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Quantum Black Holes, Wall Crossing, and Mock Modular Forms

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Abstract: We show that the quantum degeneracies of single-centered black holes in $\mathcal{N} = 4$ theories are coefficients of a *mock modular form*. The failure of modularity of such a form is of a very special type and is governed by a holomorphic modular form called the *shadow*, such that the sum of the mock modular form and a simple non-holomorphic, real-analytic function obtained from the shadow transforms like a modular form and is called its *completion*. The shadow can be viewed as a holomorphic anomaly associated with the completion. The spectral-flow invariant partition function of these black holes is a *mock Jacobi form*.

Mock modularity is a consequence of the meromorphy of the generalized elliptic genus which is closely related to the wall-crossing phenomenon. The completion makes manifest the modular symmetries expected from holography and provides a starting point for a Rademacher-type expansion of the degeneracies with implications for the exact quantum entropy and the Poincaré series. Mock modular forms are thus expected to provide a proper framework for AdS_2/CFT_1 and AdS_3/CFT_2 holography in the context of the MSW string for $\mathcal{N} = 2$ black holes, and more generally have applications in other physical problems involving noncompact conformal field theories and meromorphic Jacobi forms.

Keywords: black holes, modular forms, superstrings, dyons.

Contents

1. Introduction

Since this paper is of possible interest to both theoretical physicists (especially string theorists) and theoretical mathematicians (especially number theorists), we give two introductions in their respective dialects.

1.1 Introduction for mathematicians

In the quantum theory of black holes in the context of string theory, the physical problem of counting the dimensions of certain eigenspaces ("the number of quarter-BPS dyonic states of a given charge") has led to the study of Fourier coefficients of certain *meromorphic* Siegel modular forms and to the question of the modular nature of the corresponding generating functions. Using the results given by S. Zwegers [75], we show that these generating functions belong to the recently invented class of functions called *mock modular forms*.

Since this notion is still not widely known, it will be reviewed in some detail (in §7.1). Very roughly, mock modular forms of weight k form a vector space $\mathbb{M}^!_k$ that fits into a short exact sequence

$$0 \longrightarrow M^!_k \longrightarrow \mathbb{M}^!_k \xrightarrow{\mathcal{S}} \overline{M_{2-k}}\,, \tag{1.1}$$

where the objects of $\mathbb{M}^!_k$ are holomorphic functions f in the upper half-plane which, after the addition of a suitable non-holomorphic integral of their "shadow function" $\overline{\mathcal{S}(f)} \in M_{2-k}$, transform like ordinary holomorphic modular forms of weight k. Functions of this type occur in several contexts in mathematics: as certain q-hypergeometric series (like Ramanujan's original mock theta functions), as generating functions of class numbers of imaginary quadratic fields, and as the Fourier coefficients of meromorphic Jacobi forms. It is this last occurrence, studied by Zwegers in his thesis [75], which is at the origin of the connection to black hole theory, because the Fourier coefficients of meromorphic Jacobi forms have the same wall-crossing behavior as the one exhibited by the degeneracies of BPS states.

The specific meromorphic Jacobi forms which will be of interest to us will be the Fourier-Jacobi coefficients $\psi_m(\tau, z)$ of the meromorphic Siegel modular form

$$\frac{1}{\Phi_{10}(\Omega)} \;=\; \sum_{m=-1}^{\infty} \psi_m(\tau,z)\,p^m\,, \qquad \left(\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix},\; p = e^{2\pi i\sigma}\right), \tag{1.2}$$

the reciprocal of the Igusa cusp form of weight 10, which arises as the partition function of quarter-BPS dyons in the type II compactification on the product of a $K3$ surface and an elliptic curve. These coefficients, after multiplication by the discriminant function $\Delta(\tau)$, are meromorphic Jacobi forms of weight 2 with a double pole at $z = 0$ and no others (up to translation by the period lattice). We will study functions with these properties, and show that, up to the addition of ordinary weak Jacobi forms, they can all be obtained as linear combinations of certain forms $\mathcal{Q}_D$ and their images under Hecke operators, where D ranges over all products of an even number of distinct primes. Each $\mathcal{Q}_D$ corresponds to a mock modular form of weight 3/2, the first of these ($D = 1$) being the generating functions of class numbers mentioned above, and the second one ($D = 6$) the mock theta function of weight 3/2 with shadow $\eta(\tau)$ given in [74].

1.2 Introduction for physicists

The microscopic quantum description of supersymmetric black holes in string theory usually starts with a brane configuration of given charges and mass at weak coupling, which is localized at a single point in the noncompact spacetime. One then computes an appropriate indexed partition function in the world-volume theory of the branes. At strong coupling, the same configuration gravitates and the indexed partition function is expected to count the microstates of these macroscopic gravitating configurations. Assuming that the gravitating configuration is a single-centered black hole then gives a way to obtain a statistical understanding of the entropy of the black hole in terms of its microstates, in accordance with the Boltzmann relation[1].

One problem that one often encounters is that the macroscopic configurations are no longer localized at a point and include not only a single-centered black hole of interest but also several multi-centered ones. Moreover, the degeneracy of the multi-centered configurations typically jumps upon crossing walls of marginal stability in the moduli space where the multi-centered configuration breaks up into its single-centered constituents.

If one is interested in the physics of the horizon or the microstates of a single black hole, the multi-centered configurations and the 'wall-crossing phenomenon' of jumps in the indexed degeneracies are thus something of a nuisance. It is desirable to have a mathematical characterization that singles out the single-centered black holes directly at the microscopic level. One distinguishing feature of single-centered black holes is that they are *immortal* in that they exist as stable quantum states for all values of the moduli. We will use this property later to define the counting function for immortal black holes.

The wall-crossing phenomenon raises important conceptual questions regarding the proper holographic formulation of the near horizon geometry of a single-centered black hole. The near

[1] It is usually assumed that the index equals the absolute number, following the dictum that whatever can get paired up will get paired up. For a justification of this assumption see [60, 61, ?].

horizon geometry of a BPS black hole is the anti de Sitter space AdS_2 which is expected to be holographically dual to a one-dimensional conformal field theory CFT_1 [1]. In many cases, the black hole can be viewed as an excitation of a black string. The near horizon geometry of a black string is AdS_3 which is expected to be holographically dual to a two-dimensional conformal field theory CFT_2. The conformal boundary of Euclidean AdS_3 is a 2-torus with a complex structure parameter τ. A physical partition function of AdS_3 and of the boundary CFT_2 will be a function of τ. The $SL(2,\mathbb{Z})$ transformations of τ can be identified geometrically with global diffeomorphisms of the boundary of AdS_3 space. The partition function is expected to have good modular properties under this geometric symmetry. This symmetry has important implications for the Rademacher-type expansions of the black hole degeneracies for understanding the quantum entropy of these black holes via the AdS_2/CFT_1 holography [67, 60, 61]. It has implications also for the Poincaré series and the associated Farey tail expansion [24, 22, 50]. As we will see, implementing the modular symmetries and other symmetries presents several conceptual subtleties in situations when there is wall-crossing.

The wall-crossing phenomenon has another important physical implication for the invariance of the spectrum under large gauge transformations. Large gauge transformations lead to the 'spectral flow symmetry' of the partition function of the black string. Since these transformations act both on the charges and the moduli, degeneracies of states with a charge vector Γ at some point ϕ in the moduli space get mapped to the degeneracies to of states with charge vector Γ' at some other point ϕ' in the moduli space. Typically, there are many walls separating the point ϕ' and the original point ϕ. As a result, the degeneracies extracted from the black string at a *given* point ϕ in the moduli space do not exhibit the spectral-flow symmetry. On the other hand, the spectrum of immortal black holes is independent of asymptotic moduli and hence must exhibit the spectral-flow symmetry. This raises the question as to how to make the spectral-flow symmetry manifest for the degeneracies of immortal black holes in the generic situation when there is wall-crossing.

With these motivations, our objective will be to isolate the partition functions of the black string and of immortal black holes and investigate their transformation properties under the boundary modular group and large gauge transformations. More precisely, we would like to investigate the following four questions.

1. Can one define a *microscopic* counting function that clearly separates the microstates of immortal black holes[2] from those of multi-centered black configurations?

[2] In addition to the multi-centered configurations, there can also be contributions from the 'hair' degrees of freedom, which are degrees of freedom localized outside the black hole horizon. In this paper, we will not explicitly analyze the hair contributions and will refer to the moduli-independent part of the degeneracies as the degeneracies of immortal black holes. In certain frames where the black hole is represented entirely in terms of D-branes, the only hair modes are expected to be the zero modes which are already taken into account.

2. What are the modular properties of the counting function of immortal black holes in situations where the spectrum exhibits the wall-crossing phenomenon?
3. Can this counting function be related to a quantity that is properly modular as might be expected from the perspective of AdS_3/CFT_2 holography?
4. Can one define a partition function of the immortal black holes that manifestly exhibits the spectral-flow symmetry resulting from large gauge transformations?

The main difficulties in answering these questions stem from the complicated moduli dependence of the black hole spectrum which is often extremely hard to compute. To address the central conceptual issues in a tractable context, we consider the compactification of Type-II on $K3 \times T^2$ with $\mathcal{N} = 4$ supersymmetry in four dimensions. The spectrum of quarter-BPS dyonic states in this model is exactly computable [26, 32, 62, 63, 44, 21] and by now is well understood at *all* points in the moduli space [59, 14, 10] and for *all* possible duality orbits [14, 6, 7, 5, 16]. Moreover, as we shall see, this particular model exhibits almost all of the essential issues that we wish to address. The $\mathcal{N} = 4$ black holes have the remarkable property that even though their spectrum is moduli-dependent, the partition function itself is moduli-independent. The entire moduli dependence of the black hole degeneracy is captured by the moduli dependence of the choice of the Fourier contour [59, 14, 10].

The number of microstates of quarter-BPS black holes is given by a Fourier coefficient of a meromorphic Jacobi form with a moduli- dependent contour. The Jacobi form itself can be identified with a generalized elliptic genus of the dual CFT_2 which is a specific solvable $(0, 4)$ superconformal field theory (SCFT), but with a target space that becomes non-compact at certain walls in the moduli space[3]. The noncompactness of the target space is what is responsible for the poles in the elliptic genus. The partition function (1.2) referred to earlier is the generating function for these elliptic genera. Using this simplicity of the moduli dependence and the knowledge of the exact spectrum, it is possible to give very precise answers to the above questions in the $\mathcal{N} = 4$ framework. They turn out to naturally involve mock modular forms as we summarize below.

1. One can define a holomorphic function for counting the microstates of immortal black holes as a Fourier coefficient of the partition function of the black string for a specific choice of the Fourier contour [59, 14, 10]. The contour corresponds to choosing the asymptotic moduli of the theory in the attractor region of the single- centered black hole.
2. Because the black string partition function is a *meromorphic* Jacobi form, the counting function of immortal black holes is a *mock* modular form in that it fails to be modular

[3] For an SCFT with a compact target manifold, the trace of a certain operator in its Hilbert space can be interpreted as the elliptic genus of the target manifold. See (4.40) for the definition. We will denote such a trace in any SCFT as 'elliptic genus' even when the target manifold is noncompact. See §**??**.

but in a very specific way. The failure of modularity is governed by a *shadow*, which is a holomorphic modular form.

3. Given a mock modular form and and its shadow, one can define its *completion* which is a non-holomorphic modular form. The failure of holomorphy can be viewed as a 'holomorphic anomaly' which is also governed by the shadow.
4. The partition function of immortal black holes with manifest spectral-flow invariance is a mock Jacobi form– a new mathematical object defined and elaborated upon in §7.3.

The main physical payoff of the mathematics of mock modular forms in this context is the guarantee that one can still define a non- holomorphic partition function as in (3) which *is* modular. As mentioned earlier, the modular transformations on the τ parameter can be identified with global diffeomorphisms of the boundary of the near horizon AdS_3. This connection makes the mathematics of mock modular forms physically very relevant for AdS_3/CFT_2 holography in the presence of wall-crossing and holomorphic anomalies.

Modular symmetries are very powerful in physics applications because they relate strong coupling to weak coupling, or high temperature to low temperature. Since the completion of a mock modular form is modular, we expect this formalism to be useful in more general physics contexts. As we will explain, in the present context, mock modularity of the counting function is a consequence of meromorphy of the generalized elliptic genus. Meromorphy in turn is a consequence of noncompactness of the target space of the boundary SCFT. Now, conformal field theories with a noncompact target space occur naturally in several physics contexts. For example, a general class of four-dimensional BPS black holes obtained as supersymmetric D-brane configuration in Type-II compactification on a Calabi Yau three-fold X_6. In the M-theory limit, these black holes can be viewed as excitations of the MSW black string [49, 52]. The microscopic theory describing the low energy excitations of the MSW string is the $(0, 4)$ MSW SCFT. The target space of this SCFT will generically be noncompact and hence its elliptic genus can be a meromorphic Jacobi form[4]. Very similar objects [73, 38] have already made their appearance in the context of topological supersymmetric Yang-Mills theory on $\mathbb{CP}^2$ [69]. Other possible examples include quantum Liouville theory and E-strings [51] where the CFT is noncompact. We expect that the framework of mock modular forms and in particular the definitions and theorems discussed in §7 will be relevant in these varied physical contexts.

1.3 Organization of the paper

In §2, we review the physics background concerning the string compactification on $K3 \times T^2$ and the classification of BPS states corresponding to the supersymmetric black holes in this theory. In sections §3, §4, and §5, we review the basic mathematical definitions of various types

[4] We should emphasize that meromorphy of the generalized elliptic genus does not follow from noncompactness alone and is determined by the dynamics of the noncompact coordinates. See §**??**.

of classical modular forms (elliptic, Jacobi, Siegel) and illustrate an application to the physics of quantum black holes in each case by means of an example. In §6, we review the moduli dependence of the Fourier contour prescription for extracting the degeneracies of quarter-BPS black holes in the $\mathcal{N} = 4$ theory from the partition function which is a meromorphic Siegel modular form. In §7, we review the properties of mock modular forms and define the notion of a mock Jacobi form. In §8, we review a theorem due to Zwegers for Fourier coefficients of meromorphic Jacobi forms with a single pole. We reformulate his result in the language of mock modular forms, and then generalize this theorem to the physically relevant case of Fourier coefficients of meromorphic Jacobi forms with a double pole. In §10, we apply the theorem in the physical context to compute the counting function for single centered black hole, its shadow, and its modular completion. We briefly discuss the relation to the holomorphic anomaly and comment upon possible applications to $\mathcal{N} = 2$ and AdS/CFT holography.

2. Review of Type-II superstring theory on $K3 \times T^2$

Superstring theories are naturally formulated in ten- dimensional Lorentzian spacetime $\mathcal{M}_{10}$. A 'compactification' to four-dimensions is obtained by taking $\mathcal{M}_{10}$ to be a product manifold $\mathbb{R}^{1,3} \times X_6$ where X_6 is a compact Calabi-Yau threefold and $\mathbb{R}^{1,3}$ is the noncompact Minkowski spacetime. We will focus in this paper on a compactification of Type-II superstring theory when X_6 is itself the product $X_6 = K3 \times T^2$. A highly nontrivial and surprising result from the 90s is the statement that this compactification is quantum equivalent or 'dual' to a compactification of heterotic string theory on $T^4 \times T^2$ where T^4 is a four-dimensional torus [39, 71]. One can thus describe the theory either in the Type-II frame or the heterotic frame.

The four-dimensional theory in $\mathbb{R}^{1,3}$ resulting from this compactification has $\mathcal{N} = 4$ supersymmetry[5]. The massless fields in the theory consist of 22 vector multiplets in addition to th supergravity multiplet. The massless moduli fields consist of the S-modulus λ taking values in the coset

$$SL(2,\mathbb{Z})\backslash SL(2;\mathbb{R},)/O(2;\mathbb{R},), \tag{2.1}$$

and the T-moduli μ taking values in the coset

$$O(22,6;\mathbb{Z})\backslash O(22,6;\mathbb{R},)/O(22;\mathbb{R},) \times O(6;\mathbb{R}). \tag{2.2}$$

The group of discrete identifications $SL(2,\mathbb{Z})$ is called S-duality group. In the heterotic frame, it is the electro-magnetic duality group [55, 56], whereas in the type-II frame, it is simply

[5]This supersymmetry is a super Lie algebra containing $ISO(1,3) \times SU(4)$ as the bosonic subalgebra where $ISO(1,3)$ is the Poincaré symmetry of the $\mathbb{R},^{1,3}$ spacetime and $SU(4)$ is an internal symmetry called R-symmetry in physics literature. The odd generators of the superalgebra are called supercharges. With $\mathcal{N} = 4$ supersymmetry, there are eight complex supercharges which transform as a spinor of $ISO(1,3)$ and a fundamental of $SU(4)$.

the group of area- preserving global diffeomorphisms of the T^2 factor. The group of discrete identifications $O(22,6;\mathbb{Z})$ is called the T-duality group. Part of the T-duality group $O(19,3;\mathbb{Z})$ can be recognized as the group of geometric identifications on the moduli space of K3; the other elements are stringy in origin and have to do with mirror symmetry.

At each point in the moduli space of the internal manifold $K3 \times T^2$, one has a distinct four- dimensional theory. One would like to know the spectrum of particle states in this theory. Particle states are unitary irreducible representations, or supermultiplets, of the $\mathcal{N} = 4$ superalgebra. The supermultiplets are of three types which have different dimensions in the rest frame. A long multiplet is 256- dimensional, an intermediate multiplet is 64-dimensional, and a short multiplet is 16- dimensional. A short multiplet preserves half of the eight supersymmetries (*i.e.* it is annihilated by four supercharges) and is called a half-BPS state; an intermediate multiplet preserves one quarter of the supersymmetry (*i.e.* it is annihilated by two supercharges), and is called a quarter-BPS state; and a long multiplet does not preserve any supersymmetry and is called a non-BPS state. One consequence of the BPS property is that the spectrum of these states is 'topological' in that it does not change as the moduli are varied, except for jumps at certain walls in the moduli space [72].

An important property of the BPS states that follows from the superalgebra is that their mass is determined by the charges and the moduli [72]. Thus, to specify a BPS state at a given point in the moduli space, it suffices to specify its charges. The charge vector in this theory transforms in the vector representation of the T-duality group $O(22,6;\mathbb{Z})$ and in the fundamental representation of the S-duality group $SL(2,\mathbb{Z})$. It is thus given by a vector $\Gamma^{I\alpha}$ with integer entries

$$\Gamma^{I\alpha} = \begin{pmatrix} N^I \\ M^I \end{pmatrix} \qquad \text{where} \qquad I = 1,2,\ldots 28; \quad \alpha = 1,2 \tag{2.3}$$

transforming in the $(2,28)$ representation of $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$. The vectors N and M can be regarded as the quantized electric and magnetic charge vectors of the state respectively. They both belong to an even, integral, self-dual lattice $\Pi^{22,6}$. We will assume in what follows that $\Gamma = (N, M)$ in (2.3) is primitive in that it cannot be written as an integer multiple of (N_0, M_0) for N_0 and M_0 belonging to $\Pi^{22,6}$. A state is called purely electric if only N is non-zero, purely magnetic if only M is non- zero, and dyonic if both M and N are non-zero.

To define S-duality transformations, it is convenient to represent the S-modulus as a complex field S taking values in the upper half plane. An S-duality transformation

$$\gamma \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z}) \tag{2.4}$$

acts simultaneously on the charges and the S-modulus by

$$\begin{pmatrix} N \\ M \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix} , \qquad S \to \frac{aS+b}{cS+d} \, . \tag{2.5}$$

To define T-duality transformations, it is convenient to represent the T-moduli by a 28×28 matrix μ_I^A satisfying

$$\mu^t \, L \, \mu = L \tag{2.6}$$

with the identification that $\mu \sim k\mu$ for every $k \in O(22;\mathbb{R}) \times O(6;\mathbb{R})$. Here L is the 28×28 matrix

$$L_{IJ} = \begin{pmatrix} -\mathbf{C}_{16} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_6 \\ \mathbf{0} & \mathbf{I}_6 & \mathbf{0} \end{pmatrix} , \tag{2.7}$$

with $\mathbf{I}_s$ the $s \times s$ identity matrix and $\mathbf{C}_{16}$ is the Cartan matrix of $E_8 \times E_8$. The T-moduli are then represented by the matrix

$$\mathcal{M} \; = \; \mu^t \mu \tag{2.8}$$

which satisifies

$$\mathcal{M}^t \; = \; \mathcal{M} \, , \qquad \mathcal{M}^t L \mathcal{M} = L \, . \tag{2.9}$$

In this basis, a T-duality transformation can then be represented by a 28×28 matrix R with integer entries satisfying

$$R^t L R \; = \; L \, , \tag{2.10}$$

which acts simultaneously on the charges and the T-moduli by

$$N \to RN; \quad M \to RM \, ; \quad \mu \to \mu R^{-1} \tag{2.11}$$

Given the matrix μ_I^A, one obtains an embedding $\Lambda^{22,6} \subset \mathbb{R}^{22,6}$ of $\Pi^{22,6}$ which allows us to define the moduli-dependent charge vectors Q and P by

$$Q^A = \mu_I^A N_I \, , \qquad P^A = \mu_I^A M_I \, . \tag{2.12}$$

The matrix L has a 22-dimensional eigensubspace with eigenvalue -1 and a 6- dimensional eigensubspace with eigenvalue $+1$. Given Q and P, one can define the 'right-moving' and 'left-moving' charges[6] $Q_{R,L}$ and $P_{L,R}$ as the projections

$$Q_{R,L} \; = \; \frac{(1 \pm L)}{2} Q \, ; \qquad P_{R,L} \; = \; \frac{(1 \pm L)}{2} \, P \, . \tag{2.13}$$

[6]The right-moving charges couple to the graviphoton vector fields associated with the right-moving chiral currents in the conformal field theory of the dual heterotic string.

