



2226-1

## School and Conference on Modular Forms and Mock Modular Forms and their Applications in Arithmetic, Geometry and Physics

28 February - 18 March, 2011

**Group Actions** 

Vikram B. Mehta

Tata Institute of Fundamental Research (TIFR)

School of Mathematics

Mumbai

India

Page 1 Group Actions Let & be an affire algebrain group, G C G L (n, C) for some M. Then & is Said to be reductive if any me of the following equivalent conditions hold: I) for any f.d. representation Vof &, V = DVi, each Vi sa Simple (irreducible) & - module. 2) The rinipotent radical of G = ( the largest closed, connected, normal unipotent subroug Q G) in trivial.

Pages Estamples: G=GL(n,Q, SL(n,C) So(2n, C), So(2n+1, C), Sp(2n, C)or the other exceptional goups. Only concerned with GL(n, C) and SL(n, C). If V is a possibly infinite - dimensional Visa rational G-module if V= UVn where V is an increasing region of the Vn's , where each Vn ba finte-dim. G- Subspace V. Exemple: G = GZ(n) acting on b[X, Xn] by Substitution. Tet how I be a rational G-module.

We can write  $V=V \oplus V_G$  where  $V=V \oplus V_G$  and V are G embandales 2) (VG) 30, i.e. V has ro invariants. This complement & of V is unque. Now let A be a f.g. C. algebra and let Gast on A by C. algebra G automorphisms. We write  $A = A \oplus A_{\mathcal{S}}$ as alone. A is a C- Subalgebra of A. It R: A > AG be the projection of Honto A. One can prove R(xy) = x R/y) YXEA and yEA.

So P: A > A bak module map. R is called the Roynald's operator. We non prone A is also a f.g. C- algebra (Hilbert's therem): Einst assume that Gads on a f.d. Vector 8 para V, and consider the action of G on S(V\*): = \$5(V\*). We have S'(V#)= C[X-Xn] n=dim V. Define B = S(V#) and let R be the Reignald operator: S'(V\*) ->B Is R is B-hoear, S(V#) is a module over B. For any ideal Jof B, we have JS'(V#) DB=Jilsell.

Pox5 But S'(V#)= [[X-Xn] is a Wetherian Ring (Hilbert Basis Theorem) 3 hence satisfies the A. C. C. So Balso Satisfies the A. C. C. Port B is a graded ring, B = D Bd, where each By = Sd (V#) F. Now a graded ring, which is Northerian is J. g. as an algebra over C, hence B = 5° (V#) & is a f-g-algebra. Now ht A be any f. g. algobra over C, on which G act. Chosing generation 8. -- 8 n la A, we get a Surfection of G-algebras,

Pag 6 e[x,-x,) >> A>>0 Taking (5 - invariant, (which preserve surgediens), we get C[X,--X] G >>> A ->>> O. b.g., So A also f. g. Properties of the map TI: Spac(A) -> Spac(A). 1) II is Surgetive, and offine, 12. of UC Sped (AG) is affine. then TT'(21) also affine and

T'(21,02) = T'(21), OT'(21)

2) 9/3 and y are points of X = Spec(A),

Page 7 then TI(x)=TI(y) if and only if 0(x) n o(y) + 6. L Very rordy is TI: X > Y anorbit map, that is TI (si)=4 = y = gx for some 8 = G. There exist non-closed orbits!!) In gened, points in y parametrize the does orbits in X. Now let 5 act in a vector space V let P(V) = lines in V, the associated Prof. Space. Fax & P(V), let it be any in V fighting to Z. The following are equipment: 1) Those exists some f c s(V),

d 31, such that 3(2) +0 2) The orbit closure G(31) in V does not contain o, the origin in V. Such a piant Din V ( or x in P(V)) is colled semi-stable to the action of Gon V. Suppose I I in V Such that the orbit map TA: G -> V, g -> g'x 's proper as a map of Varieties. Then & ( or x in P(V)) is Called properly stable a just stable

Poge 9 It G act on V, and herce on P(V). Theorem: (Mumpod): ) Both P(V): = Stable Points and P(V) = semistable Points are open and G-invariant in P(V). 2) P(V)35/ & 'b a good quotient and P(V)35/Gio a Rigedine Variety. This means: dente P(VIS) & byM. Then P(VISS >M is surjective, affine, &- unariant map. And be some N >>0, the line burdles & (N) descerds to a line hurdle Lor M, heree Mis Rrogective.

Poge (0 I an open subset 111 9 M S.L. if we take T: P(V) -> M, then TT'(M3)= P(V), and P(V) -> M3 ba geometric quitant in it is a good quotient and the orbits under G are closed in P(V).

So M's is just the space of all orbits in P(V). Hilbert - Mumbad criterion: Tet >: Gm > G be a 1- Parameter Subgroup (1 PS), We write V as a direct Som of eigenspaces for the action of &

Page 11 V= OVi, where on end Vi X(t)v = tv ga any v c Vi. Tet VEV and V= VitViti-Le the decomposition qu'into eigenviertos. Define  $\mu(\lambda, v) = i$ . Hilbert-Mumbad: v'is semistable (Stable, properly stable) (=> for any non-trivial 1 PS: > G\_=>G, he have u(2, v) <0 (<0).

Stable and semistable burdles on curres Tet X be a nonsirgular projective curre Jenus g = 2. Let V be a vector burdle on X of nach r. Define deg V = deg N'V ( N'V line burdle on X), denoted by d. Define slope of V:= u(v) = d/r. Defor: Vio Stable (semistable) we have  $\mu(W) \subset (\leq) \mu(V)$ . Langlise hundle of degree I on X, define V(n) = V & L". Red xy(n) = JCT V(n)] = deg V(n) + nk V(n) (1-8) = deg V+rn+r(1-8) Now let X be a noningular projective Variety of dimd, Ha very ample line

on X. Let V be a topion-free sheaf on X Ore defines a line burdle C, (V) on X Define slope of V: = u(v) = c,(v). Hd-1 Defri. V is u- Stable ( u- Senistable) If y proper subsheaves Wof V, we have  $\mu(W) < (\leq) \mu(V)$ . If V is any torsion- free sheaf on X define of (n) = x[V&H], a poly. Adequa d'in n.  $X_{V}(n) = \underbrace{\xi}_{i=0}^{d} q_{i} (n+i), q_{i} integers$ Define Vto be X- Stable (x- semistable) If y Proper Substeaves Wef V, we have  $\frac{\mathcal{Y}_{W}(n)}{NkW} \times (\leq) \frac{\mathcal{Y}_{V}(n)}{NkV} + n >> 0$ 

Page 14

M-Stable => X-Stable => X semistable

> M semistable, and no implication

Can be reversed.

