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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Scaling of stress differences to elevations (in isostasy), and simple problems illustrating force per unit length

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Processes responsible for elevating and sustaining mountain belts

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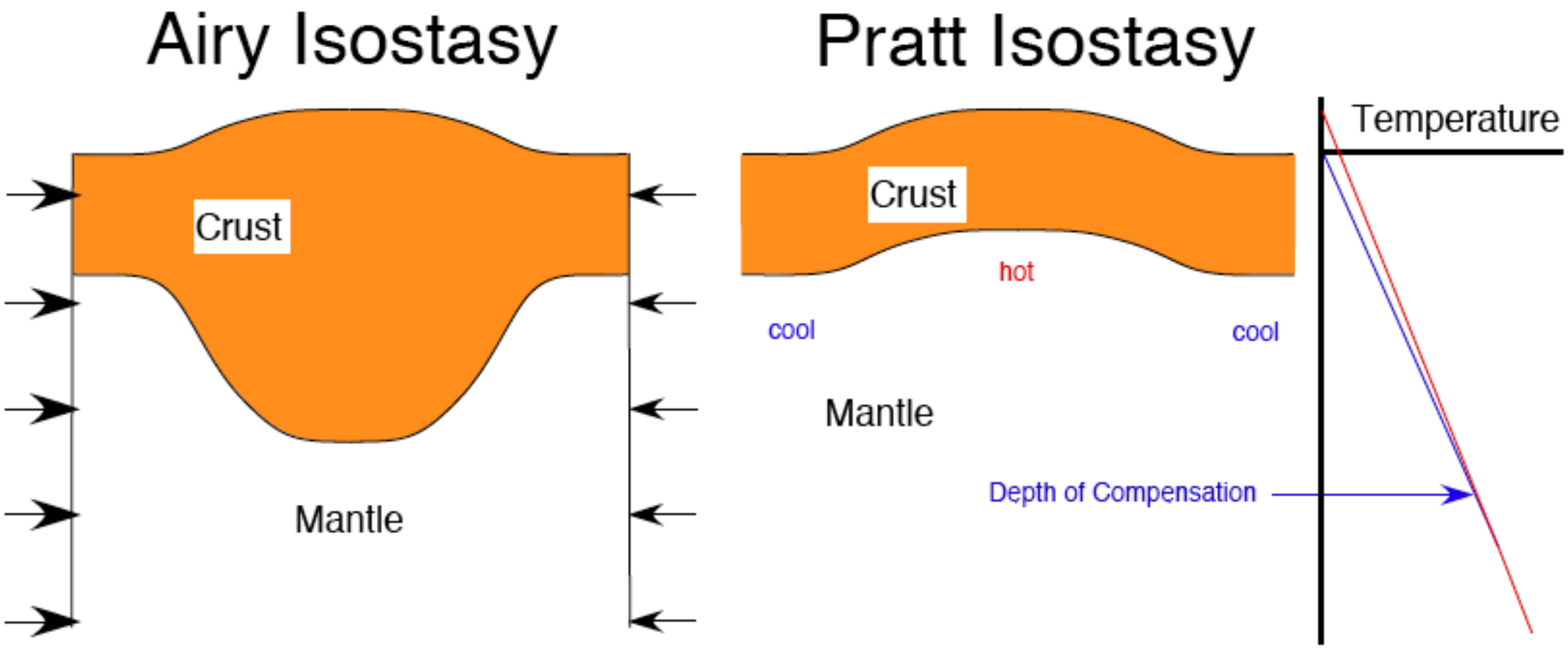
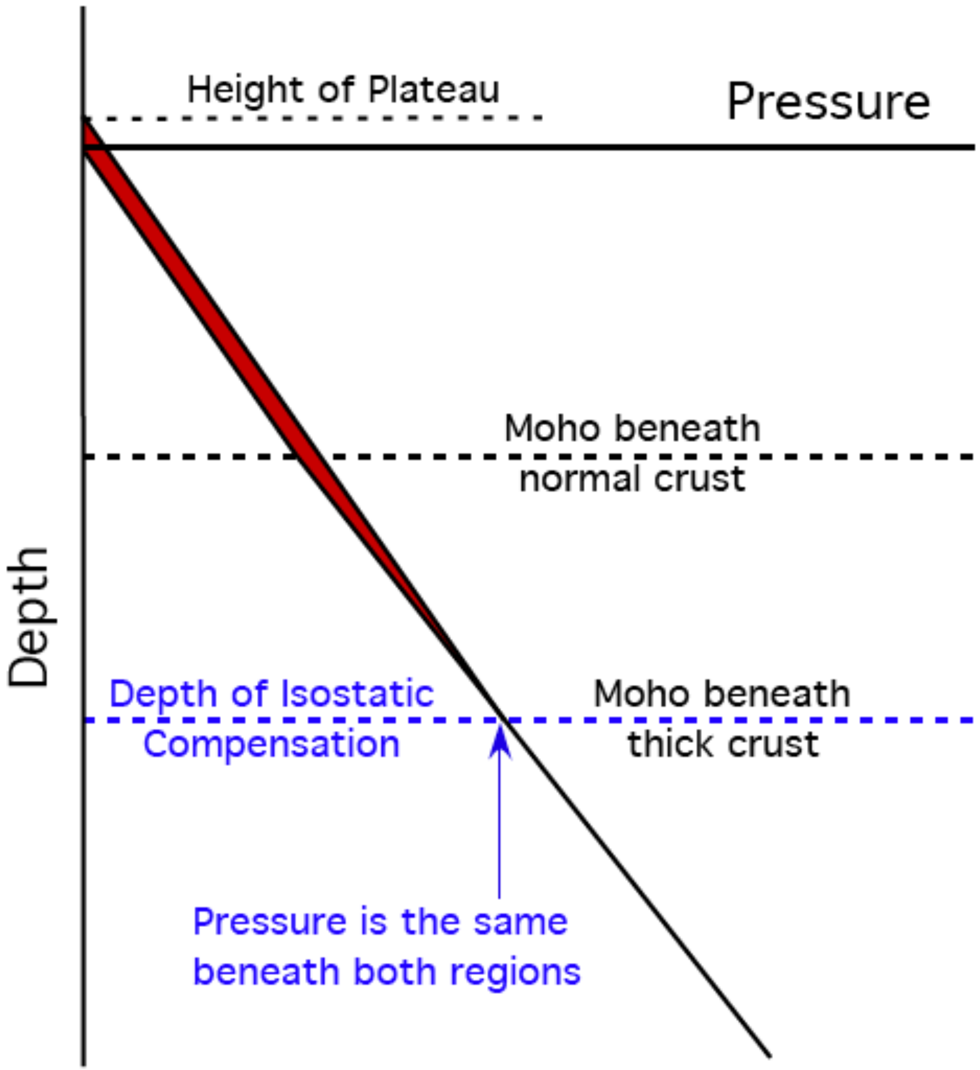
1. Isostasy: Airy and Pratt reminder
2. Potential energy per unit area (= force per unit length that adjacent regions apply to one another)
3. Prelude to the thin viscous sheet

Isostatic compensation



For Airy isostasy, the density difference between crust and mantle dictates elevation differences.

For Pratt isostasy, lateral differences density (associated mostly with lateral temperature differences in the mantle) affect surface elevations.