If the vectors N and M are nonparallel, then the state is quarter-BPS. On the other hand, if $N = pN_0$ and $M = qN_0$ for some $N_0 \in \Pi^{22,6}$ with p and q relatively prime integers, then the state is half-BPS.

An important piece of nonperturbative information about the dynamics of the theory is the exact spectrum of all possible dyonic BPS-states at all points in the moduli space. More specifically, one would like to compute the number $d(\Gamma)|_{S,\mu}$ of dyons of a given charge Γ at a specific point (S, μ) in the moduli space. Computation of these numbers is of course a very complicated dynamical problem. In fact, for a string compactification on a general Calabi-Yau threefold, the answer is not known. One main reason for focusing on this particular compactification on $K3 \times T^2$ is that in this case the dynamical problem has been essentially solved and the exact spectrum of dyons is now known. Furthermore, the results are easy to summarize and the numbers $d(\Gamma)|_{S,\mu}$ are given in terms of Fourier coefficients of various modular forms.

In view of the duality symmetries, it is useful to classify the inequivalent duality orbits labeled by various duality invariants. This leads to an interesting problem in number theory of classification of inequivalent duality orbits of various duality groups such as $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$ in our case and more exotic groups like $E_{7,7}(\mathbb{Z})$ for other choices of compactification manifold X_6. It is important to remember though that a duality transformation acts simultaneously on charges and the moduli. Thus, it maps a state with charge Γ at a point in the moduli space (S, μ) to a state with charge Γ' but at some other point in the moduli space (S', μ'). In this respect, the half-BPS and quarter-BPS dyons behave differently.

- For half-BPS states, the spectrum does not depend on the moduli. Hence $d(\Gamma)|_{S',\mu'} = d(\Gamma)|_{S,\mu}$. Furthermore, by an S-duality transformation one can choose a frame where the charges are purely electric with $M = 0$ and $N \neq 0$. Single-particle states have N primitive and the number of states depends only on the T-duality invariant integer $n \equiv N^2/2$. We can thus denote the degeneracy of half-BPS states $d(\Gamma)|_{S',\mu'}$ simply by $d(n)$.

- For quarter-BPS states, the spectrum does depend on the moduli, and $d(\Gamma)|_{S',\mu'} \neq d(\Gamma)|_{S,\mu}$. However, the partition function turns out to be independent of moduli and hence it is enough to classify the inequivalent duality orbits to label the partition functions. For the specific duality group $SL(2,\mathbb{Z}) \times O(22,6;\mathbb{Z})$ the partition functions are essentially labeled by a single discrete invariant [14, 4, 5].

$$I = \gcd(N \wedge M)\,, \tag{2.14}$$

The degeneracies themselves are Fourier coefficients of the partition function. For a given value of I, they depend only on[7] the moduli and the three T-duality invariants

[7] There is an additional dependence on arithmetic T-duality invariants but the degeneracies for states with

$(m, n, \ell) \equiv (M^2/2, N^2/2, N \cdot M)$. Integrality of (m, n, ℓ) follows from the fact that both N and M belong to $\Pi^{22,6}$. We can thus denote the degeneracy of these quarter-BPS states $d(\Gamma)|_{S,\mu}$ simply by $d(m, n, l)|_{S,\mu}$. For simplicity, we consider only $I = 1$ in this paper.

Given this classification, it is useful to choose a representative set of charges that can sample all possible values of the three T-duality invariants. For this purpose, we choose a point in the moduli space where the torus T^2 is a product of two circles $S^1 \times \tilde{S}^1$ and choose the following charges in a Type-IIB frame.

- For electric charges, we take n units of momentum along the circle S^1, and $\tilde{K}$ Kaluza-Klein monopoles associated with the circle $\tilde{S}^1$.
- For magnetic charges, we take Q_1 units of D1-brane charge wrapping S^1, Q_5 D5-brane wrapping $K3 \times S^1$ and l units of momentum along the $\tilde{S}^1$ circle.

We can thus write

$$\Gamma = \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} 0, & n; & 0, & \tilde{K} \\ Q_1, & \tilde{n}; & Q_5, & 0 \end{bmatrix} . \tag{2.15}$$

The T-duality quadratic invariants can be computed using a restriction of the matrix (2.7) to a $\Lambda^{(2,2)}$ Narain lattice of the form

$$L = \begin{pmatrix} \mathbf{0} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{pmatrix} , \tag{2.16}$$

to obtain

$$M^2/2 = Q_1 Q_5 \, , \quad N^2/2 = n\tilde{K} \, , \quad N \cdot M = \tilde{n}\tilde{K} \, . \tag{2.17}$$

We can simply the notation further by choosing $\tilde{K} = Q_5 = 1$, $Q_1 = m$, $\tilde{n} = l$ to obtain

$$M^2/2 = m, \quad N^2/2 = n, \quad N \cdot M = l \, . \tag{2.18}$$

For this set of charges, we can focus our attention on a subset of T-moduli associated with the torus T^2 parametrized by

$$\mathcal{M} = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix} , \tag{2.19}$$

where G_{ij} is the metric on the torus and B_{ij} is the antisymmetric tensor field. Let $U = U_1 + iU_2$ be the complex structure parameter, A be the area, and ϵ_{ij} be the Levi-Civita symbol with $\epsilon_{12} = -\epsilon_{21} = 1$, then

$$G_{ij} = \frac{A}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} \quad \text{and} \quad B_{ij} = AB\epsilon_{ij} \, , \tag{2.20}$$

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nontrivial values of these T-duality invariants can be obtained from the degeneracies discussed here by demanding S-duality invariance [5].

and the complexified Kähler modulus $U = U_1 + iU_2$ is defined as $U := B + iA$. The S-modulus $S = S_1 + S_2$ is defined as $S := a + i\exp(-2\phi)$ where a is the axion and ϕ is the dilaton field in the four dimensional heterotic frame. the relevant moduli can be parametrized by three complex scalars S, T, U which define the so-called 'STU' model in $\mathcal{N} = 2$ supergravity. Note that these moduli are labeled naturally in the heterotic frame which are related to the S_B, T_B, and U_B moduli in the Type-IIB frame by

$$S \;=\; U_B, \quad T = S_B, \quad U = T_B\,. \tag{2.21}$$

3. Modular forms in one variable

Before discussing mock modular forms, it is useful to recall the variety of modular objects that have already made their appearance in the context of counting black holes. In the following sections we give the basic definitions of modular forms, Jacobi forms, and Siegel forms, using the notations that are standard in the mathematics literature, and then in each case illustrate a physics application to counting quantum black holes by means of an example.

In the physics context, these modular forms arise as generating functions for counting various quantum black holes in string theory. The structure of poles of the counting function is of particular importance in physics, since it determines the asymptotic growth of the Fourier coefficients as well as the contour dependence of the Fourier coefficients which corresponds to the wall crossing phenomenon. These examples will also be relevant later in §10 in connection with mock modular forms. We suggest chapters I and III of [40] respectively as a good general reference for classical and Siegel modular forms and [28] for Jacobi modular forms.

3.1 Basic definitions and properties

Let $\mathbb{H}$ be the upper half plane, *i.e.*, the set of complex numbers τ whose imaginary part satisfies $\mathrm{Im}(\tau) > 0$. Let $SL(2,\mathbb{Z})$ be the group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with integer entries such that $ad - bc = 1$.

A *modular form* $f(\tau)$ of weight k on $SL(2,\mathbb{Z})$ is a holomorphic function on $\mathbb{H}$, that transforms as

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k\, f(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \tag{3.1}$$

for an integer k (necessarily even if $f(0) \neq 0$). It follows from the definition that $f(\tau)$ is periodic under $\tau \to \tau + 1$ and can be written as a Fourier series

$$f(\tau) \;=\; \sum_{n=-\infty}^{\infty} a(n)\, q^n q \qquad \left(q := e^{2\pi i \tau}\right), \tag{3.2}$$

and is bounded as $\mathrm{Im}(\tau) \to \infty$. If $a(0) = 0$, then the modular form vanishes at infinity and is called a *cusp form*. Conversely, one may weaken the growth condition at ∞ to $f(\tau) = \mathcal{O}(q^{-N})$ rather than $\mathcal{O}(1)$ for some $N \geq 0$; then the Fourier coefficients of f have the behavior $a(n) = 0$ for $n < -N$. Such a function is called a *weakly holomorphic modular form*.

The vector space over $\mathbb{C}$ of holomorphic modular forms of weight k is usually denoted by M_k. Similarly, the space of cusp forms of weight k and the space of weakly holomorphic modular forms of weight k are denoted by S_k and $M_k^!$ respectively. We thus have the inclusion

$$S_k \subset M_k \subset M_k^! \,. \tag{3.3}$$

The growth properties of Fourier coefficients of modular forms are known:

1. $f \in M_k^! \Rightarrow a_n = \mathcal{O}(e^{C\sqrt{n}})$ as $n \to \infty$ for some $C > 0$;
2. $f \in M_k \Rightarrow a_n = \mathcal{O}(n^{k-1})$ as $n \to \infty$;
3. $f \in S_k \Rightarrow a_n = \mathcal{O}(n^{k/2})$ as $n \to \infty$.

Some important modular forms on $SL(2,\mathbb{Z})$ are:

1. The *Eisenstein series* $E_k \in M_k$ $(k \geq 4)$. The first two of these are

$$E_4(\tau) \;=\; 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240q + \ldots \,, \tag{3.4}$$

$$E_6(\tau) \;=\; 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} = 1 - 504q + \ldots \,. \tag{3.5}$$

2. The *discriminant function* Δ. It is given by the product expansion

$$\Delta(\tau) \;=\; q \prod_{n=1}^{\infty} (1-q^n)^{24} \;=\; q - 24q^2 + 252q^3 + ... \tag{3.6}$$

   or by the formula $\Delta = (E_4^3 - E_6^2)/1728$. We mention for later use that the function $E_2(\tau) = \dfrac{1}{2\pi i}\dfrac{\Delta'(\tau)}{\Delta(\tau)} = 1 - 24 \sum_{n=1}^{\infty} \dfrac{nq^n}{1-q^n}$ is also an Eisenstein series, but is not modular. (It is a so-called *quasimodular form*, meaning in this case that the non-holomorphic function $\widehat{E}_2(\tau) = E_2(\tau) - \dfrac{3}{\pi \mathrm{Im}(\tau)}$ transforms like a modular form of weight 2.)

The two forms E_4 and E_6 generate the ring of modular forms on $SL(2,\mathbb{Z})$, so that any modular form of weight k can be written (uniquely) as a sum of monomials $E_4^\alpha E_6^\beta$ with $4\alpha + 6\beta = k$. We also have $M_k = \mathbb{C} \cdot E_k \oplus S_k$ and $S_k = \Delta \cdot M_{k-12}$, so that any $f \in M_k$ also has a unique expansion as $\sum_{0\le n\le k/12} \alpha_n \, E_{k-12n} \, \Delta^n$ (with $E_0 = 1$). From either representation, we see that a modular form is uniquely determined by its weight and first few Fourier coefficients.

Given two modular forms (f, g) of weight (k, l), one can produce a sequence of modular forms of weight $k + l + 2n$, $n \ge 0$ using the *Rankin-Cohen bracket*

$$[f, g]_n \;=\; [f, g]_n^{(k,l)} \;=\; \sum_{r+s=n} (-1)^r \binom{k+n-1}{r} \binom{\ell+n-1}{s} f^{(s)}(\tau) g^{(r)}(\tau) \tag{3.7}$$

where $f^{(m)} := \left(\frac{1}{2\pi i}\frac{d}{d\tau}\right)^m f$. For $n = 0$, this-1 is simply the product of the two forms, and for $n > 0$ $[f, g]_n \in S_{k+l+2n}$. Some examples are

$$[E_4, E_6]_1 \;=\; -3456\, \Delta \,, \qquad [E_4, E_4]_2 = 4800\, \Delta \,. \tag{3.8}$$

As well as modular forms on the full modular group $SL(2, \mathbb{Z})$, one can also consider modular forms on subgroups of finite index, with the same transformation law (3.1) and suitable conditions on the Fourier coefficients to define the notions of holomorphic, weakly holomorphic and cusp forms. The weight k now need no longer be even, but can be odd or even half integral, the easiest way to state the transformation property when $k \in \mathbb{Z} + \frac{1}{2}$ being to say that $f(\tau)/\theta(\tau)^{2k}$ is invariant under some congruence subgroup of $SL(2, \mathbb{Z})$, where $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$. The graded vector space of modular forms on a fixed subgroup $\Gamma \subset SL(2, \mathbb{Z})$ is finite dimensional in each weight, finitely generated as an algebra, and closed under Rankin-Cohen brackets. Important examples of modular forms of half-integral weight are the *unary theta series*, i.e., theta series associated to a quadratic form in one variable. They come in two types:

$$\sum_{n\in\mathbb{Z}} \varepsilon(n)\, q^{\lambda n^2} \qquad \text{for some } \lambda \in \mathbb{Q}_+ \text{ and some even periodic function } \varepsilon \tag{3.9}$$

and

$$\sum_{n\in\mathbb{Z}} n\, \varepsilon(n)\, q^{\lambda n^2} \qquad \text{for some } \lambda \in \mathbb{Q}_+ \text{ and some odd periodic function } \varepsilon \,, \tag{3.10}$$

the former being a modular form of weight 1/2 and the latter a cusp form of weight 3/2. A theorem of Serre and Stark says that in fact every modular form of weight 1/2 is a linear combination of form of the type (3.9), a simple example being the identity

$$\eta(\tau) \;:=\; q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \;=\; \sum_{n=1}^{\infty} \chi_{12}(n)\, q^{n^2/24} \,, \tag{3.11}$$

proved by Euler for the so-called *Dedekind eta function* $\eta(t) = \Delta(\tau)^{1/24}$. Here χ_{12} is the function of period 12 defined by

$$\chi_{12}(n) \;=\; \begin{cases} +1 & \text{if } n \equiv \pm 1 \pmod{12} \\ -1 & \text{if } n \equiv \pm 5 \pmod{12} \\ 0 & \text{if } (n, 12) > 1 \,. \end{cases} \tag{3.12}$$

3.2 Quantum black holes and modular forms

Modular forms occur naturally in the context of counting the Dabholkar-Harvey (DH) states [17, 15], which are states in the string Hilbert space that are dual to perturbative BPS states. The spacetime helicity supertrace counting the degeneracies reduces to the partition function of a chiral conformal field theory on a genus-one worldsheet. The τ parameter above becomes the modular parameter of the genus one Riemann surface. The degeneracies are given by the Fourier coefficients of the partition function.

A well-known simple example is the partition function $Z(\tau)$ which counts the half-BPS DH states for the Type-II compactification on $K3 \times T^2$ considered here. In the notation of (2.3) these states have zero magnetic charge $M = 0$, but nonzero electric charge N with the T-duality invariant $N^2 = 2n$, which can be realized for example by setting $Q_1 = Q_5 = l = 0$ in (2.15). They are thus purely electric and perturbative in the heterotic frame[8]. The partition function is given by the partition function of the chiral conformal field theory of 24 left-moving transverse bosons of the heterotic string. The Hilbert space $\mathcal{H}$ of this theory is a unitary Fock space representation of the commutation algebra

$$[a_{in}, a^{\dagger}_{jm}] \;=\; \delta_{ij}\, \delta_{n+m,0} \qquad (i,\, j = 1, \ldots, 24\,, \quad n,\, m = 1,\, 2, \ldots, \infty) \tag{3.13}$$

of harmonic modes of oscillations of the string in 24 different directions. The Hamiltonian is

$$H \;=\; \sum_{i=1}^{24} n\, a^{\dagger}_{in}\, a_{in} \;-\; 1\,, \tag{3.14}$$

and the partition function is

$$Z(\tau) \;=\; \mathrm{Tr}_{\mathcal{H}}(q^H)\,. \tag{3.15}$$

This can be readily evaluated since each oscillator mode of energy n contributes to the trace

$$1 + q^n + q^{2n} + \ldots \;=\; \frac{1}{1-q^n}\,. \tag{3.16}$$

The partition function then becomes

$$Z(\tau) \;=\; \frac{1}{\Delta(\tau)}\,, \tag{3.17}$$

where Δ is the cusp form (3.6). Since Δ has a simple zero at $q = 0$, the partition function itself has a pole at $q = 0$, but has no other poles in $\mathbb{H}$. Hence, $Z(\tau)$ is a weakly holomorphic

[8]Not all DH states are half-BPS. For example, the states that are perturbative in the Type-II frame correspond to a Type-II string winding and carrying momentum along a cycle in T^2. For such states both M and N are nonzero and nonparallel, and hence the state is quarter- BPS.

modular form of weight -12. This property is essential in the present physical context since it determines the asymptotic growth of the Fourier coefficients.

The degeneracy $d(n)$ of the state with electric charge N depends only on the T-duality invariant integer n and is given by

$$Z(\tau) \ = \ \sum_{n=-1}^{\infty} d(n)\, q^n \, . \tag{3.18}$$

For the Fourier integral

$$d(n) \ = \int_C e^{-2\pi i \tau n} Z(\tau) d\tau \, , \tag{3.19}$$

one can choose the contour $\mathcal{C}$ in $\mathbb{H}$ to be

$$0 \leq \mathrm{Re}(\tau) < 1 \, , \tag{3.20}$$

for a fixed imaginary part $\mathrm{Im}(\tau)$. Since the partition function has no poles in $\mathbb{H}$ except at $q = 0$, smooth deformations of the contour do not change the Fourier coefficients and consequently the degeneracies $d(n)$ are uniquely determined from the partition function. This reflects the fact that the half-BPS states are immortal and do not decay anywhere in the moduli space. As a result, there is no wall crossing phenomenon, and no jumps in the degeneracy.

In number theory, the partition function above is well-known in the context of the problem of partitions of integers. We can therefore identify

$$d(n) \ = \ p_{24}(n+1) \qquad (n \geq 0) \, . \tag{3.21}$$

where $p_{24}(I)$ is the number of colored partitions of a positive integer I using integers of 24 different colors.

These states have a dual description in the Type-II frame where they can be viewed as bound states of Q_1 number of D1-branes and Q_5 number of D5-branes with $M^2/2 = Q_1 Q_5 \equiv m$. This corresponds to setting $n = \tilde{K} = l = 0$ in (2.15). In this description, the number of such bound states $d(m)$ equals the orbifold Euler character $\chi(\mathrm{Sym}^{m+1}(K3))$ of the symmetric product of $(m+1)$ copies of $K3$-surface [69]. The generating function for the orbifold Euler character

$$\hat{Z}(\sigma) = \sum_{m=-1}^{\infty} \chi(\mathrm{Sym}^{m+1}(K3))\, p^m \qquad \left(p := e^{2\pi i \sigma}\right) \tag{3.22}$$

can be evaluated [35] to obtain

$$\hat{Z}(\sigma) = \frac{1}{\Delta(\sigma)} \, . \tag{3.23}$$

Duality requires that the number of immortal BPS-states of a given charge must equal the number of BPS-states with the dual charge. The equality of the two partition functions (3.17) and (3.23) coming from two very different counting problems is consistent with this expectation. This fact was indeed one of the early indications of a possible duality between heterotic and Type-II strings [69].

The DH-states correspond to the microstates of a small black hole [57, 12, 19] for large n. The macroscopic entropy $S(n)$ of these black holes should equal the asymptotic growth of the degeneracy by the Boltzmann relation

$$S(n) = \log d(n); \quad n \gg 1\,. \tag{3.24}$$

In the present context, the macroscopic entropy can be evaluated from the supergravity solution of small black holes [45, 48, 47, 46, 12, 19]. The asymptotic growth of the microscopic degeneracy can be evaluated using the Hardy-Ramanujan expansion (Cardy formula). There is a beautiful agreement between the two results [12, 42]

$$S(n) \;=\; \log d(n) \sim 4\pi\sqrt{n} \quad n \gg 1\,. \tag{3.25}$$

Given the growth properties of the Fourier coefficients mentioned above, it is clear that, for a black hole whose entropy scales as a power of n and not as $\log(n)$, the partition function counting its microstates can be only weakly holomorphic and not holomorphic.