Assume Xis a curve or a polarized Variety (X, H), with dim X = 2. Let V be a Vector-burdle (or a torsion bee sheef) which is not semistable ( M definition). Then I a filtration 0=V0 CV, --- CV CVn = V with 1) each VilVi, M-semistable y i = J, (8 lopes are strictly decreasing 3) each VilVi-, is torsion-free

Souppose Vis not X-semistable Then there exists a ringue filtration 0=10c1,-.- CVn=V with 1) each VilVi, is tosion-free 2) each Vil Vi- is X - semistable 3) X VilVi (n) / Ne VilVi-1 > 3 N2 | N2 -1 (W) ly N / N-1 y icJ.

These are called the Harder. Marabirlan biltrations for M and I nonsemistability respectively.

Due to ringuenes, the H-N filtrations are rational: if V and X are defined over  $k \neq k$ , and over  $\overline{k}$   $\overline{V}$  is not semistable over  $\overline{X}$ , then  $H.N.(\overline{V})$  is already defined over k (go both M and X).

Let X be a curve or a polarized variety (X, H) with dim X = 2. Suppose we are given a collection of torsion-free sheaves & V & on X. Then & V3 is Said to be a bounded family on X if any one of the 3 equivalent conditions hold:

1) There exists a Scheme T of finite type one C and a sheaf & on X x T such that Is and V & & V & , I t & T such that V & & P | X × (t).

- 2) There exists a fixed sheaf For X such that every V is a qubtient of X and the Hilbert Polynomide X, (n), V E & V & one finite.
  - 3) I a positive integer mo such that for all m = mo, and for all V in & V} we have  $H^i(X, V(m)) = 0$  foods i > 0 and  $H^i(X, V(m))$  generates V(m). And the Hilbert Rolynomials  $X_{ij}(n)$ ,  $V \in \{V\}$  are finite.

Definition-Construction of Dust Schemes: Let (X, H) be any polarized Variety. Let V be any coherent sheaf on X. We Parametrize all quotients (or subs)

91, V > F > 0, Such that Page 18 Hilbert poly. of F, Jc (n) is fised. Eise a poly. P(n) and let 0 -> S -> V-> F->0 be a quotient of V, with  $\chi_{V}(n) = P(n)$ . Then we show I mo such that fall m > mo and for all i >0 ) H'(X,S(m))=H'(X,F(m)) = # (X, V(m)) = 0 and H°(S(m), H°(V(m)) H°(F(m)) generate 5(m), V/m) and F(m) resp. As X (n) and X (n) are from, X (n) is also fixed. So H'S(m) and H'F(m) are constant dimensional vector speaces for any quotient V >> F and fixed m z mo

Page 19 Rut dim H S(m)=m, dim F(m)=m2 Then dim H° V(m) = M: = M, + M2 Each quotient FQV with x(n)=P(n) gives a point of Ger (m, m2), the Grassmannian of Mz dimensional quotients of a fixed m-dimensional Verta Space. This we get a projective Scheme Quot (X, V, P) or Just Quot of X, Vard P are fixed. Proporties: On Xx Quet, there exists a surgection: p, (V) -> G, with Go a sheaf on X x Dust, flat over Quest. And for any Scheme T, and for any surgection p, (V) -> H on X xT, with H flat over T, there exists on

unique mophism: f: T -> Quest
such that (id x f) & = H.

Quest (X, V, P) is a representable
functor.

Construction of Moduli Spaces of Semistable Burdles on Curves: Jet X be a curve and let (V) be the set of all semistable burdles of nank r and degree d. For any V E {V}, XV(m) = X[V(m)] = deg V(m)+r(1-g) = d+rm +r(1-g). We show that & v & is a bounded family: Im. s.t & m = m. and all VE {V}, we have

H(V(m)) = 0 and H(V(m)) generates V(m). Then N=dim H'(V(m)) is constant lo V E & V & . Consider Quot (X,Ox, P) where P(m) = d+rm+r(1-8), the Hilbert Polynomial of V E & V} Tet R8 ( R83) be the open Subset B Quot s.t. 1) 9 x e R3 ( R33) then the quotient Vx of 0 is stable (semistable) H'(on) = c' -> It(Vx) is an isonaphern. Iten G:=GL(n) acts on Quet and on R's R's

Poge 22

It R be the open subset of Quat s.t x ER (=) H(0") -> H(Vx) is an iso. ( no condition of stability for XER) I a map ( ("the covariant") C:R -> P(W), W Some Vector space on which G acts S.t. ) C 6 & LW. 2) gx ER, then C(x) EP(W)

3) Faxer, then C(se) eP(w)"

>> XER33

Page 23 3) shows that

Ross C> P(W) 83 P(W) 83 ( & exists as a good quotient. Sook 30/ Fe exists as a good quotient. 2) also 8hows that, Since P(W)/G exists as a geometric quotient (orbit Space), R3 G exists as a geometric quotient (orbit space) Defn: PBG = M(r,d+rm) the coarse moduli space of semistable burdles of rank r and degree d+rm RS(G) = M'(r, d+rm), the coarse

Page 24 moduli space of stable budles "I nank r and degree d+rm. Troperties: Let Von XXT be a family I stable burdles of rank r and degree dorm. We get a set-theretic map: c T -> M3 given by t >> iso. dass of V/ X = Et). i) c is a morphism T -> M°, functorially in T 2) ptr. 2 m3 classify the iso. classes of stable burdles of rank r and degree d+rm Defire F: Schames -> Sets F(T) = families of stable burdles nank v, degree d+rm on XxT

Page 25 G(T) = Hom(T, M3) Ne get a morphism of furctors. F -> G that is, for every T, E(T) -> G (T) furctorially in T. 2) And la T = C, F(C) is bijective with G(C). Fa the whole M, semistable of rank r and degree d+rm we get maps F(T) -> G(T). But F(c) -> G(d) is not a bitedion!

