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Exercises on Simple problems with channel flow

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Exercise: Channel Flow

In a plane-strain channel flow the stress balance equation:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0 \qquad (1)$$

reduces to:

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial p}{\partial y} = 0 \qquad (2)$$

if we ignore gravity, assume 2D plane-strain, and consider a vertical conduit with side walls $x = \pm 1$.

1. Using

$$\sigma_{xy} = \eta \frac{\partial v}{\partial x} \qquad (3)$$

assume that the viscosity and pressure gradient are constant, and integrate (2) to obtain an expression for $v(x)$. Determine the integration constants from boundary conditions that the velocity gradient is zero in the centre ($x = 0$), and the velocity is zero on the wall ($x = 1$). What is the velocity in the center of the channel ($x = 0$) if $\eta = 1$ and $\partial p/\partial y = 1$?

2. Using *sybil*, examine the numerical solution (*chan*1) for this test problem solved in a conduit which is 3 units high and 2 units wide (only the right half of it is shown, so $x = 0$ is the middle of the conduit) the right hand boundary is rigid, a pressure of +3 is applied to the base of the channel, and a pressure of zero is applied to the top boundary at $y = 3$ to obtain $\partial p/\partial y = 1$, gravity is set to zero and viscosity to 1 for this test problem.
   - plot Uy and Ux
   - plot the velocity arrows
   - plot the shear strain rate (edxy)
   - plot pressure (pres)
   - plot the principal strain-rate orientations (Arrow → Strain → pstd)
   - plot profiles of Uy vs *x* and edxy at half-height (y = 1.5)

   How accurate is the numerical solution ? What is the cause of any inaccuracy ?

3. What if we have a pipe rather than a vertical layer ($x = 0$ is the axis of symmetry)? Use *sybil* to compare the solutions *chax*1 (axisymmetric) and *chan*1 (plane-strain). For axisymmetry, we interpret x as the radius r, and (2) is replaced by:

$$\frac{\partial (r\,\sigma_{ry})}{\partial r} + \frac{\partial (r\,p)}{\partial y} = 0 \qquad (4)$$

   Integrate (4) and compare result with answer to Q1. what is the velocity at the center of the pipe ?

4. Now suppose we have a non-Newtonian viscosity (strain-rate proportional to *n*th power of stress difference). We'll consider *n* = 3 and 5, values that apply to rock deformation in some situations. We parameterize the viscosity using:

$$\eta = \frac{B}{2}\dot{E}^{(1-n)/n} \qquad \text{where} \qquad \dot{E} = \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} \qquad (5)$$

The only component of strain-rate in the idealized problem is the vertical shear, so

$$\eta = 2^{-(n+1)/(2n)} B \left(\frac{\partial v}{\partial x}\right)^{(1-n)/n} \qquad (6)$$

Substitute (6) into (2) (plane-strain) and integrate, what should the centre velocity be if *n* = 3 or *n*= 5 ? Compare with the *sybil* solutions (*chan*3, *chan*5). Note that the flow becomes more plug-like as *n* increases.

For 2D channel flow:

$$\frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial p}{\partial y}=0$$

$$\sigma_{xy}=\eta\frac{\partial v}{\partial x}$$

Therefore

$$\eta\frac{d^2 v}{d x^2}=-\frac{d p}{d y}$$

Therefore

$$\frac{d^2 v}{d x^2}=-\frac{1}{\eta}\frac{d p}{d y}=-C$$

$$\frac{d v}{d x}=-Cx$$ where the integration constant is zero by zero shear at x = 0.

Therefore

$$v=-\frac{1}{2}Cx^2+C_1$$

and the boundary condition at x = a gives

$$v=\frac{1}{2}C\left(a^2-x^2\right)$$

With a = 1 and C = 1 in the test case,

$$v=\frac{1}{2}\left(1-x^2\right)$$

At x = 0, v should be equal to 0.5

For axisymmetry:

$$\frac{\partial\left(r\sigma_{ry}\right)}{\partial r}+\frac{\partial\left(r p\right)}{\partial y}=0$$

or

$$\frac{d\left(r\sigma_{ry}\right)}{d r}=-r\frac{d p}{d y}$$

Integrating:

$$r\sigma_{ry}=-\frac{1}{2}r^2\frac{d p}{d y}$$

therefore, assuming as before that stress is zero at r = 0

$$\sigma_{ry}=\eta\frac{dv}{dr}=-\frac{1}{2}r\frac{dp}{dy}$$

Therefore

$$\frac{dv}{dr}=-\frac{1}{2}\frac{r}{\eta}\frac{dp}{dy}$$

therefore

$$v=-\frac{1}{4\eta}r^2\frac{dp}{dy}=-\frac{1}{4}r^2C+C_1$$

and using the boundary condition

$$v=\frac{C}{4}\left(a^2-r^2\right)$$

with a = C = 1, at r = 0, v = 0.25.

Now the non-Newtonian problem, with 2D plane-strain

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial p}{\partial y}=0$$

$$\eta=2^{(1-3n)/(2n)}B\left(\frac{\partial v}{\partial x}\right)^{(n-1)/n}$$

$$\sigma_{xy}=\eta\frac{\partial v}{\partial x}$$

The first integration is as before:

$$\eta\frac{dv}{dx}=-\frac{dp}{dy}x$$

Therefore:

$$2^{(1-3n)/(2n)}B\left(\frac{dv}{dx}\right)^{(n-1)/n}\frac{dv}{dx}=-\frac{dp}{dy}x$$

$$\left(\frac{dv}{dx}\right)^{(2n-1)/n}=-2^{(3n-1)/(2n)}\frac{1}{B}\frac{dp}{dy}x$$

Raising both sides to the power

$$\frac{dv}{dx}=-2^{(3n-1)/(4n-2)}\left[\frac{1}{B}\frac{dp}{dy}\right]^{\frac{n}{(2n-1)}}x^{\frac{n}{(2n-1)}}$$

and integrating

$$v=-2^{(3n-1)/(4n-2)}\frac{2n-1}{(3n-1)}\left[\frac{1}{B}\frac{dp}{dy}\right]^{\frac{n}{(2n-1)}}x^{\frac{3n-1}{(2n-1)}}+C_1$$

When x = 1 , v = 0, so

$$v = 2^{(3n-1)/(4n-2)} \frac{2n-1}{(3n-1)} \left[\frac{1}{B} \frac{dp}{dy} \right]^{\frac{n}{(2n-1)}} \left[a^{\frac{3n-1}{(2n-1)}} - x^{\frac{3n-1}{(2n-1)}} \right]$$

When C = 1, B = 1, a = 1, then

$$v = 2^{(3n-1)/(4n-2)} \frac{2n-1}{(3n-1)} \left[1 - x^{\frac{3n-1}{(2n-1)}} \right]$$

when n = 3,

$$v = 2^{4/5} \frac{5}{8} \left[1 - x^{8/5} \right]$$

In the middle of the layer v should be = 1.088188

when n = 5,

$$v = 2^{7/9} \frac{9}{14} \left[1 - x^{14/9} \right]$$

In the middle of the layer v should be = 1.10217