These considerations generalize in a straightforward way to congruence subgroups of $SL(2,\mathbb{Z})$ which are relevant for counting the DH-states in various orbifold compactifications with $\mathcal{N}=4$ or $\mathcal{N}=2$ supersymmetry [13, 58, 18].

4. Jacobi forms

4.1 Definitions

Consider a holomorphic function $\varphi(\tau,z)$ fvrom $\mathbb{H}\times\mathbb{C}$ to $\mathbb{C}$ which is "modular in τ and elliptic in z" in the sense that it transforms under the modular group as

$$\varphi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) \;=\; (c\tau+d)^k\, e^{\frac{2\pi i m c z^2}{c\tau+d}}\,\varphi(\tau,z) \qquad \forall \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2;\mathbb{Z}) \tag{4.1}$$

and under the translations of z by $\mathbb{Z}\tau+\mathbb{Z}$ as

$$\varphi(\tau, z+\lambda\tau+\mu) \;=\; e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(\tau,z) \qquad \forall \quad \lambda,\,\mu\in\mathbb{Z}\,, \tag{4.2}$$

where k is an integer and m is a positive integer.

These equations include the periodicities $\varphi(\tau+1,z)=\varphi(\tau,z)$ and $\varphi(\tau,z+1)=\varphi(\tau,z)$, so φ has a Fourier expansion

$$\varphi(\tau,z) \;=\; \sum_{n,r} c(n,r)\,q^n\,y^r\,, \qquad\qquad (q:=e^{2\pi i\tau},\ y:=e^{2\pi i z})\,. \tag{4.3}$$

Equation (4.2) is then equivalent to the periodicity property

$$c(n,r) \;=\; C(4nm-r^2;\,r)\;, \qquad \text{where } C(d;r) \text{ depends only on } r\,(\mathrm{mod}\,2m)\;. \tag{4.4}$$

The function $\varphi(\tau,z)$ is called a *holomorphic Jacobi form* (or simply a *Jacobi form*) of weight k and index m if the coefficients $C(d;r)$ vanish for $d<0$, *i.e.* if

$$c(n,r) \;=\; 0 \qquad \text{unless} \qquad 4mn \geq r^2\,. \tag{4.5}$$

It is called a *Jacobi cusp form* if it satisfies the stronger condition that $C(d;r)$ vanishes unless d is strictly positive, *i.e.*

$$c(n,r)=0 \qquad \text{unless} \qquad 4mn > r^2\;, \tag{4.6}$$

and conversely, it is called a *weak Jacobi form* if it satisfies the weaker condition

$$c(n,r) \;=\; 0 \qquad \text{unless} \qquad n \geq 0 \tag{4.7}$$

rather than (4.5).

4.2 Theta expansion and Taylor expansion

A Jacobi form has two important representations, the *theta expansion* and the *Taylor expansion*. In this subsection, we explain both of these and the relation between them.

If $\varphi(\tau,z)$ is a Jacobi form, then the transformation property (4.2) implies its Fourier expansion with respect to z has the form

$$\varphi(\tau,z) \;=\; \sum_{\ell\in\mathbb{Z}} q^{\ell^2/4m}\, h_\ell(\tau)\, e^{2\pi i\ell z} \tag{4.8}$$

where $h_\ell(\tau)$ is periodic in ℓ with period $2m$. In terms of the coefficients (4.4) we have

$$h_\ell(\tau) \;=\; \sum_d C(d;\ell)\, q^{d/4m} \qquad\qquad (\ell\,\in\,\mathbb{Z}/2m\mathbb{Z})\;. \tag{4.9}$$

Because of the periodicity property, equation (4.8) can be rewritten in the form

$$\varphi(\tau,z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\vartheta_{m,\ell}(\tau,z)\,, \tag{4.10}$$

where $\vartheta_{m,\ell}(\tau,z)$ denotes the standard index m theta function

$$\vartheta_{m,\ell}(\tau,z) \;:=\; \sum_{\substack{r\in\mathbb{Z}\\ r\equiv\ell\,(\mathrm{mod}\,2m)}} q^{r^2/4m}\, y^r \;\;=\; \sum_{n\in\mathbb{Z}} q^{(\ell+2mn)^2/4m}\, y^{\ell+2mn} \tag{4.11}$$

(which is a Jacobi form of weight $\frac{1}{2}$ and index m on some subgroup of $SL(2,\mathbb{Z})$). This is the theta expansion of φ. The coefficiens $h_\ell(\tau)$ are modular forms of weight $k-\frac{1}{2}$ and are weakly holomorphic, holomorphic or cuspidal if φ is a weak Jacobi form, a Jacobi form or a Jacobi cusp form, respectively. More precisely, the vector $h := (h_1, \ldots, h_{2m})$ transforms like a modular form of weight $k-\frac{1}{2}$ under $SL(2,\mathbb{Z})$.

A Jacobi form $\varphi \in J_{k,m}$ (strong or weak) also has a Taylor expansion in z which for k even takes the form

$$\varphi(\tau,z) \;=\; \xi_0(\tau) + \left(\frac{\xi_1(\tau)}{2} + \frac{m\xi_0'(\tau)}{k}\right)(2\pi i z)^2 + \left(\frac{\xi_2(\tau)}{24} + \frac{m\xi_1'(\tau)}{2\,(k+2)} + \frac{m^2\xi_0''(\tau)}{2k(k+1)}\right)(2\pi i z)^4 + \cdots \tag{4.12}$$

with $\xi_\nu \in M_{k+2\nu}(SL(2,\mathbb{Z}))$ and the prime denotes $\frac{1}{2\pi i}\frac{d}{d\tau}$ as before. In terms of the Fourier coefficients of φ,

$$\frac{(k+2\nu-2)!}{(k+\nu-2)!}\,\xi_\nu(\tau) \;=\; \sum_{n=0}^{\infty}\left(\sum_r P_{\nu,k}(nm,r^2)c(n,r)\right)q^n \tag{4.13}$$

where $P_{\nu,k}$ is a homogeneous polynomial of degree ν in r^2 and n with coefficients depending on k and m, the first few being

$$\begin{aligned} P_{0,k} &= 1\ , \\ P_{1,k} &= kr^2 - 2nm\ , \\ P_{2,k} &= (k+1)(k+2)r^4 - 12(k+1)r^2mn + 12m^2n^2\ . \end{aligned} \tag{4.14}$$

The Jacobi form φ is determined by the first $m+1$ coefficients ξ_ν, and the map $\varphi \mapsto (\xi_0, \ldots, \xi_m)$ is an isomorphism from $J^{\text{weak}}_{k,m}$ to $M_k \oplus M_{k+2} \oplus \cdots \oplus M_{k+2m}$. For k odd, the story is similar except that (4.12) must be replaced by

$$\varphi(\tau,z) \;=\; \xi_0(\tau)\,(2\pi i z) \;+\; \left(\frac{\xi_1(\tau)}{6} + \frac{m\xi_0'(\tau)}{k+2}\right)(2\pi i z)^3 \;+\; \cdots \tag{4.15}$$

with $\xi_\nu \in M_{k+2\nu+1}(SL(2,\mathbb{Z}))$, and the map $\varphi \mapsto (\xi_0, \ldots, \xi_{m-2})$ gives an isomorphism from $J^{\text{weak}}_{k,m}$ to $M_{k+1} \oplus M_{k+3} \oplus \cdots \oplus M_{k+2m-3}$.

We also observe that even if φ is weak, so that the individual coefficients $c(n,r)$ grow like $C^{\sqrt{4nm-r^2}}$, the coefficients $\sum_r P_{\nu,k}(nm,r^2)\,c(n,r)$ of ξ_ν still have only polynomial growth. We thus have the following descriptions (analogous to those given for classical modular forms in §3.1) of holomorphic and weak Jacobi forms in terms of their asymptotic properties:

$$\begin{aligned}
\text{Holomorphic Jacobi} \quad &\iff \quad c(n,r)=0 \text{ for } 4mn-r^2<0\\
&\iff \quad \text{the function } q^{m\alpha^2}\varphi(\tau,\alpha\tau+\beta) \text{ (which is a modular form of weight } k\\
&\qquad\quad \text{and some level) is bounded as } \tau_2\to\infty \text{ for every } \alpha,\beta\in\mathbb{Q}\\
&\iff \quad \text{all } h_j(\tau) \text{ in (4.10) are bounded as } \tau_2\to\infty\\
&\iff \quad c(n,r) \text{ have polynomial growth.}
\end{aligned}$$

$$\begin{aligned}
\text{Weak Jacobi} \quad &\iff \quad c(n,r)=0 \text{ for } n<0\\
&\iff \quad \varphi(\tau,\alpha\tau+\beta) \text{ is bounded as } \tau_2\to\infty \text{ for any fixed } z\in\mathbb{C}\\
&\iff \quad \text{all } h_j(\tau)=O(q^{-j^2/4m}) \text{ as } \tau_2\to\infty\\
&\iff \quad \textstyle\sum_r P_{\nu,k}(nm,r^2)c(n,r) \text{ have polynomial growth.}
\end{aligned}$$

Finally, the relation between the Taylor expansion (4.12) of a Jacobi form and its theta expansion (4.10) is given by

$$\xi_\nu(\tau) \;=\; (4m)^\nu \binom{k+2\nu-2}{\nu}^{-1} \sum_{\ell\,(\mathrm{mod}\,2m)} \bigl[h_\ell(\tau),\,\vartheta^0_{m,\ell}(\tau)\bigr]_\nu\,, \tag{4.16}$$

where $[\ ,\ ]_\nu=[\ ,\ ]_\nu^{(k-\frac12,\frac12)}$ denotes the Rankin-Cohen bracket (which, as we mentioned above, also works in half-integral weight), and $\vartheta^0_{m,\ell}(\tau)=\vartheta_{m,\ell}(\tau,0)$ (Thetanullwerte). There is a similar formula in the odd case, but with $\vartheta^0_{m,\ell}(\tau)$ replaced by

$$\vartheta^1_{m,\ell}(\tau) \;=\; \frac{1}{2\pi i}\frac{\partial}{\partial z}\vartheta_{m,\ell}(\tau,z)\Big|_{z=0} \;=\; \sum_{r\equiv\ell\,(\mathrm{mod}\ 2m)} r\,q^{r^2/4m}\,. \tag{4.17}$$

4.3 Hecke-like operators

In [28] Hecke operators T_ℓ acting on $J_{k,m}$ were introduced, but also various "Hecke-like" operators, again defined by the action of certain combinations of elements of $GL(2,\mathbb{Q})\ltimes\mathbb{Q}^2$, which send Jacobi forms to Jacobi forms, but now possibly changing the index. We describe three of these which will be needed later.

The first is the very simple operator U_t $(t\ge1)$ which sends $\varphi(\tau,z)$ to $\varphi(\tau,tz)$, i.e.,

$$U_t\;:\quad \sum_{n,r} c(n,r)\,q^n\,y^r \quad\mapsto\quad \sum_{n,r} c(n,r)\,q^n\,y^{tr}\,, \tag{4.18}$$

This operator maps $J_{k,m}$ to J_{k,t^2m}.

The second operator, V_t $(t\ge1)$, sends $J_{k,m}$ to $J_{k,tm}$. It is given in terms of its action on Fourier coefficients by

$$V_t\;:\quad \sum_{n,r} c(n,r)\,q^n\,y^r \quad\mapsto\quad \sum_{n,r}\Biggl(\sum_{d|(n,r,t)} d^{k-1}c\Bigl(\frac{nt}{d^2},\frac{r}{d}\Bigr)\Biggr)q^n\,y^r\,. \tag{4.19}$$

Finally, for each decomposition $m = m_1 m_2$ with $(m_1, m_2) = 1$ we have an involution W_{m_1} on $J_{k,m}$ defined in terms of the theta expansion of Jacobi forms by

$$W_{m_1} \;:\; \sum_{\ell \,(\mathrm{mod}\, 2m)} h_\ell(\tau)\, \vartheta_{m,\ell}(\tau, z) \quad \mapsto \quad \sum_{\ell \,(\mathrm{mod}\, 2m)} h_{\ell^*}(\tau)\, \vartheta_{m,\ell}(\tau, z) \tag{4.20}$$

(or equivalently by $C(d; r) \mapsto C(d; r^*)$), where the involution $\ell \mapsto \ell^*$ on $\mathbb{Z}/2m\mathbb{Z}$ is defined by

$$\ell^* \equiv -\ell \,(\mathrm{mod}\, 2m_1), \qquad \ell^* \equiv +\ell \,(\mathrm{mod}\, 2m_2)\ . \tag{4.21}$$

These operators commute and satisfy $W_{m/m_1} = (-1)^k W_{m_1}$, so that we get an eigenspace decomposition

$$J_{k,m} \;=\; \bigoplus_{\substack{\varepsilon_1,\dots,\varepsilon_t \in \{\pm 1\} \\ \varepsilon_1 \cdots \varepsilon_t \,=\, (-1)^k}} J_{k,m}^{(\varepsilon_1,\dots,\varepsilon_t)}\ , \tag{4.22}$$

where $m = p_1^{r_1} \cdots p_t^{r_t}$ is the prime power decomposition of m and ε_i is the eigenvalue of $W_{p_i^{r_i}}$.

4.4 Example: Jacobi forms of index 1

If $m = 1$, (4.4) reduces to $c(n, r) = C(4n - r^2)$ where $C(d)$ is a function of a single argument, because $4n - r^2$ determines the value of $r \,(\mathrm{mod}\, 2)$. So any $\varphi \in J_{k,1}^{\mathrm{weak}}$ has an expansion of the form

$$\varphi(\tau, z) \;=\; \sum_{n, r \in \mathbb{Z}} C(4n - r^2)\, q^n\, y^r\ . \tag{4.23}$$

It also follows that k must be even, since in general, $C(d; -r) = (-1)^k C(d; r)$.

One has the isomorphisms $J_{k,1} \cong M_k \oplus S_{k+2}$ and $J_{k,1}^{\mathrm{weak}} \cong M_k \oplus M_{k+2}$. If $\varphi \in J_{k,1}^{\mathrm{weak}}$ with an expansion as in (4.23), then

$$\varphi(\tau, 0) \;=\; \sum_{n=0}^{\infty} a(n)\, q^n\ , \qquad \frac{1}{2(2\pi i)^2} \varphi''(\tau, 0) \;=\; \sum_{n=1}^{\infty} b(n)\, q^n\ , \tag{4.24}$$

where

$$a(n) \;=\; \sum_{r \in \mathbb{Z}} C(4n - r^2)\ , \qquad b(n) \;=\; \sum_{r > 0} r^2\, C(4n - r^2)\ , \tag{4.25}$$

and the isomorphisms are given (if $k > 0$) by the map $\varphi \mapsto (\mathcal{A}, \mathcal{B})$ with

$$\mathcal{A}(\tau) \;=\; \sum a(n)\, q^n \in M_k\ , \qquad \mathcal{B}(\tau) \;=\; \sum \bigl(k b(n) - n a(n)\bigr)\, q^n \in M_{k+2}\ . \tag{4.26}$$

For $J_{k,1}$ one also has the isomorphism $J_{k,1} \cong M_{k-\frac{1}{2}}^{+}(\Gamma_0(4))$ given by

$$\varphi(\tau, z) \leftrightarrow g(\tau) \;=\; \sum_{\substack{d \geq 0 \\ d \equiv 0,\, 3 \,\mathrm{mod} 4}} C(d)\, q^d\ . \tag{4.27}$$

We have four particularly interesting examples $\varphi_{k,1}$

$$\varphi_{k,1}(\tau, z) \;=\; \sum_{n,\, r\in\mathbb{Z}} C_k(4n - r^2)\, q^n\, y^r\,, \qquad k = -2,\, 0,\, 10,\, 12\,, \tag{4.28}$$

which have the properties (defining them uniquely up to multiplication by scalars)

- $\varphi_{10,1}$ and $\varphi_{12,1}$ are the two index 1 Jacobi cusp forms of smallest weight;
- $\varphi_{-2,1}$ and $\varphi_{0,1}$ are the unique weak Jacobi forms of index 1 and weight ≤ 0;
- $\varphi_{-2,1}$ and $\varphi_{0,1}$ generate the ring of weak Jacobi forms of even weight freely over the ring of modular forms of level 1, so that

$$J^{\text{weak}}_{k,m} \;=\; \bigoplus_{j=0}^{m} M_{k+2j}\,(SL(2,\mathbb{Z}))\cdot \varphi^{j}_{-2,1}\,\varphi^{m-j}_{0,1} \qquad (k\text{ even})\,; \tag{4.29}$$

- $\varphi_{-2,1} = \varphi_{10,1}/\Delta$, $\varphi_{0,1} = \varphi_{12,1}/\Delta$, and the quotient $\varphi_{0,1}/\varphi_{-2,1} = \varphi_{12,1}/\varphi_{10,1}$ is a multiple of the Weierstrass $\wp$ function.

The Fourier coefficients of these functions can be computed from the above recursions, since the pairs $(\mathcal{A}, \mathcal{B})$ for $\varphi = \varphi_{-2,1}$, $\varphi_{0,1}$, $\varphi_{10,1}$ and $\varphi_{12,1}$ are proportional to $(0,1)$, $(1,0)$, $(0,\Delta)$ and $(\Delta, 0)$, respectively. The results for the first few Fourier coefficients are given in Table 1 below. In particular, the Fourier expansions of $\varphi_{-2,1}$ and $\varphi_{0,1}$ begin

$$\varphi_{-2,1} \;=\; \frac{(y-1)^2}{y} \;-\; 2\,\frac{(y-1)^4}{y^2}\,q \;+\; \frac{(y-1)^4(y^2-8y+1)}{y^3}\,q^2 \;+\; \cdots\;, \tag{4.30}$$

$$\varphi_{0,1} \;=\; \frac{y^2+10y+1}{y} \;+\; 2\frac{(y-1)^2\,(5y^2-22y+5)}{y^2}\,q \;+\; \cdots\;. \tag{4.31}$$

Table 1: Some Fourier coefficients

k	$C_k(-1)$	$C_k(0)$	$C_k(3)$	$C_k(4)$	$C_k(7)$	$C_k(8)$	$C_k(11)$	$C_k(12)$	$C_k(15)$
-2	1	-2	8	-12	39	-56	152	-208	513
0	1	10	-64	108	-513	808	-2752	4016	-11775
10	0	0	1	-2	-16	36	99	-272	-240
12	0	0	1	10	-88	-132	1275	736	-8040

The functions $\varphi_{k,1}$ ($k = 10,\, 0,\, -2$) can be expressed in terms of the Dedekind eta function (3.11) and the Jacobi theta functions ϑ_1, ϑ_2, ϑ_3, ϑ_4 by the formlas

$$\varphi_{10,1}(\tau, z) \;=\; \eta^{18}(\tau)\,\vartheta_1^2(\tau, z)\,, \tag{4.32}$$

$$\varphi_{-2,1}(\tau,z) \;=\; \frac{\vartheta_1^2(\tau,z)}{\eta^6(\tau)} \;=\; \frac{\varphi_{10,1}(\tau,z)}{\Delta(\tau)}\,. \tag{4.33}$$

$$\varphi_{0,1}(\tau,z) \;=\; 4\left(\frac{\vartheta_2(\tau,z)^2}{\vartheta_2(\tau)^2}+\frac{\vartheta_3(\tau,z)^2}{\vartheta_3(\tau)^2}+\frac{\vartheta_4(\tau,z)^2}{\vartheta_4(\tau)^2}\right)\,, \tag{4.34}$$

Finally, we say a few words about Jacobi forms of odd weight. Such a form cannot have index 1, as we saw. In index 2, the isomorphisms $J_{k,2} \cong S_{k+1}$ and $J_{k,2}^{\mathrm{weak}} \cong M_{k+1}$ show that the first examples of holomorphic and weak Jacobi forms occur in weights 11 and -1, respectively, and are related by $\varphi_{-1,2} = \varphi_{11,2}/\Delta$. The function $\varphi_{-1,2}$ is given explicitly by

$$\varphi_{-1,2}(\tau,z) \;=\; \frac{\vartheta_1(\tau,2z)}{\eta^3(\tau)}\,, \tag{4.35}$$

with Fourier expansion beginning

$$\varphi_{-1,2} \;=\; \frac{y^2-1}{y} \;-\; \frac{(y^2-1)^3}{y^3}\,q \;-\; 3\,\frac{(y^2-1)^3}{y^3}\,q^2 \;+\;\cdots\,, \tag{4.36}$$

and its square is related to the index 1 Jacobi forms defined above by

$$432\,\varphi_{-1,2}^2 \;=\; \varphi_{-2,1}\left(\varphi_{0,1}^3 \;-\; 3\,E_4\,\varphi_{-2,1}^2\,\varphi_{0,1} \;+\; 2\,E_6\,\varphi_{-2,1}^3\right)\,. \tag{4.37}$$

(In fact, $\varphi_{-1,2}/\varphi_{-2,1}^2$ is a multiple of $\wp'(\tau,z)$ and this equation, divided by $\varphi_{-2,1}^4$, is just the usual equation expressing ${\wp'}^2$ as a cubic polynomial in $\wp$.) It is convenient to introduce abbreviations

$$A \;=\; \varphi_{-2,1}\,, \qquad B \;=\; \varphi_{0,1}\,, \qquad C \;=\; \varphi_{-1,2}\,. \tag{4.38}$$

With these notations, the structure of the full bigraded ring of weak Jacobi forms is given by

$$J_{*,*}^{\mathrm{weak}} \;=\; \mathbb{C}[E_4,E_6,A,B,C]\,/\,(432C^2 - AB^3 + 3E_4A^3B - 2E_6A^4)\,. \tag{4.39}$$

4.5 Quantum black holes and Jacobi forms

Jacobi forms usually arise in string theory as elliptic genera of two-dimensional superconformal field theories (SCFT) with $(2,2)$ or more worldsheet supersymmetry[9]. We denote the superconformal field theory by $\sigma(\mathcal{M})$ when it corresponds to a sigma model with a target manifold $\mathcal{M}$. Let H be the Hamiltonian in the Ramond sector, and J be the left-moving $U(1)$ R-charge. The elliptic genus $\chi(\tau,z;\mathcal{M})$ is then defined as [70, 2, 3, 54] a trace over the Hilbert space $\mathcal{H}_R$ in the Ramond sector

$$\chi(\tau,z;\mathcal{M}) \;=\; \mathrm{Tr}_{\mathcal{H}_R}\left(q^H y^J (-1)^F\right)\,, \tag{4.40}$$

where F is the fermion number.