Page 26 Does there exist a "runiversal family" Um XxM° S.K: for every T, and for every family of on XXT Por XxT = (id x c)\*(U), where c: T > m is the classifying This is true (Y,d) = 1 (>)

This is true  $\Longrightarrow$   $(Y,d) = 1 \Longrightarrow$   $M = M^3$ , levery semistable is Stable).  $\Longrightarrow$  if there exists a GL(n) line bundle L on  $R^3$  such that the centre  $\{\chi \text{Id}\}$  of GL(n) acts by weight I on the files of the remisersal bundle on  $X \times R^3$ .

Construction of Moduli of Semistable Theaves on Surfaces

Jet (X, H) a smooth projective polarized over \$. H is a very ample line burdle on X. Jet & Vz be the set off all X-semistable torsion bee sheares on X. Recall that V is X-stable (X-semistable) (>>> for every NCV, Xw(n) < ( \le Xv(n) \rightarrow xv(n) \rightarrow (n) \ri

We may assume that each  $V \in \{v\}$  has been twisted with m > 70 so that

i) H'(V) generates V

2) Hi(V) = 0 lai=1,2.

Then as (nhV, c1, c2) are fisied, we H'(V) 'so constant, & V E & V3

Page 29 Jet n = dim H (V) for V E & V B We further assume that c, (V) is a constant line bundle, for VE & VZ, call generated, we get a map T(V): N[H(V)] -> H° (det V) = H(L) Jaeach V. Wote that H(V) is a constant Vecta Space, say W, of dim n. So we get T(V): N(W) -> H(L) for end V The group & = SL(r) [or GL(r)] acts on Hom (N(W), H(L), Let Z be the projectivization of this space. Tet H = Hill (X, Ox, P) be the Quotient scheme of all quotients of  $Q_{\chi}$ , with Helbert polynomial  $P = \mathcal{X}_{\chi}(n)$ 

Jet RORSOR be as before i.e. R = x & Quest such that ho of -> F\_ -> 0, Fx is tosion-hel and HO(Ox) = HO(For). For R38 ( nesp. R8), we further demand that Fris x-semistable (X-stable). SoT: R >> Z Basic properties of T i) T'is proper, ingestive map. 2) T(R3) CZ3 3) T (V) E 2 33 if and only if V is x-semistable.

Page 31 Again, by general theorems, the good quotient R85/ G escists and the geometric questient R' & escisto, and R' & is open in R33 6. Here R8 6 parametrizes the set of isomorphism classes of X- Stable sheaves with fixed rkr, fixed det. L and fixed Cz. Suppose V is strictly semistable, i.e. semistable but not stable: Then I a filtration o = Vo C V, - - - C Vm-, C Vm = V Such that

Swehthat

1) Each Vi/Vi-, 's torsion - free

Page 32 2) faeich i, X Vi (Vi-1 (n) / nk Vi / Vi-1 = Xy(n) rkV. Such a filtration is called the Stable biltration a the Jordan-Holder filtration. It is not ringine, but gr (V): = @ (Vi | Vi-1) is ringue. 2 sheaves V, and V2 are colled 5-equivalent if gr (V,) ~ gr (V2) as 8 heaves on X. R33 & Ranametrijes the set of S-equivalence classes

JX - semistable sheaves on X. Note that Vis Stable to V= gr(V). RS G:= M is the coarse moduli space of X-Stable Sheaves on X, with num invariants r, det V, C2. Let The any Scheme and V = sheef on XxT such for each teT, V is semistable with num. invariants r, det V, Cz. Then we get a map c: T > M, the classofying map. Similiarly we get a map c: T > m for each family of Stable Sheaves on X, parametrized by T,

Page 34 Caisterce of Unicersal-families on M3 Corender M3 with r, c, and c2 fixed.
Write  $X_{v}(n) = a_{o} + a_{i} \binom{n+1}{i} + a_{2} \binom{n+2}{2}$ with the a: integers. Prop: 8f g.c.d. (a,a,,a2)=1 then there exists a reniverbal or a Poincaré family Von X x M. mi is a fine moduli space. Unlike the situation for curve, the Converse does not hold in general ( not known).

Tocal Properties of M.

We write  $M^3 = R^3 | G$ . Jet V be a stable burdle in  $R^3$  and dente by V again to image in  $M^3$ . By Juna's etalé stree theorem  $\exists S \subset R^3$ , with  $V \in T$  and a map  $T : S \to T$ , which is étale at V i.e.  $T^{H}$  induces an isomorphism:

 $(0_{T,V})^n = (8_{N,V})^n \sim (8_{S,V})^n$ 

Put R:= (ens, y) = (es, y)

If It is the reniversal family on XXR denote by It again the restriction to XXS.

