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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Diagnostics of thin sheet deformation

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Advanced School on
Scaling Laws in Geophysics:
Mechanical and Thermal Processes
in Geodynamics

Length Scales for Boundary-driven Deformation

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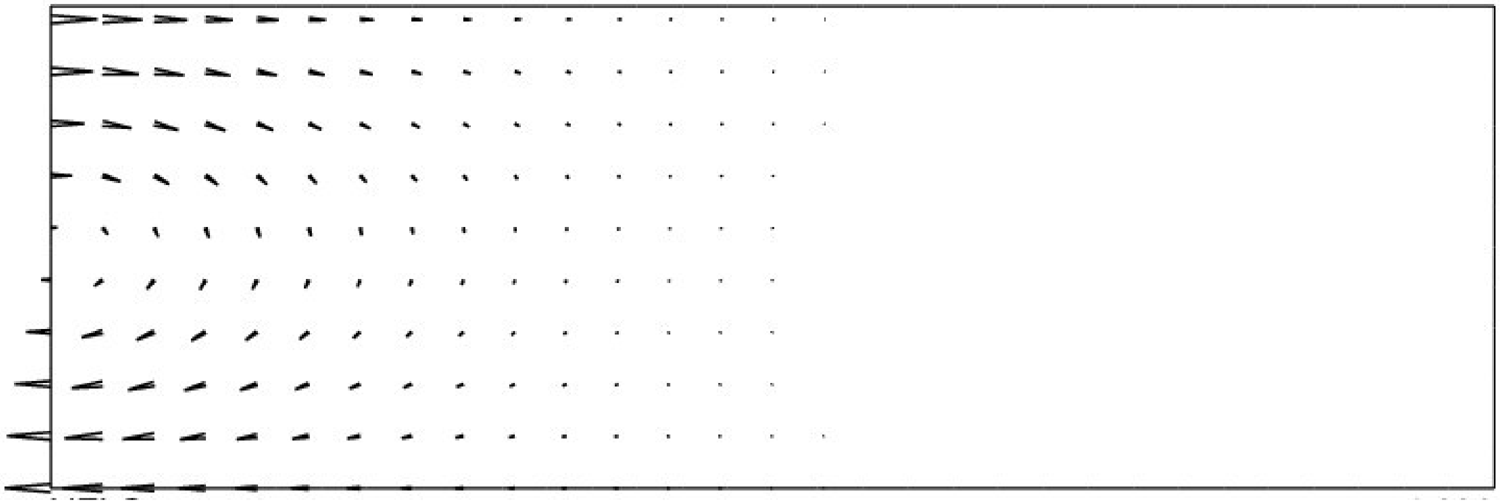
Indentation of a Thin Viscous sheet



Assuming a harmonic variation of the indenting component of velocity on the boundary of a half-space (only half a wavelength is shown):

Velocity decays away from the boundary, how exactly ?

From the thin sheet equations with no GPE variations and *n* = 1:

$$\nabla \cdot \underline{\tau} - \nabla \bar{\tau}_{zz} = 0$$

$$\eta \nabla^2 \mathbf{u} + \eta \nabla (\nabla \cdot \mathbf{u}) + 2\eta \nabla (\nabla \cdot \mathbf{u}) = 0$$

Indentation of a Thin Viscous sheet

If the velocity is specified on the boundary, the velocity field is independent of viscosity:

$$\nabla^2 \mathbf{u} + 3\nabla(\nabla \cdot \mathbf{u}) = 0$$

If we take the divergence of this equation:

$$\nabla^2(\nabla \cdot \mathbf{u}) = 0 \quad \text{or} \quad \nabla^2(\tau_{zz}) = 0$$

Assume the divergence has the same horizontal wavelength as the boundary condition, then Laplace's equation will have a solution of the form:

$$\nabla \cdot \mathbf{u} = f(x)\cos\left(\frac{2\pi y}{\lambda}\right)$$

Substitution gives a solution which decays exponentially in the x direction:

$$\left[\frac{d^2}{dx^2} - \left(\frac{2\pi}{\lambda}\right)^2\right] f(x) = 0$$

so:

$$\nabla \cdot \mathbf{u} = A\cos\left(\frac{2\pi y}{\lambda}\right)\exp\left(\frac{-2\pi x}{\lambda}\right)$$

Indentation of a Thin Viscous sheet

Then we can solve separately for the u and v components of the velocity field which each satisfy a Poisson equation:

$$\nabla^2 u_x = \left(\frac{6A\pi}{\lambda}\right)\cos\left(\frac{2\pi y}{\lambda}\right)\exp\left(\frac{-2\pi x}{\lambda}\right)$$

and:

$$\nabla^2 u_y = \left(\frac{6A\pi}{\lambda}\right)\sin\left(\frac{2\pi y}{\lambda}\right)\exp\left(\frac{-2\pi x}{\lambda}\right)$$

So with more algebra, and boundary conditions, we get solutions:

$$u_x = -U_0 \cos\left(\frac{2\pi y}{\lambda}\right)\exp\left(\frac{-2\pi x}{\lambda}\right)\left[1+\frac{6\pi x}{5\lambda}\right]$$

$$u_y = -U_0 \sin\left(\frac{2\pi y}{\lambda}\right)\exp\left(\frac{-2\pi x}{\lambda}\right)\left[\frac{6\pi x}{5\lambda}\right]$$

and the other aspects of the equation (strain-rates, deviatoric stresses, etc,) can be derived from these equations.

Shear of a Thin Viscous sheet

Suppose now that the deformation field is driven by a harmonic boundary velocity parallel to the boundary:

$$u_x = -U_0 \cos\left(\frac{2\pi y}{\lambda}\right) \exp\left(\frac{-2\pi x}{\lambda}\right)\left[\frac{6\pi x}{5\lambda}\right]$$

$$u_y = U_0 \sin\left(\frac{2\pi y}{\lambda}\right) \exp\left(\frac{-2\pi x}{\lambda}\right)\left[1 - \frac{6\pi x}{5\lambda}\right]$$

Note that the major component of velocity (shear) decays much faster away from the boundary than in the case of indentation.

Indentation Response of Thin Viscous sheet ($n > 1$)

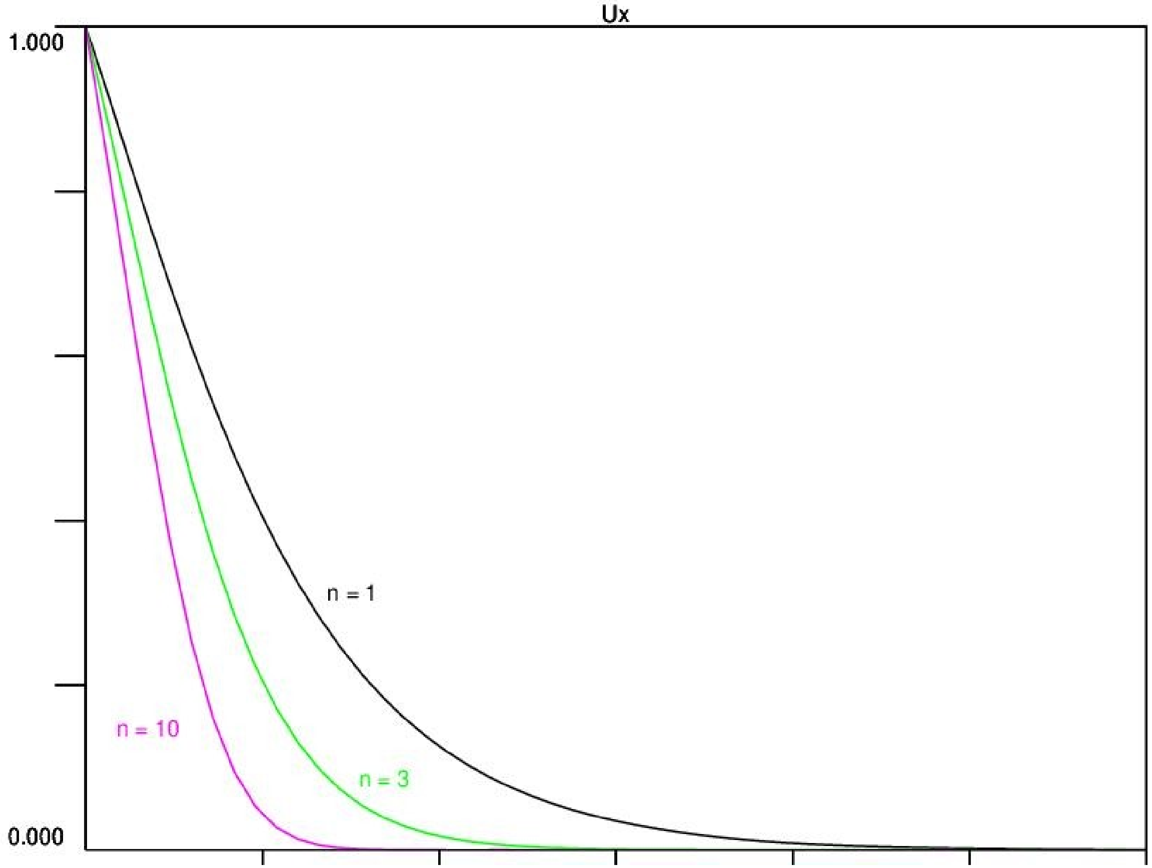
If n > 1

For indenting or shear boundary velocity on a half-space:

the major component of velocity decays faster as n increases.