[9] An SCFT with (r,s) supersymmetries has r left-moving and s right-moving supersymmetries.

An elliptic genus so defined satisfies the modular transformation property (4.1) as a consequence of modular invariance of the path integral. Similarly, it satisfies the elliptic transformation property (4.2) as a consequence of spectral flow. Furthermore, in a unitary SCFT, the positivity of the Hamiltonian implies that the elliptic genus is a weak Jacobi form. The decomposition (4.10) follows from bosonizing the $U(1)$ current in the standard way so that the contribution to the trace from the momentum modes of the boson can be separated into the theta function (4.11). See, for example, [41, 53] for a discussion. This notion of the elliptic genus can be generalized to a $(0, 2)$ theory using a left-moving $U(1)$ charge J which may not be an R- charge. In this case spectral flow is imposed as an additional constraint and follows from gauge invariance under large gauge transformations [22, 33, 43, 23].

A particularly useful example in the present context is $\sigma(K3)$, which is a $(4, 4)$ SCFT whose target space is a $K3$ surface. The elliptic genus is a topological invariant and is independent of the moduli of the $K3$. Hence, it can be computed at some convenient point in the $K3$ moduli space, for example, at the orbifold point where the $K3$ is the Kummer surface. At this point, the $\sigma(K3)$ SCFT can be regarded as a $\mathbb{Z}_2$ orbifold of the $\sigma(T^4)$ SCFT, which is an SCFT with a torus T^4 as the target space. A simple computation using standard techniques of orbifold conformal field theory yields [34]

$$\chi(\tau, z; K3) \;=\; 2\,\varphi_{0,1}(\tau, z) \;=\; 2\sum C_0(4n - l^2)\, q^n\, y^l\,. \tag{4.41}$$

Note that for $z = 0$, the trace (4.40) reduces to the Witten index of the SCFT and correspondingly the elliptic genus reduces to the Euler character of the target space manifold. In our case, one can readily verify from (4.41) and (4.34) that $\chi(\tau, 0; K3)$ equals 24, which is the Euler character of $K3$.

A well-known physical application of Jacobi forms is in the context of the five-dimensional Strominger-Vafa black hole[68], which is a bound state of Q_1 D1-branes, Q_5 D5-branes, n units of momentum and l units of five-dimensional angular momentum [9]. The degeneracies $d_m(n, l)$ of such black holes depend only on $m = Q_1Q_5$. They are given by the Fourier coefficients $c(n, l)$ of the elliptic genus $\chi(\tau, z; \mathrm{Sym}^{m+1}(K3))$ of symmetric product of $(m+1)$ copies of $K3$-surface.

Let us denote the generating function for the elliptic genera of symmetric products of K3 by

$$\widehat{Z}(\sigma, \tau, z) := \sum_{m=-1}^{\infty} \chi(\tau, z; \mathrm{Sym}^{m+1}(K3))\, p^m \tag{4.42}$$

where $\chi_m(\tau, z)$ is the elliptic genus of $\mathrm{Sym}^m(K3)$. A standard orbifold computation [25] gives

$$\widehat{Z}(\sigma, \tau, z) = \frac{1}{p} \prod_{n>0,\, m\geq 0,\, l} \frac{1}{(1 - p^n q^m y^l)^{2C_0(nm,\ell)}} \tag{4.43}$$

in terms of the Fourier coefficients $2C_o$ of the elliptic genus of a single copy of $K3$.

For $z = 0$, it can be checked that, as expected, the generating function (4.43) for elliptic genera reduces to the generating function (3.23) for Euler characters

$$\widehat{Z}(\sigma, \tau, 0) = \hat{Z}(\sigma) = \frac{1}{\Delta(\sigma)} . \tag{4.44}$$

5. Siegel modular forms

5.1 Definitions and examples of Siegel modular forms

Let $Sp(2, \mathbb{Z})$ be the group of (4×4) matrices g with integer entries satisfying $gJg^t = J$ where

$$J \equiv \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \tag{5.1}$$

is the symplectic form. We can write the element g in block form as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.2}$$

where A, B, C, D are all (2×2) matrices with integer entries. Then the condition $gJg^t = J$ implies

$$AB^t = BA^t, \qquad CD^t = DC^t, \qquad AD^t - BC^t = \mathbf{1} , \tag{5.3}$$

Let $\mathbb{H}_2$ be the (genus two) Siegel upper half plane, defined as the set of (2×2) symmetric matrix Ω with complex entries

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \tag{5.4}$$

satisfying

$$\mathrm{Im}(\tau) > 0, \quad \mathrm{Im}(\sigma) > 0, \quad \det(\mathrm{Im}(\Omega)) > 0 . \tag{5.5}$$

An element $g \in Sp(2, \mathbb{Z})$ of the form (5.2) has a natural action on $\mathbb{H}_2$ under which it is stable:

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}. \tag{5.6}$$

The matrix Ω can be thought of as the period matrix of a genus two Riemann surface[10] on which there is a natural symplectic action of $Sp(2, \mathbb{Z})$.

A Siegel form $F(\Omega)$ of weight k is a holomorphic function $\mathbb{H}_2 \to \mathbb{C}$ satisfying

$$F\big((A\Omega + B)(C\Omega + D)^{-1}\big) \;=\; \det(C\Omega + D)^k \, F(\Omega) . \tag{5.7}$$

[10]See [30, 20, 8] for a discussion of the connection with genus-two Riemann surfaces.

A Siegel modular form can be written in terms of its Fourier series

$$F(\Omega) \;=\; \sum_{\substack{n,\,r,\,m\,\in\,\mathbb{Z}\\ r^2\le 4mn}} a(n,r,m)\, q^n\, y^r\, p^m\,. \tag{5.8}$$

If one writes this as

$$F(\Omega) = \sum_{m=0}^{\infty} \varphi^F_m(\tau,z)\, p^m \tag{5.9}$$

with

$$\varphi^F_m(\tau,z) \;=\; \sum_{n,\,r} a(n,r,m)\, q^n\, y^r\,, \tag{5.10}$$

then each $\varphi^F_m (m \ge 0)$ is a Jacobi form of weight k and index m.

An important special class of Siegel forms were studied by Maass which he called the *Spezialschar*. They have the property that $a(n,r,m)$ depends only on the discriminant $4mn-r^2$ if (n,r,m) are coprime, and more generally

$$a(n,r,m) \;=\; \sum_{d|(n,r,m),\;\, d>0} d^{k-1} C\Big(\frac{4mn-r^2}{d^2}\Big) \tag{5.11}$$

for some coefficients $C(N)$. Specializing to $m=1$, we can see that these numbers are simply the coefficients associated via (4.23) to the Jacobi form $\varphi = \varphi^F_1 \in J_{k,1}$, and that (5.11) says precisely that the other Fourier-Jacobi coefficients of F are given by $\varphi^F_m = \varphi^F_1 | V_m$ with V_m as in (4.19). Conversely, if φ is any Jacobi form of weight k and index 1 with Fourier expansion (4.23), then the function $F(\Omega)$ defined by (5.8) and (5.11) or by $F(\Omega) = \sum_{m=0}^{\infty} \big(\varphi|V_m\big)(\tau,z)\, p^m$ is a Siegel modular form of weight k with $\varphi^F_1 = \varphi$. The resulting map from $J_{k,1}$ to the Spezialschar is called the Saito-Kurokawa lift or *additive lift* since it naturally gives the sum representation of a Siegel form using the Fourier coefficients of a Jacobi form as the input. (More information about the additive lift can be found in [28].)

The example of interest to us is the Igusa cusp form Φ_{10} (the unique cusp form of weight 10) which is the Saito-Kurokawa lift of the Jacobi form $\varphi_{10,1}$ introduced earlier, so that

$$\Phi_{10}(\Omega) = \sum_{n,\,r,\,m} a_{10}(n,r,m)\, q^n\, y^r\, p^m\,, \tag{5.12}$$

where a_{10} is defined by (5.11) with $k=10$ in terms of the coefficients $C_{10}(d)$ given in Table 1.

A Siegel modular form sometimes also admits a product representation, and can be obtained as Borcherds lift or *multiplicative lift* of a weak Jacobi form of weight zero and index one. This procedure is in a sense an exponentiation of the additive lift and naturally results in the product representation of the Siegel form using the Fourier coefficients of a Jacobi form as the input.

Several examples of Siegel forms that admit product representation are known but at present there is no general theory to determine under what conditions a given Siegel form admits a product representation.

For the Igusa cusp form Φ_{10}, a product representation does exist. It was obtained by Gritsenko and Nikulin [37, 36] as a multiplicative lift of the elliptic genus $\chi(\tau, z; K3) = 2\varphi_{0,1}(\tau, z)$ and is given by

$$\Phi_{10}(\Omega) \;=\; pqy \prod_{(m,n,l)>0} \left(1 - p^m q^n y^l\right)^{2C_0(4mn-l^2)}, \tag{5.13}$$

in terms of C_0 given by (4.34, 4.28). Here the notation $(m, n, l) > 0$ means that $m,\, n,\, l\, \in \mathbb{Z}$ with either $m > 0$ or $m = 0$, $n > 0$, or $m = n = 0$, $l < 0$.

5.2 The physics of Siegel modular forms

Siegel forms occur naturally in the context of counting of quarter-BPS dyons. The partition function for these dyons depends on three (complexified) chemical potentials (σ, τ, z), conjugate to the three T-duality invariant integers (m, n, ℓ) respectively and is given by

$$\mathrm{Z}(\Omega) \;=\; \frac{1}{\Phi_{10}(\Omega)}\;. \tag{5.14}$$

The product representation of the Igusa form is particularly useful for the physics application because it is closely related to the generating function for the elliptic genera of symmetric products of $K3$ introduced earlier. This is a consequence of the fact that the multiplicative lift of the Igusa form is obtained starting with the elliptic genus of $K3$ as the input. Comparing the product representation for the Igusa form (5.13) with (4.43), we get the relation:

$$\mathrm{Z}(\sigma, \tau, z) \;=\; \frac{1}{\Phi_{10}(\sigma, \tau, z)} \;=\; \frac{\widehat{Z}(\sigma, \tau, z)}{\varphi_{10,1}(\tau, z)}\;. \tag{5.15}$$

This relation to the elliptic genera of symmetric products of $K3$ has a deeper physical significance based on what is known as the 4d-5d lift [31]. The main idea is to use the fact that the geometry of the Kaluza-Klein monopole in the charge configuration (2.15) reduces to five-dimensional flat Minkowski spacetime in the limit when the radius of the circle $\tilde{S}^1$ goes to infinity. In this limit, the charge l corresponding to the momentum around this circle gets identified with the angular momentum l in five dimensions. Our charge configuration (2.15) then reduces essentially to the Strominger-Vafa black hole [68] with angular momentum [9] discussed in the previous subsection. Assuming that the dyon partition function does not depend on the moduli, we thus essentially relate $Z(\Omega)$ to $\hat{Z}(\Omega)$. The additional factor in (5.15) involving $\Phi_{10}(\sigma, \tau, z)$ comes from bound states of momentum n with the Kaluza-Klein monopole and from the center of mass motion of the Strominger-Vafa black hole in the Kaluza-Klein geometry [30, 21].

The Igusa cusp form has double zeros at $z = 0$ and its images. The partition function is therefore a *meromorphic* Siegel form (5.7) of weight -10 with double poles at these divisors. This fact is responsible for much of the interesting physics of wall-crossings in this context as we explain in the next section.

6. Walls and contours

Given the partition function (5.14), one can extract the black hole degeneracies from the Fourier coefficients. The three quadratic T-duality invariants of a given dyonic state can be organized as a 2×2 symmetric matrix

$$\Lambda \;=\; \begin{pmatrix} N\cdot N & N\cdot M \\ N\cdot M & M\cdot N \end{pmatrix} \;=\; \begin{pmatrix} 2n & \ell \\ \ell & 2m \end{pmatrix} , \tag{6.1}$$

where the dot products are defined using the $O(22, 6; \mathbb{Z})$ invariant metric L. The matrix Ω in (5.14) and (5.4) can be viewed as the matrix of complex chemical potentials conjugate to the charge matrix Λ. The charge matrix Λ is manifestly T-duality invariant. Under an S-duality transformation (2.4), it transforms as

$$\Lambda \to \gamma \Lambda \gamma^t \tag{6.2}$$

There is a natural embedding of this physical S-duality group $SL(2, \mathbb{Z})$ into $Sp(2, \mathbb{Z})$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} (\gamma^t)^{-1} & \mathbf{0} \\ \mathbf{0} & \gamma \end{pmatrix} \;=\; \begin{pmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \in Sp(2, \mathbb{Z}) \, . \tag{6.3}$$

The embedding is chosen so that $\Omega \to (\gamma^T)^{-1} \Omega \gamma^{-1}$ and $\mathrm{Tr}(\Omega \cdot \Lambda)$ in the Fourier integral is invariant. This choice of the embedding ensures that the physical degeneracies extracted from the Fourier integral are S-duality invariant if we appropriately transform the moduli at the same time as we explain below.

To specify the contours, it is useful to define the following moduli-dependent quantities. One can define the matrix of right-moving T-duality invariants

$$\Lambda_R = \begin{pmatrix} Q_R \cdot Q_R & Q_R \cdot P_R \\ Q_R \cdot P_R & P_R \cdot P_R \end{pmatrix} , \tag{6.4}$$

which depends both on the integral charge vectors N, M as well as the T-moduli μ. One can then define two matrices naturally associated to the S-moduli $S = S_1 + iS_2$ and the T-moduli μ respectively by

$$\mathcal{S} = \frac{1}{S_2} \begin{pmatrix} |S|^2 & S_1 \\ S_1 & 1 \end{pmatrix} \quad , \quad \mathcal{T} \;=\; \frac{\Lambda_R}{|\det(\Lambda_R)|^{\frac{1}{2}}} \, . \tag{6.5}$$

Both matrices are normalized to have unit determinant. In terms of them, we can construct the moduli-dependent 'central charge matrix'

$$\mathcal{Z} = |\det(\Lambda_R)|^{\frac{1}{4}} \left(\mathcal{S} + \mathcal{T}\right), \tag{6.6}$$

whose determinant equals the BPS mass

$$M_{Q,P} \;=\; |\det \mathcal{Z}| \,. \tag{6.7}$$

We define

$$\tilde{\Omega} \equiv \begin{pmatrix} \sigma & -z \\ -z & \tau \end{pmatrix} \,. \tag{6.8}$$

This is related to Ω by an $SL(2,\mathbb{Z})$ transformation

$$\tilde{\Omega} \;=\; \hat{S}\Omega\hat{S}^{-1} \quad \text{where} \quad \hat{S} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{6.9}$$

so that, under a general S-duality transformation γ, we have the transformation $\tilde{\Omega} \to \gamma\tilde{\Omega}\gamma^T$ as $\Omega \to (\gamma^T)^{-1}\Omega\gamma^{-1}$.

With these definitions, $\Lambda, \Lambda_R, \mathcal{Z}$ and $\tilde{\Omega}$ all transform as $X \to \gamma X \gamma^T$ under an S-duality transformation (2.4) and are invariant under T-duality transformations. The moduli-dependent Fourier contour can then be specified in a duality-invariant fashion by[10]

$$\mathcal{C} = \{\mathrm{Im}\tilde{\Omega} = \varepsilon^{-1}\mathcal{Z}; \quad 0 \le \mathrm{Re}(\tau), \mathrm{Re}(\sigma), \mathrm{Re}(z) < 1\}, \tag{6.10}$$

where $\varepsilon \to 0^+$. For a given set of charges, the contour depends on the moduli S, μ through the definition of the central charge vector (6.6). The degeneracies $d(m,n,l)|_{S,\mu}$ of states with the T-duality invariants (m,n,l) at a given point (S,μ) in the moduli space are then given by[11]

$$d(m,n,l)|_{S,\mu} = \int_{\mathcal{C}} e^{-i\pi \mathrm{Tr}(\Omega\cdot\Lambda)}\; \mathrm{Z}(\Omega)\, d^3\Omega \,. \tag{6.11}$$

This contour prescription thus specifies how to extract the degeneracies from the partition function for a given set of charges and in any given region of the moduli space. In particular, it also completely summarizes all wall-crossings as one moves around in the moduli space for a fixed set of charges. Even though the indexed partition function has the same functional form throughout the moduli space, the spectrum is moduli dependent because of the moduli dependence of the contours of Fourier integration and the pole structure of the partition function. Since the degeneracies depend on the moduli *only* through the dependence of the contour $\mathcal{C}$,

[11] The physical degeneracies have an additional multiplicative factor of $(-1)^{\ell+1}$ which we omit here for simplicity of notation in later chapaters.

moving around in the moduli space corresponds to deforming the Fourier contour. This does not change the degeneracy except when one encounters a pole of the partition function. Crossing a pole corresponds to crossing a wall in the moduli s pace. The moduli space is thus divided up into domains separated by 'walls of marginal stability'. In each domain the degeneracy is constant but it jumps upon crossing a wall as one goes from one domain to the other. The jump in the degeneracy has a nice mathematical characterization. It is simply given by the residue at the pole that is crossed while deforming the Fourier contour in going from one domain to the other.

We now turn to the degeneracies of single-centered black holes. Given the T-duality invariants Λ, a single centered black hole solution is known to exist in all regions of the moduli space as long as $\det(\Lambda)$ is large and positive. The moduli fields can take any values $(\lambda_\infty, \mu_\infty)$ at asymptotic infinity far away from the black hole but the vary in the black hole geometry. Because of the attractor phenomenon [29, 66], the moduli adjust themselves so that near the horizon of the black hole of charge Λ they get attracted to the values $(\lambda_*(\Lambda), \mu_*(\Lambda))$ which are determined by the requirement that the central charge $\mathcal{Z}_*(\Lambda)$ evaluated using these moduli becomes proportional to Λ. These attractor values are independent of the asymptotic values and depend only on the charge of black hole. We call these moduli the attractor moduli. This enables us to define the *attractor contour* for a given charge Λ by fixing the asymptotic moduli to the attractor values corresponding to this charge. In this case

$$\mathcal{Z}(\lambda_\infty, \mu_\infty) \; = \; \mathcal{Z}(\lambda_*(\Lambda), \mu_*(\Lambda)) \sim \Lambda \tag{6.12}$$

and we have the attractor contour

$$\mathcal{C}_* \; = \; \{\mathrm{Im}\tilde{\Omega} \; = \; \varepsilon^{-1}\Lambda; \quad 0 \leq \mathrm{Re}(\tau), \mathrm{Re}(\sigma), \mathrm{Re}(z) < 1\} \tag{6.13}$$

which depends only on the integral charges and not on the moduli. The significance of the attractor moduli in our context stems from the fact if the asymptotic moduli are tuned to these values for given (m, n, l), then only single-centered black hole solution exists. The degeneracies $d^*(m, n, l)$ obtained using the attractor contour

$$d^*(m, n, l) = \int_{\mathcal{C}_*} e^{-i\pi \mathrm{Tr}(\Omega\cdot\Lambda)} \; \mathrm{Z}(\Omega)\, d^3\Omega \tag{6.14}$$

are therefore expected to be the degeneracies of the immortal single-centered black holes.

7. Mock modular forms

Mock modular forms are a relatively new class of modular objects (although isolated examples had been known for some time). They were first isolated explicitly by S. Zwegers in his thesis [75] as the explanation of the "mock theta functions" introduced by Ramanujan in his famous last letter to Hardy. An expository account of this work can be found in [74].