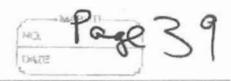
So we get a family on X = 0

So we get a family on X x Os, v

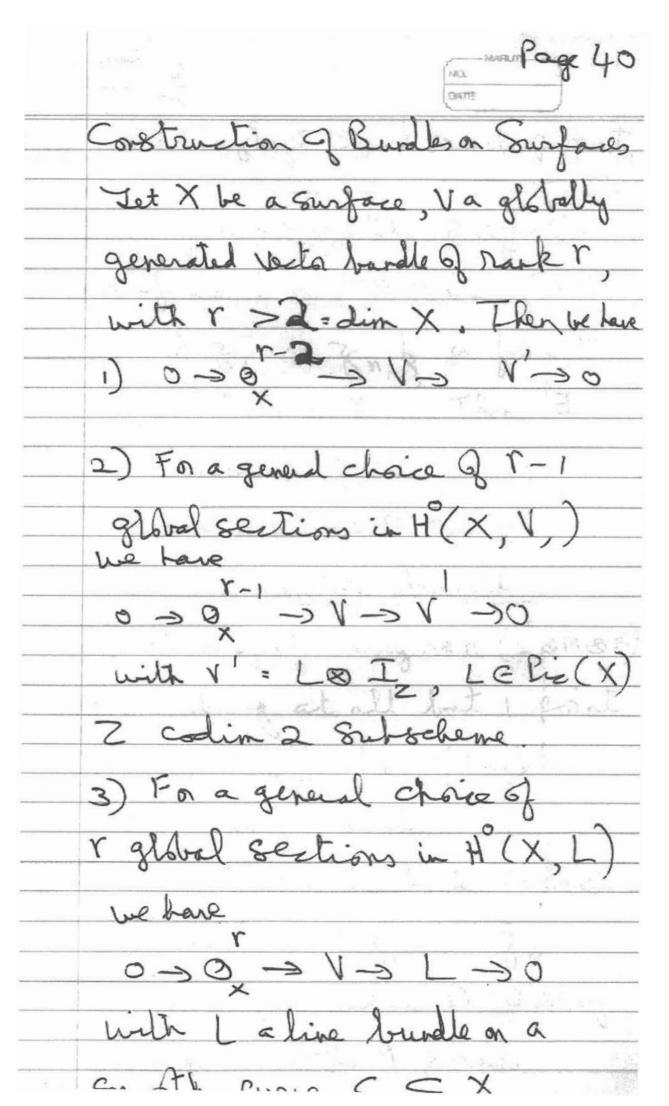
Page 36 and hence on  $X \times (\Theta_{S,V}) = X \times R$ . Duestion: When is R smooth or non-Singular Rio Smooth ( ) for all Artin Joeal rings C, for any ideal J C C, any map &: R -> C; = C/J lifts to a map f: R -> C & gives a map Spea (C/J) -> Spec R So a family T(2) on X x Spec (C/J) So & extendo to amop g: R->C (2) esterds to a family

Page 37 Say Wo on X x Spec (c). In general, the obstructions to lifting burdles "modulo nilpotents" lie in the abstruction space H (X, End(V)). Soiy H(X, EN(V)) =0, then M' is smooth at V, or Bms, is regular. Scinibiarly, we way consider modet fixed, the fixed - determinant modula space. Now the obstructions to lifting fixed - determinant families of Vector burdles "modulo rilpotento lies in H(X, End(V)),

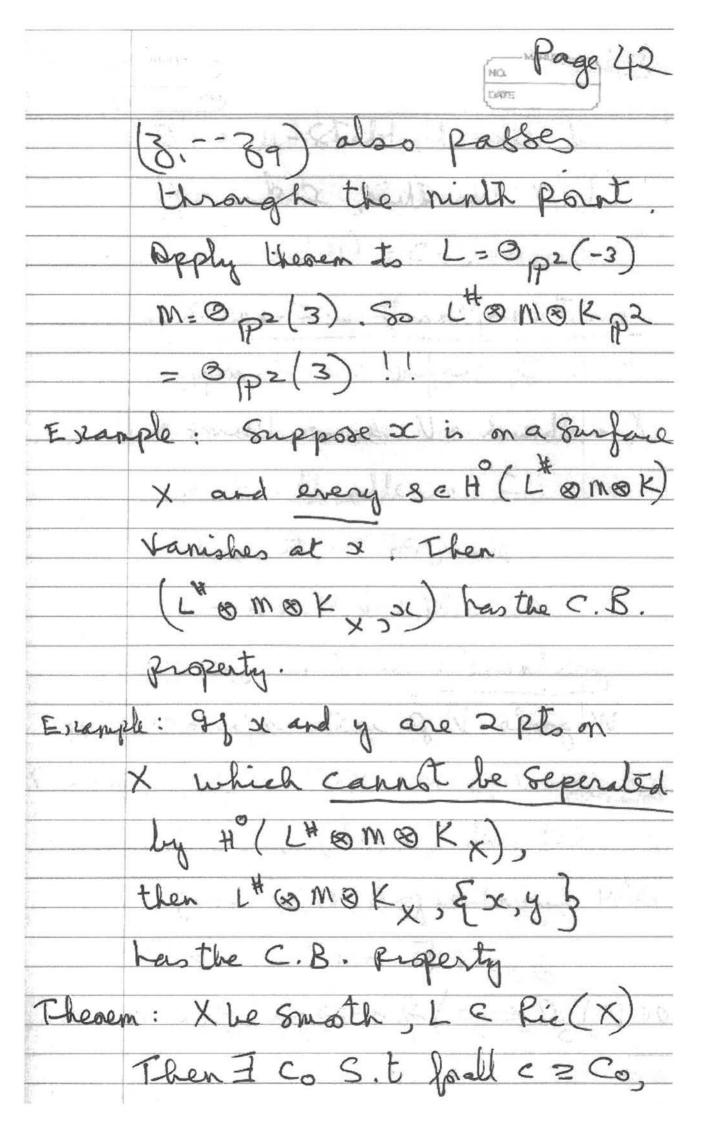
Page 38 where End (V) is the burdle of trace o endonorphisms, have 0 > End(V) -> End(V) -> 0 (this sequence splits on X). Now in general, the Zariski targest space to V in M' is given by H(X, Erd V). Similarly, the targent space to Vin Mfixed, det is H(X, End V). Now consider the map: det: mb -> lie (x), given by V -> det V

