lambda f}} \int_{\infty}^{\infty} t(x) \exp\left(\frac{-2j\pi x u}{\lambda f}\right) dx, \tag{27}$$

which apart from a leading quadratic phase is a scaled Fourier transform of $t(x)$ where $f_x = u/(\lambda f)$. The leading quadratic phase factor means that Eq. (27) is not a true Fourier transform of the function $t(x)$ however since it is usually the intensity that is measured in this instance the phase distribution does not effect the observable result.

Before we turn our attention to the scene depicted in Fig. 1(b), let us first consider the Fresnel transform from another point of view. From Ref. [1] we note that the convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) *_u g(x) \quad = \quad \int_{\infty}^{\infty} f(x)g(u-x)dx. \tag{28}$$

Comparing with Eq. (25) we see that the Fresnel transform can be described as a convolution, i.e.

$$U_f(u) \quad = \quad \frac{1}{\sqrt{j\lambda z}} U_l'(x) *_u \exp\left(\frac{j\pi x^2}{\lambda z}\right). \tag{29}$$

One more trick, we note that the Fourier transform of $\exp\left(\frac{j\pi x^2}{\lambda z}\right)$ is $\exp\left(-j\pi\lambda z u^2\right)$. Hence we can interpret the Fresnel transform in the Fourier domain. Let us make use of this new information to analyze the situation depicted in Fig. 1(b). Again a normally incident unit amplitude plane wave is incident on a transparency, $t(x)$, located a distance d from our converging focal lens. Let us define

$$F_0(f_x) = FT\left\{t(x)\right\}(f_x), \tag{30}$$

where FT is the Fourier transform operator (i.e. $FT\left\{r(x)\right\}(f_x) = \int_{-\infty}^{\infty} r(x)\exp(-j2\pi f_x x)dx$). Making use of the convolution property of the FT we note that the FT of the field incident on the lens, $F_l(f_x)$ may be written as

$$F_l(f_x) = F_0(f_x)\exp(-j\pi\lambda d f_x^2), \tag{31}$$

where d is the distance between the transparency and the lens surface. To ease analysis we assume that the lens is effectively infinite in extent and thus write the field in the back focal plane as

$$U_f(u) = \frac{\exp\left(\frac{j\pi}{\lambda f}u^2\right)}{\sqrt{j\lambda f}} F_l\left(\frac{u}{\lambda f}\right). \tag{32}$$

Subbing Eq. (31) into Eq. (32)

$$U_f(u) = \frac{\exp\left[\frac{j\pi}{\lambda f}\left(1-\frac{d}{f}\right)u^2\right]}{\sqrt{j\lambda f}} F_0\left(\frac{u}{\lambda f}\right), \tag{33}$$

which can be rewritten as

$$U_f(u) = \frac{\exp\left[\frac{j\pi}{\lambda f}\left(1-\frac{d}{f}\right)u^2\right]}{\sqrt{j\lambda f}} \int_{\infty}^{\infty} t(x)\exp\left(\frac{-j2\pi}{\lambda f}xu\right)dx \tag{34}$$

So we can see that at the back focal plane of the lens again we have a scaled FT relationship between the input transparency, $t(x)$ and $U_f(u)$. Again we note the presence of a leading quadratic phase factor. For the special case when $d = f$ we see that this factor disappears yielding a true FT relationship between input and output planes.

