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Quantum field theory for mathematicians

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QFT for Mathematicians

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1) Intro
2) Quantum mechanics
3) QFT via funct. int.
4) Partition functions

What this is not:

A mathematically rigorous treatment of QFT.

What this tries to be:

An attempt to communicate some ideas behind QFT to interested mathematicians.

There are 2 places where QFTs appear in nature

- effective theories in condensed matter
- interactions of fundamental particles

1) Introduction

1.1) Ising model in 2d (ref : [MW] Ch. XI)

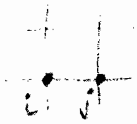
Finite lattice , $L > 0$

$$\Lambda_L = \{ (m,n) \in \mathbb{Z} \times \mathbb{Z} \mid |m|, |n| \leq L \}$$

spin config : $\sigma : \Lambda_L \to \{\pm 1\}$

(+ periodic bnd. cond.)

energy : $E(\sigma) = - \sum_{\langle i,j \rangle} \sigma_i \, \sigma_j$ ← adjacent sites



partition function , $\beta > 0$ (inverse temp.)

$$Z_L(\beta) = \sum_{\sigma} e^{-\beta E(\sigma)}$$

2pt. correlator

$$\langle \sigma_i \, \sigma_j \rangle_\beta = \lim_{L \to \infty} \frac{1}{Z_L(\beta)} \sum_{\sigma} \sigma_i \, \sigma_j \, e^{-\beta E(\sigma)}$$

Can show ($s = \sinh(2\beta)$)

$$\langle \sigma_{0,0} \, \sigma_{N,N} \rangle_\beta \underset{s<1}{=} (\pi N)^{-\frac{1}{2}} \frac{s^{2N}}{(1-s^4)^{1/4}} \cdot \left(1 + O(N^{-1}) \right)$$

[[MW] p. 257 (2.46) , with $E_1 = E_2 = 1$ according to p. 77 (2.1)]

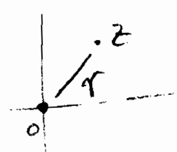
($s = 1$: $\sim$ const $N^{-1/4}$, $s > 1$: $\sim$ const [[MW] p 260 (3.77) , p 265 (4.43)])

Continuum limit ? $z \in \mathbb{C}$

$\langle \sigma(z) \sigma(0) \rangle$ from $\lim_{a \to 0} \langle \sigma_{M,N} \, \sigma_{0,0} \rangle$ s.t. $a \cdot (M,N) \approx z$

↑ lattice spacing

Set $z = r \cdot \frac{1}{\sqrt{2}} (1,1)$



<u>Try 1</u> $\quad N \approx \frac{r}{\sqrt{2}\, a}$

$$\langle \sigma_{00}\ \sigma_{NN} \rangle = (\text{const}(\beta))\ r^{-\frac{1}{2}}\ a^{\frac{1}{2}}\ \exp\Big(\frac{\sqrt{2}\ \overbrace{\ln s}^{<0}}{a} \cdot r\Big)\big(1 + O(a)\big) \xrightarrow[a \to 0]{} 0$$

<u>Try 2</u>

Aim for $\langle \sigma(z)\, \sigma(0) \rangle \approx (mr)^{-\frac{1}{2}}\ e^{-mr}\ \big(1 + O(\tfrac{1}{mr})\big)$

<u>Choose β</u> s.t. $m = -\frac{\sqrt{2}}{a} \ln \sinh(2\beta)$, $s = e^{-\frac{am}{\sqrt{2}}}$

<u>Normalise</u> $\sigma(x,y) \hat{=}$ " $\left(\frac{\sqrt{2}\,\pi^2}{m\,a}\right)^{\frac{1}{8}} \cdot \sigma_{[\frac{x}{a}, \frac{y}{a}]}$ for $a \to 0$ "

Then

$$\langle \sigma(z)\, \sigma(0) \rangle_m = \lim_{a \to 0} \left(\frac{\sqrt{2}\,\pi^2}{m a}\right)^{\frac{1}{4}} \langle \sigma_{00}\ \sigma_{NN} \rangle = (*)$$

⚠ cont. limit needed <u>new parameter</u> m <u>not fixed</u> by lattice model.

1.2) Free boson in 2d

Ref: [DMS] Ch 2.3

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ smooth, decaying

action (Euclidean) $S(\phi) \ \frac{1}{8\pi} \int d^2x \left((\partial_1\phi)^2 + (\partial_2\phi)^2 + m^2\phi^2 \right)$

2pt - correlator

"funct. integral", ill-defined

$$\langle \phi(x)\, \phi(y) \rangle_m \quad \left(= \frac{\int D\phi \left(\phi(x)\, \phi(y)\, e^{-S(\phi)} \right)}{\int D\phi \ e^{-S(\phi)}} \right)$$

$$= 2\, K_0(m\, |x-y|) \quad \text{modified Bessel fn, order } 0$$

[[DMS] Ch. 2.3.4 with $g = \frac{1}{4\pi}$, $K_0(x) = \int_0^\infty dt \, \frac{\cos(xt)}{\sqrt{t^2+1}} \quad (x>0)$]

Since $K_0(x) \underset{x\to\infty}{\sim} \sqrt{\frac{\pi}{2x}} \cdot e^{-x}$ have $\langle \phi(x)\, \phi(0) \rangle_m \underset{x\to\infty}{\sim} (\text{const})(mx)^{-\frac{1}{2}} e^{-mx}$.

$r = |x-y|$

$$= (\text{const})\, (mr)^{-\frac{1}{2}}\, e^{-mr} \left(1 + O\!\left(\tfrac{1}{mr}\right) \right)$$

In fact

$$\langle \phi(x)\, \phi(y) \rangle = (\text{const})\, (mx)^{-\frac{1}{2}}\, e^{-mx} \left(1 - \frac{1}{8mx} + \frac{9}{128\,(mx)^2} + \cdots \right)$$

1.3) Briefly: 3 sets of axioms for QFT

a) correlation functions (Minkowski, Euclidean)
b) algebras of observables (Mink.)
c) "bordism amplitudes" (mostly Euclid.)

To a)

Let $V = \mathbb{R}^d$ (Euclidean space) or $V = \mathbb{R}^{1,d-1}$ (Minkowski space)

$\mathcal{F}$: finite dim rep$^{\underline{n}}$ of $\mathrm{Spin}\, V$ "space of fundamental fields"

c_n $(n \in \mathbb{N})$ functions (really: distributions) "correlation functions"

$$c_n : V \times \ldots \times V \times \mathcal{F} \times \ldots \times \mathcal{F} \longrightarrow \mathbb{C}$$

Usual notation

$$c_n(x_1, \ldots, x_n, \phi_1, \ldots, \phi_n) \equiv \langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle$$

$V = \mathbb{R}^d$: Schwinger functions
$V = \mathbb{R}^{1,d-1}$: Wightman functions

Conditions:
- $\mathrm{Spin}\, V \ltimes V$ - covariance (double cover of Poincaré / Euclid. group, conn. comp of id,
- $\underbrace{\mathbb{Z}_2\text{-graded}}_{\text{Fermions / Bosons}}$ S_n-permutation sym of c_n
- positivity, growth conditions, ...

Refs: [Ha] Ch. II, [Ka] §1.2, 2.2, [RS] Ch. IX.8

E.g. in

Ising : $\mathcal{F} = \mathbb{C} \cdot \sigma$, $V = \mathbb{R}^2$, $\langle \sigma(x) \sigma(y) \rangle = \ldots$

F. B. : $\mathcal{F} = \mathbb{C} \cdot \phi$, $V = \mathbb{R}^2$, $\langle \phi(x) \phi(y) \rangle = \ldots$

omit ?