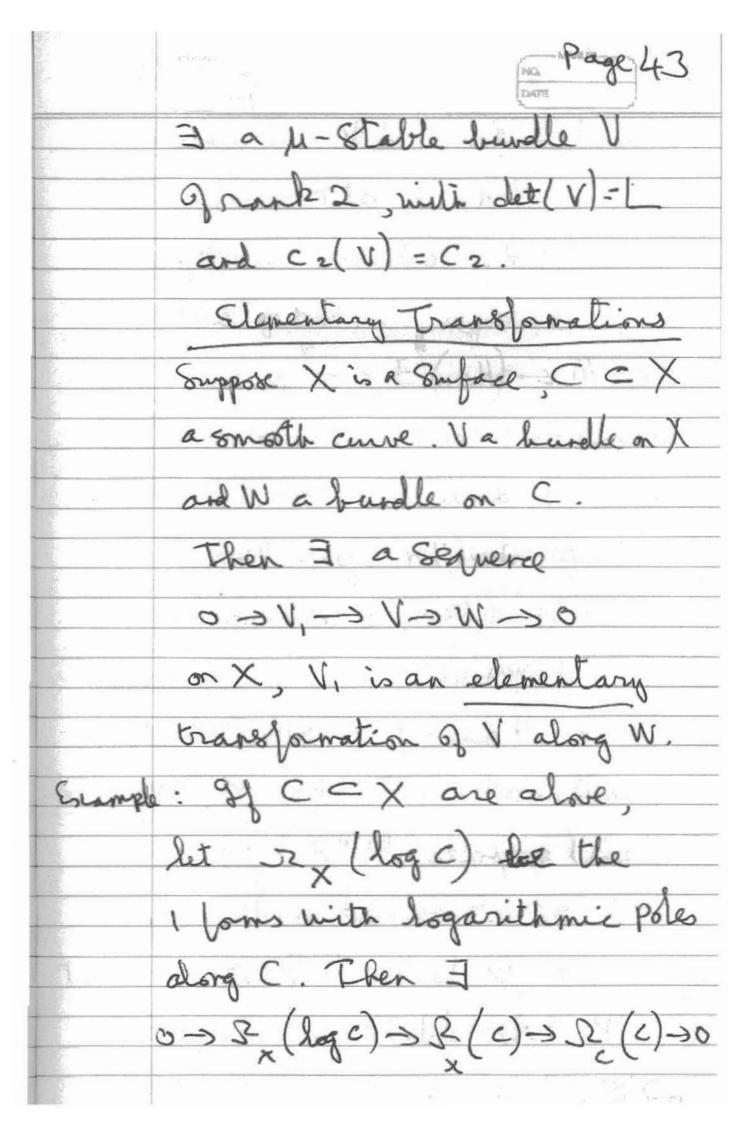


The fibre over L & Ric (X) is M3 = {V E M3 | Let V = L}. A Similian reasoning Shows that if H2(X, EndoV)=0 then M3 is Somosth at V. Suppose this holds, i.e. H2(X, End V)=0. Then Ms is smooth at V. But Pic(x) is always smooth. So we have Theorem: 91 H2 (X, End V) =0 VVEM then M's is smooth at all points. In general we have: dim H (End V) = dim (m et V) > dim H'(ENV) - dim H'(ENV).



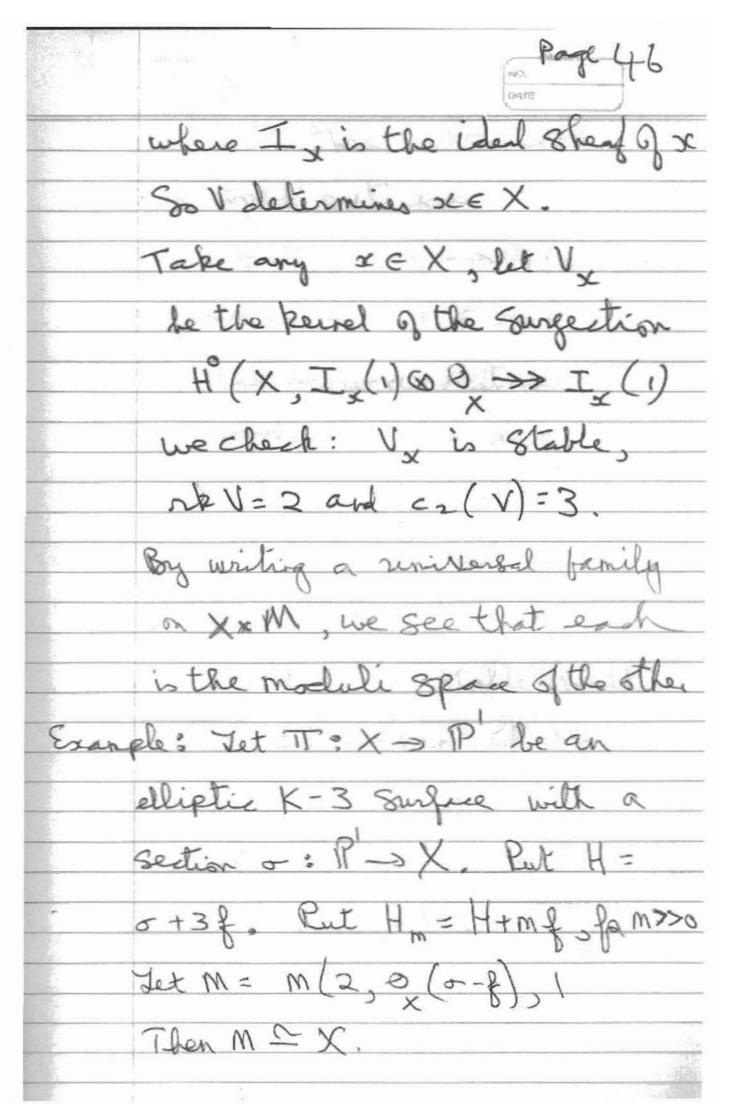
The Cayley - Bacharach property Tet X be a Smooth projective Surface, Land M & Ric (X), Z = { z, - ... z n } a finite set of points on X. Then I an extension: O ->L -> V -> MOIZ with V a vector burdle if and only if every set (L'OMOK) sanishing at all but I point of gr. -- 3 n & also Vanishes at that point. Example: Let C, and C2 be 2 Cubics in P, meeting at 3, -.. 39. Then any other cubic D passing through 8 of the 9 points