4-f imaging system

We are now in a position to analyze a commonly used optical system, called the 4-f imaging system, as depicted in Fig. 2. For simplicity we assume this system is a unit-magnificaiton system where both lenses have focal length f. From the previous section, see Eq. (34), we note that for the special case when $d = f$ we have a Fourier transform relation. Suppose that our input field is

$$u(X) \quad = \quad p_{D_I} \left[\cos(2\pi f_{x1} x) + \cos(2\pi f_{x2} x) \right]. \tag{35}$$

We note that the input function consists of two spatial frequency components whose spatial extent in the input plane is limited by an aperture function of size $\pm D_I$. From the theoretical results discussed thus far we are now in a position to have a go at analyzing the behavior of this system. Lets look at the Fourier plane first. Each cosine functions (associated with frequencies $f_x 1$ and $f_x 2$) will map to two Dirac delta functions located at $u = \pm f_x 1/(\lambda f)$ and $u = \pm f_x 2/(\lambda f)$ respectively. What is the effect of the aperture function? From Eq. (3), we expect it will be mapped to a sinc function and remembering to scale our output variable, since we are analyzing an optical system, we expect the following result: $P_{D_I}(u) = K \mathrm{sinc}(2 D_I u/(\lambda f))$, where K is a complex scaling constant. We also remember that a multiplication operation in one domain is a convolution operation in its Fourier domain and hence we expect that each of the Dirac delta functions will be broadened, leading to 4 spatially separated sinc functions in the Fourier plane.

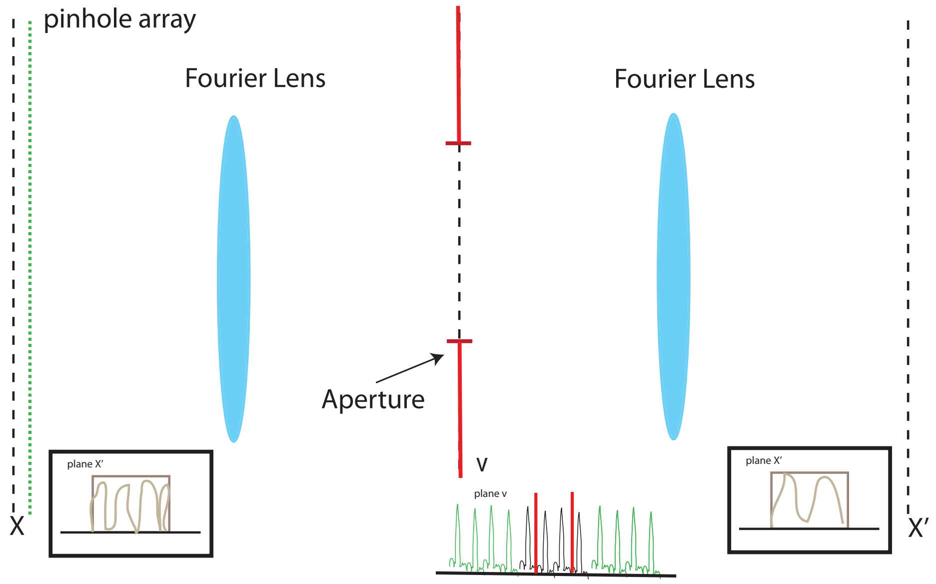

Figure 2: 4-f imaging system, with a limiting aperture in the Fourier plane. In Fourier plane distribution red lines indicate location of aperture, the green distributions represent the replicas that arise when a pinhole array is placed over the input function. The spatial separation is given by $\lambda f/T$. Finally the distriubution in output/image plane only contains the lower spatial frequency component.

$$U(u) \quad = \quad P_{D_I}(u - f_{x2}) + P_{D_I}(u - f_{x1}) + P_{D_I}(u + f_{x1}) + P_{D_I}(u + f_{x2}) \tag{36}$$

This distribution is now incident on the Fourier plane aperture, see Fig. 2, and its extent is limited to the range $-D_{FP} < u < D_{FP}$. In the particular situation depicted in Fig. 2 we see that the Fourier plane filtering operation acts to largely remove the contribution of the f_{x2} frequency component. This truncated FP distribution is then subject to a second scaled FT operation to yield the output image. We would expect from Fig. 2 that the output image will not contain contributions from the f_{x2} frequency component. We have seen this type of filtering operation earlier in our section on *Convolution* and explored some results with the Mathematica file: ‘convolution example.nbp’. We now wish to consider the effect of sampling. In order to model this effect we introduce a regularly spaced pinhole grating (see Fig. 2), where the spacing between the holes is again given by T. Thus our input function is modified according to the following expression:

$$\begin{aligned} u_S(nT) &= u(X)\delta_T(X) \\ &= \sum_{n=-\infty}^{\infty} u(X)\exp\left(j2\pi\frac{n}{T}X\right). \end{aligned} \tag{37}$$