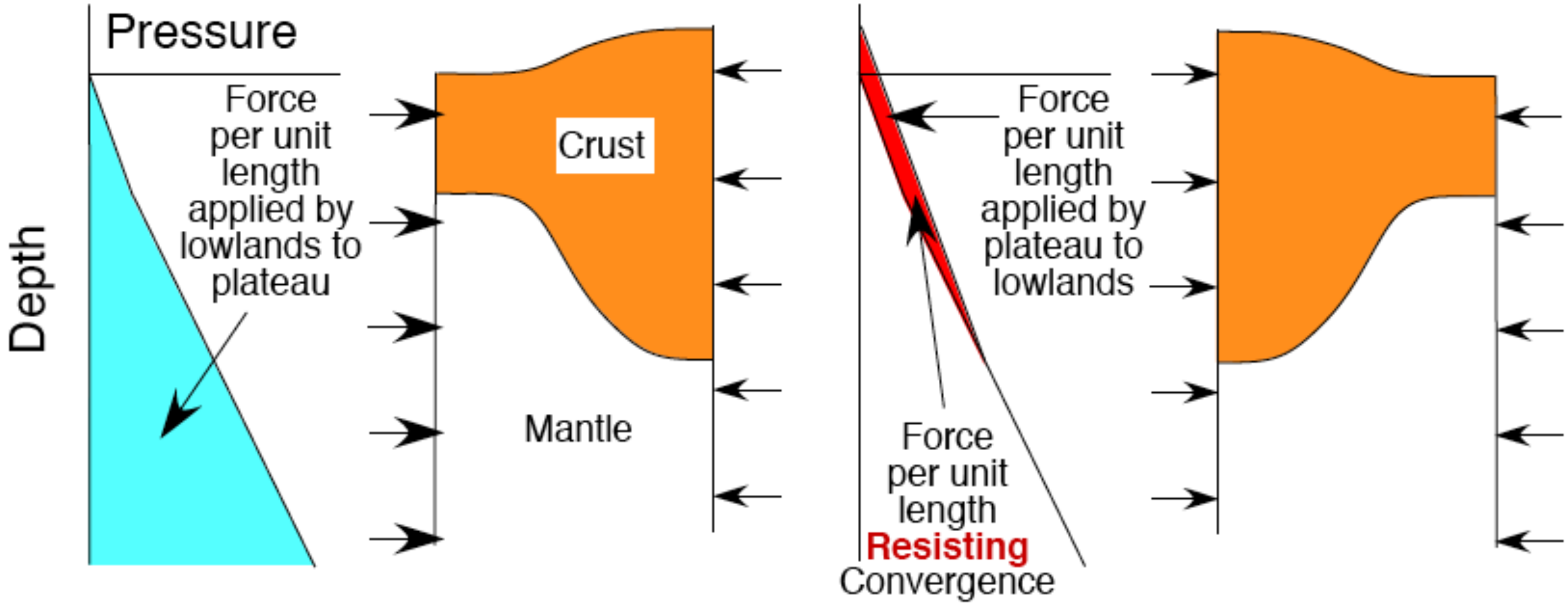
Red area shows Force per unit length that plateau and lowlands apply to one another, or equivalently, the difference in Potential Energy per unit area beneath the two regions

Isostasy:

at the Depth of Compensation, the **Pressure** is the same beneath columns of mass

Airy Isostasy

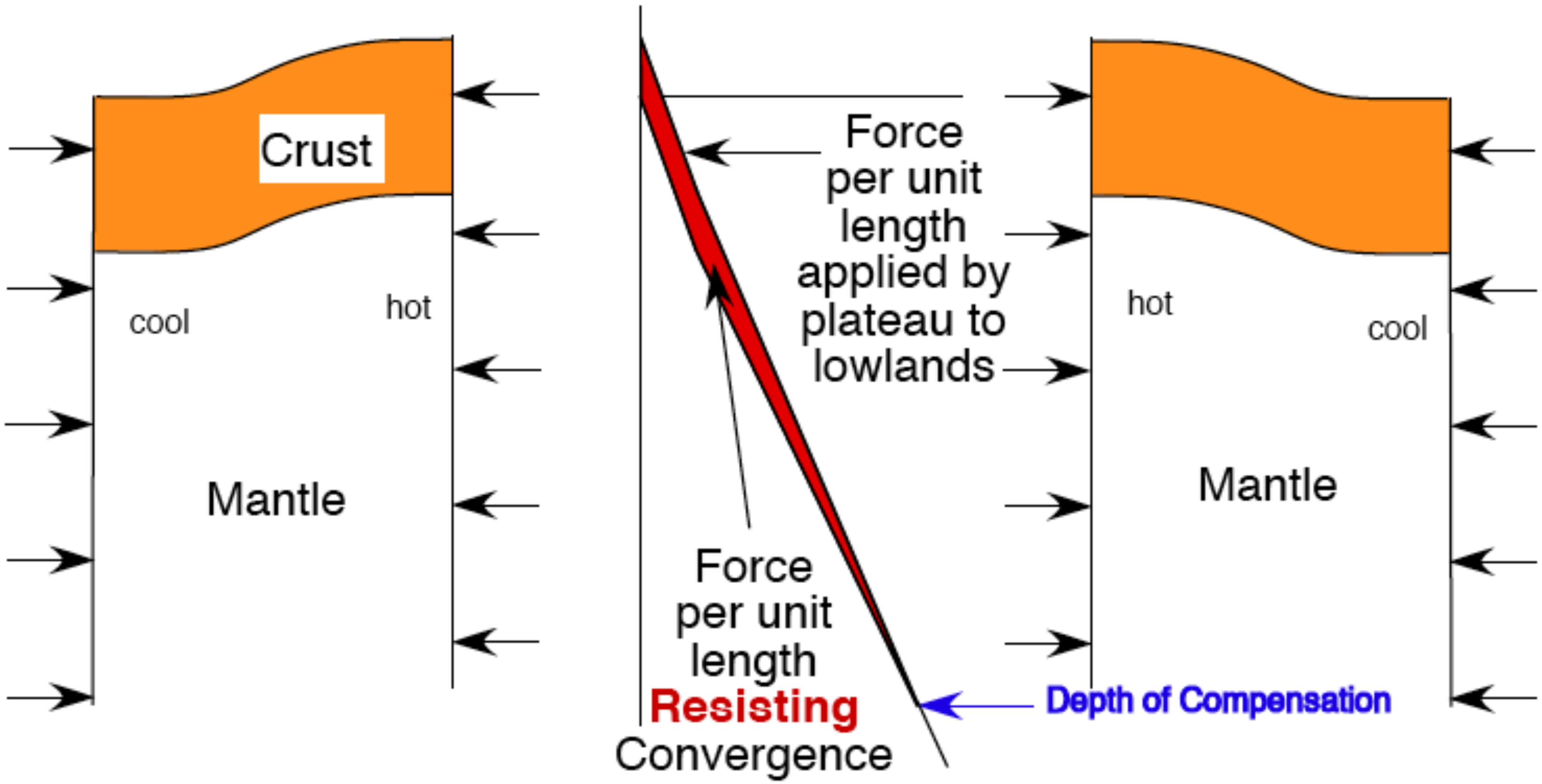
Forces per Unit Length Resisting Convergence
(Available Potential Energy)



To show differences, we must exaggerate pressure differences. (*Be careful*.)

Pratt Isostasy

Force per Unit Length Resisting Convergence
(Available Potential Energy)



Compensation occurs deeper than for Airy isostasy.

Consequently, potential energy differences and forces per unit length can be greater for Pratt than Airy compensation of the same elevation.

Equation of Equilibrium

$$\nabla \cdot \sigma_{ij} - \rho g \hat{z} = 0$$

The gradient in the stress tensor plus the body force is zero (no acceleration).

(For now, compressive stresses are positive; z increases downward.)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

Simple Case: 2 dimensions (*x*,*z*)

no *y*-dependence

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

The equations simplify.

Simple Case: 2 dimensions (*x*,*z*)

no *y*-dependence

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

Moreover, for two dimensions, we may ignore y-components of stress.

Simple Case: 2 dimensions (*x*,*z*)
boundary conditions

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

1. No shear or normal stress on the top.
2. No shear stress on the bottom (no traction from flow in the asthenosphere).

For 2-D: the horizontal component of the force balance.

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Integrate this over depth; second term vanishes.

$$\int_{Bottom}^{Top} \frac{\partial \sigma_{xz}}{\partial z} dz = \sigma_{xz}(top) - \sigma_{xz}(Bottom) = 0$$

$$\int_{Bottom}^{Top} \frac{\partial \sigma_{xx}}{\partial x} dz = \frac{\partial \left(L \overline{\sigma}_{xx} \right)}{\partial x} = 0$$