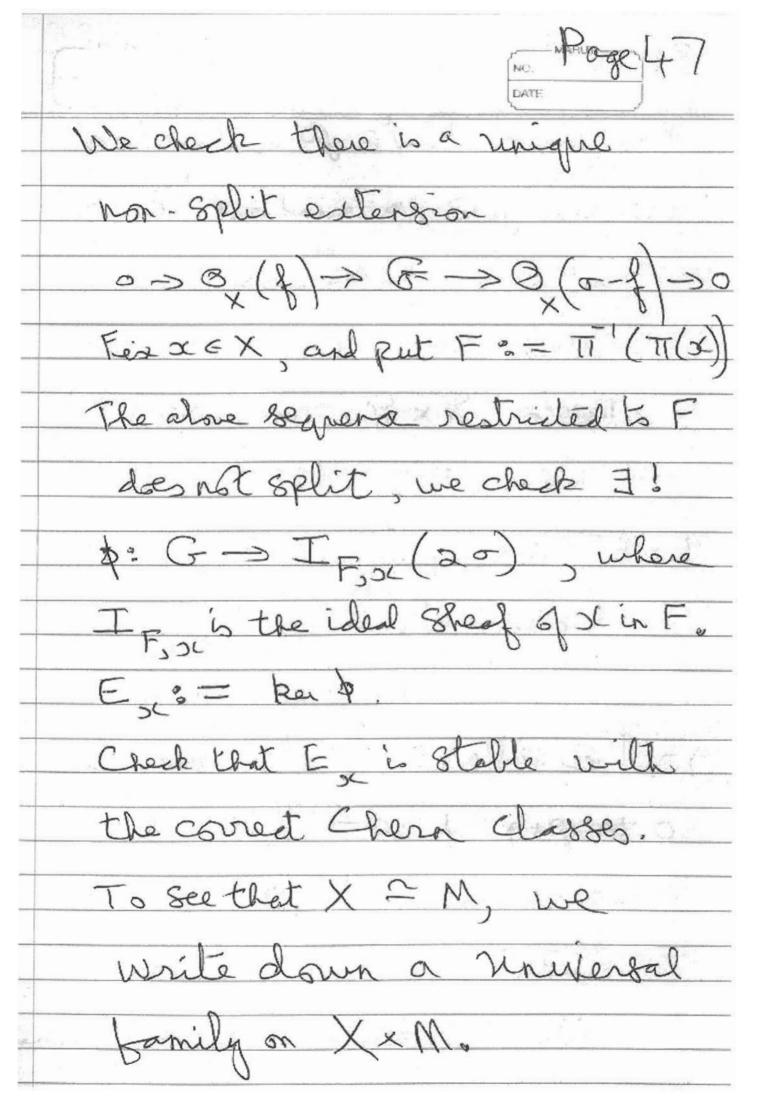




MO MARTHAGE 44 Now let V be a rk. r burelle on X. Then E\* (nH) is globally generated for Y N >>0. We get 0 -> @ -> E (nH) -> M -> 0 Ma line burdle on some curve C = X. Dualize and twist by (nH) to get 0 -> V -> 0 (nH) -> L -> 0 where LE Ric (C). Thesem: Every Vomankr is an elementary transformation of o (nH)Or along a line burdle on some curve. Thesen: Given L & Lie (X), and inleger Co, Jau-Stable skr V with det(V)= Land c2(V) = Co

MARIN Page 45 Examples of Moduli Spaces Jet X = P be a general hypersurface of degree H. Then X is a K-3 Surface: 1) H (X, 0x) = 0 Kx is trivial, bence H(X, 0x) has dim. I . Rie (X) generated by 3x(1) [restriction of 3p3(1) Yet M = M(2, 3 (-1), C2) be the moduli spora of 4- stable Sheaves grank 2, det Bx (-1) and c2(V)=C2 Example: M(2, 8, (-1), 3) = X Sketch: Let VEM. X & X and an estact Sequence 0=(1) = I = VGO