From

$$\nabla \cdot \underline{\tau} - \nabla \bar{\tau}_{zz} = 0$$



$$\eta \nabla^2 \mathbf{u} + 3\eta \nabla (\nabla \cdot \mathbf{u}) + 2(\nabla \eta) \cdot \underline{\varepsilon} + 2(\nabla \eta)(\nabla \cdot \mathbf{u}) = 0$$

of course such an equation does not have an exact analytical solution for n >1, so we try to simplify by removing small terms

Approximate solution for n > 1 indentation

Complete equation for non-Newtonian thin viscous sheet is:

$$\eta \nabla^2 \mathbf{u} + 3\eta \nabla(\nabla \cdot \mathbf{u}) + 2(\nabla \eta) \cdot \underline{\varepsilon} + 2(\nabla \eta)(\nabla \cdot \mathbf{u}) = 0$$

considering the x-component

$$\eta\left[4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 3\frac{\partial^2 v}{\partial x \partial y}\right] + \left[2\frac{\partial \eta}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \eta}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\frac{\partial \eta}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] = 0$$

We first look for a way to simplify the second group of terms which include the viscosity gradients. If we divide through by

$$\eta = (B/2)\,\dot{E}^{(1-n)/n}$$

then

$$\frac{\partial(\ln \eta)}{\partial x} = \frac{(1-n)}{n}\frac{\partial(\ln \dot{E})}{\partial x}$$

At this point we need to make some simplifying assumption, and we assume that the second invariant of the strain rate is dominated by the convergent strain in front of the indenter

$$\dot{E} \approx \frac{\partial u}{\partial x}$$

Approximate solution for n > 1 indentation

Therefore
$$\frac{\partial(\ln\eta)}{\partial x}=\frac{(1-n)}{n}\frac{\partial(\ln\dot{E})}{\partial x}=\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x^2}\right)/\left(\frac{\partial u}{\partial x}\right)$$

and
$$\frac{\partial(\ln\eta)}{\partial y}=\frac{(1-n)}{n}\frac{\partial(\ln\dot{E})}{\partial y}=\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)/\left(\frac{\partial u}{\partial x}\right)$$

The terms which include the *x* viscosity gradients then are:

$$2\frac{\partial(\ln\eta)}{\partial x}\frac{\partial u}{\partial x}+2\frac{\partial(\ln\eta)}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=$$

$$2\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x^2}\right)+2\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x^2}\right)\left(1+\cancel{\left(\frac{\partial v}{\partial y}\right)/\left(\frac{\partial u}{\partial x}\right)}\right)=4\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

The final term on the right is neglected because *v* < *u*, in the spirit of the approximation used for second invariant.

Approximate solution for n > 1 indentation

The terms which include the y viscosity gradients then are:

$$\frac{\partial(\ln\eta)}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)=\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x\partial y}\right)\cancel{\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)}/\left(\frac{\partial u}{\partial x}\right)$$

we can justify ignoring all of these terms in the region directly in front of the indenter, where shear strain rate is negligible. Collecting then the terms that appear to be significant in the stress balance:

$$4\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\cancel{3\frac{\partial^2 v}{\partial x\partial y}}+4\frac{(1-n)}{n}\left(\frac{\partial^2 u}{\partial x^2}\right)=0$$

we discard the remaining term in v, and eventually get:

$$\frac{4}{n}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\approx 0$$

If we again assume solution like $u=f(x)\cos\left(\frac{2\pi y}{\lambda}\right)$

then $\left[\frac{d^2}{dx^2}-n\left(\frac{\pi}{\lambda}\right)^2\right]f(x)=0$ and solution is $u=U_0\exp\left(\frac{-x\pi\sqrt{n}}{\lambda}\right)$

Approximate solution for n > 1 sheared boundary

Use the approximation that strain dominated by: $\dot{E} \approx \frac{\partial v}{\partial y}$

whose validity can be assessed after solution

to obtain:

$$\frac{\partial^2 v}{\partial x^2} + 4n \frac{\partial^2 v}{\partial y^2} \approx 0$$

If we again assume solution like $v = f(x) \sin\left(\frac{2\pi y}{\lambda}\right)$

Then

$$\left[\frac{d^2}{dx^2} - n\left(\frac{4\pi}{\lambda}\right)^2\right] f(x) = 0$$

And the solution is

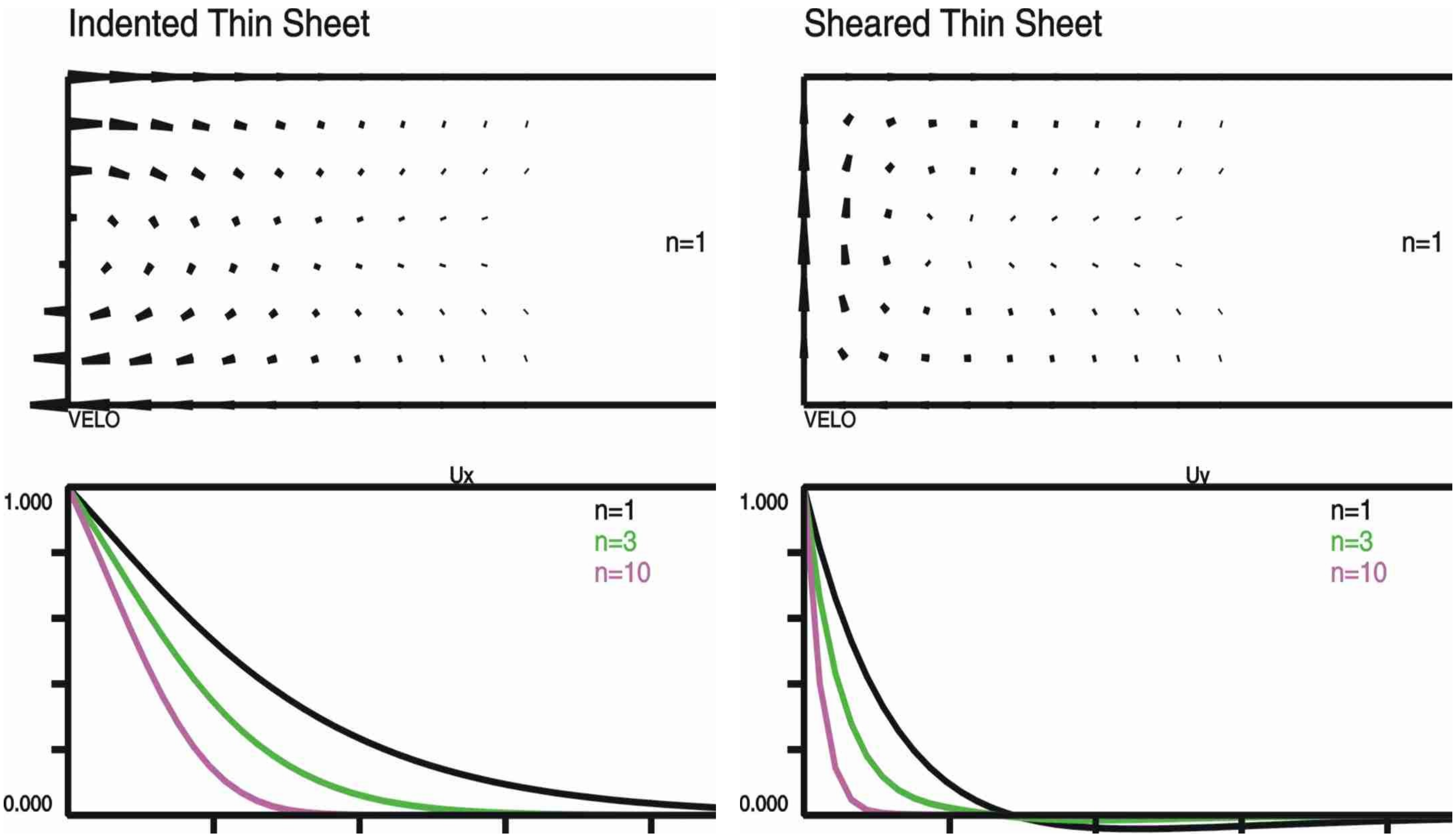
$$v = V_0 \exp\left(\frac{-4\pi x \sqrt{n}}{\lambda}\right)$$