A **simple implication** of $L\overline{\sigma}_{xx} = \text{Constant}$

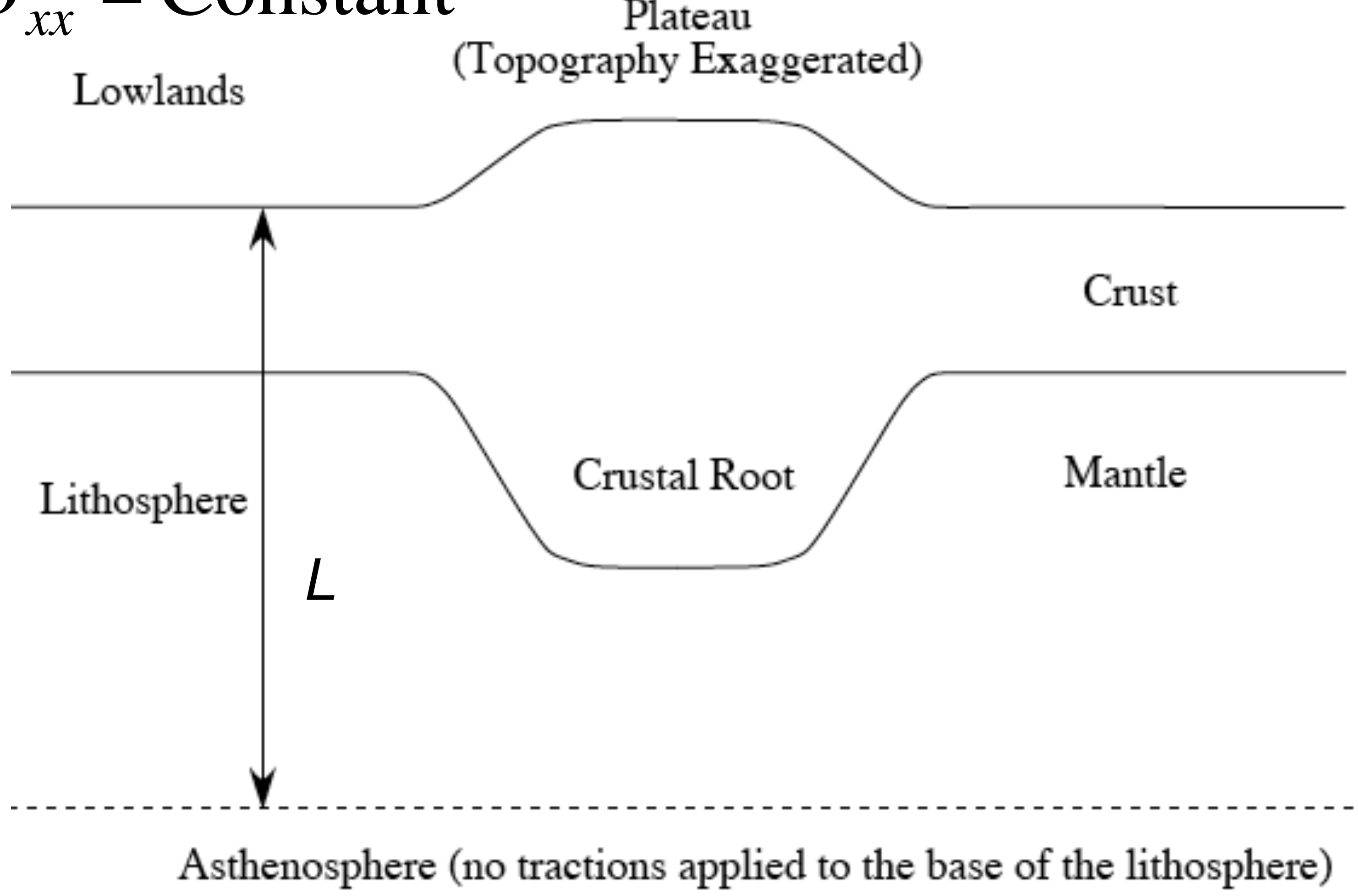
As L, thickness of the lithosphere, is ~ constant, the average **horizontal compressive stress** is (virtually) **constant**:

$$\overline{\sigma}_{xx} \approx \text{Constant}$$

Thus (*insofar as the assumptions made so far are reasonable*) variations in tectonic style depend **not** on the horizontal compressive stress, but **instead on the variations in the vertical compressive stress**, and hence **on elevation** and **how isostatic equilibrium is maintained**.

The average horizontal stress and the horizontal force per unit length are constant

$$L\overline{\sigma}_{xx} = \text{Constant}$$



Simple Case: 2 dimensions (*x*,*z*)
boundary conditions

1. No shear or normal stress on the top.
2. No shear stress on the bottom (no traction from flow in the asthenosphere).
3. **Still simpler assumption**: negligible shear stresses on horizontal or vertical planes.

Vertical component of force balance

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

1. First term vanishes (by assumption)

$$\frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

2. Integrate remaining terms.

$$\sigma_{zz} = \int_{z}^{Top} \rho g dz = \rho g z$$

3. Vertical normal stress = lithostatic pressure

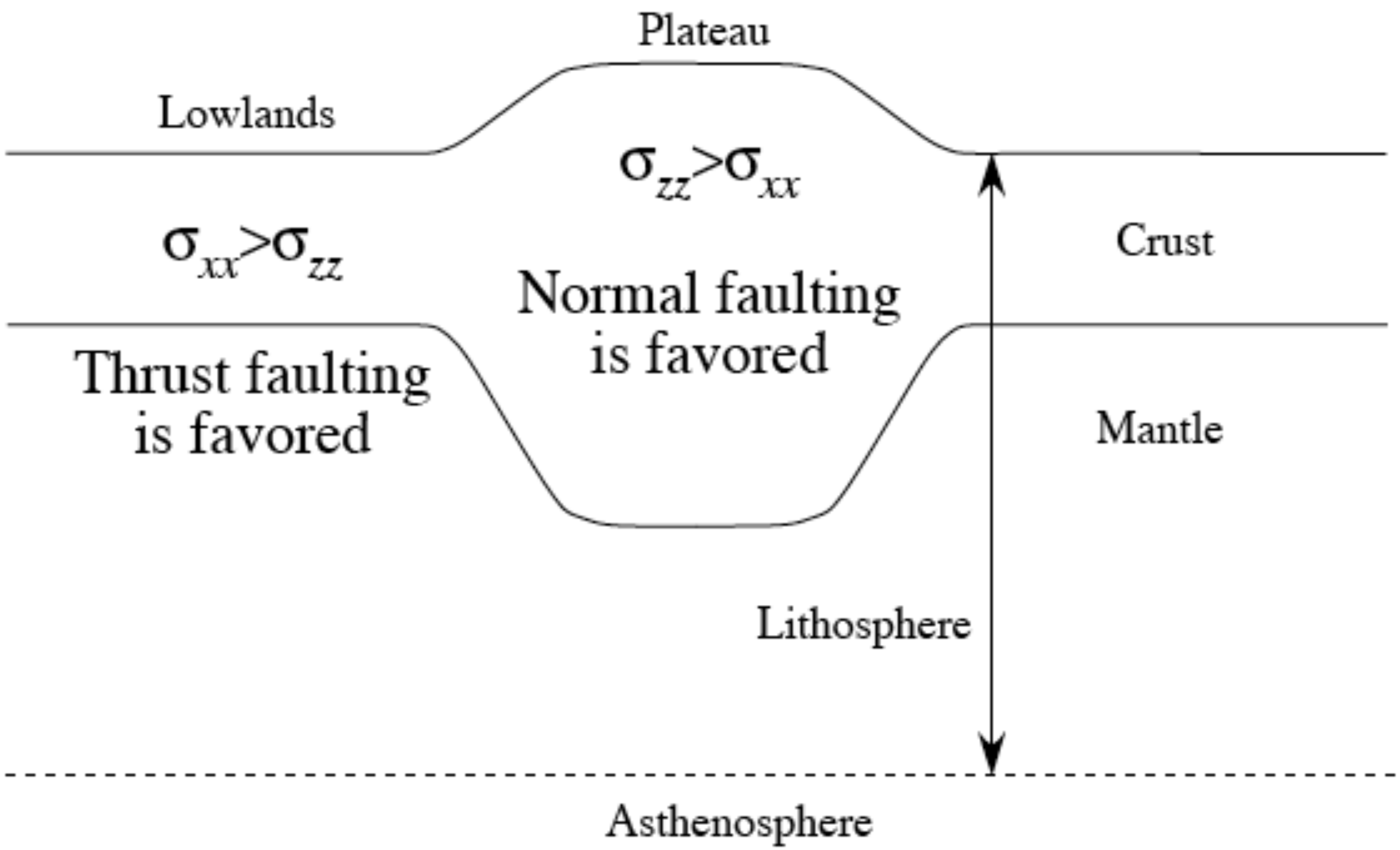
Thrust and normal faulting

σ_{zz} is larger beneath high regions than beneath low regions.

Beneath high regions, σ_{zz} can be greater than σ_{xx}, which favors normal faulting there.

Beneath lower regions, where σ_{zz} is the smaller, for a constant value of σ_{xx} , we may find: $\sigma_{xx} > \sigma_{zz}$. Hence thrust faulting is likely, but **neither normal faulting in high areas, nor thrust faulting in low areas is required**.