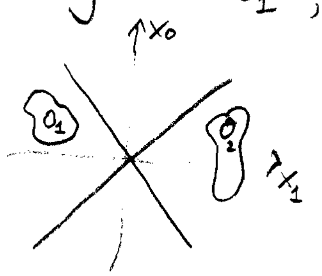
To b)

nets of observables : $\mathcal{O}$ open bounded subset of $\mathbb{R}^{1,d-1}$ $\longmapsto$ $\mathcal{A}(\mathcal{O})$, a C^*-algebra

s.t. $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$

self-adj elements in $\mathcal{A}(\mathcal{O}) \hat{=}$ quantities that can be measured in $\mathcal{O}$

locality : $\mathcal{O}_1, \mathcal{O}_2$ space-like sep.



$\Rightarrow [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0$

more ...

Refs [Ha] Ch. III , [Ar] Ch 4 .

To c)

d-dim. bordism : $A \xrightarrow{M} B$

M : d-dim smooth mfld, can have non-empty bnd
poss. extra structure :
- orientation
- metric
- spin-structure
- ...

A, B : (d-1)-dim smooth mfld, empty bnd,
with "small d-dim nbhood" $(A \times]-\varepsilon, \varepsilon[\ )$

embeddings
$$A \times [0,\varepsilon[\xrightarrow{\imath} M \xleftarrow{\omega} B \times]-\varepsilon, 0]$$
s.t. $\partial M \simeq A \cup B$

Compose by gluing
$$C \xleftarrow{N} B \xleftarrow{M} A \quad \rightsquigarrow \quad C \xleftarrow{N \circ M} A$$

d-dim QFT as (sym, monoidal, ..) functor

$$\tau : \text{d-Bord} \dashrightarrow \text{top. vector sp.}$$
amplitude
$$\left(A \xrightarrow{M} B\right) \longmapsto \left(\tau(A) \xrightarrow{\tau(M)} \tau(B)\right)$$
"space of states on A"

Ref [At], [Se], [ST]

Idea of relation

c) $\dashrightarrow$ a) Eucl.   a) Mink $\dashrightarrow$ b)

$\mathbb{R}^d \setminus (B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n))$

$\underbrace{\tau(S^{d-1}(\varepsilon))^{\otimes n}}_{\supset F} \xrightarrow{\tau(\;)} \mathbb{C}$, $\varepsilon \to 0$

construct. Hilb. sp. $\mathcal{H}$

- for $\phi \in F$ an operator $\phi(x)$ (distrib.) on $\mathcal{H}$
- smear with f supp in O

  $\int_O f(x)\phi(x)$

4) Briefly: Perturbations of CFT

Let F, V be a QFT as in 3 a). Take $V = \mathbb{R}^d$ (Eucl.)

Pick $\mathrm{Spin}\,V$-inv. vector $\Psi \in F$

The QFT perturbed by Ψ has correlation functions

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle_\lambda := \langle \phi_1(x_1) \cdots \phi_n(x_n)\, e^{-\lambda \int_V \Psi(x)\,dx} \rangle \cdot \frac{1}{\langle e^{-\lambda \int \Psi} \rangle}$$

where

$$\langle \phi_1(x_1) \cdots e^{-\lambda \int_V \Psi(y)\,dy} \rangle$$

$$\stackrel{\text{def}}{:=} \langle \phi_1(x_1) \cdots \rangle + (-\lambda) \int_V dy_1 \, \langle \phi_1(x_1) \cdots \Psi(y_1) \rangle$$

$$+ \frac{1}{2}(-\lambda)^2 \int_V dy_1\, dy_2 \, \langle \phi_1(x_1) \cdots \Psi(y_1)\Psi(y_2) \rangle$$

$$+ \cdots$$

⚠ Problems:
- Integrand may be too singular as $y_i \to x_k$ or $y_i \to y_j$. (UV divergence)
- Integral may diverge for $|y_i| \to \infty$. (IR divergence)
- $\sum_n \lambda^n \cdots$ may diverge.

Remark: Why $\exp(\lambda \int \Psi)$?

1) path integral, later

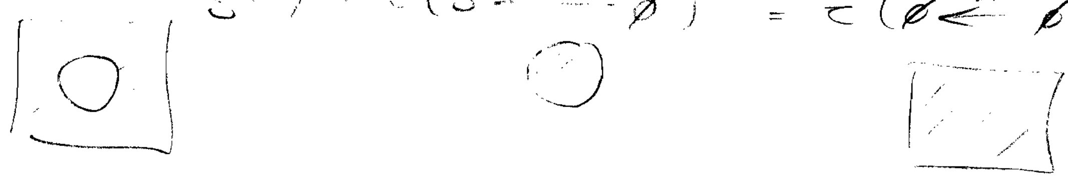
2) "$e^{a+b} = e^a e^b$" : Write $\mathbb{R}^d = A \cup B$ disj.

$$\sum_{m=0}^{\infty} \frac{1}{m!} (-\lambda)^m \int dy_1 \dots dy_m \langle \phi_1(x_1) \dots \psi(y_1) \dots \psi(y_m) \rangle$$

$$= \sum_{k,\ell=0}^{\infty} \frac{1}{k!} (-\lambda)^k \frac{1}{\ell!} (-\lambda)^\ell \int_A dy_1 \dots dy_k \int_B dz_1 \dots dz_\ell \langle \phi_1(x_1) \dots \psi(y_1) \dots \psi(z_1) \dots \rangle \quad (*)$$

In QFT 3) version a) : e.g.

$$\tau\left(\emptyset \xleftarrow{\mathbb{R}^d \setminus B^d} S^d\right) \circ \tau\left(S^d \xleftarrow{B^d} \emptyset\right) = \tau\left(\emptyset \xleftarrow{\mathbb{R}^d} \emptyset\right)$$



holds in pert. theory by $(*)$.

i.e. set

$$\tau_\lambda(B^d) = \tau(B^d) + (-\lambda) \int_{B^d} dy_1 \, \tau\left(\times y_1\right)(\psi) + \frac{1}{2}(-\lambda)^2 \int_{B^d} dy_1 \, dy_2 \, \tau\left(\times y_1 \; \times y_2\right)(\psi, \psi) + \dots$$

2) Classical and quantum mechanics

2.1) Eqn of motion

E.g.: particle moving in $\mathbb{R}^n$, potential $V: \mathbb{R}^n \to \mathbb{R}$ C^1-Fn

classical

state of particle
- position $x \in \mathbb{R}^n$
- momentum $p \in \mathbb{R}^n$

Eqn. of motion: Newton

$\varphi: [a,b] \longrightarrow \mathbb{R}^n$ C^2-Fn

$m\ddot{\varphi} = -\nabla V$

initial cond.

$\varphi(a) = x_0$

$\dot{\varphi}(a) = \frac{1}{m} p_0$

Interpret.: at each time particle has unique pos & mom., which can be measured.

E.g. $V = \frac{1}{2} m \omega^2 x^2$

$\ddot{\varphi} = -\omega^2 \varphi$

$\varphi(t) = a \cos(\omega t + b)$

quantum

state of particle: wave function

$\psi \in L^2(\mathbb{R}^n)$ ("$\psi: \mathbb{R}^n \to \mathbb{C}$")

EOM: Schrödinger eqn

$\psi: [a,b] \to L^2(\mathbb{R}^n)$

$$\dot{\psi}(t,x) = \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}(\Delta\psi)(x,t) + V(x)\,\psi(x,t)\right)$$

initial cond.