In §7.1 and §7.2, we present the definition and general properties of mock modular forms and give a number of examples. In §7.3, we introduce a notion of mock Jacobi forms (essentially, holomorphic functions of τ and z with theta expansions like that of usual Jacobi forms, but in which the coefficients $h_\ell(\tau)$ are mock modular forms) and show how the examples given in §7.2 occur naturally as pieces of mock Jacobi forms.

7.1 Mock modular forms

A mock modular form is a holomorphic function $h(\tau)$ which transforms under modular transformations almost but not quite as a modular form. The non-modularity is of a very special nature and is governed by another holomorphic function called its *shadow* which is itself an ordinary modular form.

More precisely, a (weakly holomorphic) mock modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ is the first member of a pair (h, g), where

1. h is a holomorphic function in $\mathbb{H}$ with at most exponential growth as $\tau \to \alpha$ for any $\alpha \in \mathbb{Q}$;

2. $g(\tau)$, the *shadow* of h, is an holomorphic modular form of weight $2-k$, assumed cuspidal if $k \le 1$, and

3. the sum $\widehat{h} =: h + \widetilde{g}$, called the *completion* of h, transforms like a holomorphic modular form of weight k, *i.e.* $\widehat{h}(\tau)/\theta(\tau)^{2k}$ is invariant under $\tau \to \gamma\tau$ for all $\tau \in \mathbb{H}$ and for all γ in some congruence subgroup of $SL(2,\mathbb{Z})$.

Here $\widetilde{g}(\tau)$, called the *non-holomorphic Eichler integral* of g, is the function of τ defined by

$$\widetilde{g}(\tau) = \left(\frac{i}{2\pi}\right)^{k-1} \int_{-\overline{\tau}}^{\infty} (z+\tau)^{-k}\, \overline{g(-\overline{z})}\, dz \tag{7.1}$$

(notice that the integral is independent of the path chosen, because the integrand is holomorphic in all of $\mathbb{H}$) or alternatively by

$$\widetilde{g}(\tau) \;=\; \overline{b_0}\,\frac{(4\pi\tau_2)^{-k+1}}{k-1} + \sum_{n=1}^{\infty} n^{k-1}\,\overline{b_n}\, q^{-n}\, \Gamma(1-k, 4\pi n\tau_2) \qquad \text{if} \quad g(\tau) = \sum_{n=0}^{\infty} b_n\, q^n\,, \tag{7.2}$$

where $\tau_2 = \mathrm{Im}(\tau)$ and $\Gamma(1-k,x) = \int_x^\infty t^{-k}\, e^{-t}\, dt$ is the incomplete gamma function and where the series is convergent despite the factor q^{-n} because $\Gamma(1-k,x) = O(x^{-k}e^{-x})$. The function $\widetilde{g}(\tau)$ satisfies

$$(4\pi\tau_2)^k\, \frac{\partial\widetilde{g}(\tau)}{\partial\overline{\tau}} \;=\; -2\pi i\, \overline{g(\tau)}\,, \tag{7.3}$$

and since h is holomorphic, we find that also

$$(4\pi\tau_2)^k\, \frac{\partial\widehat{h}(\tau)}{\partial\overline{\tau}} \;=\; -2\pi i\, \overline{g(\tau)}\,. \tag{7.4}$$

In the special case when the shadow g is a unary theta series as in (3.9) or (3.10) (which can only hapen if k equals $3/2$ or $1/2$, respectively), the mock modular form h is called a *mock theta function.* All of Ramanujan's examples, and all of ours in this paper, are of this type. In these cases the incomplete gamma functions in (7.2) reduce to the complementary error function:

$$\Gamma\bigl(-\frac{1}{2},\, x\bigr) \;=\; \frac{2}{\sqrt{x}}\, e^{-x} \;-\; 2\,\sqrt{\pi}\, \mathrm{erfc}\bigl(\sqrt{x}\bigr)\;, \qquad \Gamma\bigl(\frac{1}{2},\, x\bigr) \;=\; \sqrt{\pi}\, \mathrm{erfc}\bigl(\sqrt{x}\bigr)\;. \tag{7.5}$$

We denote by $\mathbb{M}^!_k$ the space of weakly holomorphic mock modular forms of weight k and arbitrary level. Clearly it contains the space $M^!_k$ of ordinary weakly holomorphic modular forms (the special case $g=0$, $h=\widehat{h}$) and we have an exact sequence

$$0 \longrightarrow M^!_k \longrightarrow \mathbb{M}^!_k \xrightarrow{\;\mathcal{S}\;} \overline{M_{2-k}}\,, \tag{7.6}$$

where the *shadow map* $\mathcal{S}$ sends h to $\overline{g}$.[12]

If we replace the condition "exponential growth" in 1. above by "polynomial growth," we get the class of *strongly holomorphic mock modular forms*, which we can denote $\mathbb{M}_k$, and an exact sequence $0 \to M_k \to \mathbb{M}_k \to \overline{M_{2-k}}$. This is not very useful, however, because there are almost no examples of "pure" mock modular forms that are strongly holomorphic, essentially the only ones being the function $\mathcal{H}$ of Example 2 below and its variants. It becomes useful if we generalize to *mixed mock modular forms* of weight (k,ℓ). These are holomorphic functions $h(\tau)$, having polynomial growth near $\partial\mathbb{H}$, which have completions $\widehat{h}$ of the form $\widehat{h} = h + \sum_j f_j\, \widetilde{g}_j$ with $f_j \in M_\ell$, $g_j \in M_{2-k-\ell}$ that transform like modular forms of weight k. The space $\mathbb{M}_{k,\ell}$ of such forms thus fits into an exact sequence

$$0 \longrightarrow M_k \longrightarrow \mathbb{M}_{k,\ell} \xrightarrow{\;\mathcal{S}\;} M_\ell \otimes \overline{M_{2-k-\ell}}\,, \tag{7.7}$$

[12]We will use the word "shadow" to denote either $g(\tau)$ or $\overline{g(\tau)}$, but the shadow *map*, which should be linear over $\mathbb{C}$, always sends h to $\overline{g}$, the complex conjugate of its holomorphic shadow. We will also often be sloppy and say that the shadow of a certain mock modular form "is" some modular form g when in fact it is merely proportional to g, since the constants occurring are of no interest and are often messy.

where the shadow map $\mathcal{S}$ now sends h to $\sum_j f_j\ \bar{g}_j$. If $\ell = 0$ this reduces to the previous definition, since each f_j is constant, but with the more general notion of mock modular forms there are now plenty of strongly holomorphic examples, and, as for ordinary modular forms, they have much nicer properties (notably, polynomial growth of their Fourier coefficients) than the weakly holomorphic ones. Note that if the shadow of a mixed mock modular form $h \in \mathbb{M}_{k,\ell}$ happens to contain only one term $f(\tau)\overline{g(\tau)}$, and if $f(\tau)$ has no zeros in the upper half-plane, then $f^{-1}\,h$ is a weakly holomorphic mock modular form of weight $k-\ell$ (and in fact, all weakly holomorphic mock modular forms arise in this way). Note also that, although $\mathbb{M}^!_{k,\ell}$ (weakly holomorphic mixed mock modular forms) can be defined in the obvious way, there is little point doing so since $\mathbb{M}^!_{k,\ell}$ is just $\mathbb{M}^!_{k-\ell} \otimes M_\ell$, as one sees easily, but there is no corresponding decomposition for the strongly holomorphic mixed objects. Finally, we mention that we can also define "even more mixed" mock modular forms by replacing $\mathbb{M}_{k,\ell}$ by $\mathbb{M}_{k,*} = \bigoplus_{\ell\in\mathbb{Z}} \mathbb{M}_{k,\ell}$, i.e., by allowing functions whose shadow is a finite sum of products $f_j(\tau)\overline{g_j(\tau)}$ with the f_j of varying weights ℓ_j and g_j of weight $2-k-\ell_j$. Natural examples will occur in §8.

7.2 Examples

One somewhat artificial example of a mock modular form is the weight 2 Eisenstein series $E_2(\tau)$ mentioned in §3.1, which was a quasimodular form: here the shadow $g(\tau)$ is a constant and the corresponding non-holomorphic Eichler integral $\widetilde{g}(\tau)$ a multiple of $\mathrm{Im}(\tau)^{-1}$. (This example, however, is exceptional. Most quasimodular forms, like $E_2(\tau)^2$, are not mock modular forms.) In this subsection we give several less trivial examples. Many more will occur later in the paper.

Example 1. In Ramanujan's famous last letter to Hardy in 1920, he gives 17 examples of mock theta functions, though without giving any complete definition of this term. All of them have weight 1/2 and are given as q-hypergeometric series. A typical example (Ramanujan's second mock theta function of "order 7"—a notion that he also does not define) is

$$\mathcal{F}_{7,2}(\tau) = -q^{-25/168}\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^n)\cdots(1-q^{2n-1})} \;=\; -q^{143/168}\left(1+q+q^2+2q^3+\cdots\right). \tag{7.8}$$

This is a mock theta function of weight 1/2 on $\Gamma_0(4)\cap\Gamma(7)$ with shadow the unary theta series

$$\sum_{n\equiv 2\ (\mathrm{mod}\ 7)} \chi_{12}(n)\, n\, q^{n^2/168}\,, \tag{7.9}$$

with $\chi_{12}(n)$ as in (3.12). The product $\eta(\tau)\mathcal{F}_{7,2}(\tau)$ is a strongly holomorphic mixed mock modular form of weight $(1, 1/2)$, and by an identity of Hickerson is equal to an indefinite theta series

$$\eta(\tau)\,\mathcal{F}_{7,2}(\tau) \;=\; \sum_{r,\,s\,\in\,\mathbb{Z}+\frac{5}{14}} \frac{1}{2}\big(\mathrm{sgn}(r)+\mathrm{sgn}(s)\big)\,(-1)^{r-s}\,q^{(3r^2+8rs+3s^2)/2}\,. \tag{7.10}$$

Example 2. The second example is the generating function of the Hurwitz-Kronecker class numbers $H(N)$. These numbers are defined for $N > 0$ as the number of $PSL(2,\mathbb{Z})$-equivalence classes of integral binary quadratic forms of discriminant $-N$, weighted by the reciprocal of the number of their automorphisms (if $-N$ is the discriminant of an imaginary quadratic field K other than $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, this is just the class number of K), and for other values of N by $H(0) = -1/12$ and $H(N) = 0$ for $N < 0$. It was shown in [73] that the function

$$\mathcal{H}(\tau) \;:=\; \sum_{N=0}^{\infty} H(N)\, q^N \;=\; -\frac{1}{12} \,+\, \frac{1}{3}q^3 \,+\, \frac{1}{2}q^4 \,+\, q^7 \,+\, q^8 \,+\, q^{11} \,+\, \cdots \tag{7.11}$$

is a mock modular form of weight $3/2$ on $\Gamma_0(4)$, with shadow the classical theta function $\theta(\tau) = \sum q^{n^2}$. Here $\mathcal{H}(\tau)$ itself is strongly holomorphic, and one does not need to multiply it by anything or to consider mixed objects.

Example 3. This example is taken from [74]. We define the function

$$F_2^{(6)}(\tau) \;=\; -\sum_{r>s>0} \chi_{12}(r^2-s^2)\, s\, q^{rs/6} \;=\; q + 2q^2 + q^3 + 2q^4 - q^5 + \cdots \tag{7.12}$$

with χ_{12} as in (3.12). Then the function $E_2(\tau) - 12F_2^{(6)}(\tau)$ is a strongly holomorphic mixed mock modular form of weight $(2, 1/2)$ on the full modular group, having the shadow $\eta(\tau)\,\overline{\eta(\tau)}$, and the quotient

$$h^{(6)}(\tau) \;=\; \frac{E_2(\tau) - 12F_2^{(6)}(\tau)}{\eta(\tau)} \;=\; q^{-1/24}\left(1 - 35q - 130q^2 - 273q^3 - 595q^4 - \cdots\right) \tag{7.13}$$

is a weakly holomorphic mock modular form of weight $3/2$ on $SL(2,\mathbb{Z})$ with shadow $\eta(\tau)$. More generally, if we define

$$F_k^{(6)}(\tau) \;=\; -\sum_{r>s>0} \chi_{12}(r^2-s^2)\, s^{k-1} q^{rs/6} \qquad (k = 2,\, 4,\, \dots)\;, \tag{7.14}$$

then for all $\nu \geq 0$ we have

$$\frac{24^\nu}{\binom{2\nu}{\nu}}\,[h^{(6)}, \eta]_\nu \;=\; E_k \;-\; 12F_k^{(6)} \;+\; \text{cusp form of weight } k\;, \quad k = 2\nu + 2\;, \tag{7.15}$$

where $[h^{(6)}, \eta]_\nu$ denotes the νth Rankin-Cohen bracket of the mock modular form h and the modular form η in weights $(3/2, 1/2)$. This statement, and the similar statements for other mock modular forms which come later, are proved by the method of holomorphic projection, which we do not explain here, and are intimately connected with the mock Jacobi forms introduced in the next subsection. (That connection will also explain the superscript "6" in (7.12)–(7.15).)

Example 4. Our last example, which is very similar to the preceding one, has recently aroused considerable interest because of the discovery by Eguchi, Ooguri and Tachikawa [27] (see also [11]) of its connection with the character table of the Mathieu group M_{24}. We now define

$$F_2^{(2)}(\tau) \;=\; -\sum_{\substack{r>s>0\\ r-s\,\text{odd}}} (-1)^r\, s\, q^{rs/2} \;=\; q+q^2-q^3+q^4-q^5+\cdots\;; \tag{7.16}$$

then $E_2(\tau)-24F_2^{(2)}(\tau)$ is a strongly holomorphic mixed mock modular form of weight $(2,3/2)$ on the full modular group, having the shadow $\eta(\tau)^3\,\overline{\eta(\tau)^3}$, and the quotient

$$h^{(2)}(\tau) \;=\; \frac{E_2(\tau)-24F_2^{(2)}(\tau)}{\eta(\tau)^3} \;=\; q^{-1/8}\left(1-45q-231q^2-770q^3-2277q^4-\cdots\right) \tag{7.17}$$

is a weakly holomorphic mock modular form of weight $1/2$ on $SL(2,\mathbb{Z})$ with shadow $\eta(\tau)^3$. As before, if we set

$$F_k^{(2)}(\tau) \;=\; -\sum_{\substack{r>s>0\\ r-s\,\text{odd}}} (-1)^r\, s^{k-1}\, q^{rs/2} \qquad (k=2,\,4,\,\dots)\;, \tag{7.18}$$

then for all $\nu\geq 0$ we have

$$\frac{8^\nu}{\binom{2\nu}{\nu}}\,[h^{(2)},\eta^3]_\nu \;=\; E_k \;-\; 24F_k^{(2)} \;+\; \text{cusp form of weight } k\;, \quad k=2\nu+2\;, \tag{7.19}$$

where $[h^{(2)},\eta^3]_\nu$ denotes the Rankin-Cohen bracket in weights $(1/2,3/2)$.

7.3 Mock Jacobi forms

By a *mock Jacobi form* (resp. a *weak mock Jacobi form*) of weight k and index m we will mean a holomorphic function φ on $\mathbb{H}\times\mathbb{C}$ which satisfies the elliptic transformation property (4.2), and hence has a Fourier expansion as in (4.3) with the periodicity property (4.4) and a theta expansion as in (4.10), and which satisfies the same cusp conditions (4.5) (resp. (4.7)) as in the classical case, but in which the modularity property with respect to the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}\times\mathbb{Z}$ is weakened: the coefficients $h_\ell(\tau)$ in (4.10) are now mock modular forms rather than modular forms of weight $k-\frac{1}{2}$, and the modularity property of φ is that the *completed* function

$$\widehat{\varphi}(\tau,z) \;=\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}} \widehat{h}_\ell(\tau)\,\vartheta_{m,\ell}(\tau,z)\,, \tag{7.20}$$

rather than φ itself, transforms according to (4.1). If g_ℓ denotes the shadow of h_l, then we have

$$\widehat{\varphi}(\tau,z) \;=\varphi(\tau,z) \;+\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}} \widetilde{g}_\ell(\tau)\,\vartheta_{m,\ell}(\tau,z)$$

with $\widetilde{g}_\ell$ as in (7.2) and hence, by (7.3),

$$\psi(\tau,z) \;:=\; \tau_2^{k-1/2}\,\frac{\partial}{\partial\overline{\tau}}\widehat{\varphi}(\tau,z) \;\doteq\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}} \overline{g_\ell(\tau)}\,\vartheta_{m,\ell}(\tau,z)\,. \tag{7.21}$$

(Here $\doteq$ indicates an omitted constant.) The function $\psi(\tau,z)$ is holomorphic in z, satisfies the same elliptic transformation property (4.2) as φ does (because each $\vartheta_{m,\ell}$ satisfies this), satisfies the heat equation $\big(8\pi i m\,\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial z^2}\big)\psi=0$ (again, because each $\vartheta_{m,\ell}$ does), and, by virtue of the modular invariance property of $\widehat{\varphi}(\tau,z)$, also satisfies the transformation property

$$\psi\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big) \;=\; |c\tau+d|\,(c\overline{\tau}+d)^{2-k}\,e^{\frac{2\pi i m c z^2}{c\tau+d}}\,\psi(\tau,z) \qquad \forall\,\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2;\mathbb{Z}) \tag{7.22}$$

with respect to the action of the modular group. These properties say precisely that ψ is a *skew-holomorphic Jacobi form* of weight $3-k$ and index m in the sense of Skoruppa [64, 65], and the above discussion can be summarized by saying that we have an exact sequence

$$0 \longrightarrow J^{\text{weak}}_{k,m} \longrightarrow \mathbb{J}^{\text{weak}}_{k,m} \xrightarrow{\;\mathcal{S}\;} J^{\text{skew}}_{3-k,m} \tag{7.23}$$

(and similarly with the word "weak" omitted), where $\mathbb{J}_{k,m}$ and $\mathbb{J}^{\text{weak}}_{k,m}$ denote the spaces of strong and weak mock Jacobi forms, respectively, and the "shadow map" $\mathcal{S}$ now sends φ to ψ.

It turns out that most of the classical examples of mock theta functions occur as the components of a vector-valued mock modular form which gives the coefficients in the theta series expansion of a mock Jacobi form. We illustrate this for the four examples introduced in the previous subsection.

Example 1. The function $\mathcal{F}_{7,2}(\tau)$ in the first example of §7.2 is actually one of three mock theta functions $\{\mathcal{F}_{7,j}\}_{j=1,2,3}$ of "order 7" defined by Ramanujan, each given by a q-hypergeometric formula like (7.8), each having a shadow $\Theta_{7,j}$ like in (7.9) but with the summation over $n\equiv j$ rather than $n\equiv 2$ modulo 7, and each satisfying an indefinite theta series identity like (7.10). We extend $\{\mathcal{F}_{7,j}\}$ to all j by defining it to be an odd periodic function of j of period 7, so that the shadow of $\mathcal{F}_{7,j}$ equals $\Theta_{7,j}$ for all $j\in\mathbb{Z}$. Then the function

$$\mathcal{F}_{42}(\tau,z) \;=\; \sum_{\ell\ (\mathrm{mod}\ 84)} \chi_{12}(\ell)\,\mathcal{F}_{7,\ell}(\tau)\,\vartheta_{42,\ell}(\tau,z) \tag{7.24}$$

belongs to $\mathbb{J}^{\text{weak}}_{1,42}$. The Taylor coefficients ξ_ν as defined in equation (4.15) are proportional to $\sum_{j=1}^{3}\big[\mathcal{F}_{7,j},\Theta_{7,j}\big]_\nu$ and have the property that their completions $\widehat{\xi}_\nu=\sum_{j=1}^{3}\big[\widehat{\mathcal{F}}_{7,j},\Theta_{7,j}\big]_\nu$ transform like modular forms of weight $2\nu+2$ on the full modular group $SL(2,\mathbb{Z})$.