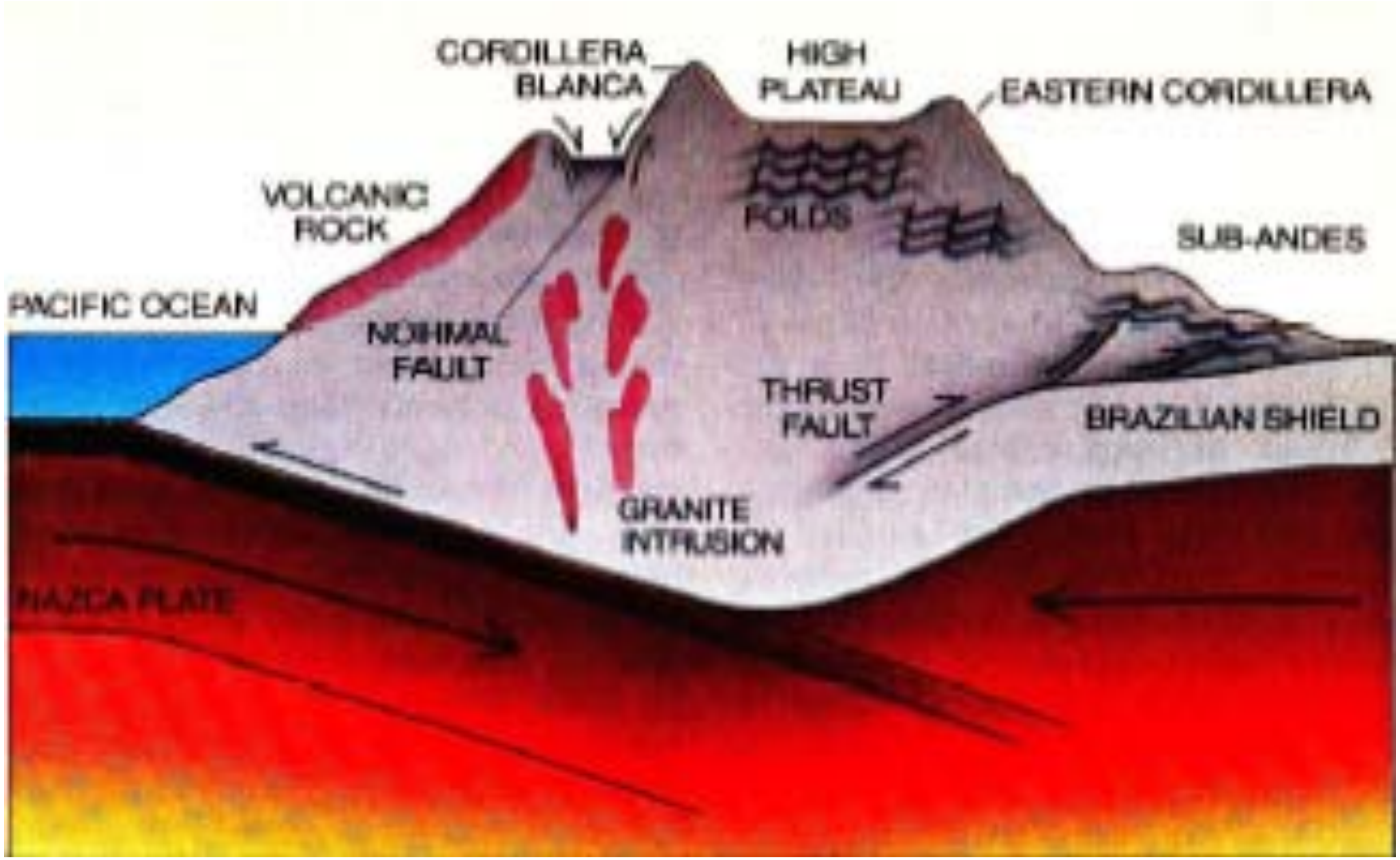
Thrust and Normal faulting



$$L\overline{\sigma}_{xx} = \text{Constant}$$

Cartoon cross section of the Andes

Vertically exaggerated cross section across the Andes of Peru



An example of a high range undergoing extension

Normal faulting in the Cordillera Blanca, Peru

[*Dalmayrac and Molnar* 1981]

Gravitational Potential Energy
(per unit area)

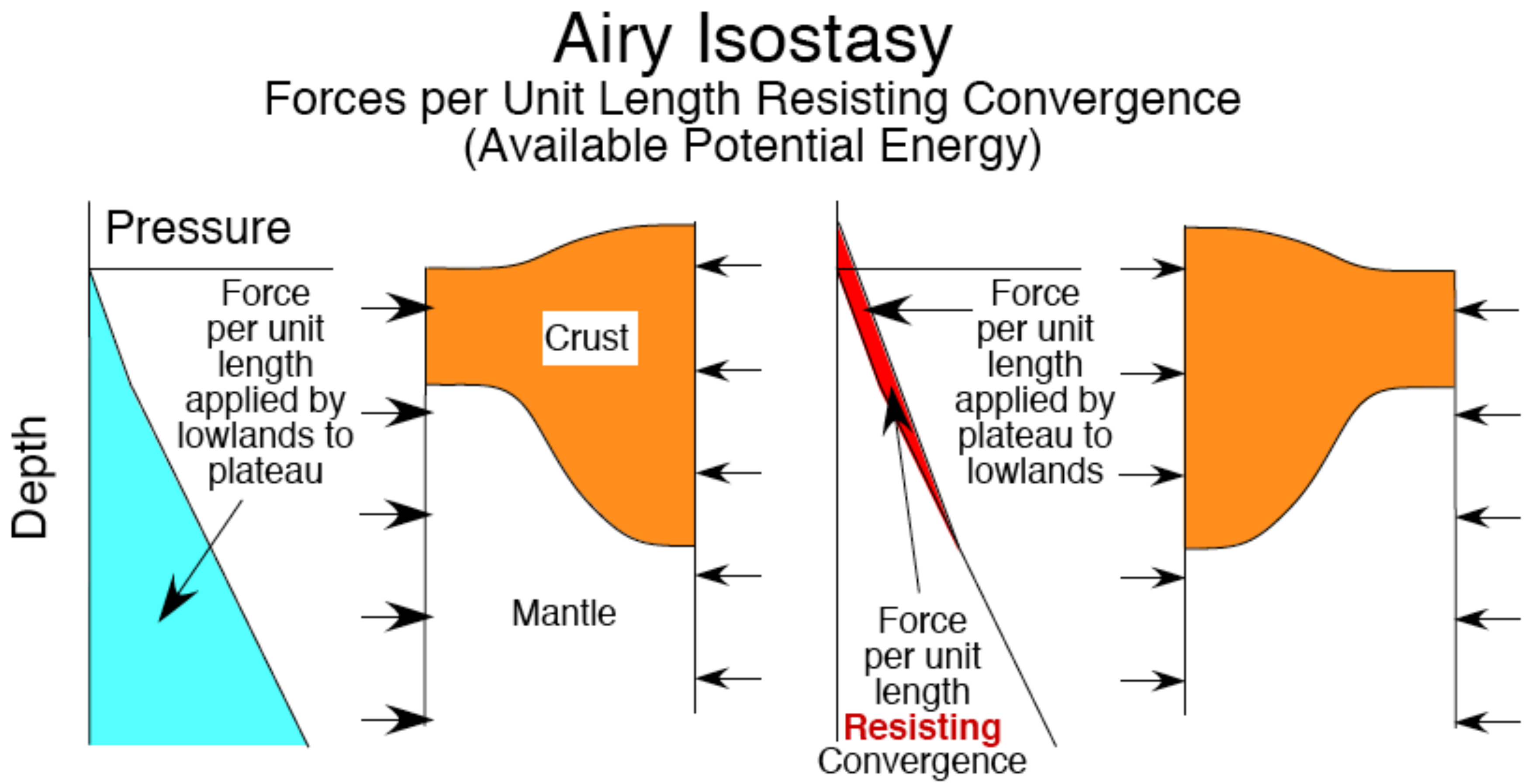
Potential energy: $U = \int \vec{F} \cdot d\vec{u}$

Potential energy per unit area in the earth:

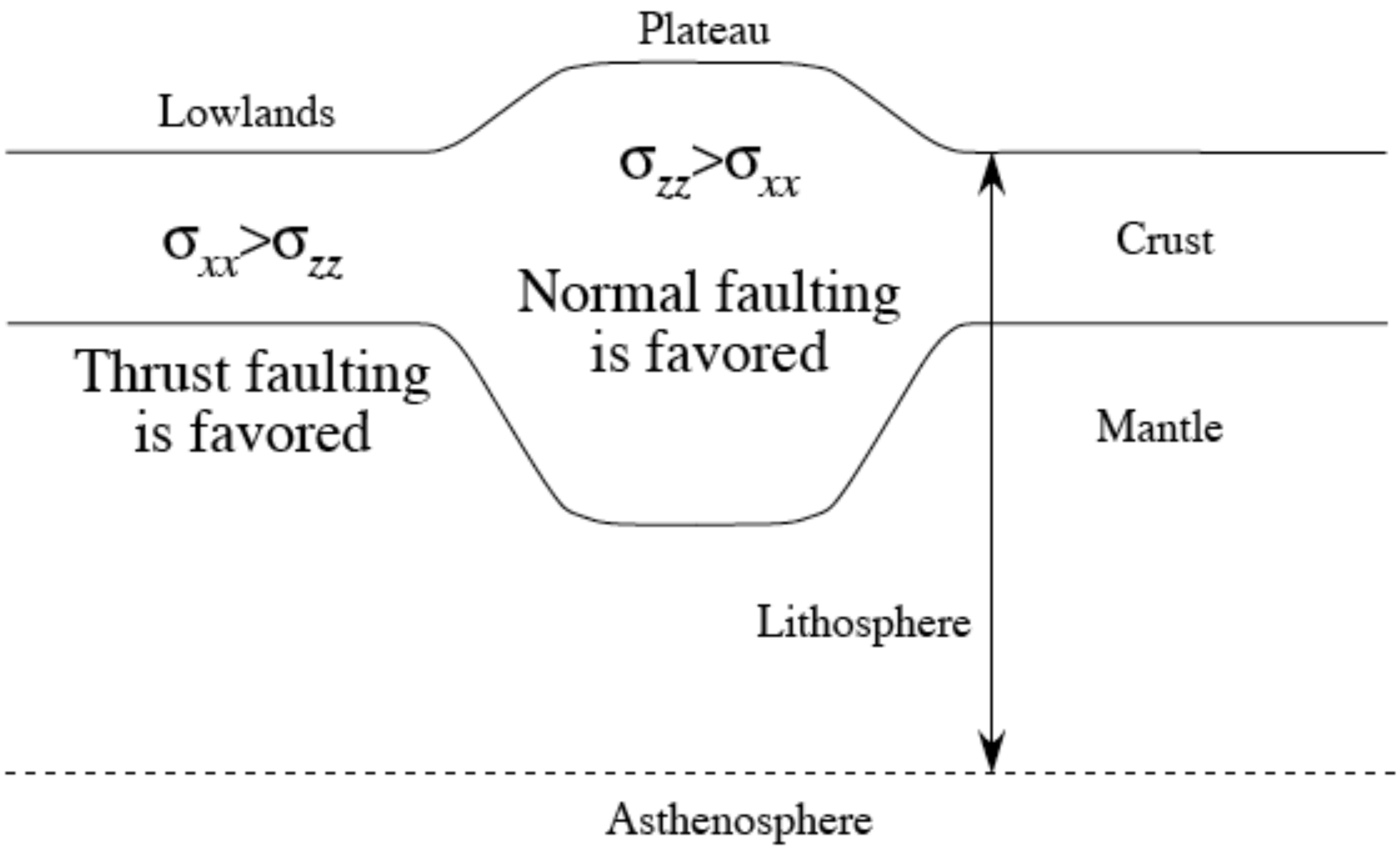
$$PE = \int_{\text{Depth of Compensation}}^{\text{Earth's Surface}} \sigma_{zz} dz$$

$$= \int_{\text{Depth of Compensation}}^{\text{Earth's Surface}} \left(\int_{z}^{\text{Earth's Surface}} \rho(z') g dz' \right) dz$$