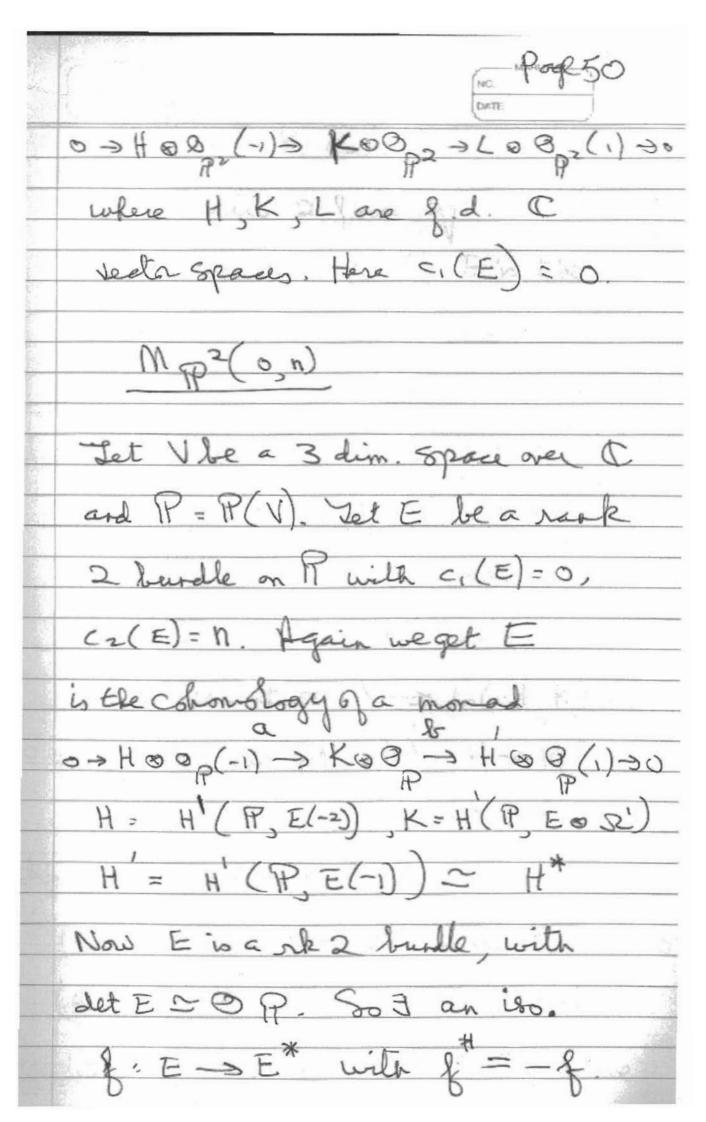


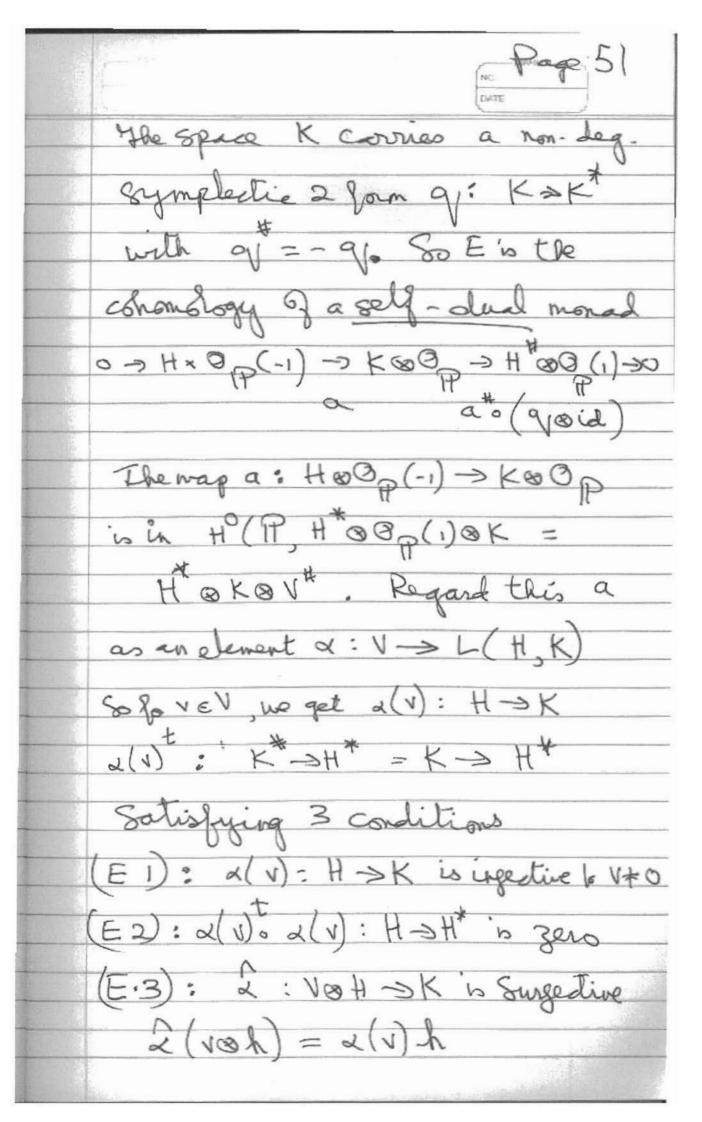
Construction of stable burdlen on P. On P we have a sequence the Euler segnera 0 > 0 -> 8(1) -> Tpn -> 0. This gives on PXP a vector Jurdle W and a section se H(W) s.t. Z(3) = diagonal A = PxP This in term gives a spectral Sequerce, Josheaf En P, E P. 9 = HV (P E (P)) & S- 9 (-P) with E = 0 if p+9 +0 and @ E-PiP is the associated graded of a filtration of E. Example: Tet E be - Nr. 2 Stable hundle on P with c1(E)=- 1 and  $0 \rightarrow E, \rightarrow E, \rightarrow E, \rightarrow 0$ 

NOT EXACT but A a Subburdle of B, and ACRO(BX):= K. K/ A is called the cohomology burdle of the monad. Continuing with our example,

-2! -1,1 01

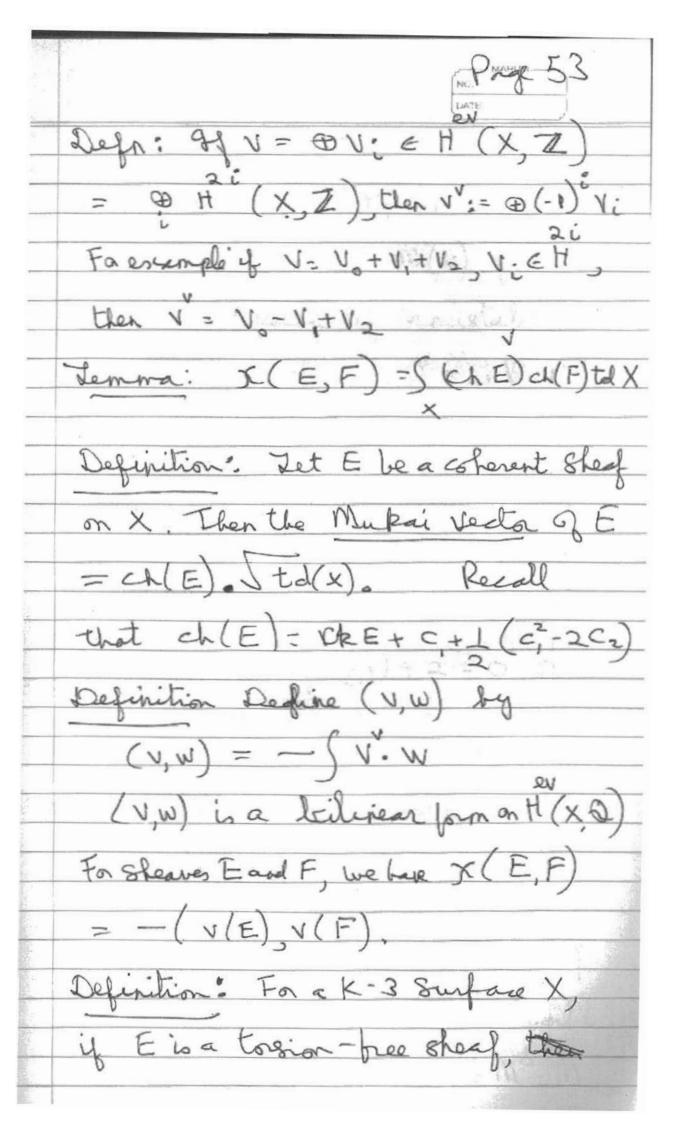
becompute E, E, and E, we get E = sip2(1). Example: E be rank 2 Stable on P <,(E)=0 we get E as the cohomology of a morad O(-1) -> 52(1) ->0 Example: Tet E be a rk. r stable on Then E's the cohomology of