Example 2. Set $\mathcal{H}_0(\tau)=\sum_{n=0}^{\infty}H(4n)q^n$ and $\mathcal{H}_1(\tau)=\sum_{n=1}^{\infty}H(4n-1)q^{n-\frac{1}{4}}$. Then the function

$$\mathcal{F}_1(\tau,z) \;=\; \mathcal{H}_0(\tau)\vartheta_{1,0}(\tau,z) \;+\; \mathcal{H}_1(\tau)\vartheta_{1,1}(\tau,z) \;=\; \sum_{n,\,r\in\mathbb{Z}} H(4n-r^2)\,q^n\,y^r\,, \tag{7.25}$$

is a mock Jacobi form of weight 2 and index 1 with shadow $\overline{\vartheta_{1,0}(\tau,0)}\,\vartheta_{1,0}(\tau,z)+\overline{\vartheta_{1,1}(\tau,0)}\,\vartheta_{1,1}(\tau,z)$. The νth Taylor coefficient ξ_ν of $\mathcal{F}_1$ is given by

$$\frac{4^\nu}{\binom{2\nu}{\nu}}\sum_{j=0}^{1}[\vartheta_{1,j},\mathcal{H}_j]_\nu \;=\; \delta_{k,2}\,E_k \;-\; F_k^{(1)} \;+\; (\text{cusp form of weight } k \text{ on } SL(2,\mathbb{Z}))\,, \tag{7.26}$$

where $k=2\nu+2$ and

$$F_k^{(1)}(\tau) \;:=\; \sum_{n>0}\Big(\sum_{d|n}\min\big(d,\frac{n}{d}\big)^{k-1}\Big)q^n \qquad (k \text{ even},\ k\ge 2)\,. \tag{7.27}$$

In fact the cusp form appearing in (7.26) is a very important one, namely (up to a factor -2) the sum of the normalized Hecke eigenforms in $S_k(SL(2,\mathbb{Z}))$, and equation (7.26) is equivalent to the famous formula of Eichler and Selberg expressing the traces of Hecke operators on $S_k(SL(2,\mathbb{Z}))$ in terms of class numbers of imaginary quadratic fields. It would be very interesting to know whether the cusp forms on $SL(2,\mathbb{Z})$ occurring in (7.15) and (7.19), and in similar examples occurring later, also have some natural arithmetic meaning.

Example 3. Write the function $h^{(6)}$ defined in (7.13) as

$$h^{(6)}(\tau) \;=\; -\sum_{\substack{D\ge -1\\ D\equiv -1 \ (\mathrm{mod}\ 24)}} C^{(6)}(D)\,q^{D/24} \tag{7.28}$$

with

D	-1	23	47	71	95	119	143	167	191
$C^{(6)}(D)$	-1	35	130	273	595	1001	1885	2925	4886

.

Then the function

$$\mathcal{F}_6(\tau,z) \;=\; \sum_{\substack{n,\,r\in\mathbb{Z}\\ 24n-r^2\ge -1}} \Big(\frac{12}{r}\Big)\,C^{(6)}(24n-r^2)\,q^n\,y^r \tag{7.29}$$

is a mock Jacobi form of index 6 (explaining the notation $h^{(6)}$). Note that, surprisingly, this is even simpler than the expansion of the index 1 mock Jacobi form just discussed, because its twelve Fourier coefficients h_ℓ are all proportional to one another, while the two Fourier coefficients h_ℓ of $\mathcal{F}_1(\tau,z)$ are not proportional. Specifically, we have $h_\ell(\tau)=\chi_{12}(\ell)h^{(6)}(\tau)$ for all ℓ, where χ_{12} is the character defined in (3.12), so that we have the factorization

$$\mathcal{F}_6(\tau,z) \;=\; h^{(6)}(\tau)\,\big(\vartheta_{6,1}(\tau,z)-\vartheta_{6,5}(\tau,z)-\vartheta_{6,7}(\tau,z)+\vartheta_{6,11}(\tau,z)\big)\,. \tag{7.30}$$

(This is related to the fact that the shadow $\eta(\tau)$ of $h^{(6)}(\tau)$ is a modular form on the full modular group, while the shadow $\theta(\tau)$ of $\mathcal{H}(\tau)$ is a modular form on a congruence subgroup.) Combining this with (4.16) and noting that the functions $\vartheta^0_{6,\ell}$ satisfy $\vartheta^0_{6,1}-\vartheta^0_{6,5}=\vartheta^0_{6,11}-\vartheta^0_{6,7}=\eta$, we see that the functions described in (7.15) are proportional to the Taylor coefficients ξ_ν of $\mathcal{F}_6$.

Example 4. The fourth example is very similar. Write the mock modular form (7.17) as

$$h^{(2)}(\tau) \;=\; -\sum_{\substack{D\geq -1\\ D\equiv -1\,(\mathrm{mod}\ 8)}} C^{(2)}(D)\, q^{D/8} \tag{7.31}$$

with

D	-1	7	15	23	31	39	47	55	63
$C^{(6)}(D)$	-1	45	231	770	2277	5796	13915	30843	65550

.

Then the function

$$\mathcal{F}_2(\tau,z) \;=\; \sum_{\substack{n,\,r\in\mathbb{Z}\\ 8n-r^2\geq -1}} \chi_4(r)\, C^{(2)}(8n-r^2)\, q^n\, y^r \;=\; h^{(2)}(\tau)\,\bigl(\vartheta_{2,1}(\tau,z)-\vartheta_{2,3}(\tau,z)\bigr)\,, \tag{7.32}$$

where $\chi_4(r) = \pm 1$ for $r \equiv \pm 1\,(\mathrm{mod}\ 4)$ and $\chi_4(r) = 0$ for r even, is a mock Jacobi form of index 2 and the functions given in (7.19) are proportional to the Taylor coefficients ξ_ν of $\mathcal{F}_2$, because $\vartheta^1_{2,1} - \vartheta^1_{2,3} = \eta^3$, where $\vartheta^1_{2,\ell}$ is defined by (4.17).

8. From meromorphic Jacobi forms to mock modular forms

In this section we consider Jacobi forms $\varphi(\tau,z)$ that are meromorphic with respect to the variable z. It was discovered by Zwegers [75] that such forms, assuming that their poles occur only at points $z \in \mathbb{Q}\tau + \mathbb{Q}$ (i.e., at torsion points on the elliptic curve $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$), have a modified theta expansion related to mock modular forms. Our treatment is based on his, but the presentation is quite different and the results go further in one key respect. We show that φ decomposes canonically into two pieces, one constructed directly from its poles and consisting of a finite linear combination of Appell-Lerch sums with modular forms as coefficients and one being a mock Jacobi form in the sense introduced in the preceding section. Each piece separately transforms like a Jacobi form with respect to elliptic transformations. Neither piece separately transforms like a Jacobi form with respect to modular transformations, but each can be completed by the addition of an explicit and elementary non-holomorphic correction term so that it does transform correctly with respect to the modular group.

In §8.1 we explain how to modify the Fourier coefficients h_ℓ defined in (4.8) when φ has poles, and use these to define a "finite part" of φ by the theta decomposition (4.10). In §8.2 we define (in the case when φ has simple poles only) a "polar part" of φ as a finite linear combination of standard Appell-Lerch sums times modular forms arising as the residues of φ at its poles, and show that φ decomposes as the sum of its finite part and its polar part. Subsection 8.3 gives the proof that the finite part of φ is a mock Jacobi form and a description of the non-holomorphic correction term needed to make it transform like a Jacobi form. This subsection also contains a summary in tabular form of the various functions that have been introduced and the relations between them. In §8.4 we describe the modifications needed in the case of double poles (the case actually needed in this paper) and in §8.5 we present a few examples to illustrate the theory. Among the “mock” parts of these are two of the most interesting mock Jacobi forms from §7 (the one related to class numbers of imaginary quadratic fields and the one conjecturally related to representations of the Mathieu group M_{24}). Many other examples will be given in §9.

Throughout the section, we use the convenient notation $\mathbf{e}(x) := e^{2\pi i x}$.

8.1 The Fourier coefficients of a meromorphic Jacobi form

As indicated above, the main problem we face is to find an analogue of the theta decomposition (4.10) of holomorphic Jacobi forms in the meromorphic case. We will approach this problem from two sides: computing the Fourier coefficients of $\varphi(\tau, z)$ with respect to z, and computing the contribution from the poles. In this subsection we treat the first of these questions.

Consider a meromorphic Jacobi form $\varphi(\tau, z)$ of weight k and index m. We assume that $\varphi(\tau, z)$ for each $\tau \in \mathbb{H}$ is a meromorphic function of z which has poles only at points $z = \alpha\tau + \beta$ wit α and β rational. In the case when φ was holomorphic, we could write its Fourier expansion in the form (4.8). By Cauchy's theorem, the coefficient $h_\ell(\tau)$ in that expansion could also be given by the integral formula

$$h_\ell^{(P)}(\tau) \;=\; q^{-\ell^2/4m} \int_P^{P+1} \varphi(\tau, z)\, \mathbf{e}(-\ell z)\, dz\,, \tag{8.1}$$

where P is an arbitrary point of $\mathbb{C}$. From the holomorphy and transformation properties of φ it follows that the value of this integral is independent of the choice of P and of the path of integration and depends only on ℓ modulo $2m$ (implying that we have the theta expansion (4.10)) and that each h_ℓ is a modular form of weight $k - \frac{1}{2}$. Here each of these properties fails: the integral (8.1) is not independent of the path of integration (it jumps when the path crosses a pole); it is not independent of the choice of the initial point P; it is not periodic in ℓ (changing ℓ by $2m$ corresponds to changing P by τ); it is not modular; and of course the expansion (4.10) cannot possibly hold since the right-hand-side has no poles in z.

To take care of the first of these problems, we specify the path of integration in (8.1) as the horizontal line from P to $P+1$. If there are poles of $\varphi(\tau, z)$ along this line, this does not make sense; in that case, we define the value of the integral as the average of the integral over a path deformed to pass just above the poles and the integral over a path just below them. (We do not allow the initial point P to be a pole of φ, so this makes sense.) To take care of the second and third difficulties, the dependence on P and the non-periodicity in ℓ, we play one of these problems off against the other. From the elliptic transformation property (4.2) we find that

$$\begin{aligned} h_\ell^{(P+\tau)}(\tau) \;&=\; q^{-\ell^2/4m} \int_P^{P+1} \varphi(\tau,\, z+\tau)\,\mathbf{e}(-\ell\,(z+\tau))\,dz \\ &=\; q^{-(\ell+2m)^2/4m} \int_P^{P+1} \varphi(\tau,\, z)\,\mathbf{e}(-(\ell+2m)\,z)\,dz \;=\; h_{\ell+2m}^{(P)}(\tau)\,, \end{aligned}$$

i.e., changing P by τ corresponds to changing ℓ by $2m$, as already mentioned. It follows that if we choose P to be $-\ell\tau/2m$ (or $-\ell\tau/2m + B$ for any $B \in \mathbb{R}$, since it is clear that the value of the integral (8.1) depends only on the height of the path of integration and not on the initial point on this line), then the quantity

$$h_\ell(\tau) \;:=\; h_\ell^{(-\ell\tau/2m)}(\tau) \tag{8.2}$$

depends only on the value of ℓ (mod $2m$). This in turn implies that the sum

$$\varphi^F(\tau,z) \;:=\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\,\vartheta_{m,\ell}(\tau,z)\,, \tag{8.3}$$

which we will call the **finite part** (or **Fourier part**) of φ, is well defined. If φ is holomorphic, then of course $\varphi^F = \varphi$, by virtue of (4.10).

Note that the definiton of $h_\ell(\tau)$ can also be written

$$h_\ell(\tau) \;=\; q^{\ell^2/4m} \int_{\mathbb{R}/\mathbb{Z}} \varphi(\tau, z - \ell\tau/2m)\;\mathbf{e}(-\ell z)\;dz\,, \tag{8.4}$$

with the same convention as above if $\varphi(\tau, z - \ell\tau/2m)$ has poles on the real line.

8.2 The polar part of φ (case of simple poles)

We now consider the contribution from the poles. To present the results we first need to fix notations for the positions and residues of the poles of our meromorphic function φ. We assume for now that the poles are all simple.

By assumption, $\varphi(\tau, z)$ has poles only at points of the form $z = z_s = \alpha\tau + \beta$ for $s = (\alpha, \beta)$ belonging to some subset S of $\mathbb{Q}^2$. The double periodicity property (4.2) implies that S is invariant under translation by $\mathbb{Z}^2$, and of course $S/\mathbb{Z}^2$ must be finite. The modular transformation property (4.1) of φ implies that S is $SL(2,\mathbb{Z})$-invariant. For each $s = (\alpha, \beta) \in S$, we set

$$D_s(\tau) \;=\; 2\pi i\;\mathbf{e}(m\alpha z_s)\,\mathrm{Res}_{z=z_s}\big(\varphi(\tau,z)\big) \qquad (s = (\alpha,\beta) \in S,\; z_s = \alpha\tau+\beta)\,. \tag{8.5}$$

The functions $D_s(\tau)$ are holomorphic modular forms of weight $k-1$ and some level, and only finitely many of them are distinct. More precisely, they satisfy

- $$D_{(\alpha+\lambda,\beta+\mu)} \;=\; \mathbf{e}(m(\mu\alpha-\lambda\beta+\lambda\mu))\, D_{(\alpha,\beta)} \quad \text{for} \quad (\lambda,\mu)\,\in\,\mathbb{Z}^2\,, \tag{8.6}$$
- $$D_s\Bigl(\frac{a\tau+b}{c\tau+d}\Bigr) \;=\; (c\tau+d)^{k-1}\, D_{s\gamma}(\tau) \quad \text{for} \quad \gamma=\bigl(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\bigr)\,\in\,SL(2,\mathbb{Z})\;, \tag{8.7}$$

as one sees from the transformation properties of φ. (The calculation is given in §8.4. It is to obtain the simple transformation equation (8.7) that we included the non-obvious factor $\mathbf{e}(m\alpha z_s)$ in (8.5).) Since we are assuming for the moment that there are no higher-order poles, all of the information about the non-holomorphy of φ is contained in these functions.

The strategy is to define a "polar part" of φ by taking the poles z_s in some fundamental parallelogram for the action of the lattice $\mathbb{Z}\tau+\mathbb{Z}$ on $\mathbb{C}$ (i.e., for $s=(\alpha,\beta)$ in the intersection of S with some square box $[A,A+1)\times[B,B+1)$) and then averaging the residues at these poles over all translations by the lattice. But we must be careful to do this in just the right way to get the desired invariance properties. For each $m\in\mathbb{N}$ we introduce the *averaging operator*

$$\mathrm{Av}^{(m)}\bigl[F(y)\bigr] \;:=\; \sum_{\lambda\in\mathbb{Z}} q^{m\lambda^2}y^{2m\lambda}F(q^\lambda y) \tag{8.8}$$

which sends any function of y ($=\mathbb{Z}$-invariant function of z) of polynomial growth in y to a function of z transforming like an index m Jacobi form under translations by the full lattice $\mathbb{Z}\tau+\mathbb{Z}$. For example, we have

$$q^{\ell^2/4m}\,\mathrm{Av}^{(m)}\bigl[y^\ell\bigr] \;=\; \sum_{\lambda\,\in\,\mathbb{Z}} q^{(\ell+2m\lambda)^2/4m}\, y^{\ell+2m\lambda} \;=\; \vartheta_{m,\ell}(\tau,z) \tag{8.9}$$

for any $\ell\in\mathbb{Z}$. If $F(y)$ itself is given as the average

$$F(y) \;=\; \mathrm{Av}_{\mathbb{Z}}\bigl[f(z)\bigr] \;:=\; \sum_{\mu\in\mathbb{Z}} f(z+\mu) \qquad (z\in\mathbb{C},\;\; y=\mathbf{e}(z)) \tag{8.10}$$

of a function $f(z)$ in $\mathbb{C}$ (of sufficiently rapid decay at infinity), then we have

$$\mathrm{Av}^{(m)}\bigl[F(y)\bigr] \;=\; \mathrm{Av}^{(m)}_{\mathbb{Z}\tau+\mathbb{Z}}\bigl[f(z)\bigr] \;:=\; \sum_{\lambda,\,\mu\in\mathbb{Z}} e^{2\pi i m(\lambda^2\tau+2\lambda z)}\, f(z+\lambda\tau+\mu)\;. \tag{8.11}$$

We want to apply the averaging operator (8.8) to the product of the function $D_s(\tau)$ with a standard rational function of y having a simple pole of residue 1 at $y=y_s=\mathbf{e}(z_s)$, but the choice of this rational function is not obvious. The right choice turns out to be $\mathcal{R}_{-2m\alpha}(y)$, where $\mathcal{R}_c(y)$ for $c\in\mathbb{R}$ is defined by

$$\mathcal{R}_c(y) \;=\; \begin{cases} \dfrac{1}{2}\,y^c\,\dfrac{y+1}{y-1} & \text{if } c\in\mathbb{Z}\,,\\[2mm] y^{\lceil c\rceil}\,\dfrac{1}{y-1} & \text{if } c\in\mathbb{R}\smallsetminus\mathbb{Z}\,. \end{cases} \tag{8.12}$$

(Here $\lceil c \rceil$ denotes the "ceiling" of c, i.e., the smallest integer $\geq c$. The right-hand side can also be written more uniformly as $\dfrac{1}{2}\,\dfrac{y^{\lfloor c \rfloor+1}+y^{\lceil c \rceil}}{y-1}$, where $\lfloor c \rfloor = -\lceil -c \rceil$ denotes the "floor" of c, i.e., the largest integer $\leq c$.) This function looks artificial, but is in fact quite natural. First of all, by expanding the right-hand side of (8.12) in a geometric series we find

$$\mathcal{R}_c(y) \;=\; \begin{cases} -\sum_{\ell\geq c}^{*} y^\ell & \text{if } |y|<1, \\ \ \ \sum_{\ell\leq c}^{*} y^\ell & \text{if } |y|>1, \end{cases} \tag{8.13}$$

where the asterisk on the summation sign means that the term $\ell = c$ is to be counted with multiplicity $1/2$ when it occurs (which happens only for $c \in \mathbb{Z}$, explaining the case distinction in (8.12)). This formula, which can be seen as the prototypical example of wall-crossing, can also be written in terms of z as a Fourier expansion (convergent for all $z \in \mathbb{C} \smallsetminus \mathbb{R}$)

$$\mathcal{R}_c(\mathbf{e}(z)) \;=\; -\sum_{\ell\in\mathbb{Z}} \frac{\operatorname{sgn}(\ell-c)\;+\;\operatorname{sgn}(z_2)}{2}\,\mathbf{e}(\ell z) \qquad (y=\mathbf{e}(z),\;\; z_2=\operatorname{Im}(z)\neq 0)\,, \tag{8.14}$$

without any case distinction. Secondly, $R_c(y)$ can be expressed in a natural way as an average:

Proposition 8.1 *For $c \in \mathbb{R}$ and $z \in \mathbb{C} \smallsetminus \mathbb{Z}$ we have*

$$\mathcal{R}_c(\mathbf{e}(z)) \;=\; Av_{\mathbb{Z}}\Big[\frac{\mathbf{e}(cz)}{2\pi i z}\Big]\,. \tag{8.15}$$

Proof: If $c \in \mathbb{Z}$, then

$$\operatorname{Av}_{\mathbb{Z}}\Big[\frac{\mathbf{e}(cz)}{2\pi i z}\Big] \;=\; \frac{y^c}{2\pi i}\sum_{n\in\mathbb{Z}}\frac{1}{z-n} \;=\; \frac{y^c}{2\pi i}\,\frac{\pi}{\tan \pi z} \;=\; \frac{y^c}{2}\,\frac{y+1}{y-1}$$

by a standard formula of Euler. If $c \notin \mathbb{Z}$ then the Poisson summation formula and (8.14) give

$$\begin{aligned} \operatorname{Av}_{\mathbb{Z}}\Big[\frac{\mathbf{e}(cz)}{2\pi i z}\Big] \;&=\; \sum_{n\in\mathbb{Z}}\frac{\mathbf{e}(c(z+n))}{2\pi i(z+n)} \;=\; \sum_{\ell\in\mathbb{Z}}\bigg(\int_{iz_2-\infty}^{iz_2+\infty}\frac{\mathbf{e}(c(z+u))}{2\pi i(z+u)}\,\mathbf{e}(-\ell u)\,du\bigg)\,\mathbf{e}(\ell z) \\ &=\; -\sum_{\ell\in\mathbb{Z}}\frac{\operatorname{sgn}(\ell-c)\;+\;\operatorname{sgn}(z_2)}{2}\,\mathbf{e}(\ell z) \;=\; \mathcal{R}_c(\mathbf{e}(z)) \end{aligned}$$

if $z_2 \neq 0$, and the formula remains true for $z_2 = 0$ by continuity. An alternative proof can be obtained by noting that $\mathbf{e}(-cz)\mathcal{R}_c(\mathbf{e}(z))$ is periodic of period 1 with respect to c and expanding it as a Fourier series in c, again by the Poisson summation formula.