Difference in potential energy (per unit area) equals a force per unit length

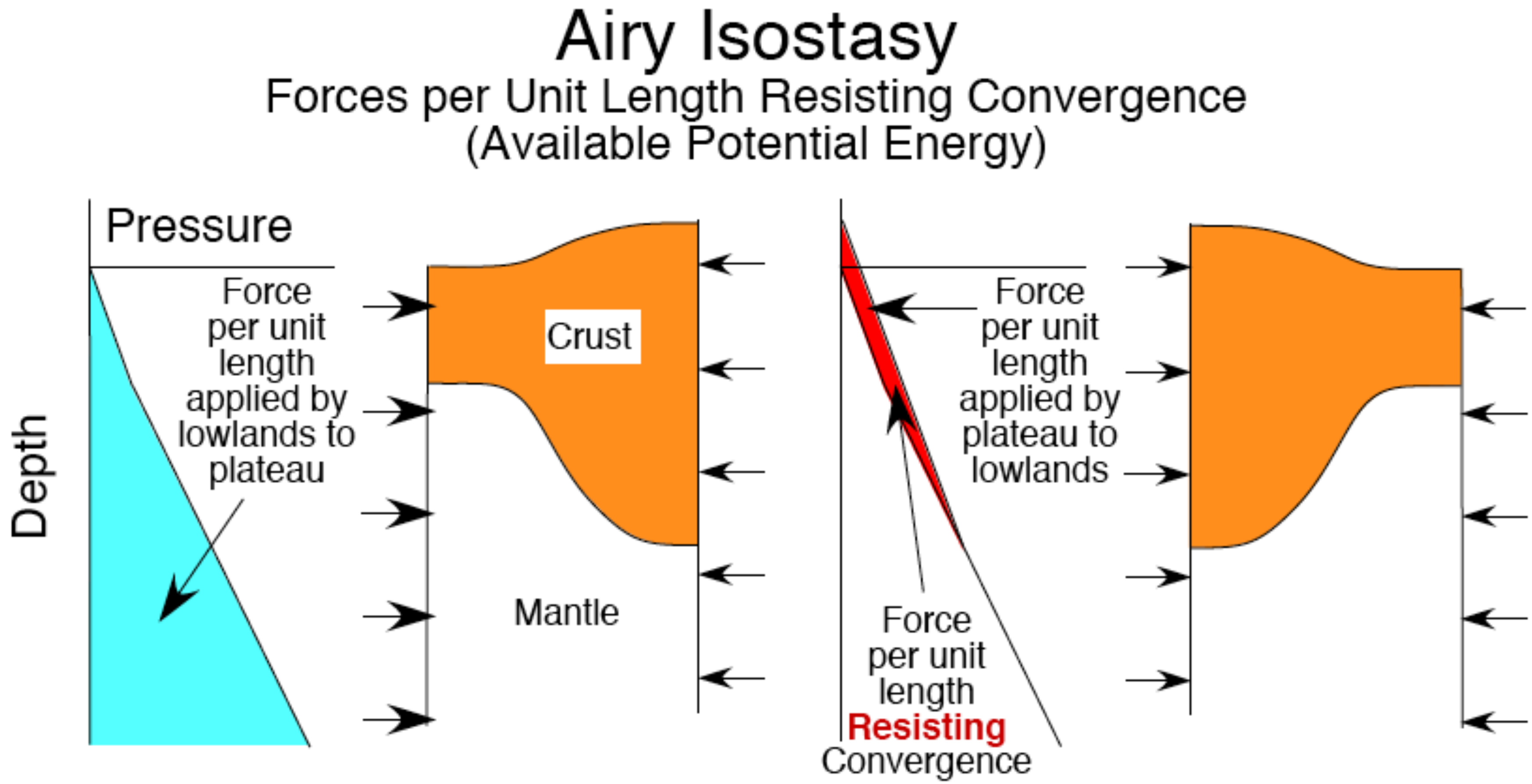


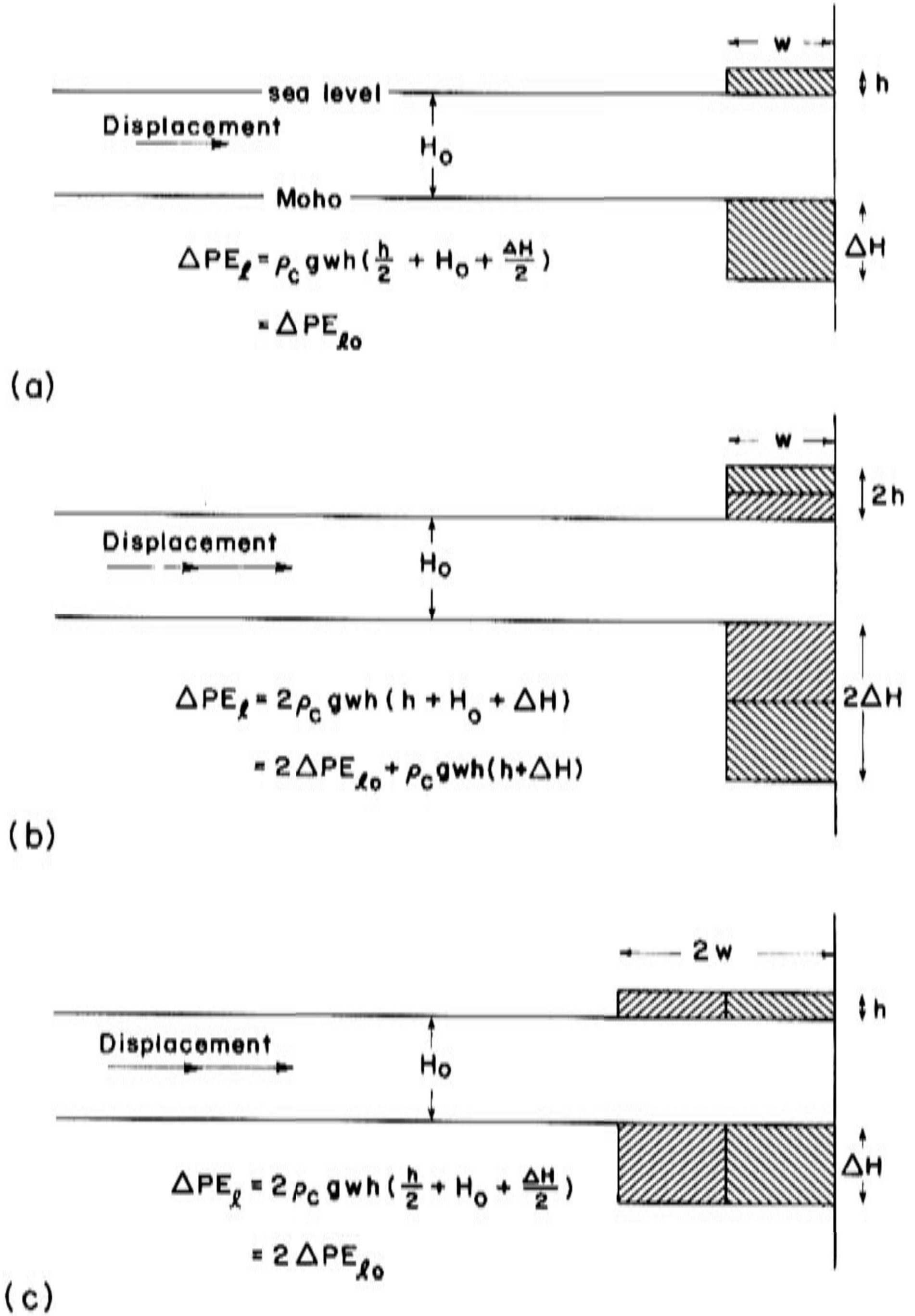
More potential energy is stored beneath higher than lower terrain. The difference in *PE*, however, does not determine whether normal or thrust faulting occurs.



$$L\overline{\sigma}_{xx} = \text{Constant}$$

The vertical compressive stress increases linearly with depth, proportional to the product of density and gravity.





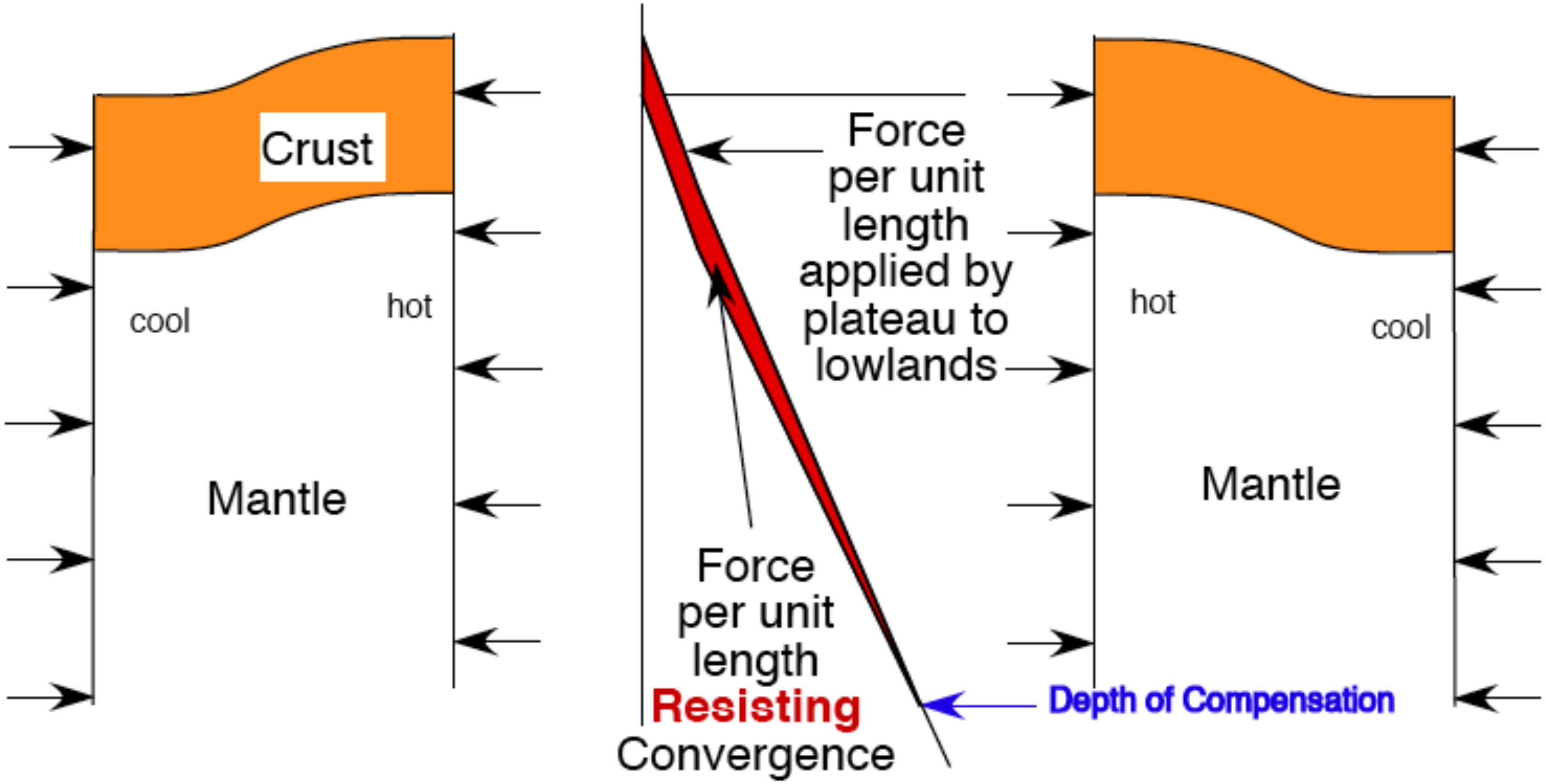
Some GPE arithmetic

Difference in GPE change between doubling the crustal thickness and doubling the width of a high region.