Predictions: (1) shear decays 4 times faster than indentation

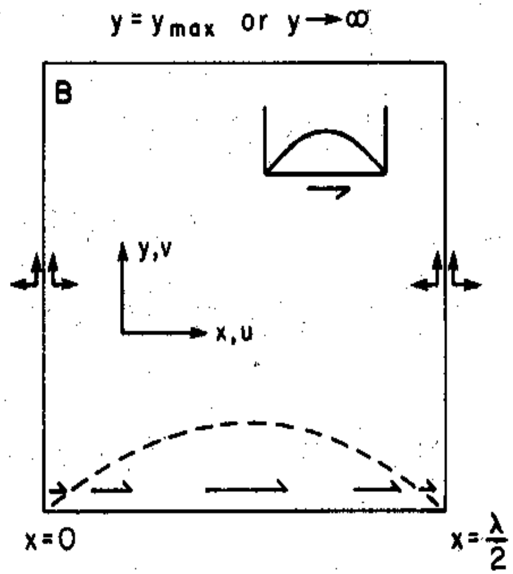
(2) indentation and shear decay at a $\text{rate} \propto \sqrt{n}$

Indented vs Sheared Boundaries

The principal components of velocity and strain decay about 4 times faster with distance from a sheared boundary:



Length Scales



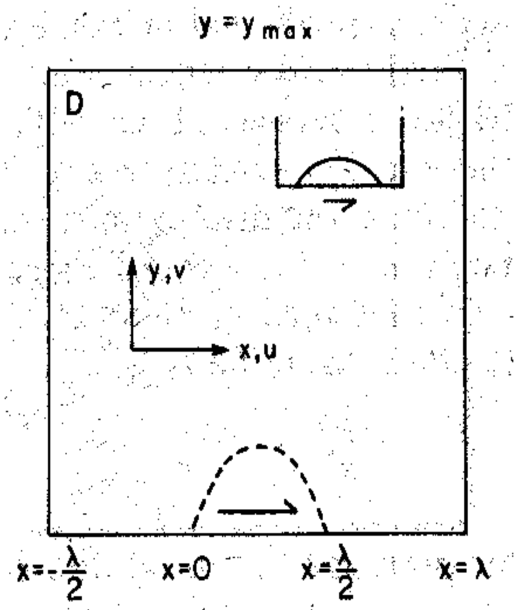
Thin sheet solutions for a transform boundary:

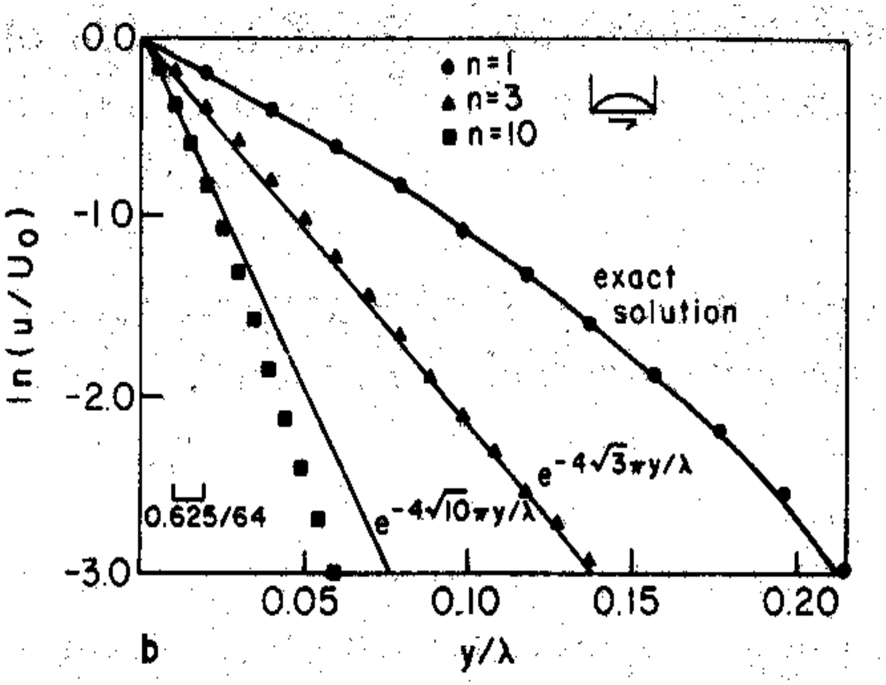
$$u = u_0 \exp\left(-4\pi y\sqrt{n}/\lambda\right)$$

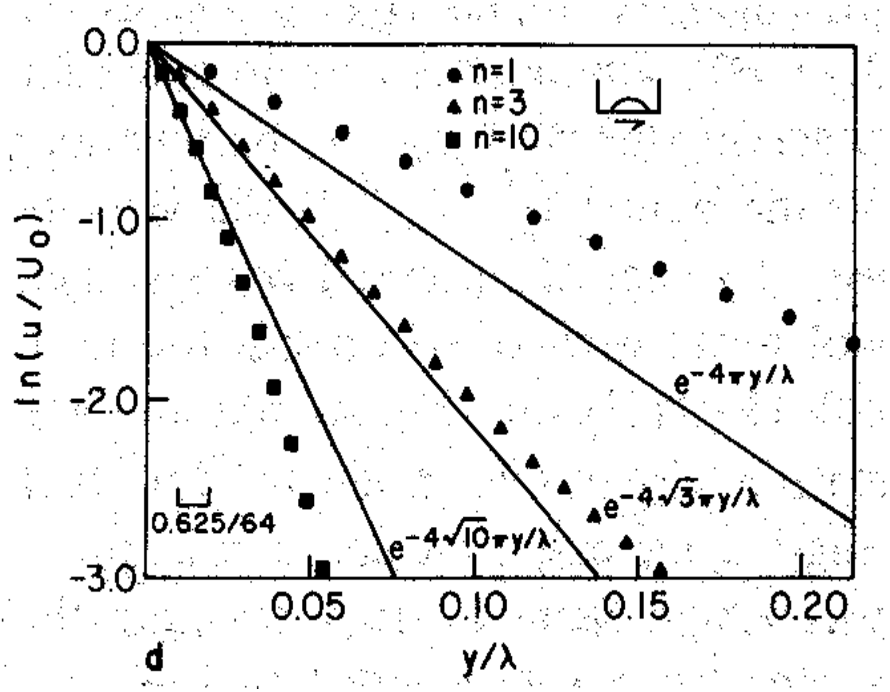
for a convergent or divergent boundary:

$$v = v_0 \exp\left(-\pi y\sqrt{n}/\lambda\right)$$

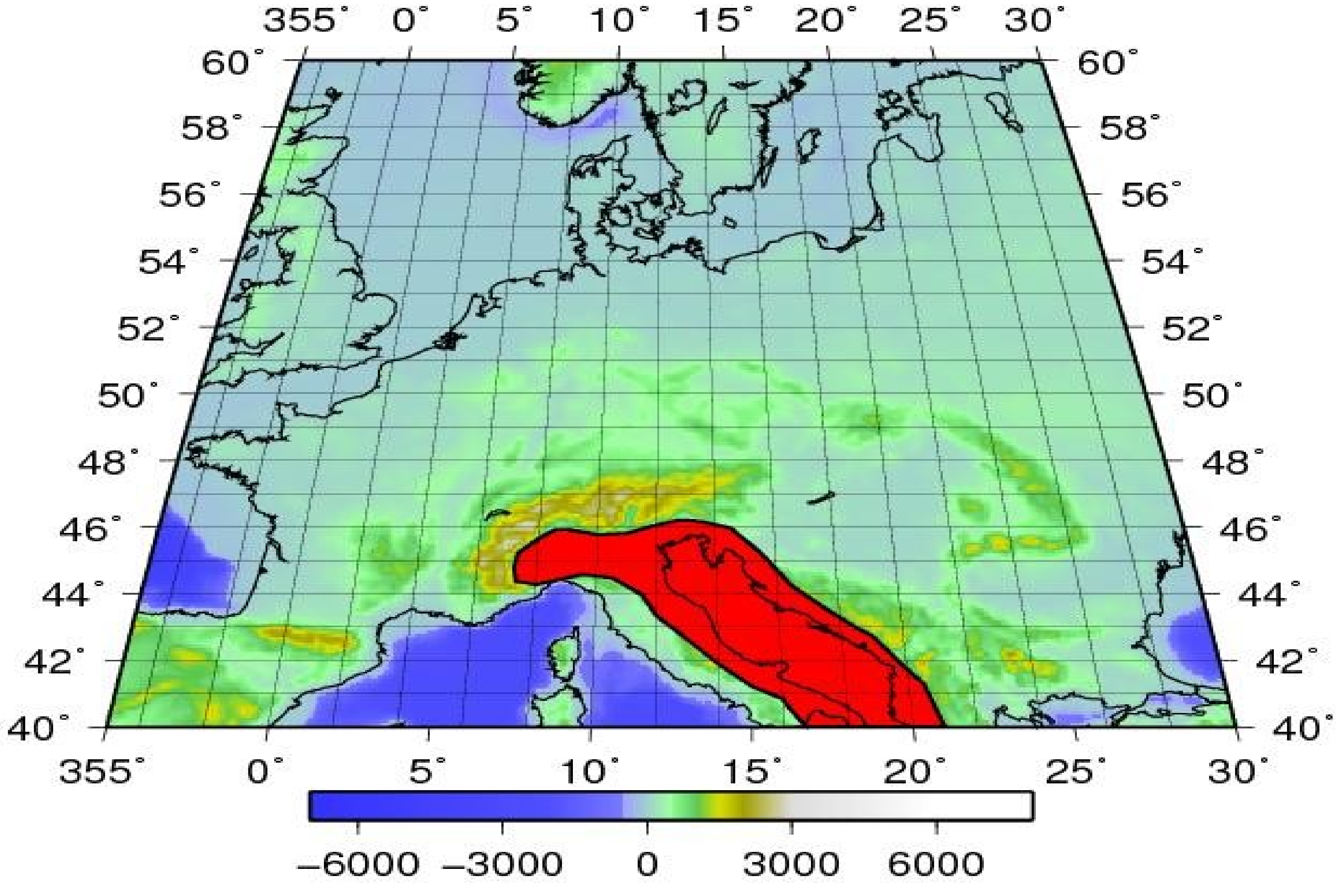
(England et al., 1985)



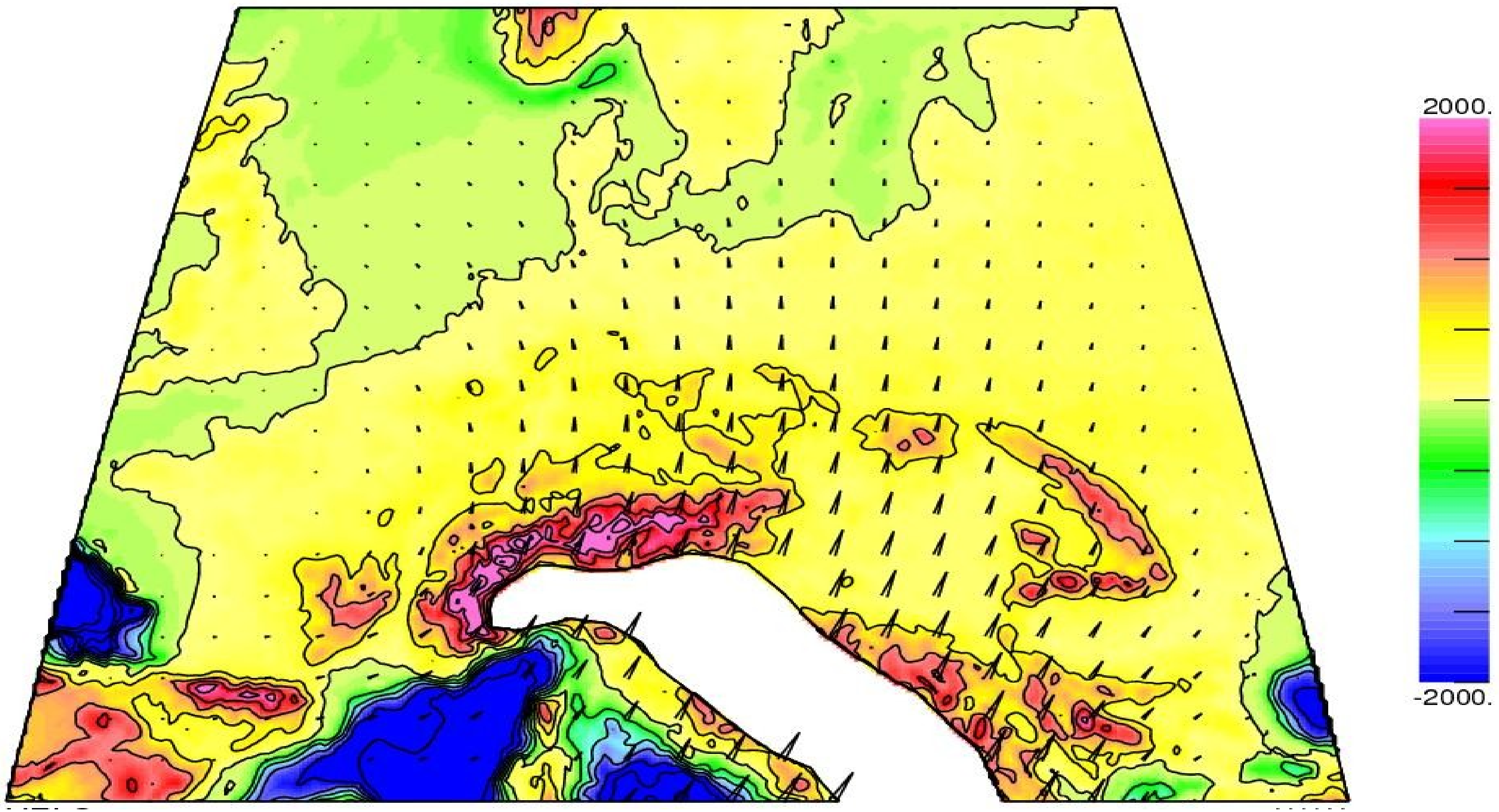




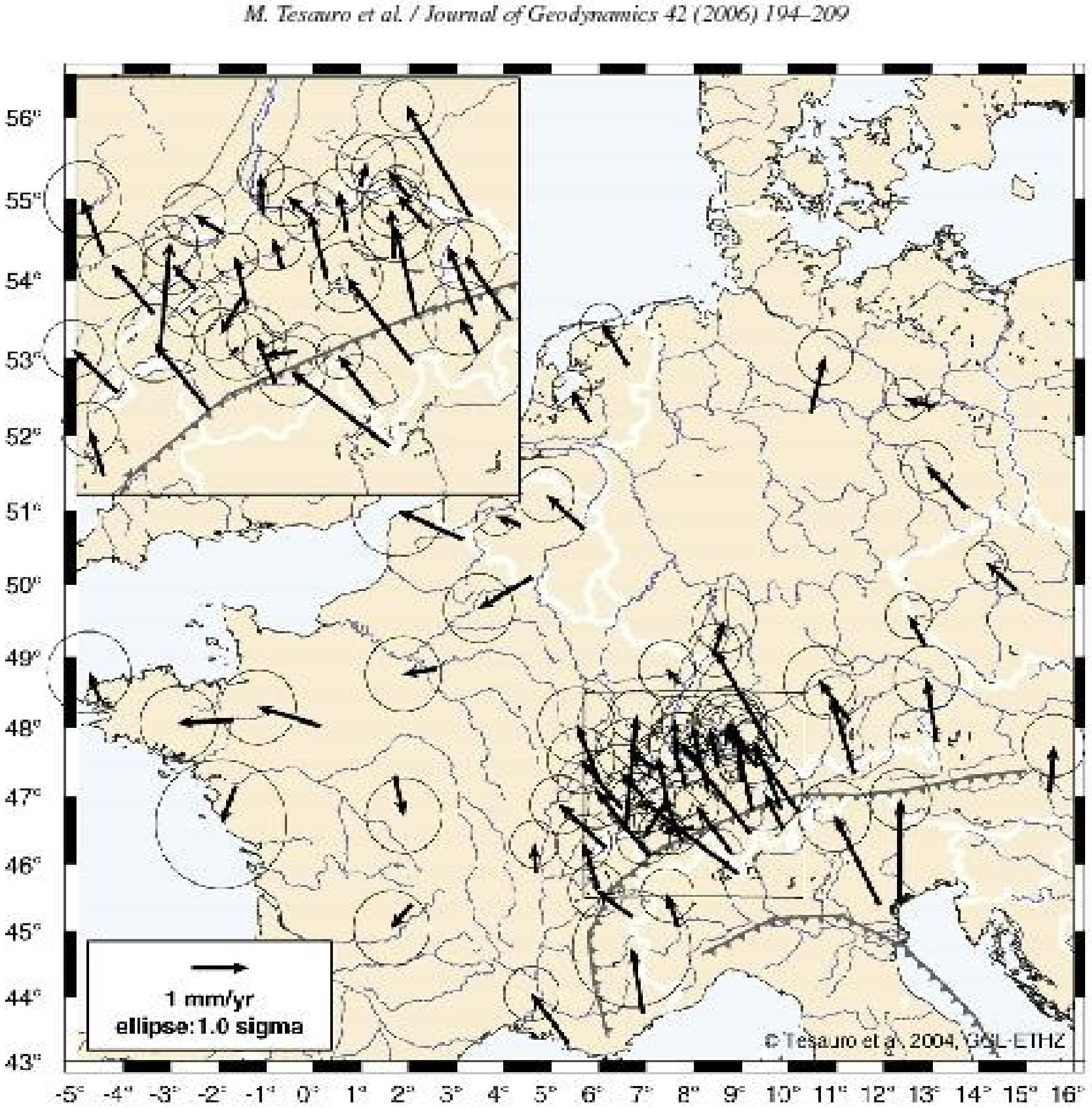
Alpine Orogeny caused by a rigid Adriatic block
(pushed and rotated by Africa)



