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Appell-Lerch sums

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Mock theta functions

Watson (1935) found identities for the third order mock theta functions.

For example, for

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2}$$

he found

$$f(q) = \frac{2}{(q)_\infty} \sum_{n \in \mathbf{Z}} \frac{(-1)^n q^{3n^2/2+n/2}}{1+q^n},$$

with $(q)_\infty = (1-q)(1-q^2)(1-q^3)\ldots = q^{-1/24}\eta(\tau)$ and $q = \exp(2\pi i \tau)$.

He used these identities to find the modular transformation properties of the mock theta functions.

Similar identities have been found for other mock theta functions.

Ranks of partitions

The rank of a partition is the largest part minus the number of parts.

The generating function that counts the number of partitions of given size and rank is given by

$$\mathcal{R}(w;q) := \sum_{\lambda} w^{\text{rank}(\lambda)} q^{||\lambda||} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-wq^k)(1-w^{-1}q^k)}$$
$$= \frac{1-w}{(q)_\infty} \sum_{n \in \mathbf{Z}} \frac{(-1)^n q^{3n^2/2+n/2}}{1-wq^n}.$$

Level 1 Appell function

The level 1 Appell function was studied by Appell (1884), Lerch (1892) and others.

For fixed $\tau \in \mathcal{H}$ we define a function μ of two complex variables u, v by

$$\mu(u,v) = \mu(u,v;\tau) := \frac{a^{1/2}}{\vartheta(v)} \sum_{n \in \mathbf{Z}} \frac{(-1)^n q^{n^2/2+n/2} b^n}{1-aq^n},$$

where $q = \exp(2\pi i \tau)$, $a = \exp(2\pi i u)$, $b = \exp(2\pi i v)$ and $\vartheta(v)$ is the Jacobi theta series

$$\vartheta(v) = \vartheta(v;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} (-1)^\nu b^\nu q^{\nu^2/2}.$$

The Mordell integral

We define the function h by

$$h(z) = h(z;\tau) := \int_{\mathbf{R}} \frac{e^{\pi i \tau x^2 - 2\pi z x}}{\cosh \pi x} dx,$$

with $z \in \mathbf{C}$ and $\tau \in \mathcal{H}$.

This function was studied by Mordell (1920).

As a function of z it is the unique holomorphic function satisfying

$$h(z) + h(z+1) = \frac{2}{\sqrt{-i\tau}} e^{\pi i (z+1/2)^2/\tau}$$
$$h(z) + e^{-2\pi i z - \pi i \tau} h(z+\tau) = 2e^{-\pi i z - \pi i \tau/4}.$$

Furthermore, it satisfies

$$h\Big(\frac{z}{\tau}; -\frac{1}{\tau}\Big) = \sqrt{-i\tau}\, e^{-\pi i z^2/\tau} h(z;\tau).$$

Properties of ν

The function μ has the symmetry property

$$\mu(u,v) = \mu(v,u),$$

the elliptic transformation properties

$$\mu(u+1,v) = -\mu(u,v)$$
$$a^{-1} b q^{-1/2} \mu(u+\tau, v) = -\mu(u,v) - i a^{-1/2} b^{1/2} q^{-1/8},$$

and the "modular" transformation properties

$$\mu(u,v;\tau+1) = \zeta_8^{-1} \mu(u,v)$$
$$\frac{1}{\sqrt{-i\tau}} e^{\pi i (u-v)^2/\tau} \mu\Big(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\Big) = -\mu(u,v) + \frac{1}{2i} h(u-v;\tau),$$

with $\zeta_N = \exp(2\pi i/N)$.

Near Jacobiness

These properties show that μ behaves *nearly* like a Jacobi form of weight 1/2 with two elliptic variables.

Its failure to transform exactly like a Jacobi form depends only on the difference $u - v$.

The non-holomorphic part

We can construct a second, but now non-holomorphic, function R whose "non-Jacobiness" matches that of μ

$$R(u;\tau) := \sum_{\nu \in \frac{1}{2}+\mathbf{Z}} \Big\{ \operatorname{sgn}(\nu) - E\big((\nu + \operatorname{Im} u/y)\sqrt{2y}\big) \Big\} (-1)^{\nu - \frac{1}{2}} a^{-\nu} q^{-\nu^2/2},$$

where $y = \operatorname{Im}\tau$ and E is the odd entire function, defined by

$$E(z) = 2\int_0^z e^{-\pi u^2} du.$$

Properties of R

This function has the elliptic transformation properties

$$R(u+1) = -R(u)$$
$$a^{-1}q^{-1/2}R(u+\tau) = -R(u) + 2a^{-1/2}q^{-1/8},$$

and the modular transformation properties

$$R(u;\tau+1) = \zeta_8^{-1}R(u;\tau)$$
$$\frac{1}{\sqrt{-i\tau}}e^{\pi i u^2/\tau}R\big(\frac{u}{\tau};-\frac{1}{\tau}\big) = -R(u;\tau) + h(u;\tau).$$

Properties of $\widehat{\mu}$

If we now set

$$\widehat{\mu}(u,v;\tau) := \mu(u,v;\tau) + \frac{i}{2}R(u-v;\tau),$$

then $\widehat{\mu}$ is symmetric in u and v and satisfies the elliptic transformation properties

$$\widehat{\mu}(u+1,v;\tau) = a^{-1}bq^{-1/2}\widehat{\mu}(u+\tau,v;\tau) = -\widehat{\mu}(u,v;\tau)$$

and the modular transformation properties

$$\widehat{\mu}(u,v;\tau+1) = \zeta_8^{-1}\widehat{\mu}(u,v;\tau),$$
$$\widehat{\mu}\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right) = -\sqrt{-i\tau}e^{-\pi i(u-v)^2/\tau}\widehat{\mu}(u,v;\tau).$$

A mock Jacobi form

So $\widehat{\mu}$ transforms like a Jacobi form of weight $1/2$ and index $\left(\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}\right)$.