$\psi(a,x) = \psi_0(x)$

Interpret.:

probability of meas. particle in $U \subset \mathbb{R}^n$ at time t is

$$\text{prob.}(U) = \int_U |\psi(t,x)|^2 dx$$

(if $\langle \psi, \psi \rangle = 1$)

E.g. $V = \frac{1}{2} m \omega^2 x^2$, one soln is

$$\psi(t, \underset{\mathbb{R}^n}{\underset{\cap}{x}}) = \underbrace{C}_{\left(\frac{m\omega}{\hbar\pi}\right)^{n/4}} \exp\Big(\frac{i}{\hbar}\underbrace{E}_{\frac{n}{2}\hbar\omega} t - \underbrace{D}_{\frac{1}{2}\frac{m\omega}{\hbar}} x^2\Big)$$

(ground state of harmonic osc.)

$\hbar = 1.054\ldots \cdot 10^{-34}\,\mathrm{J \cdot s}$, $m_e = 9.109\ldots 10^{-31}\,\mathrm{kg}$

2.2 Least action

Lagrangian functional $\quad L(x,v) = \frac{1}{2} m v^2 - V(x) \quad ; x, v \in \mathbb{R}^n \quad (*)$

Action for path $\varphi : [a,b] \to \mathbb{R}^n$

$$S(\varphi) = \int_a^b L(\varphi(t), \dot\varphi(t))\, dt$$

Variation of action (n=1 for simp.)

$$S(\varphi + \delta\varphi) = \int_a^b \underbrace{L(\varphi + \delta\varphi, \dot\varphi + \delta\dot\varphi)}\, dt$$

$$= L(\varphi, \dot\varphi) + \frac{\partial L}{\partial x} \cdot \delta\varphi + \frac{\partial L}{\partial v} \cdot \delta\dot\varphi + (\text{higher})$$

$$= S(\varphi) + \int_a^b \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial v} \right) \right)}_{\text{Euler-Lag. eqn.}} \cdot \delta\varphi\, dt + \left(\frac{\partial L}{\partial v} \cdot \delta\varphi \right)\Big|_a^b + \cdots$$

For $(*)$ got

$$-\left(\frac{\partial}{\partial x} V\right)(\varphi(t)) - \frac{d}{dt}\left(m \dot\varphi(t)\right) = 0$$

Classical path is stationary point of S within path with fixed end points.

Conjugate momentum $\quad p_i = \frac{\partial L}{\partial v_i}$ $\quad i = 1, \dots, n \quad$ (for $(*)$: $p_i = m v_i$)

Hamiltonian : $H(q,p) = \sum p_i \cdot v_i(p) - L(q, v(p))$

for $(*)$: $\quad H(q,p) = \frac{1}{2m} p^2 + V(q) \quad (**)$

Phase space : $\underbrace{\mathbb{R}^n}_{q\text{'s : position}} \times \underbrace{\mathbb{R}^n}_{p\text{'s : momenta}}$ (in general : T^*M)

Poisson bracket $f, g : \mathbb{R}^{2n} \to \mathbb{R}$ $\mathcal{C}^1$

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

e.g.

$$\{q_i, p_j\} = \delta_{ij}$$

for H in (**)

$$\{p_i, H\} = -\frac{\partial H}{\partial q_i} \overset{\downarrow}{=} -\partial_i V(q)$$

In general: For $f : \mathbb{R}^{2n} \to \mathbb{R}$, $q(t)$ sol. to EOM, $p(t) = \frac{\partial L}{\partial v}(q, \dot{q})$

$$\frac{d}{dt} f(q(t), p(t)) = \{f, H\}(q(t), p(t))$$

in part.: (for H in (**))

$$\{H, H\} = 0 \Rightarrow H \text{ conserved (energy)}$$

$$\{q, H\} = \frac{\partial H}{\partial p} = \frac{1}{m} p \Rightarrow \dot{q}(t) = \frac{1}{m} p$$

$$\{p, H\} = -\partial V(q) \Rightarrow \underset{m\ddot{q}(t)}{\underset{\shortparallel}{\dot{p}(t)}} = -\partial V(q)$$

2.3 Canonical quantization

Phase space : $\mathbb{R}^n \times \mathbb{R}^n$ (pos. $\mathbb{R}^n$, mom. $\mathbb{R}^n$)

Functions f on $\longrightarrow$ operators $\hat{f}$ on Hilb. sp. $\mathcal{H}$

s.t.

$\{\ ,\ \} \longrightarrow i\hbar[\ ,\ ]$?
- not unique
- not poss. for all f

postulate

$$q \mapsto \hat{q}, \quad p \mapsto \hat{p} \quad \text{s.t.} \quad [\hat{q},\hat{p}] = i\hbar\,\mathbb{1}$$

(note e.g. $[\hat{p}\hat{q}^2, \hat{q}^2\hat{p}] \neq 0$)

For Hamiltonian

$$H = \frac{1}{2m}p^2 + V(q) \rightsquigarrow \frac{1}{2m}\hat{p}^2 + V(\hat{q})$$

For $\mathcal{H}$ take $L^2(\mathbb{R}^n)$ (pos. space),

$$\left.\begin{aligned}(\hat{q}\psi)(x) &= x\,\psi(x) \\ (\hat{p}\psi)(x) &= -i\hbar\frac{\partial}{\partial x}\psi(x)\end{aligned}\right\} \text{on appr. domain.}$$

(more general: geometric quantization)

get $$(\hat{H}\psi)(x) = \left(-\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 + V(x)\right)\psi(x)$$

$\hookrightarrow$ Schröd. equ.

$$q^2p = pq^2 + q[q,p] + [q,p]q = pq^2 + 2i\hbar q$$

and $$[pq^2, q] = [p,q]q^2 = -i\hbar q^2 \quad \text{s.t.} \quad [pq^2, q^2p] = -i\hbar(2i\hbar)q^2 = 2\hbar^2 q^2$$

2.4 1-dim. Harmonic oscillator

$(\hbar = m = 1)$: $H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$ (do not write ^)

Set $a = \sqrt{\frac{\omega}{2}}\left(q + \frac{i}{\omega} p\right) \Rightarrow a^* = \sqrt{\frac{\omega}{2}}\left(x - \frac{i}{\omega} p\right)$

$$\Rightarrow [a, a^*] = \mathbb{1}$$

$$H = \omega\left(a^* a + \frac{1}{2}\right)$$

Note: $H\psi = E\psi \Rightarrow Ha\psi = (E-\omega)a\psi$

Suppose energy bounded below

$\rightsquigarrow$ ground state Ω with $a\Omega = 0$

$\rightsquigarrow$ $H\Omega = \frac{1}{2}\omega\,\Omega$ and $H(a^*)^n\Omega = \omega(n+\frac{1}{2})(a^*)^n\Omega$

Expectation values:

$\langle \Omega, q\,\Omega\rangle = 0$ $\quad$ but $\langle \Omega, q^2\,\Omega\rangle$

$q = \sqrt{\frac{2}{\omega}}(a+a^*)$ $\quad$ $= \frac{2}{\omega}\langle \Omega, (a+\cancel{a^*})(\cancel{a}+a^*)\,\Omega\rangle$

$= \frac{2}{\omega}$.