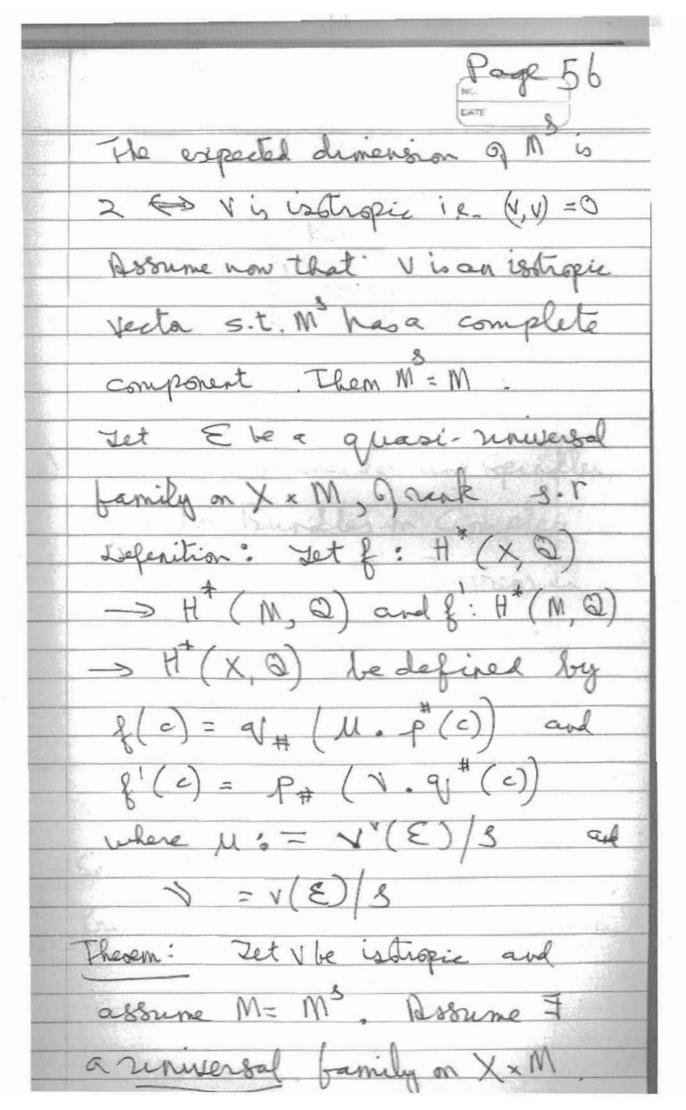
NO. MATURE 52 Now take V, H, K be Vector Spaces of dim 3, n and 2 n+2. On K fix a non-degenerate symplectic form q: K > K\*, q = - q. Tet Sp(q) be the associated group. Then G:=GL(H)+Sp(q) acts on L(V, L(H,K)) Jet P=a € V, L(H,K)), & has properties (E), (E2) (E3). Pisa & invariant Subset. Then P/G is isomorphic to Mp2 (20 n) Moduli Spaces of Sheaves on K-3 Surfaces Tet X be a K-3 Surface. We define the Euler characteristic of the pair of sheaves E, F by :=

) (E,F) = E (-1) din Est (E,F)



NE MARPHOGE 54 with nk E = r c, (E) = c, C2(E) = C2 define V(E):= (r, C1, C1/2-C2+r) We use the notation M(v) for the moduli space of semistable torsion-free sheaves of rank r, C,(E)=C, C2(E)=C2 Note M' the moduli of stable Sheaves is always smooth on a K-3 Surface! Thesem. 3f (v,v) + 2 = 0, or equivalently (V,V) =-2, then if M'is nonemply, then M consists of a single point, locally free and Stable. And M=M3 Thesen: Suppose (v,v) = 0. 99 M3 has a complete irreducible component M, then M, = Ms - M

no Prage 55 I.l. Mis irreduable and all sheaves are stable and in is smooth. Definition: Tet y be any Surface Define H (Y, Z) (a H(Y, Q)) natural Weight 2 Hodge Structure on H (Y,Z) given by H 2,0 (Y, C) = H2,0 (Y, C), H 0,2 (Y, C) = H (Y, C) and H 11 (Y, C) = H(Y, C) & H' (Y, C) BH (Y, C). delect H (Y, Z) has a pairing given by 1,w -> (V,w); = -5 v.w H(Y,Z) CH(Y,Z) is compatible with the Hodge Structure The Mukai yeata M(V) is an element of H(Y, Z) of type (1,1)



57 Let E be the reniversal family. Then:
1) Misa K-3 Surface 2) fof =1 References to pages 28-57 1) Okonek, Schneider and Spindler, "Vector Bundles on Complex Projective Spaces Progress in Mathematics, Birkhauser 1980 Orapter 2, Sections 3 and 4 2) Huybredts and Jehn, The Geometry of Moduli Spaces of Sheaves, Second Edition Cambridge University Press 2010, Part 2, Sections 5 and 6.