Computed Thin sheet displacement fields (thin viscous sheet n = 1, Ar = 0) of indentation field caused by rotating Adriatic block



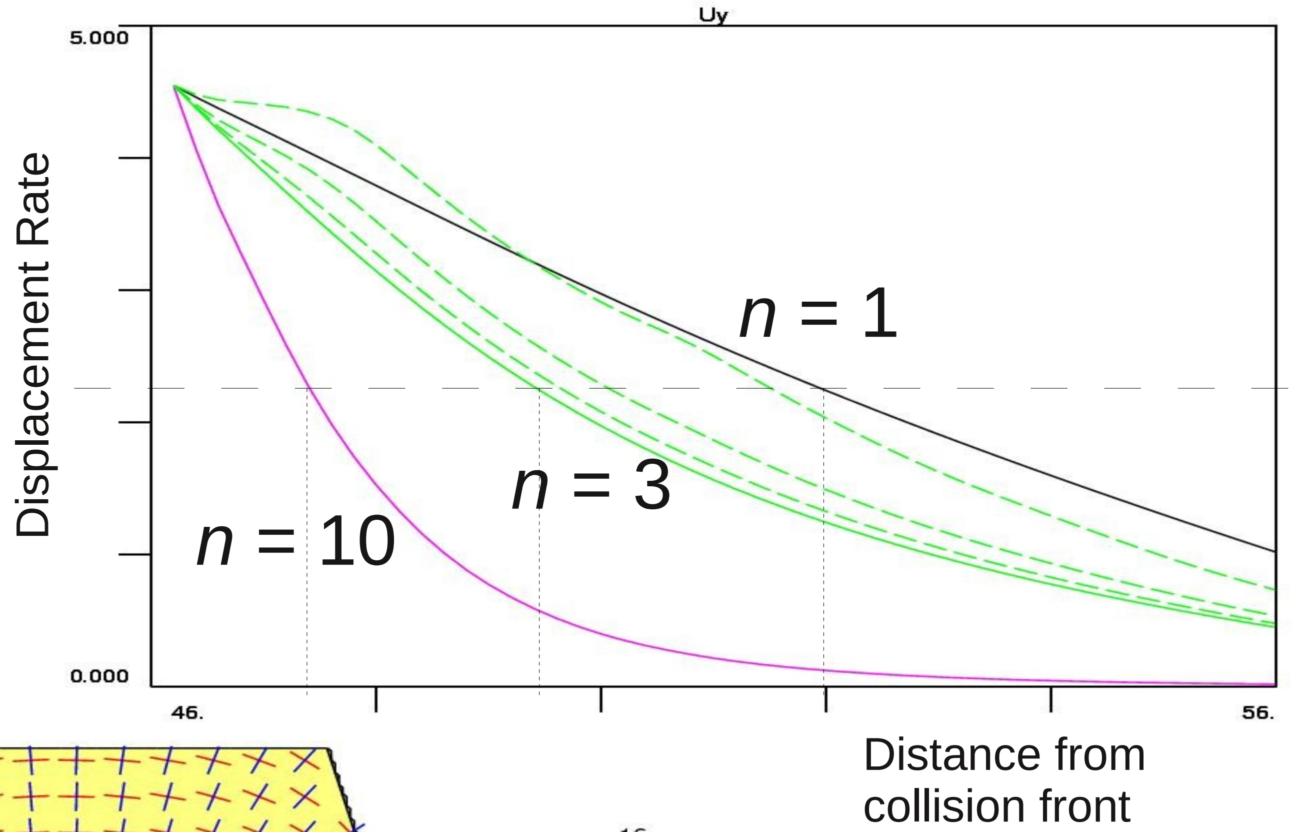
Geodetic Displacement Rates: Europe relative to Eurasia 1996-2004: (map by Tesauro et al., 2006).

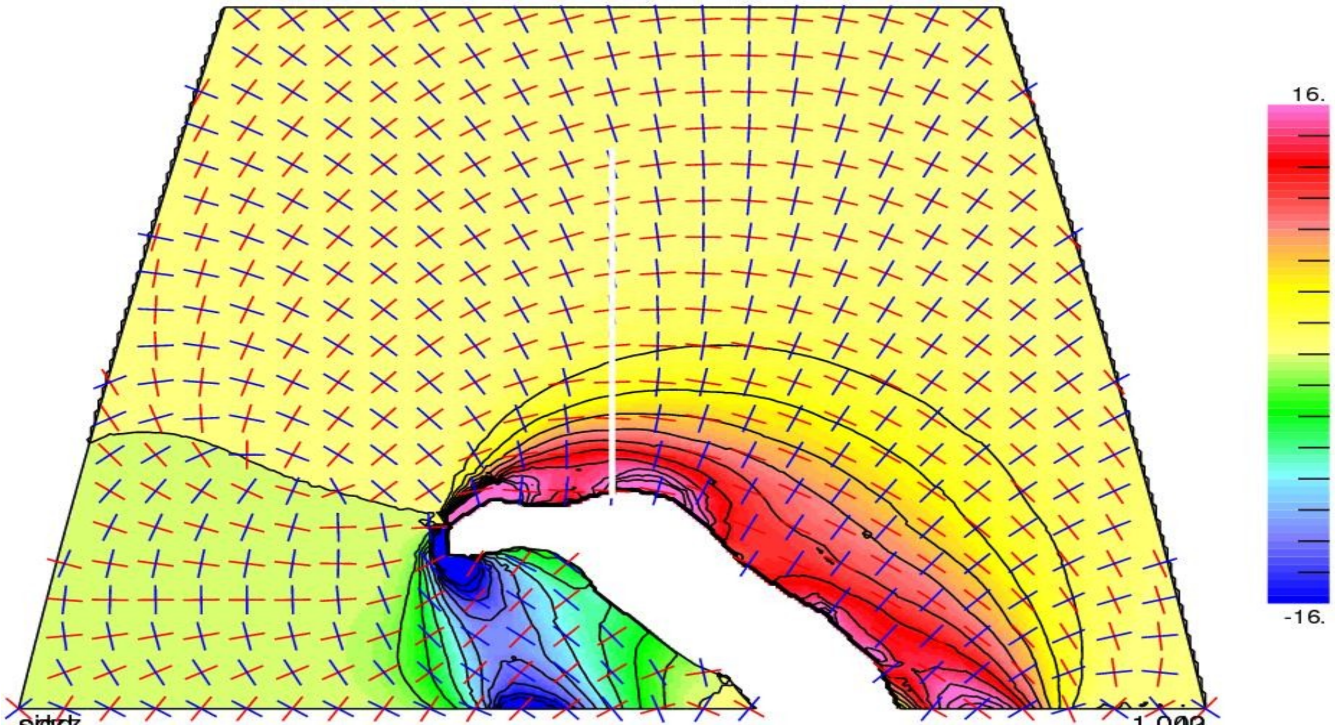


Observed geodetic displacement rates for 8 year period 1996-2004 decrease rapidly across the Alps (by a factor of two in a distance of 2º or less) – suggests a value of *n* greater than 3.