Time dependence:

$A : [a,b] \longrightarrow End(\mathcal{H})$, $A(0) = A_0$, $\dot{A}(t) = \frac{1}{i\hbar}[A(t), H]$

$[a, a^*] = \frac{\omega}{2}\left(-\frac{i}{\omega} - \frac{i}{\omega}\right)\cdot i = 1$, $\quad q = \sqrt{\frac{2}{\omega}}\left(a + a^*\right)$

$$Ha = \omega\left(\underbrace{a^* a}_{=[a^*,a]+aa^*}a + \frac{1}{2}a\right) = \omega a\left(-1 + a^* a + \frac{1}{2}\right) = a(H-\omega)$$

e.g. $$a(t) = e^{-i\omega t} a$$

$$q(t) = \sqrt{\frac{2}{\omega}} \left(e^{i\omega t} a^* + e^{-i\omega t} a \right)$$

Note : Did not need to know $\mathcal{H} = L^2(\mathbb{R})$

$$\dot{a}(t) = -i\omega\, a(t)$$

$$\frac{1}{i}\left[a(t), H\right] = \frac{1}{i} \cdot \omega\, a(t)$$

2.5 Path integrals

Thm (Trotter prod. formula)

$\mathcal{H}$: separable Hilb sp

A, B self-adj op. in $\mathcal{H}$ (meaning $(D:=)D(A) = D(A^*)$, $A = A^*$ on D and $D(A)$ dense in $\mathcal{H}$)

not necc. bounded

s.t. $A+B$ ess. self adj on $D(A) \cap D(B)$

Then for all $\psi \in \mathcal{H}$

$$\lim_{k\to\infty} \left(e^{itA/k} e^{itB/k} \right)^k \psi = e^{it(A+B)} \psi$$

(c.f. [RS] Thm $\underline{\mathrm{VIII}}$.31, Ch $\underline{\mathrm{VIII}}$.8)

Thm (free propagator)

Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $\psi \in \mathcal{H}$ $\quad \left(p = -i \frac{\partial}{\partial x} \right)$

$$\left(e^{-it\left(\frac{p^2}{2m}\right)} \psi \right)(x) = \lim_{R\to\infty} \left(\frac{2\pi i t}{m} \right)^{-n/2} \int_{|y|\le R} e^{i \frac{m|x-y|^2}{2t}} \psi(y)\, dy$$

(c.f. [RS] Ch. IX.7, example 3)

<u>Thm</u> (discrete path int.) Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be s.t. $H = \frac{1}{2m} p^2 + V$ is ess. self adj on $D(p^2) \cap D(V)$. Then for all $\psi \in \mathcal{H}$, $x_0 \in \mathbb{R}^n$:

$$\left(e^{-itH}\psi\right)(x_0)$$

$$\underset{\substack{\uparrow \\ \Delta t = \frac{t}{k}}}{=} \lim_{k\to\infty} \left(\frac{2\pi i}{m}\Delta t\right)^{-\frac{nk}{2}} \int \cdots \int \exp\Bigg(\underbrace{i \sum_{j=1}^{k} \Delta t \left\{ \frac{m}{2}\left(\frac{|x_j - x_{j-1}|}{\Delta t}\right)^2 - V(x_j)\right\}}_{(*)} \Bigg)$$

$$\times \psi(x_n) \, dx_n \ldots dx_1$$

("$\int$" $\hat{=} \lim_{R\to\infty} \int_{|x|\le R}$, all lim in L^2)

<u>Pf</u>: Apply Trotter to H, use free prop. □

(c.f. [RS] Thm X.66, Ch X.11)

(*) : discretisation of $S(\varphi) = \int_0^t \left(\frac{m}{2} \dot{\varphi}(s)^2 - V(\varphi(s)) \right) ds$

Path integral

$$\langle u, e^{-\frac{it}{\hbar}H} v \rangle \text{``}=\text{''} \int \underset{\substack{\nearrow \\ \text{all path } \varphi : [0,t] \to \mathbb{R}^n}}{\mathcal{D}\varphi} \; \underbrace{u(\varphi(t))\, v(\varphi(0))}_{\text{bnd. cond.}} \; \underbrace{e^{\frac{i}{\hbar}S(\varphi)}}_{\text{weight}}$$

(c.f. [PS] Ch 9.1)

strongest contrib. from "stationary phase" $\frac{\delta S}{\delta \varphi}(\varphi) = 0$

$\rightarrow$ classical path

3) Quantum field theory via functional integrals

3.1) Classical eucl. scalar field

$\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ C^1-fn,

Euclid. action $\quad S_E(\phi) = \int_{\mathbb{R}^d} \left(\frac{1}{2}\sum_{\mu=1}^{n} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) dx$

or subset, e.g. $[a,b]\times\mathbb{R}^{d-1}$

E.O.M.: $\quad S_E(\phi + \delta\phi) = \int \left(-\Delta\phi + m^2\phi \right) \delta\phi \, dx \overset{!}{=} 0 \quad \forall \delta\phi$

vanish at ∞ / on bnd.

$$\Rightarrow \quad (-\Delta + m^2)\,\phi(x) = 0 .$$

3.2) Propagator in QFT

Want to motivate 2 pt corr $\langle \phi(x)\,\phi(y) \rangle = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{1}{p^2+m^2} e^{i(p,x-y)}$

maybe should have called the x's here "y" or "p", to avoid confusion with coord. $x \in \mathbb{R}^d$ in 3.2.2

3.2.1) Gaussian integrals

Ref: [BB] Sec 4.1 [LMS] App. 2.A

Have
$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,x)^2} dx = (2\pi)^{n/2}$$

Let A be a symmetric $n\times n$ matrix, $A^t = A$, $A \in \mathrm{Mat}_n(\mathbb{C})$, $\mathrm{Re}(A)$ pos. def.

Write $A = U^t \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U$, change var

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x, Ax)} \, dx = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

For $J \in \mathbb{R}^n$:

$$I(J) := \int_{\mathbb{R}^n} e^{\underbrace{-\frac{1}{2}(x, Ax) - (J, x)}_{= -\frac{1}{2}\left[\left(x + A^{-1}J, A(x + A^{-1}J)\right) - (A^{-1}J, J)\right]}} \, dx = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}(A^{-1}J, J)}$$

Then $$\frac{I(J)}{I(0)} = e^{\frac{1}{2}(J, A^{-1}J)}$$

3.2.2 Free fields   Refs: [BB] Sec 4.2, [DMS] Ch 2.3, [PS] Ch 9.2, 9.3

Functional integral ($J : \mathbb{R}^d \to \mathbb{R}$)

$$I(J) = \int D\phi \; e^{-S(\phi) - \overbrace{(\phi, J)}^{\int \phi(x) J(x)}}$$

write $$S(\phi) = \frac{1}{2} \int_{\mathbb{R}^d} \phi(x) (-\Delta + m^2) \phi(x) \, dx$$

assume $\phi(x) \to 0$ $(x \to \infty)$ fast enough

Thus $$I(J) = \int D\phi \; e^{-\frac{1}{2}(\phi, A\phi) - (\phi, J)}$$ for $A = -\Delta + m^2$

Want to set

$$\frac{I(J)}{I(0)} = e^{\frac{1}{2}(J, A^{-1}J)}$$

Define A^{-1} as int f^n

$$(A^{-1}J)(x) = \int_{\mathbb{R}^d} D(x-y)\, J(y)\, dy$$

where $D(x)$ is distrib s.t.

$$(-\Delta_x + m^2)\, D(x) = \delta(x) \quad ; \quad \Delta_x = \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i}\right)^2$$

Fourier transform : $D(x) = \displaystyle\int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d}\, e^{-i(p,x)}\, \hat{D}(p)$

$$(p^2 + m^2)\, \hat{D}(p) = 1$$

Thus

$$D(x) = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{1}{p^2 + m^2}\, e^{-i(p,x)}$$

Define

$$\boxed{\frac{I(J)}{I(0)} := e^{\frac{1}{2}\iint_{\mathbb{R}^d \mathbb{R}^d} J(x)\, D(x-y)\, J(y)\, dx\, dy}}$$