For any $s = (\alpha,\beta) \in \mathbb{Q}^2$ and $m \in \mathbb{N}$ we now define

$$F_m^s(\tau,z) \;=\; \mathbf{e}(-m\alpha z_s)\,\operatorname{Av}^{(m)}\big[\mathcal{R}_{-2m\alpha}(y/y_s)\big] \qquad (y_s=\mathbf{e}(z_s)=\mathbf{e}(\beta)q^\alpha)\,, \tag{8.16}$$

an Appell-Lerch sum. It is easy to check that this function satisfies

$$F_m^{(\alpha+\lambda,\beta+\mu)} \;=\; \mathbf{e}(-m(\mu\alpha-\lambda\beta+\lambda\mu))\,F_m^{(\alpha,\beta)} \qquad (\lambda,\,\mu\in\mathbb{Z}) \tag{8.17}$$

and hence, in view of the corresponding property (8.6) of D_s, that the product $D_s(\tau)F_m^s(\tau,z)$ depends only on the class of s in $S/\mathbb{Z}^2$. We can therefore define the **polar part** of φ by the formula

$$\varphi^P(\tau,z) \;:=\; \sum_{s\in S/\mathbb{Z}^2} D_s(\tau)\,F_m^s(\tau,z)\,, \tag{8.18}$$

and it is obvious from the above discussion that this function satisfies the index m elliptic transformation property (4.2) and has the same poles and residues as φ, so that the difference $\varphi-\varphi^P$ is holomorphic and has a theta expansion. In fact, we have:

Theorem 8.1 *Let $\varphi(\tau,z)$ be a meromorphic Jacobi form with simple poles at $z=z_s=\alpha\tau+\beta$ for $s=(\alpha,\beta)\in S\subset\mathbb{Q}^2$, with Fourier coefficients $h_\ell(\tau)$ defined by (8.1) and (8.2) or by (8.4) and residues $D_s(\tau)$ defined by (8.5). Then φ has the decomposition*

$$\varphi(\tau,z) \;=\; \varphi^F(\tau,z)\;+\;\varphi^P(\tau,z)\,, \tag{8.19}$$

where φ^F and φ^P are defined by equations (8.3) and (8.18), respectively.

Proof: Fix a point $P=A\tau+B\in\mathbb{C}$ with $(A,B)\in\mathbb{R}^2\smallsetminus S$. Since φ, φ^F and φ^P are meromorphic, it suffices to prove the decomposition (8.19) on the horizontal line $\mathrm{Im}(z)=\mathrm{Im}(P)=A\tau_2$. On this line we have the Fourier expansion

$$\varphi(\tau,z) \;=\; \sum_{\ell\in\mathbb{Z}} q^{\ell^2/4m}\,h_\ell^{(P)}(\tau)\,y^\ell\,,$$

where the coefficients $h_\ell^{(P)}$ are defined by (8.1) (modified as explained in the text there if $A=\alpha$ for any $(\alpha,\beta)\in S$, but for simplicity we will simply assume that this is not the case, since we are free to choose A any way we want). Comparing this with (8.3) gives

$$\varphi(\tau,z)\;-\;\varphi^F(\tau,z) \;=\; \sum_{\ell\in\mathbb{Z}}\bigl(h_\ell^{(P)}(\tau)-h_\ell(\tau)\bigr)\,q^{\ell^2/4m}\,y^\ell \qquad (\mathrm{Im}(z)=\mathrm{Im}(P))\,. \tag{8.20}$$

But $q^{\ell^2/4m}(h_\ell^{(P)}(\tau)-h_\ell(\tau))$ is just $2\pi i$ times the sum of the residues of $\varphi(\tau,z)\mathbf{e}(-\ell z)$ in the parallelogram with width 1 and horizontal sides at heights $A\tau_2$ and $-\ell\tau_2/2m$, with the residues of any poles on the latter line being counted with multiplicity $1/2$ because of the way we defined h_ℓ in that case, so

$$\begin{aligned} q^{\ell^2/4m}\bigl(h_\ell^{(P)}(\tau)-h_\ell(\tau)\bigr) \;&=\; 2\pi i \sum_{s=(\alpha,\beta)\in S/\mathbb{Z}} \frac{\mathrm{sgn}(\alpha-A)-\mathrm{sgn}(\alpha+\ell/2m)}{2}\,\mathrm{Res}_{z=z_s}\bigl(\varphi(\tau,z)\mathbf{e}(-\ell z)\bigr) \\ &=\; \sum_{s=(\alpha,\beta)\in S/\mathbb{Z}} \frac{\mathrm{sgn}(\alpha-A)-\mathrm{sgn}(\ell+2m\alpha)}{2}\;D_s(\tau)\,\mathbf{e}(-(\ell+m\alpha)z_s)\,. \end{aligned}$$

(Here "$(\alpha,\beta)\in S/\mathbb{Z}$" means that we consider all α, but β only modulo 1, which is the same by periodicity as considering only the (α,β) with $B\le\beta<B+1$.) Inserting this formula into (8.20) and using (8.14), we find

$$\begin{aligned}
\varphi(\tau,z)\;-\;\varphi^F(\tau,z) \;&=\; -\sum_{s=(\alpha,\beta)\,\in\,S/\mathbb{Z}} \mathbf{e}(-m\alpha z_s)\,D_s(\tau)\sum_{\ell\in\mathbb{Z}}\frac{\operatorname{sgn}(\operatorname{Im}(z-z_s))+\operatorname{sgn}(\ell+2m\alpha)}{2}\,\Bigl(\frac{y}{y_s}\Bigr)^{\ell}\\
&=\;\sum_{s=(\alpha,\beta)\,\in\,S/\mathbb{Z}} \mathbf{e}(-m\alpha z_s)\,D_s(\tau)\,\mathcal{R}_{-2m\alpha}(y/y_s)\\
&=\;\sum_{s=(\alpha,\beta)\,\in\,S/\mathbb{Z}^2}\;\sum_{\lambda\in\mathbb{Z}}\mathbf{e}\bigl(-m(\alpha-\lambda)(z_s-\lambda\tau)\bigr)\,D_{(\alpha-\lambda,\beta)}(\tau)\,\mathcal{R}_{-2m(\alpha-\lambda)}(q^\lambda y/y_s)\\
&=\;\sum_{s=(\alpha,\beta)\,\in\,S/\mathbb{Z}^2} D_s(\tau)\,\mathbf{e}(-m\alpha z_s)\,\sum_{\lambda\in\mathbb{Z}}q^{m\lambda^2}\,y^{2m\lambda}\,\mathcal{R}_{-2m\alpha}(q^\lambda y/y_s)\,,
\end{aligned}$$

where in the last line we have used the periodicity property (8.6) of $D_s(\tau)$ together with the obvious periodicity property $\mathcal{R}_{c+n}(y)=y^n\mathcal{R}_c(y)$ of $\mathcal{R}_c(y)$. But the inner sum in the last expression is just $\mathrm{Av}^{(m)}\bigl[\mathcal{R}_{-2m\alpha}(y/y_s)\bigr]$, so from the definition (8.16) we see that this agrees with $\varphi^P(\tau,z)$, as claimed.

8.3 Mock modularity of the Fourier coefficients

In subsections §8.1 and §8.2 we introduced a canonical splitting of a meromorphic Jacobi form φ into a finite part φ^F and a polar part φ^P, but there is no reason yet (apart from the simplicity of equation (8.2)) to believe that the choice we have made is the "right" one: we could have defined periodic Fourier coefficients $h_\ell(\tau)$ in many other ways (for instance, by taking $P=P_0-\ell/2m\tau$ with any fixed $P_0\in\mathbb{C}$ or more generally $P=P_\ell-\ell\tau/2m$ where P_ℓ depends only on ℓ modulo $2m$) and obtained other functions φ^F and φ^P. What makes the chosen decomposition special is that, as we will now show, the Fourier coefficients defined in (8.2) are (mixed) mock modular forms and the function φ^F therefore a (mixed) mock Jacobi form in the sense of §7.3. The corresponding shadows will involve theta series which we now introduce.

For $m\in\mathbb{N}$, $\ell\in\mathbb{Z}/2m\mathbb{Z}$ and $s=(\alpha,\beta)\in\mathbb{Q}^2$ we define the unary theta series

$$\Theta^s_{m,\ell}(\tau)\;=\;\mathbf{e}(-m\alpha\beta)\sum_{\lambda\,\in\,\mathbb{Z}+\alpha+\ell/2m}\lambda\,\mathbf{e}(2m\beta\lambda)\,q^{m\lambda^2} \tag{8.21}$$

of weight 3/2 and its Eichler integral[13]

$$\widetilde{\Theta}^s_{m,\ell}(\tau)\;=\;\frac{\mathbf{e}(m\alpha\beta)}{2}\sum_{\lambda\,\in\,\mathbb{Z}+\alpha+\ell/2m}\operatorname{sgn}(\lambda)\,\mathbf{e}(-2m\beta\lambda)\operatorname{erfc}\bigl(2|\lambda|\sqrt{\pi m\tau_2}\bigr)\,q^{-m\lambda^2} \tag{8.22}$$

[13]Strictly speaking, the Eichler integral as defined by equation (7.2) with $k=1/2$ would be this multiplied by $2\sqrt{\pi/m}$, but this normalization will lead to simpler formulas and, as already mentioned, there is no good universal normalization for the shadows of mock modular forms.

(cf. (7.2) and (7.5)). One checks easily that these functions transform by

$$\Theta_{m,\ell}^{(\alpha+\lambda,\beta+\mu)}(\tau) \;=\; \mathbf{e}(m(\mu\alpha-\lambda\beta+\lambda\mu))\,\Theta_{m,\ell}^{(\alpha,\beta)}(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,, \tag{8.23}$$

$$\widetilde{\Theta}_{m,\ell}^{(\alpha+\lambda,\beta+\mu)}(\tau) \;=\; \mathbf{e}(-m(\mu\alpha-\lambda\beta+\lambda\mu))\,\widetilde{\Theta}_{m,\ell}^{(\alpha,\beta)}(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,. \tag{8.24}$$

with respect to translations of s by elements of $\mathbb{Z}^2$. From this and (8.6) it follows that the products $D_s\overline{\Theta_{m,\ell}^s}$ and $D_s\widetilde{\Theta}_{m,\ell}^s$ depend only on the class of s in $S/\mathbb{Z}^2$, so that the sums over s occurring in the following theorem make sense.

Theorem 8.2 *Let φ, h_ℓ and φ^F be as in Theorem 8.1. Then each h_ℓ is a mixed mock modular form of weight $(k-1,1/2)$, with shadow $\sum_{s\in S/\mathbb{Z}^2} D_s(\tau)\,\overline{\Theta_{m,\ell}^s(\tau)}$, and the function φ^F is a mixed mock Jacobi form. More precisely, for each $\ell\in\mathbb{Z}/2m\mathbb{Z}$ the completion of h_ℓ defined by*

$$\widehat{h}_\ell(\tau) \;:=\; h_\ell(\tau) \,+\, \sum_{s\in S/\mathbb{Z}^2} D_s(\tau)\,\widetilde{\Theta}_{m,\ell}^s(\tau)\,, \tag{8.25}$$

with $\widetilde{\Theta}_{m,\ell}^s$ as in (8.22), tranforms like a modular form of weight $k-1/2$ with respect to some congruence subgroup of $SL(2,\mathbb{Z})$, and the completion of φ^F defined by

$$\widehat{\varphi}^F(\tau,z) \;:=\; \sum_{\ell \pmod{2m}} \widehat{h}_\ell(\tau)\,\vartheta_{m,\ell}(\tau,z) \tag{8.26}$$

transforms like a Jacobi form of weight k and index m with respect to the full modular group.

The key property needed to prove this theorem is the following proposition, essentially due to Zwegers, which says that the functions $F_m^s(\tau,z)$ defined in §8.2 are (meromorphic) mock Jacobi forms of weight 1 and index m, with shadow $\sum_{\ell \pmod{2m}} \overline{\Theta_{m,\ell}^s(\tau)}\,\vartheta_v m,\ell(\tau,z)$ (more precisely, that each F_m^s is a meromorphic mock Jacobi form of this weight, index and shadow with respect to some congruence subgroup of $SL(2,\mathbb{Z})$ depending on s and that the collection of all of them is a vector-valued mock Jacobi form on the full modular group):

Proposition 8.2 *For $m\in\mathbb{N}$ and $s\in\mathbb{Q}^2$ the completion $\widehat{F}_m^s$ of F_m^s defined by*

$$\widehat{F}_m^s(\tau,z) \;:=\; F_m^s(\tau,z) \;-\; \sum_{\ell \pmod{2m}} \widetilde{\Theta}_{m,\ell}^s(\tau)\,\vartheta_{m,\ell}(\tau,z)\;. \tag{8.27}$$

satisfies

$$\widehat{F}_m^{(\alpha+\lambda,\beta+\mu)}(\tau,z) \;=\; \mathbf{e}(-m(\mu\alpha-\lambda\beta+\lambda\mu))\,\widehat{F}_m^{(\alpha,\beta)}(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,, \tag{8.28}$$

$$\widehat{F}_m^s(\tau,z+\lambda\tau+\mu) \;=\; \mathbf{e}(-m(\lambda^2\tau+2\lambda z))\,\widehat{F}_m^s(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,, \tag{8.29}$$

$$\widehat{F}_m^s\Big(\frac{a\tau+b}{c\tau+d},\,\frac{z}{c\tau+d}\Big) \;=\; (c\tau+d)\,\mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\,\widehat{F}_m^{s\gamma}(\tau,z) \qquad \Big(\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\,\in\,SL(2,\mathbb{Z})\Big)\,. \tag{8.30}$$

Proof: The first two properties are easy to check because they hold for each term in (8.27) separately. The modular transformation property, considerably less obvious, is essentially the content of Proposition 3.5 of [75], but the functions he studies are different from ours and we need a small calculation to relate them. Zwegers defines two functions $f_u^{(m)}(z;\tau)$ and $\widetilde{f}_u^{(m)}(z;\tau)$ ($m\in\mathbb{N},\ \tau\in\mathbb{H},\ z,\,u\in\mathbb{C}$) by

$$f_u^{(m)}(z;\tau)=\mathrm{Av}^{(m)}\Big[\frac{1}{1-y/\mathbf{e}(u)}\Big],\qquad \tilde{f}_u^{(m)}(z;\tau)=f_u^{(m)}(z;\tau)-\frac{1}{2}\sum_{\ell\ (\mathrm{mod}\ 2m)}R_{m,\ell}(u;\tau)\,\vartheta_{m,\ell}(\tau,z)$$

(here we have rewritten his Definition 3.2 in our notation), where

$$R_{m,\ell}(u;\tau)\;=\;\sum_{r\in\ell+2m\mathbb{Z}}\Big\{\mathrm{sgn}\big(r+\frac{1}{2}\big)\;-\;\mathrm{erf}\Big(\sqrt{\pi}\,\frac{r\tau_2+2mu_2}{\sqrt{m\tau_2}}\Big)\Big\}\;q^{-r^2/4m}\;\mathbf{e}(-ru)\,, \tag{8.31}$$

and shows (Proposition 3.5) that $\widetilde{f}_u^{(m)}$ satisfies the modular transformation property

$$\tilde{f}^{(m)}_{u/(c\tau+d)}\Big(\frac{z}{c\tau+d};\,\frac{a\tau+b}{c\tau+d}\Big)\;=\;(c\tau+d)\;\mathbf{e}\Big(\frac{mc(z^2-u^2)}{c\tau+d}\Big)\;\tilde{f}_u^{(m)}(z;\tau) \tag{8.32}$$

for all $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})$. Noting that $\mathrm{erf}(x)=\mathrm{sgn}(x)(1-\mathrm{erfc}(|x|))$, we find that

$$\frac{1}{2}\,R_{m,\ell}(z_s;\tau)\;=\;\sum_{r\equiv\ell\ (\mathrm{mod}\ 2m)}\frac{\mathrm{sgn}(r+\frac{1}{2})-\mathrm{sgn}(r+2m\alpha)}{2}\;q^{-r^2/4m}\;y_s^{-r}\;+\;\mathbf{e}(m\alpha z_s)\,\widetilde{\Theta}_{m,l}(\tau)$$

in our notation. On the other hand, from (8.12) we have

$$\mathcal{R}_{-2m\alpha}(y)\;=\;\frac{1}{y-1}\;+\;\sum_{r\in\mathbb{Z}}\frac{\mathrm{sgn}(r+\frac{1}{2})-\mathrm{sgn}(r+2m\alpha)}{2}\;y^r$$

(note that the sum here is finite). Replacing y by y/y_s and applying the operator $\mathrm{Av}^{(m)}$, we find (using (8.9))

$$\mathbf{e}(m\alpha z_s)\,F_m^s(\tau,z)\;=\;-f_{z_s}^{(m)}(z;\tau)\;+\;\sum_{r\in\mathbb{Z}}\frac{\mathrm{sgn}(r+\frac{1}{2})-\mathrm{sgn}(r+2m\alpha)}{2}\;q^{-r^2/4m}\;y_s^{-r}\;\vartheta_{m,r}(\tau,z)\,.$$

Combining these two equations and rearranging, we obtain **[signs still to be checked!]**

$$\widehat{F}_m^s(\tau,z)\;=\;-\,\mathbf{e}(-m\alpha z_s)\,\tilde{f}_{z_s}^{(m)}(z;\tau)\,,$$

and the modularity property(8.30) then follows from (8.32) after a short calculation.

The proof of Theorem 8.2 follows easily from Proposition 8.2. We define the completion of the function φ^P studied in §8.2 by

$$\widehat{\varphi}^P(\tau,z) \;:=\; \sum_{s\in S/\mathbb{Z}^2} D_s(\tau)\; \widehat{F}^s_m(\tau,z)\,. \tag{8.33}$$

The sum makes sense by (8.28), and from the transformation equations (8.29)–(8.30) together with the corresponding properties (8.6)–(8.7) of the residue functions $D_s(\tau)$ it follows that $\widehat{\varphi}^P(\tau,z)$ transforms like a Jacobi form of weight k and index m with respect to the full modular group. Comparing equations (8.33) and (8.27) with equations (8.26) and (8.25), we find that

$$\varphi^P(\tau,z) \,-\, \widehat{\varphi}^P(\tau,z) \;=\; \sum_{\ell\,\in\,\mathbb{Z}/2m\mathbb{Z}}\;\sum_{s\,\in\,S/\mathbb{Z}^2} D_s(\tau)\;\widetilde{\Theta}^s_{m,\ell}(\tau)\,\vartheta_{m,\ell}(\tau,z) \;=\; \widehat{\varphi}^F(\tau,z) \,-\, \varphi^F(\tau,z)$$

and hence, using Theorem 8.1, that

$$\widehat{\varphi}^F(\tau,z) \,+\, \widehat{\varphi}^P(\tau,z) \;=\; \varphi^F(\tau,z) \,+\, \varphi^P(\tau,z) \;=\; \varphi(\tau,z)\,.$$

Since both $\varphi(\tau,z)$ and $\widehat{\varphi}^P(\tau,z)$ transform like Jacobi forms of weight k and index m, it follows that $\widehat{\varphi}^F(\tau,z)$ also does, and then the fact that each $\widehat{h}_\ell$ transforms like a modular form of weight $k-1/2$ (and hence that each h_ℓ is a mixed mock modular form with the weight and shadow given in the theorem) follows by the same argument that proves the modularity of the coefficients h_ℓ in the theta expansion (4.10) in the classical case.

Summary. For the reader's convenience, we give a brief summary of the results given up to now. We have the following six functions of two variables $(\tau,z)\in\mathbb{H}\times\mathbb{C}$:

- $\varphi(\tau,z)$, a meromorphic Jacobi form of weight k and index m, assumed to have only simple poles at $z=z_s=\alpha\tau+\beta$ for $s=(\alpha,\beta)$ in some discrete subset $S\subset\mathbb{Q}^2$;
- $\varphi^F(\tau,z)$, the finite part of φ, defined by the theta expansion $\sum_{\ell \pmod{2m}} h_\ell(\tau)\vartheta_{m,\ell}(\tau,z)$ where $h_\ell(\tau)$ is $q^{\ell^2/4m}$ times the ℓth Fourier coefficient of $\varphi(\tau,z-\ell\tau/2m)$ on the real line;
- $\varphi^P(\tau,z)$, the polar part of φ, defined as $\sum_{s\in S/\mathbb{Z}^2} D_s(\tau)F^s_m(\tau,z)$, where F^s_m is an explicit Appell-Lerch sum having simple poles at $z\in z_s+\mathbb{Z}\tau+\mathbb{Z}$;
- $C(\tau,z)$, a non-holomorphic correction term, defined as $\sum_s\sum_\ell D_s(\tau)\widetilde{\Theta}^s_{m,\ell}(\tau)\vartheta_{m,\ell}(\tau,z)$ where the $\widetilde{\Theta}_{m,\ell}$ are the Eichler integrals of explicit unary theta series of weight $3/2$;
- $\widetilde{\varphi}^F(\tau,z)$, the completed finite part, defined as $\varphi^F(\tau,z)+C(\tau,z)$;
- $\widetilde{\varphi}^P(\tau,z)$, the completed polar part, defined as $\varphi^P(\tau,z)-C(\tau,z)$.

These functions are related by

$$\varphi^F \,+\, \varphi^P \;=\; \varphi \;=\; \widehat{\varphi}^F \,+\, \widehat{\varphi}^P\,, \qquad \widehat{\varphi}^F \,-\, \varphi^F \;=\; C \;=\; \varphi_P \,-\, \widehat{\varphi}_P\,. \tag{8.34}$$

Each of them is real-analytic in τ, meromorphic in z, satisfies the elliptic transformation property (4.2) with respect to z, and has precisely two of four further desirable properties of such a function (note that $6 = \binom{4}{2}$), as shown in the following table

	φ	φ^F	φ^P	C	$\widehat{\varphi}^F$	$\widehat{\varphi}^P$
holomorphic in τ?	✓	✓	✓	–	–	–
transforms like a Jacobi form?	✓	–	–	–	✓	✓
holomorphic in z?	–	✓	–	✓	✓	–
determined by the poles of φ?	–	–	✓	✓	–	✓

in which the three checked entries in each row correspond to one of the equations (8.34). Each Fourier coefficient h_ℓ of φ is a mixed mock modular form of weight $(k-1, 1/2)$, and the finite part φ^F is a mixed mock Jacobi form. In the holomorphic case, the functions φ, φ^F and $\widehat{\varphi}^F$ coincide and the functions φ^P, C and $\widehat{\varphi}^P$ vanish.