Again invoking the shift operator we see the resulting distribution in the Fourier plane is given by

$$U_S(u) = \sum_{n=-\infty}^{\infty} U(u - \frac{n}{T}) \tag{38}$$

When $f_{x2} > 1/(2T)$ the sinc function associated with this frequency moves outside the sampling bandwidth, while at the same time higher order replicas move into the central sampling order, masquerading as lower frequencies.

Fresnel based DH systems

Our optical system here is depicted in Fig. 3. Again we assume that a function similar to Eq. (35), is input to our system. This field propagates to the camera plane where its extent is limited by the finite support of the camera aperture, and the field is sampled (we ignore the filtering effect of the finite size pixels [4]). The reconstruction is now performed numerically by a computer. We can represent these two operations graphically, (i) introduction of a pinhole array, and (ii) numerical propagation of the complex amplitude.

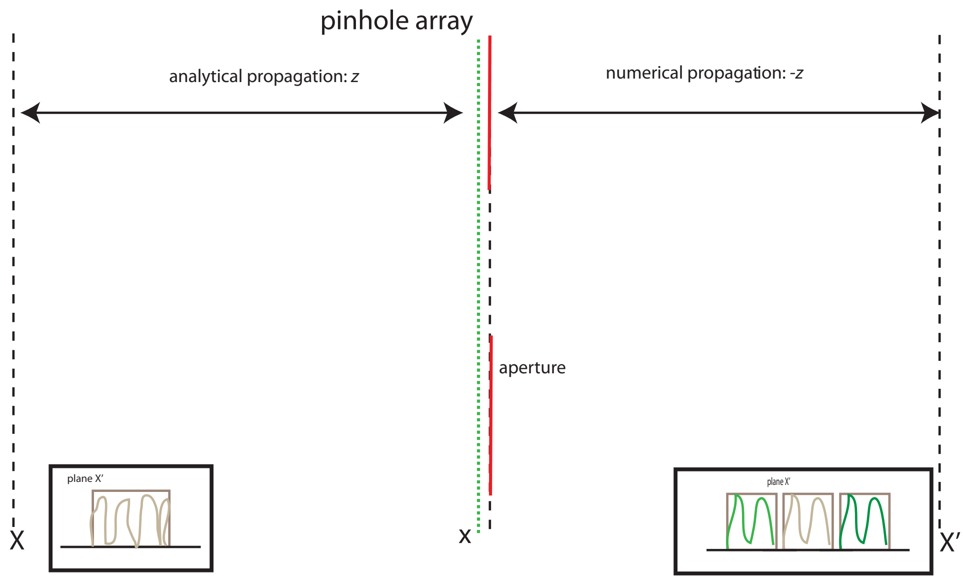

Figure 3: A Fresnel based digital holographic system. The input field propagates to the camera plane where the higher spatial frequency component is again filtered out by the finite camera aperture. The second half of the optical system consists of numerical back-propagation of the complex wavefield. The sampling operation is accounted for with the pinhole array and generates an infinte series of replicas spaced a distance $\lambda z/T$. Finally the distriubution in output/image plane only contains the lower spatial frequency component.

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Fourier transform pairs and properties[1]

[1] I am grateful to John Sheridan for providing this Appendix

Functions

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1/2 \end{cases}$$

$$\sin\mathrm{c}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

$$\mathrm{tri}(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{+\infty} \delta(x - n)$$

$$circ(\sqrt{x^2 + y^2}) = \begin{cases} 1 & \sqrt{x^2 + y^2} \leq 1 \\ 0 & \sqrt{x^2 + y^2} > 1 \end{cases}$$

Q : Sketch these functions.