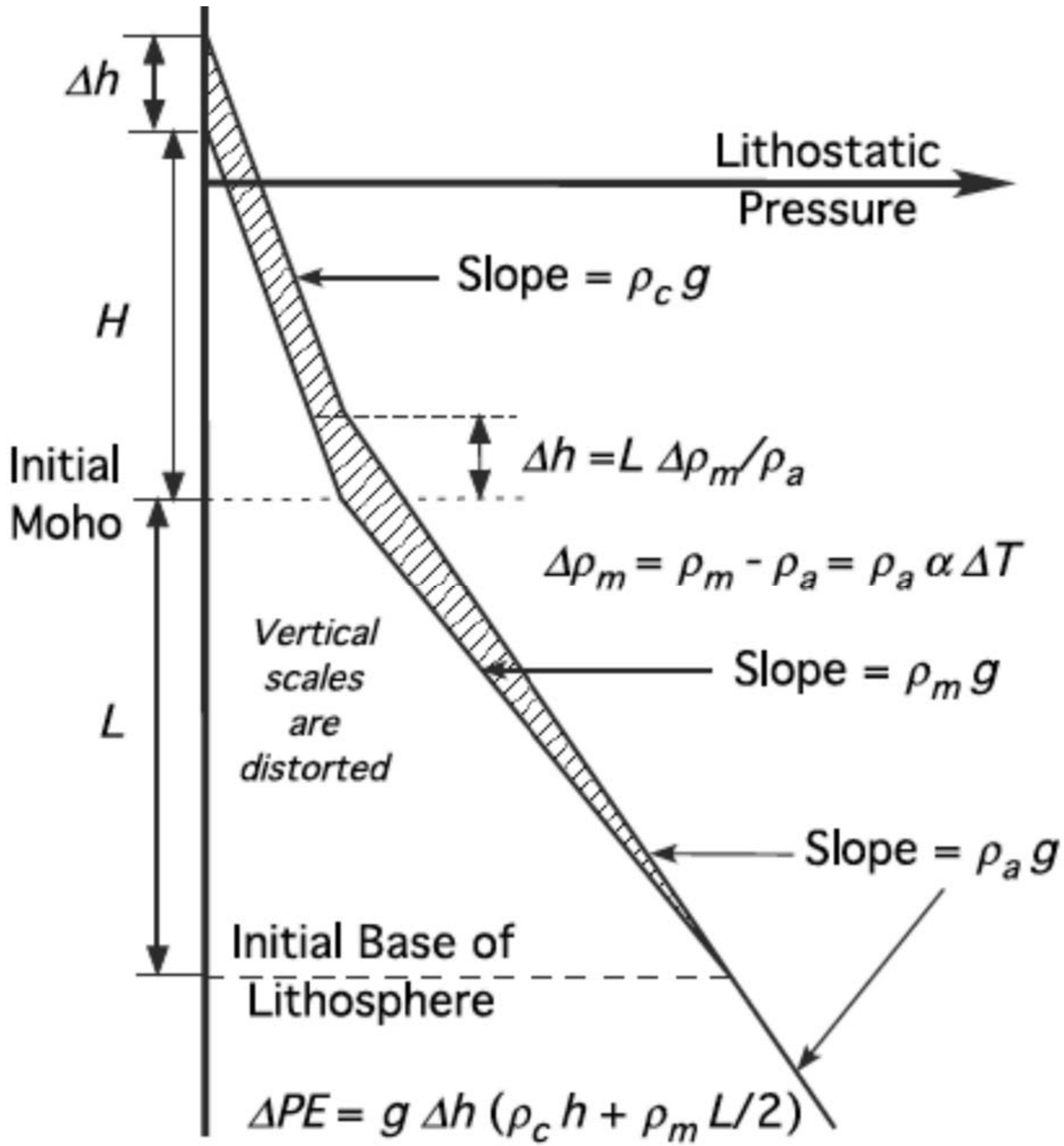
[*Molnar and Lyon-Caen* 1988]

Pratt Isostasy

Force per Unit Length Resisting Convergence
(Available Potential Energy)



Again, difference in potential energy (per unit area) equals force per unit length



GPE gain associated with removal of mantle lithosphere.

This is not small!

[*Molnar and Stock* 2009]

Some possible misunderstandings

1. "Transmission of stress" is a non concept
2. "Regional stress field"

$$L\overline{\sigma}_{xx} = \mathrm{Constant}$$

Horizontal normal stresses are subject to the equation of equilibrium, but regional variations need not be large.

Deformation results from **deviatoric** stress.

Stress and Deviatoric Stress
(*a source of confusion*)

$$\sigma_{ij} = P\delta_{ij} + \tau_{ij}$$

σ_{ij} is the stress tensor (*positive if compressive*)

P is pressure

δ_{ij} is the Kronecker delta: $\delta_{ij} = 1, i = j; \delta_{ij} = 0, i \neq j$

τ_{ij} is the deviatoric stress tensor.

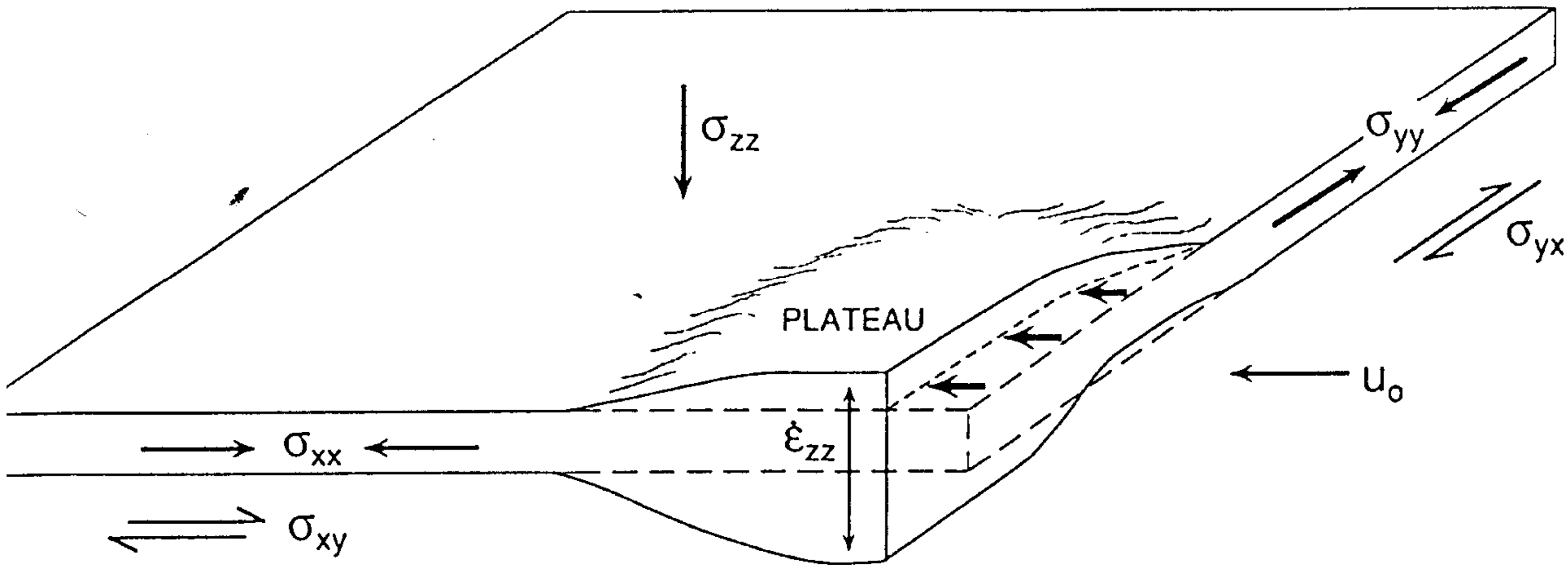
Stress does work.

Deviatoric stress causes deformation.

Some possible misunderstandings

1. “Transmission of stress” is a non concept.
2. “Regional stress field” is a misleading concept because:
3. Deformation results from **deviatoric** stress, and $L\overline{\sigma}_{xx} = \text{Constant}$
4. The *deviatoric* stress field depends on lateral variations in crustal thickness or in the thermal structure of the upper mantle (the body force in the equation of equilibrium).

Thin Viscous Sheet



Basic assumptions

1. No shear or normal stresses on the top surface. (No wind)
2. No shear stress on the bottom surface. (Mantle dynamics apply no basal traction.)
3. Applicable to deformation averaged over distances comparable to the thickness of the lithosphere.
4. Vertically average properties, stresses, strains, and velocities describe well regional fields.

Equation of Equilibrium (again)

$$\nabla \cdot \sigma_{ij} - \rho g \hat{z} = 0$$

(Now ***tensile stress is positive; z increases upward***.)

The gradient in the stress tensor plus the body force is zero (no acceleration).

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g$$

Pressure and deviatoric stress

$$P = -\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)/3 \qquad \tau_{ij} = \sigma_{ij} + P\delta_{ij}$$

Thus,

$$P = \tau_{xx} - \sigma_{xx} = \tau_{yy} - \sigma_{yy} = \tau_{zz} - \sigma_{zz}$$

Hence,

$$\sigma_{xx} = \tau_{xx} - \tau_{zz} + \sigma_{zz} \qquad \sigma_{yy} = \tau_{yy} - \tau_{zz} + \sigma_{zz}$$

and, of course:

$$\sigma_{xy} = \tau_{xy} \qquad \sigma_{xz} = \tau_{xz} \qquad \sigma_{yz} = \tau_{yz}$$

The horizontal components

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \qquad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\boldsymbol{\sigma}_{xx} = \boldsymbol{\tau}_{xx} - \boldsymbol{\tau}_{zz} + \boldsymbol{\sigma}_{zz} \qquad \boldsymbol{\sigma}_{yy} = \boldsymbol{\tau}_{yy} - \boldsymbol{\tau}_{zz} + \boldsymbol{\sigma}_{zz}$$

$$\frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{zz}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_{zz}}{\partial x}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \frac{\partial \tau_{zz}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_{zz}}{\partial y}$$

The vertical component: $\sigma_{zz}(z) = \int_{z'}^{top} \rho(z') g dz'$

Integrate the two horizontal component equations over depth

$$\int_{-L}^{0} \frac{\partial \tau_{xx}}{\partial x} dz = L \frac{\partial \overline{\tau}_{xx}}{\partial x} ; \int_{-L}^{0} \frac{\partial \tau_{xz}}{\partial z} dz = \tau_{xz}(z=0) - \tau_{xz}(z=-L) = 0$$

and the same for the others.