Of course, $\widehat{\mu}$ is no longer holomorphic.

A period integral

In applications we usually specialize the elliptic variables u and v to torsion points: elements of $\mathbf{Q}\tau + \mathbf{Q}$.

This kind of specialization done on Jacobi forms, gives functions of τ, which are modular forms up to a rational power of q.

For $u = \alpha\tau - \beta$, with $\alpha, \beta \in \mathbf{R}$, $|\alpha| < 1/2$, we get

$$e^{-\pi i\alpha^2\tau + 2\pi i\alpha(\beta+1/2)}R(\alpha\tau-\beta) = -\int_{-\overline{\tau}}^{i\infty}\frac{g_{\alpha+1/2,\beta+1/2}(z)}{\sqrt{-i(z+\tau)}}dz,$$

with

$$g_{\alpha,\beta}(z) := \sum_{\nu\in\alpha+\mathbf{Z}}\nu q^{\nu^2/2}e^{2\pi i\nu\beta},$$

a unary theta function of weight $3/2$.

Higher level Appell functions

For $l \in \mathbf{Z}_{>0}$ the *level l Appell function* A_l is defined by

$$A_l(u,v) = A_l(u,v;\tau) := a^{l/2} \sum_{n\in\mathbf{Z}} \frac{(-1)^{ln} q^{ln(n+1)/2} b^n}{1 - aq^n},$$

where as usual $a = \exp(2\pi i u)$, $b = \exp(2\pi i v)$ and $q = \exp(2\pi i \tau)$.

For $l = 1$ we have

$$A_1(u,v) = \vartheta(v)\mu(u,v).$$

Reduction to level 1

We can reduce the study of these functions to the case $l = 1$, with the equations

$$A_l(u,v;\tau) = \sum_{k=0}^{l-1} a^k A_1(lu, v + k\tau + (l-1)/2; l\tau),$$

$$A_l(u,v;\tau) = \frac{1}{l} a^{(l-1)/2} \sum_{k \bmod l} A_1(u, (v+k)/l + (l-1)\tau/2l; \tau/l).$$

The first one follows from

$$\frac{1}{1-x} = \frac{1 + x + \ldots + x^{l-1}}{1 - x^l},$$

and the second one from

$$\frac{1}{l} \sum_{k \bmod l} e^{2\pi i n k/l} = \begin{cases} 1 & \text{if } n \equiv 0 \bmod l, \\ 0 & \text{otherwise.} \end{cases}$$

Transformation properties of $\widehat{A}_l$

Using the correction term for μ we can now find a correction term for A_l to get $\widehat{A}_l$ which has the elliptic transformation properties

$$\begin{aligned}
\widehat{A}_l(u+1,v) &= (-1)^l \widehat{A}_l(u,v), \\
\widehat{A}_l(u,v+1) &= \widehat{A}_l(u,v), \\
\widehat{A}_l(u+\tau,v) &= (-1)^l a^l b^{-1} q^{l/2} \widehat{A}_l(u,v), \\
\widehat{A}_l(u,v+\tau) &= a^{-1} \widehat{A}_l(u,v),
\end{aligned}$$

and the modular transformation properties

$$\begin{aligned}
\widehat{A}_l(u,v;\tau+1) &= \widehat{A}_l(u,v;\tau), \\
\widehat{A}_l\Big(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\Big) &= \tau e^{\pi i (2v - lu)u/\tau} \widehat{A}_l(u,v;\tau).
\end{aligned}$$

We see that $\widehat{A}_l$ transforms as a Jacobi form of weight 1 and index $\left(\begin{smallmatrix} -l & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and we could call A_l a mixed mock Jacobi form.

Examples

Using these results we can find the transformation properties of the mock theta function f.

We set

$$\widehat{f}(\tau) = q^{-1/24} f(q) + \frac{i}{\sqrt{3}} \int_{-\bar{\tau}}^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}} dz,$$

with

$$g(\tau) = \sum_{n \equiv 1 \bmod 6} n\, q^{n^2/24}.$$

Then $\widehat{f}$ is a harmonic weak Maaß form of weight $1/2$ on $\Gamma_0(2)$.

The rank generating function

We can also apply these results to the rank generating function

$$\mathcal{R}(w;q) = \frac{1-w}{(q)_\infty}\sum_{n\in\mathbf{Z}}\frac{(-1)^n q^{3n^2/2+n/2}}{1-wq^n}.$$

We take $w=\zeta\neq 1$ a root of unity and add a correction term

$$\widehat{\mathcal{R}}(\zeta;q) = q^{-1/24}\mathcal{R}(\zeta;q) - \frac{i}{12}(\zeta^{1/2}-\zeta^{-1/2})\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{g(z)}{\sqrt{-i(z+\tau)}}dz,$$

with $g(\tau)=\sum_{n\in\mathbf{Z}} n\left(\frac{12}{n}\right)\zeta^{n/2}q^{n^2/24}$.

Then $\widehat{\mathcal{R}}(\zeta;q)$ is a harmonic weak Maaß form of weight $1/2$ on some congruence subgroup of $SL_2(\mathbf{Z})$ (of finite index).

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