3.2.3 Generating functions

Want

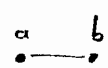
$$\frac{1}{I(0)} \int_{\mathbb{R}^n} (x_a\, x_b \cdots)\, e^{-\frac{1}{2}(x, Ax)}\, dx \qquad (*)$$

$$= \left(\frac{\partial}{\partial J_a} \frac{\partial}{\partial J_b} \cdots\right) \frac{1}{I(0)} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x, Ax) - (x, J)}\, dx \Bigg|_{J=0}$$

$$= e^{\frac{1}{2}(J, A^{-1}J)}$$

write

$$(*) = \langle x_a\, x_b \cdots \rangle$$

E.g. $\langle x_a\, x_b \rangle = (A^{-1})_{ab} =$ 

$\langle x_a\, x_b\, x_c\, x_d \rangle =$ 

$$= (A^{-1})_{ab}(A^{-1})_{cd} + (A^{-1})_{ac}(A^{-1})_{bd} + (A^{-1})_{ad}(A^{-1})_{bc}$$

3.2.4 Green's functions

Want

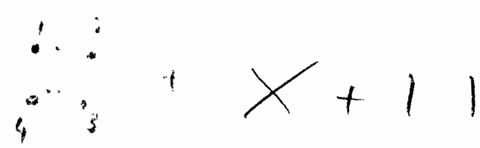
$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{1}{I(0)} \int \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) \, e^{-S(\phi)}$$

Define

$$\langle \phi(x_1) \dots \rangle = \left(\frac{\delta}{\delta J(x_1)} \dots \right) e^{\frac{1}{2}(J, DJ)}$$

e.g.

$\langle \phi(x)\phi(y) \rangle = D(x-y)$. 

$\langle \phi(x_1) \dots \phi(x_4) \rangle =$ 

$\dots \qquad = D(x_1 - x_2) D(x_3 - x_4) + \dots$

3.3 Perturbations

3.3.1 0-dim QFT

Ref [BB]

Consider $I_\lambda(J) = \int_{\mathbb{R}} e^{-\left(\frac{1}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 + J\varphi\right)} d\varphi$

- no λ-expansion around 0 (diverges for $\lambda < 0$)
- look at

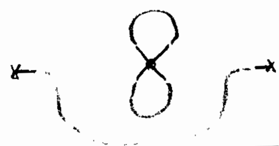
$$\tilde{I}_\lambda(J) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!} \left(\frac{(-\lambda)}{4!} \right)^k \frac{1}{m!}(-J)^m \int_{\mathbb{R}} (\varphi^4)^k \varphi^m e^{-\frac{1}{2}\varphi^2} d\varphi$$

e.g. ($\bullet\!\!-\!\!\bullet = 1$, as here the matrix A is just $A=(1)$)

* $\lambda \cdot J^0$ term $\quad \frac{1}{4!}\Big($  $+$ 2 more $\Big)$

$$= \frac{1}{24}\left(1 + 1 + 1\right)\cdot I(0)$$

shorter : $\quad \frac{1}{\mathrm{Aut}\,\Gamma}\cdot \Gamma$, $\quad \frac{1}{8}\cdot$ 

* $\lambda \cdot J^2$ term :  $\frac{1}{2}\frac{1}{8}$ $+$  $\frac{1}{2}\frac{1}{2}$

$$= \frac{5}{16}\, I(0)$$

By def., the group Aut Γ is generated by:
- permutation of pts :: within each 4-vertex ✗ : an S_4
- permutation of 4 vertices ✗, an S_k
- permutation of ends ×—, an S_m

(for $(-\lambda)^k(-J)^m$). In 2nd diag. above have  and 

3.3.2 d-dim QF

Consider

$$I_\lambda(J) = \int \mathcal{D}\phi\; e^{-\left(S(\phi) + (\phi, J) + \frac{\lambda}{4!}\int \phi^4(x)\,dx\right)}$$

Take this to mean

$$\tilde{I}_\lambda(J) = \left\langle e^{-\int \left(J(x)\phi(x) + \frac{\lambda}{4!}\phi^4(x)\right)dx} \right\rangle \cdot I(0)$$

$$= \sum_{k,m=0}^{\infty} \int dx_1 \ldots dx_k\, dy_1 \ldots dy_m\, \frac{1}{k!}\frac{(-\lambda)^k}{(4!)^k}\cdot\frac{1}{m!}\, J(y_1)\cdots J(y_m)$$

$$\left\langle \phi(y_1) \ldots \phi(y_m)\, \phi^4(x_1) \ldots \phi^4(x_k) \right\rangle$$

Still ill-defined.

<u>Problem</u>

1) $\langle \phi^4(x) \rangle$ means $\langle \phi(x)\phi(x)\phi(x)\phi(x) \rangle$ and involves $D(x-x)$

2) Integral $\int D(x_1, x_2)^2 \, dx_1 dx_2$

diverges for $x_1 \to x_2$

and $x_1, x_2 \to \infty$

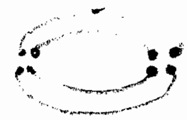
3) Correlator dep. on Λ

<u>Way out</u>

Declare $\phi^4(x)$ to mean "drop all summands with $\Delta(x-x)$"

e.g.

$\langle \phi^4(x) \rangle = 0$

$\langle \phi^4(x) \, \phi^4(y) \rangle =$ 

+ 23 more

$= 24 \, D(x-y)^4$

"<u>normal ordering</u>"

Introduce <u>cutoff</u> Λ

and say

$|x_i - x_j| \geq \frac{1}{\Lambda}$ for all $i \neq j$

$|x_i| \leq \Lambda$

"<u>regularisation</u>"

Fix "physical quantities", e.g.

fix m and demand

$$\langle \phi(r) \, \phi(0) \rangle_{\text{pert.}} \underset{x \to \infty}{\sim} (mr)^{\cdots} e^{-mr} \quad (*)$$

...

replace action

$$\tfrac{1}{2}\left(A(\Lambda)(\partial\phi)^2 + M(\Lambda)\,\phi^2 + \frac{\lambda(\Lambda)}{4!}\,\phi^4 \right)$$

Take $\Lambda \to \infty$ at each order in λ keeping $(*)$ fix

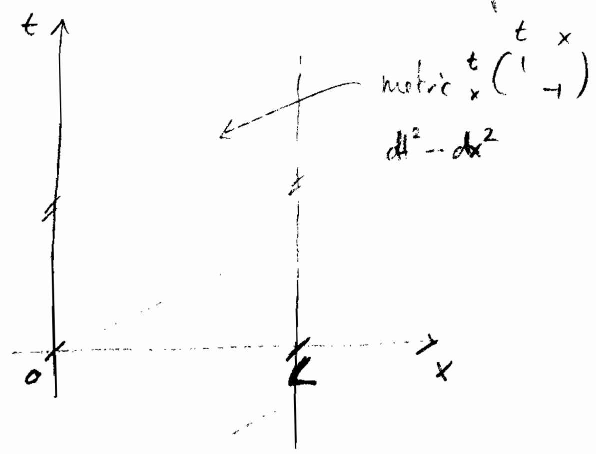
(cf. Ising model)

"renormalisation" $\leftarrow$ (not poss. for all interactions, e.g. not for gravity.)