Define gravitational potential energy per unit area as

$$\Gamma = \int_{-L}^{0} \sigma_{zz}(z) dz$$

then

$$\int_{-L}^{0} \frac{\partial \sigma_{zz}}{\partial x} dz = \frac{\partial \Gamma}{\partial x}$$

Thin viscous sheet equations

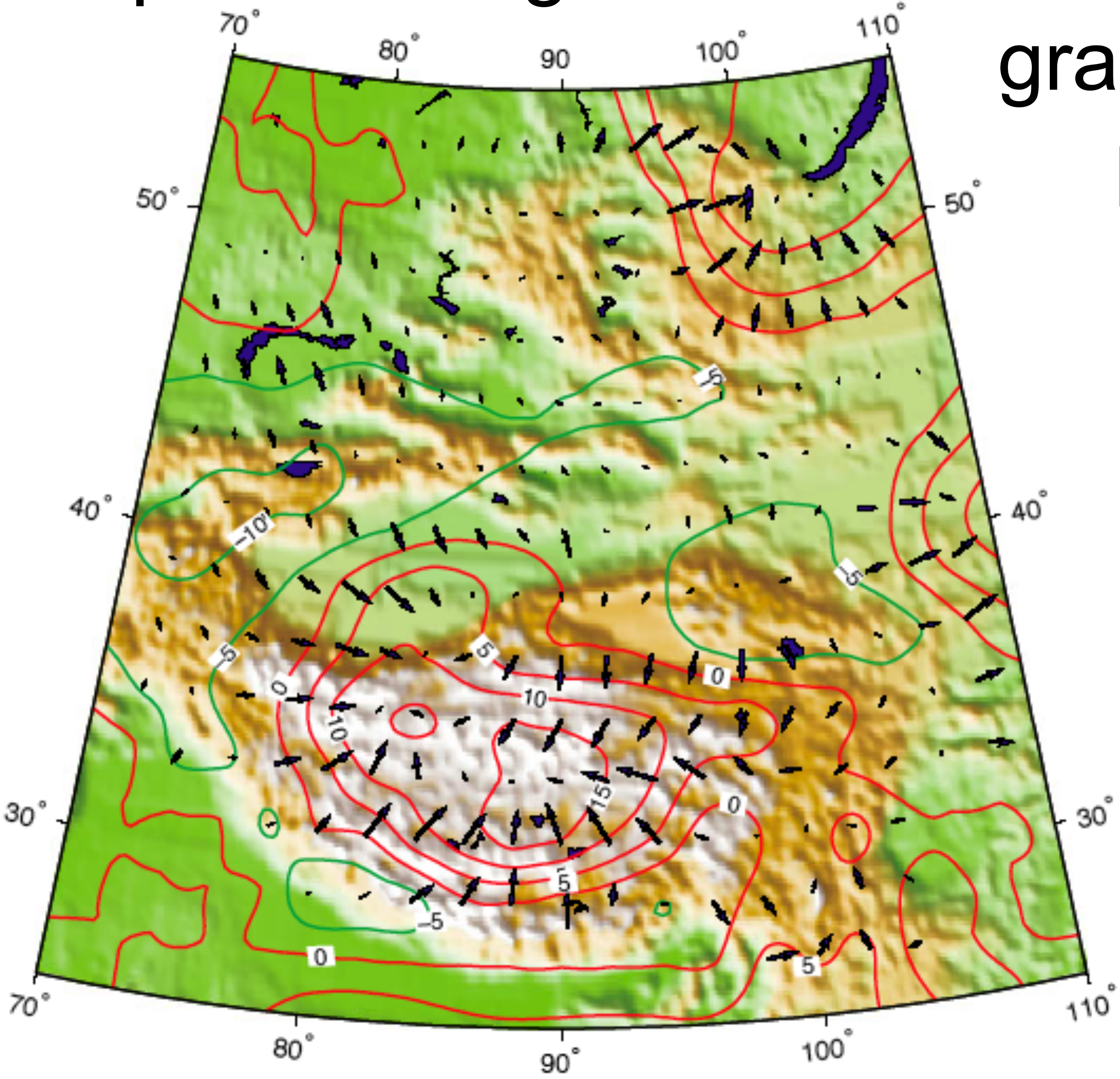
$$L\left(\frac{\partial\overline{\boldsymbol{\tau}}_{xx}}{\partial x} - \frac{\partial\overline{\boldsymbol{\tau}}_{zz}}{\partial x} + \frac{\partial\overline{\boldsymbol{\tau}}_{xy}}{\partial y}\right) = \frac{\partial\Gamma(x,y)}{\partial x}$$

$$L\left(\frac{\partial\overline{\boldsymbol{\tau}}_{yy}}{\partial y} - \frac{\partial\overline{\boldsymbol{\tau}}_{zz}}{\partial y} + \frac{\partial\overline{\boldsymbol{\tau}}_{xy}}{\partial x}\right) = \frac{\partial\Gamma(x,y)}{\partial y}$$

These state that horizontal gradients in (vertically averaged) stress (*left side*) equal horizontal gradients in potential energy per unit area, Γ, (*right side*).

(Potential energy per unit area scales approximately with mean elevation.)

Comparison of gradients in "stress" with gradients in potential energy per unit area.



[*England and Molnar* 1997]

An example where assumptions are violated

Consider lithosphere underthrust beneath a mountain belt and flexed down (effectively by elastic stresses).

Strength of the lithosphere supports the mountain range, because shear stresses on vertical planes are not negligible.

Summary

1. Isostasy: both thick crust or hot upper mantle can support mountain ranges.
2. Crustal thickening can build mountain ranges:
   a. Widespread thrust faulting and crustal shortening
   b. Underthrusting of thick lithosphere (flexure)
   c. Channel flow in the crust
3. Forces (per unit length) do work against strength of rock (friction and viscosity) and against gravity (potential energy). (*Both must be considered*)
4. Horizontal gradients in stress balanced horizontal gradients in potential energy (mean elevations).
5. Removal of mantle lithosphere can alter the thermal structure and (via Pratt isostasy) alter mean elevations.

Some possible misunderstandings

1. Transmission of stress:

$$\nabla \cdot \sigma_{ij} - \rho g \hat{z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho g \qquad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

Transmission of stress is a non concept!

Gradients in stress and body forces balance each other. Stress **must be** “transmitted.”

Some possible misunderstandings

1. Transmission of stress is a non concept
2. Regional stress field

Basic assumptions

1. No shear or normal stresses on the top surface. (No wind)
2. No shear stress on the bottom surface. (Mantle dynamics play no role)
3. Applicable to deformation averaged over distances comparable to the thickness of the lithosphere.
4. Vertically average properties, stresses, strains, and velocities describe well regional fields.

Thin viscous sheet equations

$$L\left(\frac{\partial\bar{\boldsymbol{\tau}}_{xx}}{\partial x}-\frac{\partial\bar{\boldsymbol{\tau}}_{zz}}{\partial x}+\frac{\partial\bar{\boldsymbol{\tau}}_{xy}}{\partial y}\right)=\frac{\partial\Gamma(x,y)}{\partial x}$$

$$L\left(\frac{\partial\bar{\boldsymbol{\tau}}_{yy}}{\partial y}-\frac{\partial\bar{\boldsymbol{\tau}}_{zz}}{\partial y}+\frac{\partial\bar{\boldsymbol{\tau}}_{xy}}{\partial x}\right)=\frac{\partial\Gamma(x,y)}{\partial y}$$

These state that horizontal gradients in (vertically averaged) stress (*left side*) equal horizontal gradients in potential energy per unit area, Γ, (*right side*).

(Potential energy per unit area, Γ, scales approximately with mean elevation.)