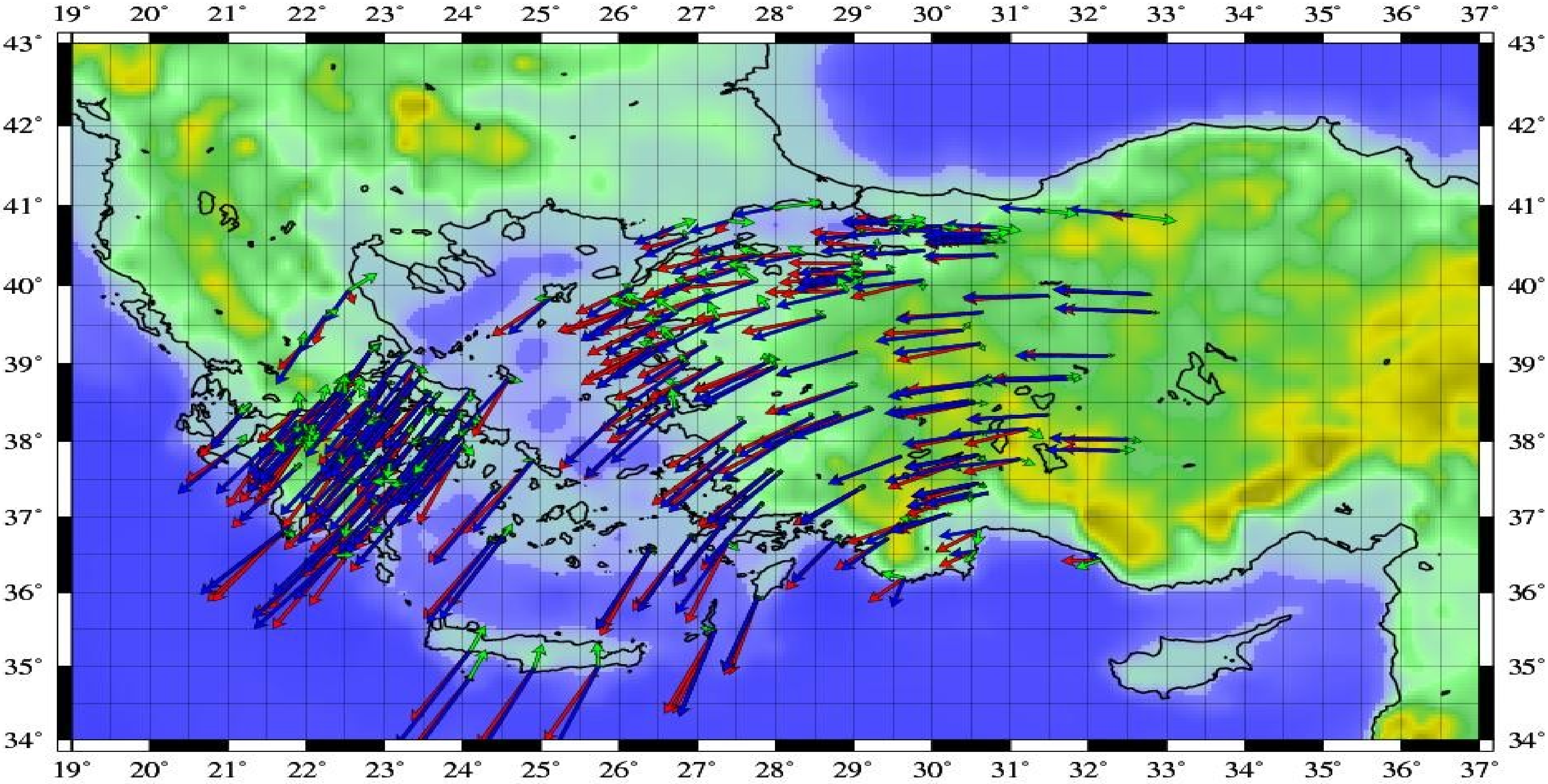
Thickening rates and Principal Stress Directions

Vertical strain-rates (10^{-9} yr^{-1}) for n = 1, 3, and 10, with buoyancy forces negligible. Principal horizontal stress directions also shown.