4) Partition functions

4.1) Free boson via canonical quantisation

Cylinder



$\mathbb{R}^{1,1} / \mathbb{Z} \cdot \begin{pmatrix} 0 \\ L \end{pmatrix}$ (t, x)

4.1.1) Classical theory

scalar field $\varphi : \mathbb{R} \times \frac{\mathbb{R}}{\mathbb{Z}L} \longrightarrow \mathbb{R}$

action $S(\varphi) = \frac{1}{2} \int \left((\partial_t \varphi)^2 - (\partial_x \varphi)^2 - M^2 \varphi^2 \right) dx\, dt$

Lagrangian

$$L(\varphi, \dot{\varphi}) = \frac{1}{2} \int_0^L (\quad) \, dx \quad , \text{ s.t. } S = \int L \, dt$$

Fourier expand φ :

$$\varphi(x,t) = \sum_{n \in \mathbb{Z}} e^{2\pi i n \frac{x}{L}} \cdot \varphi_n(t)$$

where $\varphi_n : \mathbb{R} \longrightarrow \mathbb{C}$; $\varphi_n(t)^* = \varphi_{-n}(t)$

s.t. $\varphi(x,t)$ real.

Rewrite L in terms of $\varphi_n, \dot\varphi_n$:

$$L(\{\varphi_n, \dot\varphi_n\}) = \frac{L}{2} \sum_{n\in\mathbb{Z}} \left(\dot\varphi_n \dot\varphi_{-n} - \left(\left(\frac{2\pi n}{L}\right)^2 + M^2\right)\varphi_n \varphi_{-n}\right)$$

Conjugate momenta

$$\pi_n = \frac{\partial L}{\partial \dot\varphi_n} = L\,\dot\varphi_{-n}$$

Hamiltonian

$$H = \sum_{n\in\mathbb{Z}} \pi_n \dot\varphi_n - L$$

$$= \frac{1}{2L} \sum_{n\in\mathbb{Z}} \left(\pi_n \pi_{-n} + \left((2\pi n)^2 + (LM)^2\right)\varphi_n \varphi_{-n}\right)$$

4.1.2) Quantum theory

($\hbar = 1$)

Operators φ_n, π_n ($n \in \mathbb{Z}$) on Hilb. sp $\mathcal{H}$ (below) s.t.
$\varphi_n^* = \varphi_{-n}$, $\pi_n^* = \pi_{-n}$,
$[\varphi_m, \varphi_n] = 0 = [\pi_m, \pi_n]$ and $[\varphi_m, \pi_n] = i\,\delta_{m,n}$

Start with truncated H :

$$H_K = \frac{1}{2L} \sum_{n=-K}^{K} \left(\pi_n \pi_{-n} + \left((2\pi n)^2 + (LM)^2\right)\varphi_n \varphi_{-n}\right)$$

Set

$$\omega_n = +\sqrt{\left(\frac{2\pi n}{L}\right)^2 + M^2}\,, \quad \beta_n = +\frac{L}{2}\sqrt{\omega_n} \quad (\in \mathbb{R})$$

$$a_n = \beta_n \varphi_n + \frac{i}{2\beta_n}\pi_{-n} \quad \left(\Rightarrow a_n^* = \beta_n \varphi_{-n} - \frac{i}{2\beta_n}\pi_n\right)$$

Check

$$[a_m, a_n] = 0 = [a_m^*, a_n^*]$$

$$[a_m, a_n^*] = \delta_{m,n}$$

and

$$H_K = \sum_{n=-K}^{K} \omega_n \left(a_n^* a_n + \tfrac{1}{2} \right)$$

$\rightsquigarrow$ one harmonic oscillator for each $n \in \mathbb{Z}$.

Infinite constant for $K \to \infty$. Define

$$\tilde{H} = \underbrace{\left(\sum_{n \in \mathbb{Z}} \omega_n \, a_n^* a_n \right)}_{=:\, H_0} + \underset{\mathbb{R}}{\underset{\cap}{C(L,M)}}$$

Hilb. space as in H.O. : Ω with $a_m \Omega = 0 \;\; \forall m \in \mathbb{Z}$ and

$$\mathcal{H} = \overline{\operatorname{span}_{m_1 \geq \cdots \geq m_\ell} \left(a_{m_1}^* \cdots a_{m_\ell}^* \, \Omega \right)} = \overline{\bigotimes_{n \in \mathbb{Z}} V_n}$$

with $V_n = \operatorname{span}_{k \geq 0} \left((a_n^*)^k \Omega \right)$

4.1.3) Partition function

P.F. $\triangleq$

Generating function for dim. of H-eigenspaces/values

$$Z(R,L) = \mathrm{tr}_{\mathcal{H}}\left(e^{-R\tilde{H}}\right)$$

$$= e^{-R\,d(L,M)} \prod_{m\in\mathbb{Z}} \mathrm{tr}_{V_m}\left(e^{-RH_0}\right)$$

Now

$$[H_0, a_m^*] = \sum_{n\in\mathbb{Z}} \omega_n \underbrace{[a_n^* a_n, a_m^*]}_{=\delta_{m,n}\cdot a_n^*} = \omega_m a_m^*$$

$$H_0\Omega = 0$$

$$H_0 (a_m^*)^k \Omega = k\cdot\omega_m (a_m^*)^k \Omega$$

Thus

$$\mathrm{tr}_{V_m}\left(e^{-RH_0}\right) = \sum_{k=0}^{\infty} \underbrace{e^{-R\omega_m k}}_{<1} = \frac{1}{1-e^{-R\omega_m}}$$

and

$$Z(R,L) = e^{-R\,d(L,M)} \prod_{m\in\mathbb{Z}} \left(1 - e^{-2\pi\frac{R}{L}\sqrt{m^2+\mu^2}}\right)^{-1} \quad ; \mu = \frac{ML}{2\pi}$$

Consider, for $q = e^{-2\pi t}$, $t>0$ and $\nu>0$ (cf [BGG] Sec. 3.1)

$$f_1^{(\nu)}(q) = q^{-\Delta(\nu)} (1-q^{\nu})^{\frac{1}{2}} \prod_{n=1}^{\infty} \left(1 - q^{\sqrt{n^2+\nu^2}}\right)$$

$\Delta(\nu) \in \mathbb{R}$ some (fixed) fn

Have

$$\boxed{f_1^{(\nu)}(q) = f_1^{(\nu \cdot t)}(\tilde{q}) \quad ; \quad \tilde{q} = e^{-2\pi/t}}$$

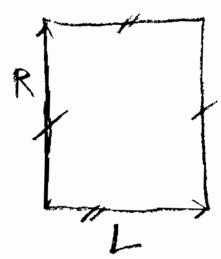
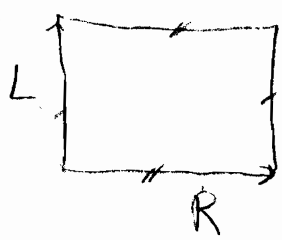
For $G(L,M) = \frac{4\pi}{L}\underbrace{\Delta(\mu)}_{\mu = \frac{ML}{2\pi}}$ have $\underbrace{e^{-2\pi\frac{R}{L}}}_{=q}{}^{\cdot \frac{LG(L,M)}{2\pi}} = q^{-2\Delta(\mu)}$

get

$$Z(R,L) = \left(f_1^{(\mu)}(q)\right)^{-2}, \quad q = e^{-2\pi\frac{R}{L}}$$

$$= \left(f_1^{(\mu\cdot t)}(\tilde{q})\right)^{-2} = f_1^{\left(\frac{ML}{2\pi}\cdot\frac{R}{L}\right)}\left(e^{-2\pi\frac{L}{R}}\right)^{-2} = Z(L,R)$$

Interpretation: euclid. QFT on torus: amplitude inv. under $R \leftrightarrow L$.