Table of 2-D Functions and their FTs

Function Name	$f(x,y)$ *separable form*	$F(u,v) = F(f_x, f_y)$ *separable form*
rectangular	$\mathrm{rect}(x,y)$ $= \mathrm{rect}(x)\mathrm{rect}(y)$	$\sin\mathrm{c}(u,v)$ $= \mathrm{sinc}(u)\mathrm{sinc}(v)$
triangular	$\mathrm{tri}(x,y)$ $= \mathrm{tri(x)tri(y)}$	$\sin\mathrm{c}^2(u,v)$ $= \mathrm{sinc}^2(u)\mathrm{sinc}^2(v)$

sinc	$\sin c^2(x,y)$	$\mathrm{tri}(u,v)$
Gaussian	$\exp\left[-\pi\left(x^2+y^2\right)\right]$	$\exp\left[-\pi\left(u^2+v^2\right)\right]$
unity	1	$\delta(u,v)$
shifted delta	$\delta(x-a,y-b)$ $=\boldsymbol{\delta}(x\text{-}a)\boldsymbol{\delta}(y\text{-}b)$	$\exp\left[-j2\pi\left(af_x+bf_y\right)\right]$
expon- -ential	$\exp\left[+j2\pi\left(ax+by\right)\right]$	$\delta(f_x-a,f_y-b)$
cosine	$\cos\left[2\pi(ax+by)\right]$	$\left[\delta(f_x+a,f_y+b)+\right.$ $\left.\delta(f_x+a,f_y+b)\right]/2$
sine	$\sin\left[2\pi(ax+by)\right]$	$\left[\delta(f_x+a,f_y+b)-\right.$ $\left.\delta(f_x+a,f_y+b)\right]/2j$
comb	$\mathrm{comb}(x,y)$ $=\mathrm{comb}(x)\mathrm{comb}(y)$	$\mathrm{comb}(f_x,f_y)$
flip	$f(-x,y)$	$F(-f_x,f_y)$
invert	$f(-x,-y)$	$F(-f_x,-f_y)$
conjugate	$f^*(x,y)$	$F^*(-f_x,-f_y)$
invert and conjugate	$f^*(-x,-y)$	$F^*(f_x,f_y)$

Properties of the 2-D FT

1. Linearity : $\Im\{\alpha f(x,y) + \beta g(x,y)\} = \alpha F(u,v) + \beta G(u,v)$

2. Similarity (Scaling) : $\Im\{f(\alpha x, \beta y)\} = \frac{1}{|\alpha\beta|} F(\frac{u}{\alpha}, \frac{v}{\beta})$

3. Shift : $\Im\{f(x-\alpha, y-\beta)\} = F(u,v)\exp[-j2\pi(\alpha u + \beta v)]$

4. Modulation :

$$\Im\{f(x,y)\exp[+j2\pi(\alpha x + \beta y)]\} = F(u-\alpha, v-\beta)$$

5. Convolution : $\Im\{f(x,y) * g(x,y)\} = F(u,v) \times G(u,v)$

6. (Cross-)Correlation :

$$\Im\{f(x,y) \otimes g(x,y)\} = F(u,v) \times G^*(u,v)$$

7. Autocorrelation : $\Im\{f(x,y) \otimes f(x,y)\} = |F(u,v)|^2$

8. Duality :

IF $\Im\{f(x,y)\} = F(u,v)$ **THEN** $\Im\{F(x,y)\} = f(-u,-v)$

9. Conservation : $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x,y)|^2 dxdy = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |F(u,v)|^2 dudv$

NOTE : Separable Functions :

If $f(x,y) = f_1(x) f_2(y)$

Then $\Im\{f(x,y)\} = \Im\{f(x)\}\Im\{f(y)\} = F_1(u)F_2(v) = F(u,v)$.