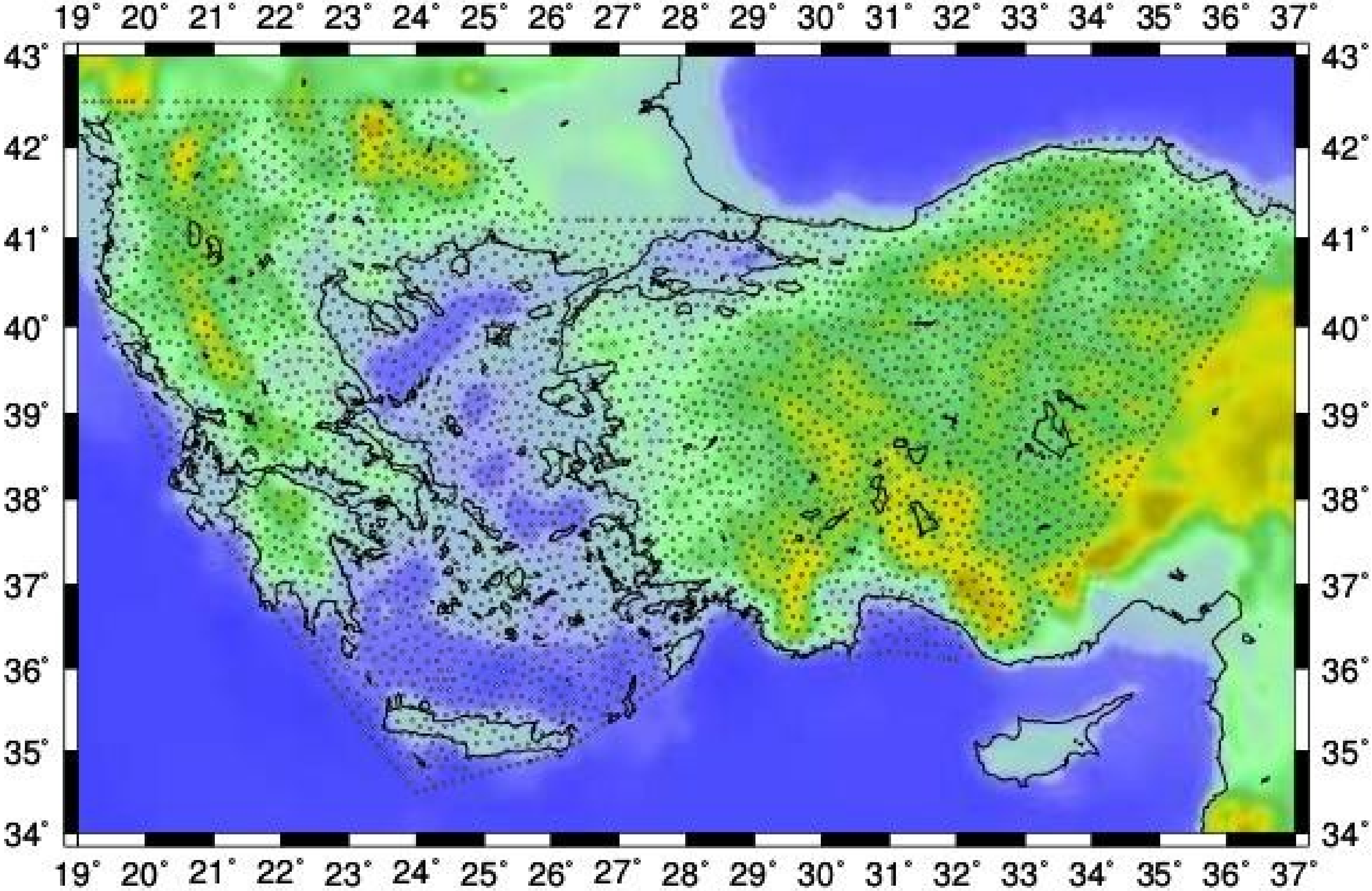




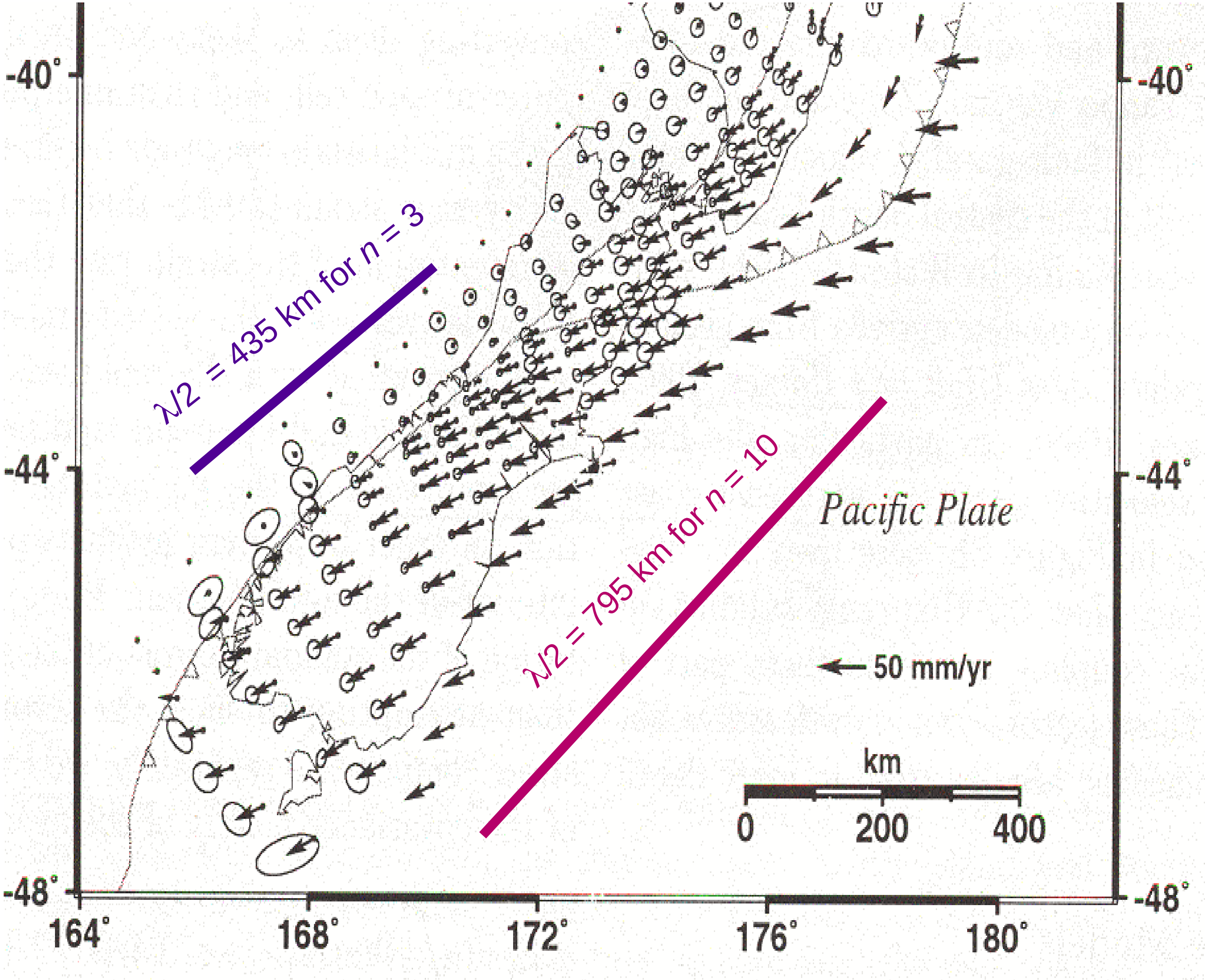
Deformation of Aegean and Anatolia



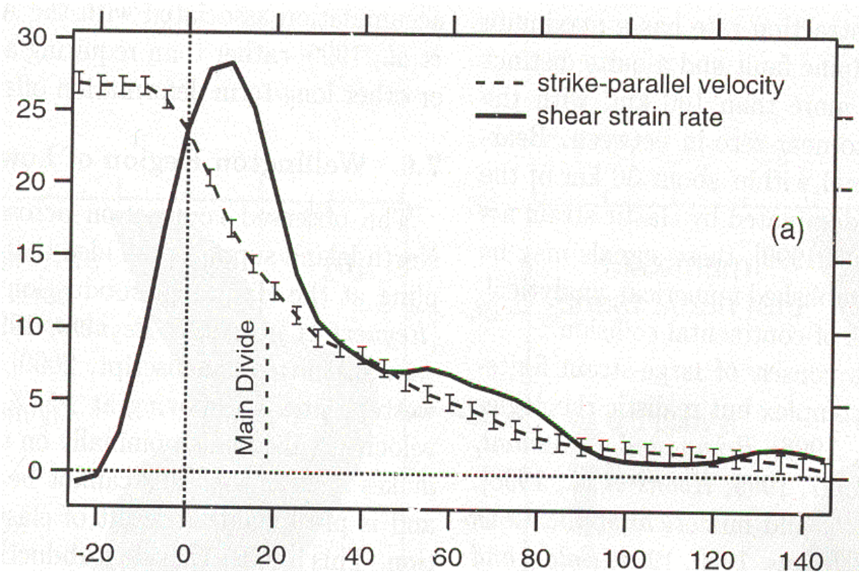
Deformation of Aegean and Anatolia

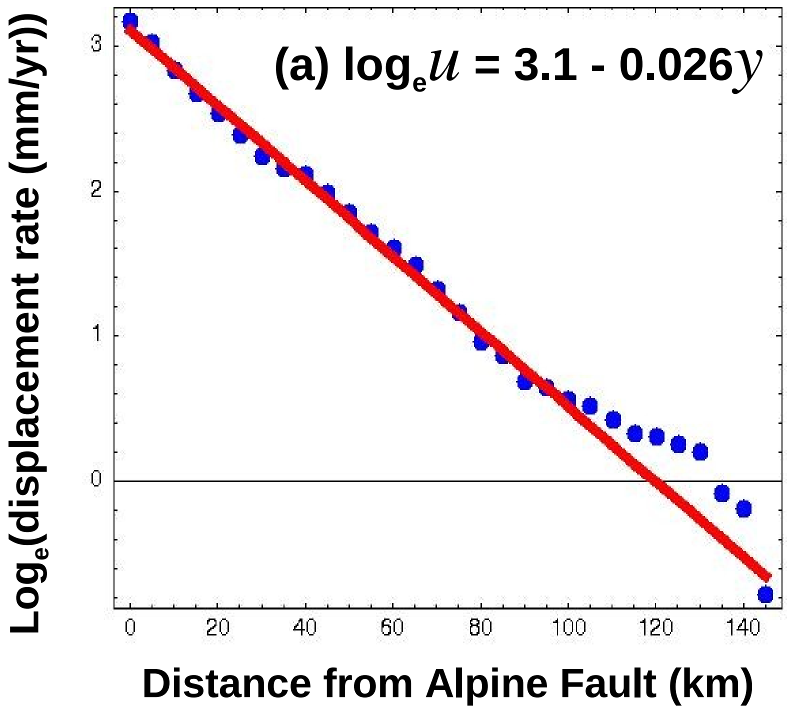


Geodetic Displacement rates in NZ



This inversion solution by Beavan and Haines (JGR, 2001) is based on 362 GPS stations with observations in the period 1991-98.





Exponential Decrease of Displacement Rate

Displacement rate parallel to the Alpine Fault, (Beavan and Haines, 2001):

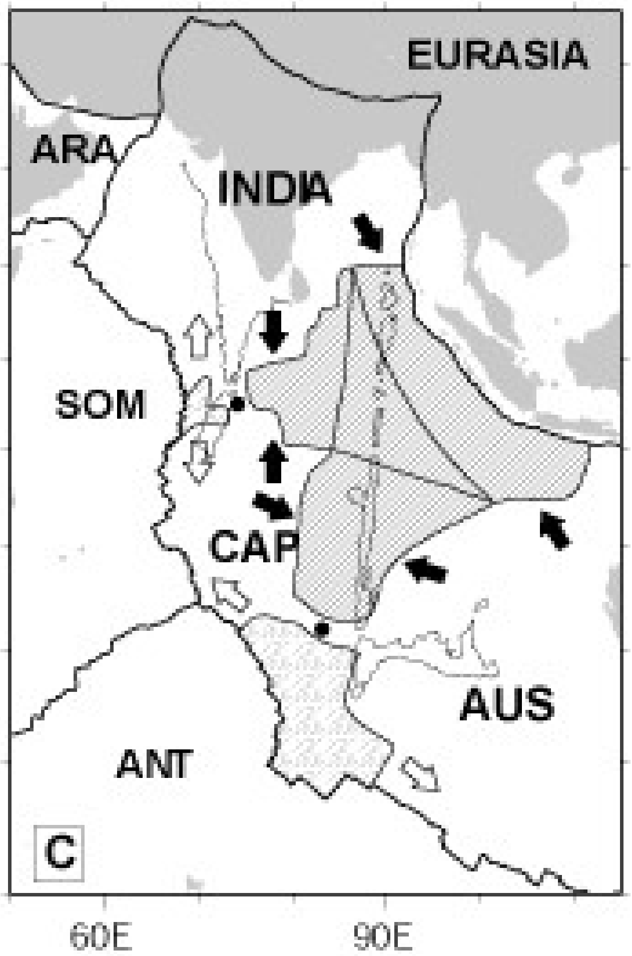
The strike-parallel displacement rate decays exponentially, to a good approximation, for a distance of about 100 km east of the Alpine Fault. Error estimates of about 2 mm/yr imply that measurements beyond 100 km from fault are probably not significant.