8.4 The case of double poles

In this subsection we extend our considerations to the case when φ is allowed to have double poles, again assumed to be at points $z = z_s = \alpha\tau + \beta$ for $s = (\alpha, \beta)$ belonging to some discrete subset S of $\mathbb{Q}^2$. The first thing we need to do is to generalize the definition (8.5) to this case. For $s \in S$ we define functions E_s and D_s on $\mathbb{H}$ by the Laurent expansion

$$\mathbf{e}(m\alpha z_s)\,\varphi(\tau, z_s + \varepsilon) \;=\; \frac{E_s(\tau)}{(2\pi i\varepsilon)^2} + \frac{D_s(\tau) - 2m\alpha\, E_s(\tau)}{2\pi i\varepsilon} + O(1) \qquad \text{as } \varepsilon \to 0. \tag{8.35}$$

(Notice that $D_s(\tau)$ is the same function as in (8.5) when the pole is simple.) It is easily checked that the behavior of these functions under translations of s by elements of $\mathbb{Z}^2$ is given by equation (8.6) and its analogue for E_s. For the modular behavior, we have:

Proposition 8.3 *The functions $E_s(\tau)$ and $D_s(\tau)$ defined by (8.35) are modular forms of weight $k-2$ and $k-1$, respectively. More respectively, for all $s \in S$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ we have*

$$E_s\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; (c\tau+d)^{k-2} E_{s\gamma}(\tau)\,, \qquad D_s\Big(\frac{a\tau+b}{c\tau+d}\Big) \;=\; (c\tau+d)^{k-1} D_{s\gamma}(\tau)\,. \tag{8.36}$$

Proof: We rewrite (8.35) as

$$\mathbf{e}(m\alpha(z_s + 2\varepsilon))\,\varphi(\tau, z_s + \varepsilon) \;=\; \frac{E_s(\tau)}{(2\pi i\varepsilon)^2} + \frac{D_s(\tau)}{2\pi i\varepsilon} + \mathrm{O}(1)\,,$$

and also write α_s and $z_s(\tau)$ instead of just α and z_s. Then using the easily checked identities

$$z_s(\gamma\tau) \;=\; \frac{z_{s\gamma}(\tau)}{c\tau+d}\,, \qquad \alpha_{s\gamma}\, z_{s\gamma}(\tau) \,-\, \alpha_s\, z_s(\gamma\tau) \;=\; \frac{c\, z_{s\gamma}(\tau)^2}{c\tau+d} \qquad \Big(\gamma\tau := \frac{a\tau+b}{c\tau+d}\Big)\,,$$

and the modular transformation equation (4.1), we find

$$\begin{aligned}\frac{(c\tau+d)^2\,E_s(\gamma\tau)}{(2\pi i\varepsilon)^2}+\frac{(c\tau+d)\,D_s(\gamma\tau)}{2\pi i\varepsilon}&\equiv\;\mathbf{e}\Big(m\alpha\Big(z_s(\gamma\tau)+\frac{2\varepsilon}{c\tau+d}\Big)\Big)\,\varphi\Big(\gamma\tau,\;z_s(\gamma\tau)+\frac{\varepsilon}{c\tau+d}\Big)\\ &\equiv\;\mathbf{e}\Big(m\alpha\frac{z_{s\gamma}(\tau)+2\varepsilon}{c\tau+d}\Big)\,\varphi\Big(\gamma\tau,\;\frac{z_{s\gamma}(\tau)+\varepsilon}{c\tau+d}\Big)\\ &\equiv\;(c\tau+d)^k\,\mathbf{e}\Big(m\alpha\frac{z_{s\gamma}+2\varepsilon}{c\tau+d}+\frac{mc(z_{s\gamma}+\varepsilon)^2}{c\tau+d}\Big)\,\varphi\Big(\tau,\;z_{s\gamma}+\varepsilon\Big)\\ &\equiv\;(c\tau+d)^k\,\mathbf{e}(m\alpha_{s\gamma}(z_{s\gamma}+2\varepsilon))\,\varphi(\tau,\;z_{s\gamma}+\varepsilon)\\ &\equiv\;(c\tau+d)^k\,\Big[\frac{E_{s\gamma}(\tau)}{(2\pi i\varepsilon)^2}+\frac{D_{s\gamma}(\tau)}{2\pi i\varepsilon}\Big]\,,\end{aligned}$$

where "$\equiv$" means equality modulo a quantity that is bounded as $\varepsilon\to 0$. The claim follows.

Next, we must define a standard Appell-Lerch sum with a double pole at $z=z_s$. We begin by defining a rational function $\mathcal{R}_c^{(2)}(y)$ with a double pole at $y=1$ for each $c\in\mathbb{R}$. Motivated by Proposition 8.1, we require

$$\mathcal{R}_c^{(2)}(\mathbf{e}(z))\;=\;\mathrm{Av}_{\mathbb{Z}}\Big[\frac{\mathbf{e}(cz)}{(2\pi iz)^2}\Big]\;=\;\frac{1}{(2\pi i)^2}\sum_{n\in\mathbb{Z}}\frac{\mathbf{e}(c(z-n))}{(z-n)^2}\,. \tag{8.37}$$

To calculate this explicitly as a rational function, we could imitate the proof of Proposition 8.1, but it is easier to note that $\mathcal{R}_c^{(2)}(\mathbf{e}(z))=\big(-\frac{1}{2\pi i}\frac{d}{dz}+c\big)\mathcal{R}_c(\mathbf{e}(z))$ and hence

$$\mathcal{R}_c^{(2)}(y)\;=\;\Big(-y\,\frac{d}{dy}+c\Big)\,\mathcal{R}_c(y)\;=\;\sum_{\ell\in\mathbb{Z}}\frac{|\ell-c|\;+\;\mathrm{sgn}(z_2)\,(\ell-c)}{2}\;y^\ell \tag{8.38}$$

$$=\;\begin{cases}\sum_{\ell\ge c}(\ell-c)\,y^\ell & \text{if } |y|<1\\ \sum_{\ell\le c}(c-\ell)\,y^\ell & \text{if } |y|>1\end{cases}\;=\;y^{\lfloor c\rfloor+1}\Big(\frac{1}{(y-1)^2}+\frac{c-\lfloor c\rfloor}{y-1}\Big)\,. \tag{8.39}$$

(Notice that in the second line neither the asterisk on the summation sign nor the case distinction for $c\in\mathbb{Z}$ and $c\notin\mathbb{Z}$ are needed, and that the function $\mathcal{R}_c^{(2)}(y)$, unlike $\mathcal{R}_c(y)$, is continuous in c.) For $s=(\alpha,\beta)\in\mathbb{Q}^2$ and $m\in\mathbb{N}$ we set

$$G_m^s(\tau,z)\;=\;\mathbf{e}(-m\alpha z_s)\,\mathrm{Av}^{(m)}\big[\mathcal{R}^{(2)}_{-2m\alpha}(y/y_s)\big]\,, \tag{8.40}$$

in analogy with (8.16). If we then define the polar part φ^P of φ by

$$\varphi^P(\tau,z)\;=\;\sum_{s\in S/\mathbb{Z}^2}\Big(E_s(\tau)\,G_m^s(\tau,z)\;+\;D_s(\tau)\,F_m^s(\tau,z)\Big)\,; \tag{8.41}$$

then the definitions of the functions D_s, E_s, F^s_m and G^s_m immediately imply that φ^P has the same singularities as φ, so that the difference

$$\varphi^F(\tau,z) \;=\; \varphi(\tau,z) \;-\; \varphi^P(\tau,z) \tag{8.42}$$

is a holomorphic function of z.

As before, the key property of the Appell-Lerch sums is that they are again mock Jacobi forms, of a somewhat more complicated type than before. We introduce the unary theta series

$$\theta^s_{m,\ell}(\tau) \;=\; \mathbf{e}(-m\alpha\beta) \sum_{\lambda\,\in\,\mathbb{Z}+\alpha+\ell/2m} \mathbf{e}(2m\beta\lambda)\, q^{m\lambda^2} \tag{8.43}$$

of weight 1/2 and its (again slightly renormalized) Eichler integral

$$\widetilde{\theta}^s_{m,\ell}(\tau) \;=\; \frac{\overline{\theta^s_{m,\ell}(\tau)}}{2\pi\sqrt{m\tau_2}} \;-\; \mathbf{e}(m\alpha\beta) \sum_{\lambda\,\in\,\mathbb{Z}+\alpha+\ell/2m} |\lambda|\, \mathbf{e}(-2m\beta\lambda)\, \mathrm{erfc}\big(2|\lambda|\sqrt{\pi m\tau_2}\big)\, q^{-m\lambda^2} \tag{8.44}$$

(cf. (7.2) and (7.5)). Then we can define the completion $\widehat{G}^s_m$ of G^s_m by

$$\widehat{G}^s_m(\tau,z) \;:=\; G^s_m(\tau,z) \;+\; m \sum_{\ell \pmod{2m}} \widetilde{\theta}^s_{m,\ell}(\tau)\, \vartheta_{m,\ell}(\tau,z)\;. \tag{8.45}$$

Proposition 8.4 *For $m \in \mathbb{N}$ and $s \in \mathbb{Q}^2$ the completion $\widehat{G}^s_m$ of G^s_m defined by(8.45) satisfies*

$$\widehat{G}^{(\alpha+\lambda,\beta+\mu)}_m(\tau,z) \;=\; \mathbf{e}(-m(\mu\alpha-\lambda\beta+\lambda\mu))\, \widehat{G}^{(\alpha,\beta)}_m(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,, \tag{8.46}$$

$$\widehat{G}^s_m(\tau,z+\lambda\tau+\mu) \;=\; \mathbf{e}(-m(\lambda^2\tau+2\lambda z))\, \widehat{G}^s_m(\tau) \qquad (\lambda,\,\mu\in\mathbb{Z})\,, \tag{8.47}$$

$$\widehat{G}^s_m\Big(\frac{a\tau+b}{c\tau+d},\,\frac{z}{c\tau+d}\Big) \;=\; (c\tau+d)^2\, \mathbf{e}\Big(\frac{mcz^2}{c\tau+d}\Big)\, \widehat{G}^{s\gamma}_m(\tau,z) \qquad \Big(\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})\Big)\,. \tag{8.48}$$

The proof is exactly similar to that of 8.2. We define functions $g^{(m)}_u(z;\tau)$ and $\tilde{g}^{(m)}_u(z;\tau)$ by applying the operator $\dfrac{1}{2\pi i}\dfrac{\partial}{\partial u} - 2m\dfrac{u_2}{\tau_2}$ to $f^{(m)}_u(z;\tau)$ and $\tilde{f}^{(m)}_u(z;\tau)$; then the transformation equation (8.32) of $\tilde{f}^{(m)}_u$ implies the same transformation equation for $\tilde{g}^{(m)}_u$ but with the initial factor $(c\tau+d)$ replaced by $(c\tau+d)^2$, and a calculation exactly similar to the one given before shows that $\mathbf{e}(m\alpha z_s)G^s_m(\tau,z)$ differs from $g^{(m)}_{z_s}(z;\tau)$ by a finite linear combination of functions $\vartheta_{m,r}(\tau,z)$ and that $\mathbf{e}(m\alpha z_s)\widehat{G}^s_m(\tau,z) = \tilde{g}^{(m)}_{z_s}(z;\tau)$. We omit the details.

Theorem 8.3 *Let φ be as above, with singularities at $z = z_s$ ($s \in S \subset \mathbb{Q}^2$) given by (8.35). Then the finite part φ^F as defined by (8.42) coincides with the finite part defined by the theta* *expansion* (8.3), *the coefficients* $h_\ell(\tau)$ *in this expansion are mixed mock modular forms, with completion given by*

$$\widehat{h}_\ell(\tau) \;=\; h_\ell(\tau) \,+\, \sum_{s\in S/\mathbb{Z}^2} \Big(E_s(\tau)\,\widetilde{\theta}^s_{m,\ell}(\tau) \,+\, D_s(\tau)\,\widetilde{\Theta}^s_{m,\ell}(\tau)\Big)\,,$$

and the completion $\widehat{\varphi}^F$ *defined by* (8.26) *transforms like a Jacobi form of weight* k *and index* m.

The proof follows the same lines as before: the equivalence of (8.3) and (8.42) is proved by expanding $\varphi(\tau,z)$ as a Fourier series along the horizontal line $\mathrm{Im}(z) = \mathrm{Im}(P)$ for some generic point $P \in \mathbb{C}$ and calculating the difference $\varphi - \varphi^F$ as a sum of residues, and the mock modularity is proved by decomposing φ as $\widehat{\varphi}^F + \widehat{\varphi}^P$ with $\widehat{\varphi}^F$ as in (8.26) and $\widehat{\varphi}^P = \sum_{s\in S/\mathbb{Z}^2}\big(E_s\widehat{G}^s_m + D_s\widehat{F}^s_m\big)$, which transforms like a Jacobi form by virtue of Proposition 8.4. Again the details are left to the reader. Note that here the mock modular forms $h_\ell(\tau)$ are of the "even more mixed" variety mentioned at the end of §7.3, since they now have a shadow that is a linear combination of two terms $\sum_s E_s\,\overline{\theta^s_{m,l}}$ and $\sum_s D_s\,\overline{\Theta^s_{m,l}}$ belonging to two different tensor products $M_{k-2}\otimes\overline{M_{1/2}}$ and $M_{k-1}\otimes\overline{M_{3/2}}$ and hence two different bi-weights$(k-2,\frac{3}{2})$ and $(k-1,\frac{1}{2})$ rather than a single bi-weight $(k-1,\frac{1}{2})$ as before.

8.5 Examples

We end this section by giving five examples of meromorphic Jacobi forms and their decompositions into a mock Jacobi form and a finite linear combination of Appell-Lerch sums. We use systematically use the notations $A = \varphi_{-2,1}$, $B = \varphi_{0,1}$, $C = \varphi_{-1,2}$ for the three basic generators of the ring of weak Jacobi forms as described in (4.30)–(4.39).

Example 1: Simple pole at $z = 0$, weight 1, index 1. As our first example we take the Jacobi form $\varphi = C/A \in J^{\mathrm{mer}}_{1,1}$, which has a simple pole of residue $1/\pi i$ at $z = 0$ and a Fourier expansion beginning

$$\frac{y+1}{y-1} \,-\, (y^2 - y^{-2})\,q \,-\, 2(y^3 - y^{-3})\,q^2 \,-\, 2(y^4 - y^{-4})\,q^3 \,-\, (2y^5 + y^4 - y^{-4} - 2y^{-5})\,q^4 \,-\, \cdots\,.$$

The Fourier expansion of the polar part $\varphi^P = \mathrm{Av}^{(1)}\big[\frac{y+1}{y-1}\big]$ begins the same way, and indeed, we must have $\varphi = \varphi^P$ because the Fourier coefficients h_ℓ all vanish identically (we have $h_{-\ell} = -h_\ell$ because the weight is odd and $h_{\ell+2} = h_\ell$ because the index is 1). So here there is no mock Jacobi form at all, but only the polar correction term given by the Appell-Lerch sum, a kind of a "Cheshire cat" example which is all smile and no cat.

Example 2: Simple pole at $z = 0$, weight 1, index 2. As a second example take $\varphi = BC/A \in J^{\mathrm{mer}}_{1,2}$. Here we find

$$\begin{aligned}\varphi^P \;&=\; 12\,\mathrm{Av}^{(2)}\Big[\frac{y+1}{y-1}\Big] \;=\; 12\,\frac{y+1}{y-1} \,-\, 12(y^4 - y^{-4})q^2 \,+\, 24(y^5 - y^{-5})\,q^3 \,+\, \cdots\\ \varphi^F \;&=\; (y - y^{-1}) \,-\, (y^3 + 45y - 45y^{-1} - y^{-3})q \,+\, (45y^3 - 231y + 231y^{-1} - 45y^{-1})q^2 \,+\, \cdots\end{aligned}$$

and we see that φ^F is precisely the mock Jacobi form $\mathcal{F}_2$ discussed in Example 4 of §7.3 that is related to the mock modular form $h^{(2)}$ and to the representations of the Mathieu group M_{24}.

Example 3: Double pole at $z = 0$, weight 2, index 0. This example is a bit of a cheat, because we did not allow $m = 0$ in the definition (8.8) of the averaging operator ($m < 0$ doesn't work at all, because the Appell-Lerch sums diverge, and $m = 0$ is less interesting since a form of index 0 is clearly determined up to a function of τ alone by its singularities, so that in our discussion we excluded that case too), but nevertheless it works. Take $\varphi = B/A$, which, as we saw in §4.4, is nothing other than a multiple of the Weierstrass $\wp$-function. Then $\varphi^P = 12\,\mathrm{Av}^{(0)}\big[\frac{y}{(1-y)^2}\big]$ and $\varphi^F = \varphi - \varphi^P$ is simply the quasimodular (and mock modular) form $E_2(\tau)$. (It has to be independent of z because it is holomorphic and of index 0.)

Example 4: Double pole at $z = 0$, weight 2, index 1. This is the basic example. Take

$$\begin{aligned}\varphi \;&=\; \frac{B^2}{A} \;=\; \frac{144y}{(1-y)^2} + y + 22 + y^{-1} + \big(22y^2 + 152y - 636 + 152y^{-1}\big)\,q \\ &\qquad + (145y^3 - 636y^2 + 3831y - 7544 + 3831y^{-1} - 636y^2 + 145y^{-3}\big)\,q^2 + \cdots, \\ \varphi^P \;&=\; 144\,\mathrm{Av}^{(1)}\Big[\frac{y}{(1-y)^2}\Big] \;=\; \frac{144y}{(1-y)^2} + 144\,(y^3 - y^{-3})q^2 + \cdots, \\ \varphi^F \;&=\; \varphi - \varphi^P \;=\; \sum_{\substack{n,\,r\in\mathbb{Z}\\ 4n-r^2\ge -1}} C(4n-r^2)\,q^n\,y^r\end{aligned}$$

with the first few $C(D)$ given by

D	-1	0	3	4	7	8
$C(D)$	1	22	152	-636	3831	-7544

If we compare this with the coefficients $C'(D)$ defined by $E_4(\tau)\varphi_{-2,1}(\tau,z) = \sum C'(4n-r^2)q^ny^r$, of which the first few are given by

D	-1	0	3	4	7	8
$C'(D)$	1	-2	248	-492	4119	-7256

then we see that they are very close to each other (actually, asymptotically the same) and that the difference $C(D) - C'(D)$ is precisely -288 times the Hurwitz-Kronecker class number $H(D)$. We thus have $\varphi^F = E_4A - 288\mathcal{H}$.

Example 5: Simple poles at the 2-torsion points, weight -5, index 1. Take $\varphi = A^3/C \in J_{1,-5}$. This function has three poles, all simple, at the three non-trivial 2-torsion points on the torus $\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$. With some trouble one calculates the three corresponding modular forms (of weight -6):

$$D_{(0,\frac12)}(\tau) \;=\; 16\,\frac{\eta(2\tau)^{12}}{\eta(\tau)^{24}}\,, \quad D_{(\frac12,0)}(\tau) \;=\; -\frac14\,\frac{\eta\big(\frac{\tau}{2}\big)^{12}}{\eta(\tau)^{24}}\,, \quad D_{(\frac12,\frac12)}(\tau) \;=\; \frac{i}{4}\,\frac{\eta\big(\frac{\tau+1}{2}\big)^{12}}{\eta(\tau)^{24}}\,.$$

One then finds **[coefficients may be slightly off]**

$$\varphi \;=\; \varphi^P \quad=\quad D_{(0,\frac12)}(\tau)\,\mathrm{Av}^{(1)}\Big[\frac12\,\frac{y-1}{y+1}\Big] \;+\; q^{1/4}\,D_{(\frac12,0)}(\tau)\,\mathrm{Av}^{(1)}\Big[\frac{1}{2y}\,\frac{y+\sqrt{q}}{y-\sqrt{q}}\Big]$$
$$+\,q^{1/4}\,D_{(\frac12,\frac12)}(\tau)\,\mathrm{Av}^{(1)}\Big[\frac{1}{2y}\,\frac{y-\sqrt{q}}{y+\sqrt{q}}\Big]\,,$$

another "Cheshire cat" example (of necessity, for the same reason as in Example 1, since again $m = 1$ and k is odd).

9. A family of meromorphic Jacobi forms

10. Quantum black holes and mock modular forms

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