$Z\Big($  $\Big) = Z\Big($  $\Big)$

$$\Delta(\nu) = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \int_0^{\infty} ds\, e^{-p^2 s}\, e^{-\pi^2\nu^2/s}$$

4.1.4 Massless Limit

For $\nu \to 0$: $\quad q^{\sqrt{n^2+\nu^2}} \to q^n$

$$\Delta(\nu) \longrightarrow -\frac{1}{24} \qquad \text{(c.f. [BGG] Sect. 3.1)}$$

$$(1-q^\nu)^{\frac{1}{2}} = \left(1-e^{-2\pi t\nu}\right)^{\frac{1}{2}} = \sqrt{2\pi t\nu}\,\left(1+O(\nu)\right)$$

Then

$$\lim_{\nu\to 0} \frac{1}{\sqrt{2\pi\nu}} \cdot f_1^{(\nu)}(q) = \sqrt{t}\cdot\eta(q) \qquad ; \quad q = e^{-2\pi t}$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n>0}(1-q^n)$$

Define

$$Z_0(R,L) = \lim_{M\to 0} \sqrt{RL}\, M\, Z(R,L)$$

this gives

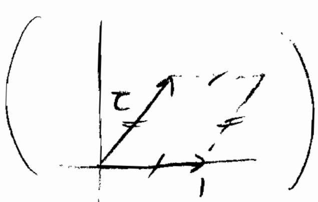
$$= \lim_{M\to 0} \sqrt{RL}\, M \left(f_1^{\left(\frac{ML}{2\pi}\right)}(q)\right)^{-2}$$

$$= \lim_{\mu\to 0} \sqrt{RL}\,\frac{1}{L}\, 2\pi \underbrace{\left(\frac{ML}{2\pi}\right)}_{=\mu} \left(f_1^{(\mu)}(q)\right)^{-2} \quad ; \quad q = e^{-2\pi\frac{R}{L}} \; ; \; t = \frac{R}{L}$$

$$= \sqrt{RL}\,\frac{1}{L}\cdot\frac{L}{R}\,\left(\eta(q)\right)^{-2} = \frac{1}{\sqrt{t}}\,\frac{1}{\eta(q)^2}$$

Can redo for $\tau = it$ in UHP

$$Z_0(\tau) = \frac{1}{\sqrt{\operatorname{Im}\tau}\cdot|\eta(q)|^2} \quad ; \quad q = e^{2\pi i\tau} \quad ; \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$



One checks:
$Z_0(\tau)$ is modular invariant, i.e.

part. fn. of massless free bos. is mod. inv.

4.2 Free boson partition function via funct. integrals

Want to extract $Z(R,L)$ from funct. int.

$$(*) \quad \int\limits_{\{\varphi : T \to \mathbb{R}\}} D\varphi \; e^{-S_E(\varphi)} \qquad \text{where} \quad T = \mathbb{R}^2 / R\mathbb{Z} \times L\mathbb{Z} \quad \text{a torus}$$

$$S_E(\varphi) = \frac{1}{2} \int_T \left((\partial_1 \varphi)^2 + (\partial_2 \varphi)^2 + m^2 \varphi^2 \right) dx$$

Euclidean action.

Rewrite

$$S_E(\varphi) = \frac{1}{2} \int_T \varphi(x) \left(-\Delta + M^2\right) \varphi(x) \, dx$$

Expand φ in eigenfns of $(-\Delta + M^2)$

$$\varphi(x_1, x_2) = \sum_{k,l \in \mathbb{Z}} \frac{c_{k,l}}{RL} e^{2\pi i \frac{x_1}{R} \cdot k} e^{2\pi i \frac{x_2}{L} \cdot l} \quad ; \quad c_{k,l}^* = c_{-k,-l}$$

$$\Rightarrow \quad S_E(\varphi) = \sum_{k,l \in \mathbb{Z}} \underbrace{c_{k,l} \, c_{-k,-l}}_{= |c_{k,l}|^2} \underbrace{\left(\left(\frac{2\pi k}{R}\right)^2 + \left(\frac{2\pi l}{L}\right)^2 + M^2 \right)}_{=: \lambda_{k,l}}$$

So replace

$$\int D\varphi \; e^{-S_E(\varphi)} \rightsquigarrow \int \prod dc_{k,l} \; e^{-\frac{1}{2} \sum |c_{k,l}|^2 \cdot \lambda_{k,l}} \propto \frac{1}{\sqrt{\prod \lambda_{k,l}}} = \frac{1}{\sqrt{\det'(-\Delta + M^2)}}$$

Interpret $(*)$ as ζ-regularised det.

$$(*) \propto \frac{1}{\sqrt{\det'(-\Delta + M^2)}}$$

where, for op. A with eval λ_i set

$$G(s) = \sum_i{}'^{\text{not } \lambda=0} \lambda_i^{-s} \Rightarrow G'(s) = -\sum_i{}' \ln \lambda_i \cdot \lambda_i^{-s}$$

and

$$\det{}' A := e^{-G'(0)}$$

Longish calc... recover result from 4.1.3, 4.1.4, see [SI] Sect. 2 and [DMS] Ch. 10.2.

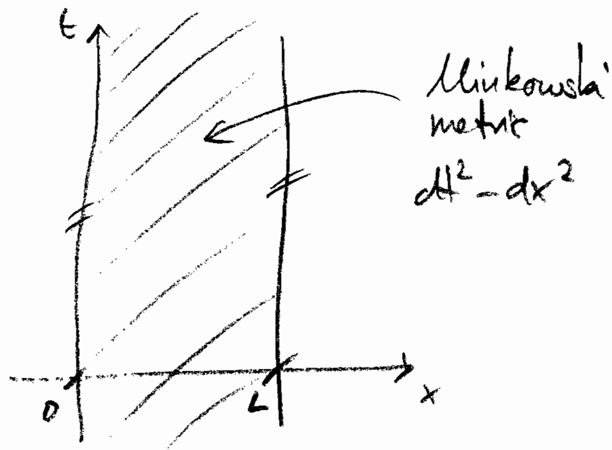
4.3 Massless free fermion on the circle

Here: start directly with quantum theory

4.3.1 Quantum theory

Space-time as in 4.1

$$\underbrace{\mathbb{R}}_{t} \times \underbrace{\frac{\mathbb{R}}{L\mathbb{Z}}}_{x}$$



Operators $b_k, \bar{b}_k$, $k \in \mathbb{Z}$ s.t. (see [DMS] Ch 6.4, [Gi] Sect. 6)

$$b_k^* = b_{-k} \qquad \bar{b}_k^* = \bar{b}_{-k}$$

$$\{b_k, b_\ell\} = \delta_{k+\ell,0} \qquad \text{dito}$$

$\{b_k, b_\ell\}$ = $b_k b_\ell + b_\ell b_k$ anticommutator (not Poisson bracket)

and
$$\{b_k, \bar{b}_\ell\} = 0 \quad \forall k, \ell \in \mathbb{Z}$$

Real fermions $\psi, \bar{\psi}$:

$$\psi(x) = \sqrt{\frac{2\pi}{L}} \sum_{k \in \mathbb{Z}+\nu} e^{2\pi i k \frac{x}{L}} b_k$$

$$\bar{\psi}(x) = \sqrt{\frac{2\pi}{L}} \sum_{k \in \mathbb{Z}+\nu} e^{-2\pi i k \frac{x}{L}} \bar{b}_k$$

$\nu \in \{0, \frac{1}{2}\}$

$\nu = 0$: periodic
Ramond sector (R)

$\nu = \frac{1}{2}$: antiperiodic
Neveu-Schwarz sector (NS)

Note : $\psi(x)^* = \psi(x)$

Hamiltonian

$$H_\nu = \frac{2\pi}{L}\Big(\sum_{\substack{k\in\mathbb{Z}+\nu,\\ k>0}} k\big(b_{-k}b_k + \bar b_{-k}\bar b_k\big) + \overset{\text{constant}}{\overset{\downarrow}{E_\nu}} \Big)$$

Hilbert space : $\mathcal{H}_\nu$

ground state Ω, $\quad b_k\Omega = 0 = \bar b_k \Omega \quad (k>0)$

excited states

$$b_{-k_1} b_{-k_2} \cdots b_{-k_\ell}\Omega \;,\quad k_1 > k_2 \cdots > k_\ell$$
$$k_i \in \mathbb{Z}+\nu$$

e.g.

$\mathcal{H}_0 \equiv \mathcal{H}_R$ has $\Omega,\ b_0\Omega,\ b_{-1}\Omega,\ b_{-1}b_0\Omega$
(and $\bar b$'s)(*)

but $b_k b_k = 0$ for $k\neq 0$.

$\mathcal{H}_{\frac12} \equiv \mathcal{H}_{NS}$ has $\Omega,\ b_{-\frac12}\Omega,\ b_{-\frac32} b_{-\frac12}\Omega, \ldots$
(and $\bar b$'s)

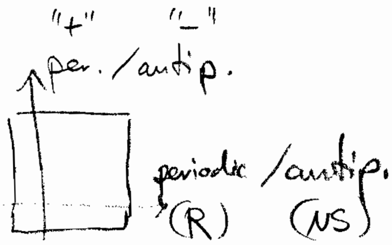
Fermion number operator F

$$F\, b_{-k_1}\cdots b_{-k_m}\bar b_{-\ell_1}\cdots \bar b_{-\ell_n}\Omega = (m+n)\, b_{-k_1}\cdots \bar b_{-\ell_n}\Omega$$

(*) ⚠ $\mathrm{span}_{\mathbb{C}}(\Omega, b_0\Omega)$ is a 2-dim rep$^{\underline{n}}$ of the 4-dim Clifford alg. $\{b_0, b_0\} = 1 = \{\bar b_0, \bar b_0\}$ and $\{b_0, \bar b_0\} = 0$.
E.g. basis $\Omega, b_0\Omega$ and $b_0 b_0\Omega = \frac12\Omega$, $\bar b_0\Omega = -i\, b_0\Omega$, $\bar b_0 b_0 \Omega = \frac{i}{2}\Omega$.

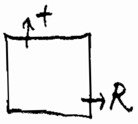
4.3.2 Partition functions

Four torus partition functions



Here only b's

$$q = e^{-2\pi R/L}$$



$: \quad \chi_{R,+}(q) = \mathrm{tr}_{\mathcal{H}_R}\left((-1)^F e^{-RH_0} \right)$

$: \quad \chi_{R,-}(q) = \mathrm{tr}_{\mathcal{H}_R}\left(e^{-RH_0} \right)$

etc...

Note

$$e^{-RH_\nu} \, b_{-k_1} \cdots b_{-k_n} \Omega$$

$$= e^{-2\pi \frac{R}{L}(k_1 + \cdots + k_n)} \, b_{-k_1} \cdots \Omega$$

Each b_k ($k \in \mathbb{Z}_{\geq 0} + \nu$) can be present 0 or 1 times. Thus

other ½ with $\bar{b}$

$$\mathrm{tr}_{\mathcal{H}_\nu}\left(e^{-RH_\nu} \right) = q^{\frac{1}{2}E_0} (1 + q^\nu)(1 + q^{\nu+1}) \cdots$$

$$= q^{\frac{1}{2}E_0} \prod_{m \in \mathbb{Z}_{\geq 0} + \nu} (1 + q^m)$$

$$\mathrm{tr}_{\mathcal{H}_\nu}\left((-1)^F e^{-RH_\nu} \right) = q^{\frac{1}{2}E_{1/2}} (1 - q^\nu)(1 - q^{\nu+1}) \cdots$$

$$= q^{\frac{1}{2}E_{1/2}} \prod_{m \in \mathbb{Z}_{\geq 0} + \nu} (1 - q^m)$$

Thus (see [DMS] Ch 10.3, 10.A)

$\nu=0$: $\chi_{R,+}(q) = 0$ $\quad (1-q^\nu = 0$ for $\nu = 0)$

$\nu=\frac{1}{2}$: $\chi_{NS,+}(q) = \sqrt{\dfrac{\Theta_4(q)}{\eta(q)}}$ if $E_{\frac{1}{2}} = -\frac{1}{24}$

as $\underset{(3)}{\Theta_4}(q) = \prod_{n=1}^{\infty} (1-q^n)(1\underset{(+)}{-}q^{n-1/2})^2$

$\eta(q) = q^{1/24} \prod_{n=1}^{\infty}(1-q^n)$

Similarly

$\chi_{R,-}(q) = \sqrt{\dfrac{2\,\Theta_2(q)}{\eta(q)}}$ if $E_0 = \frac{1}{12}$

as

$\Theta_2(q) = 2q^{1/8} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2$

(with $\bar{b}$'s: $\chi_{R,-}(q) = \sqrt{\dfrac{2\,\Theta_2(q)\Theta_2(\bar{q})}{\eta(q)\eta(\bar{q})}}$ as $\bar{b}_0\Omega$ and $\bar{b}_0 b_0 \Omega$ already counted via Ω, $b_0\Omega$, see ⚠ on p.33)

$\chi_{NS,-}(q) = \sqrt{\dfrac{\Theta_3(q)}{\eta(q)}}$

$\frac{1}{2} \cdot$ (Sum over 4 periodicities) gives modular invariant partition function of free fermion (which incidentally is the critical Ising model we started from (continuum limit taken at $\beta = \beta_c$ though, not at $\beta < \beta_c$))

$$Z_{ferm}(\tau) = \frac{1}{2}\left(0 + \left|\frac{\theta_2}{\eta}\right| + \left|\frac{\theta_3}{\eta}\right| + \left|\frac{\theta_4}{\eta}\right| \right)$$

1.1. Ising model in 2d: [MW, Ch. XI].

1.2. Free boson in 2d: [DMS, Ch.2.3].

1.3. 3 sets of axioms for QFT.

a) correlation functions: [Ha, Ch. II], [Ka, §1.2,2.2], [RS, Ch.IX.8].

b) algebras of observables: [Ha, Ch. **III],** [Ar, Ch.4].

c) "bordism amplitudes": [At, Se, ST].

2.2 Least action: [Ar, Ch.3]

2.5 Path integrals: [RS, Ch. VII1.8, IX.7, X.ll], [PS, Ch.9.1.].

3.2.1 Gaussian integrals: [BB, Sec. 4.1]' [OMS, App. 2.A].

3.2.2 Free fields: [BB, Sec. 4.2]' [OMS, Ch.2.3], [PS, Ch. 9.2,9.3].

3.2.3 Generating functions: [Ze, Ch. 1.7].

3.3.1 a-dim. QFT: [BB, Sec. 5].

4.1.2 Quantum theory: [OMS, Ch.6.3].

4.1.3 Partition function: [BGG, Sec. 3.1], lSI, Sec. 2].

4.1.4 Massless limit: [BGG, Sec.3.1].

4.2 Free boson partition function via functional integrals: [OMS, Ch. 10.2]' lSI, Sec. 2].

4.3.1 Quantum theory for free fermion: [OMS, Ch.6.4]' [Gi, Sec. 6].

4.3.2 Partition functions: [OMS, Ch. 10.3, 10.A], [Gi, Sec. 7].

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