The length scale for decay of u is α:

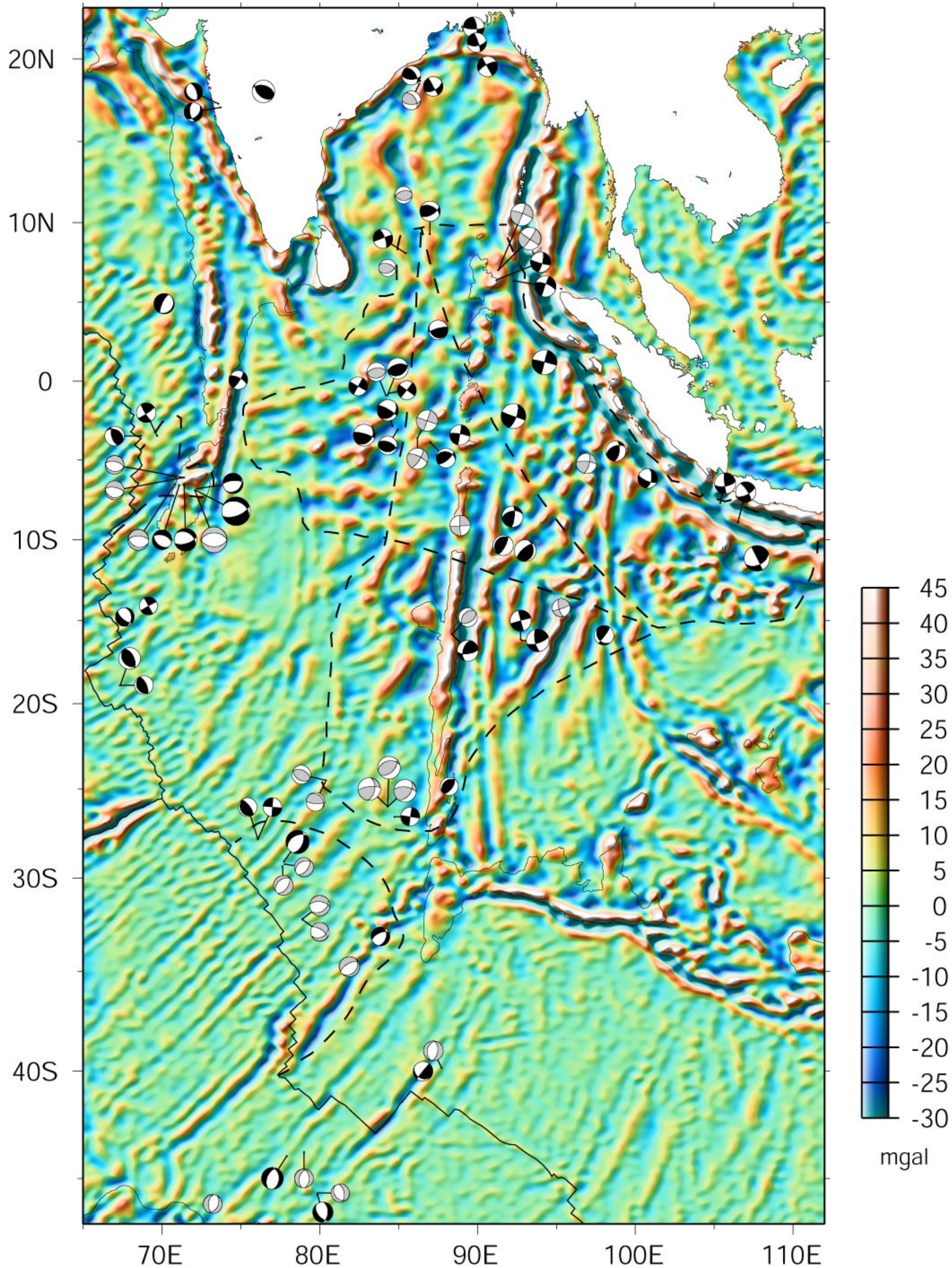
$$u = u_0 \exp\left(-y/\alpha\right)$$

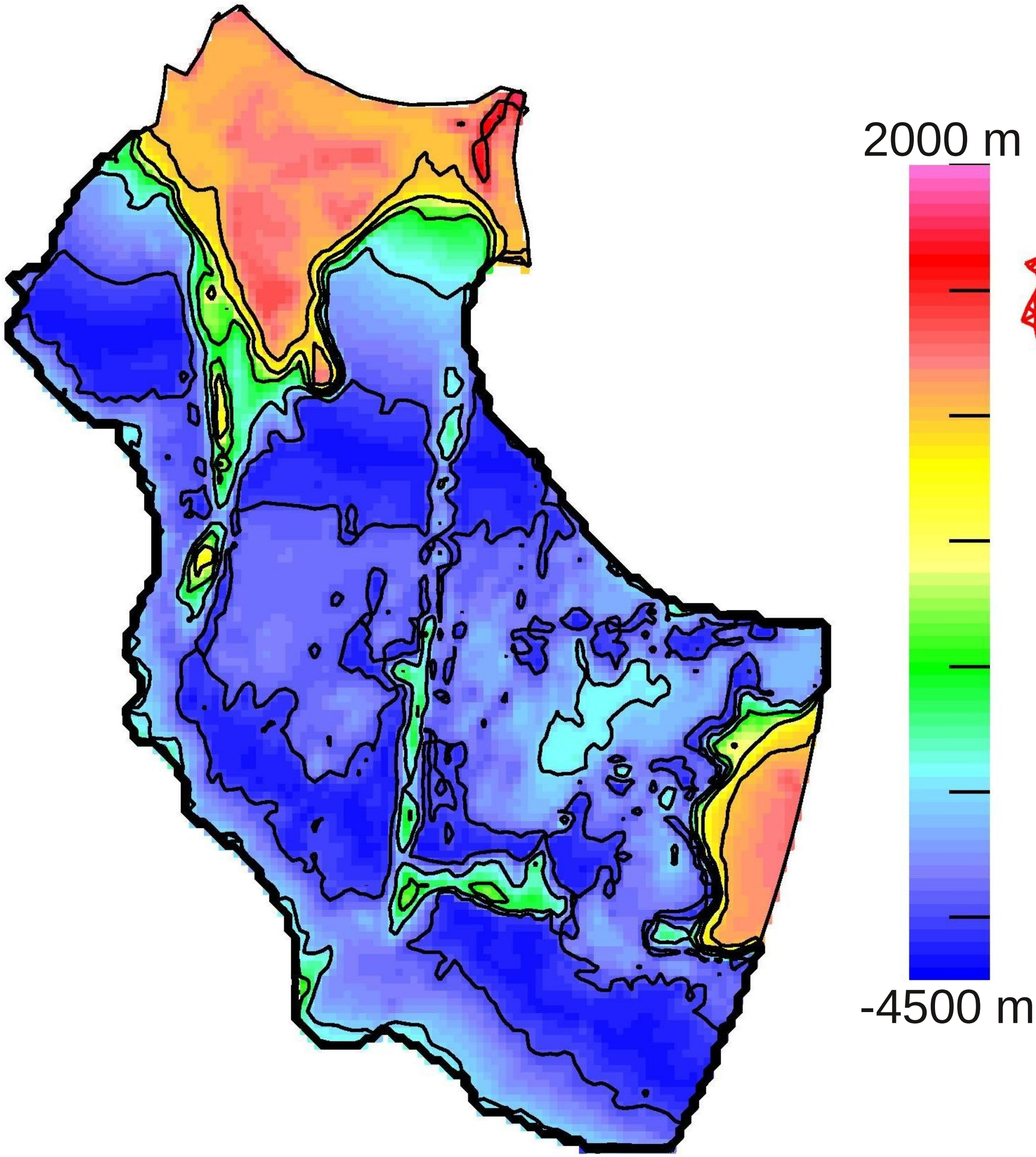
with $\alpha \approx 40$ km, consistent with a half wavelength of about 430 km for $n = 3$, or 800 km for $n = 10$.

Composite plate: Indo-Australia
Component plates: India, Capricorn, & Australia



Royer & Gordon 1997





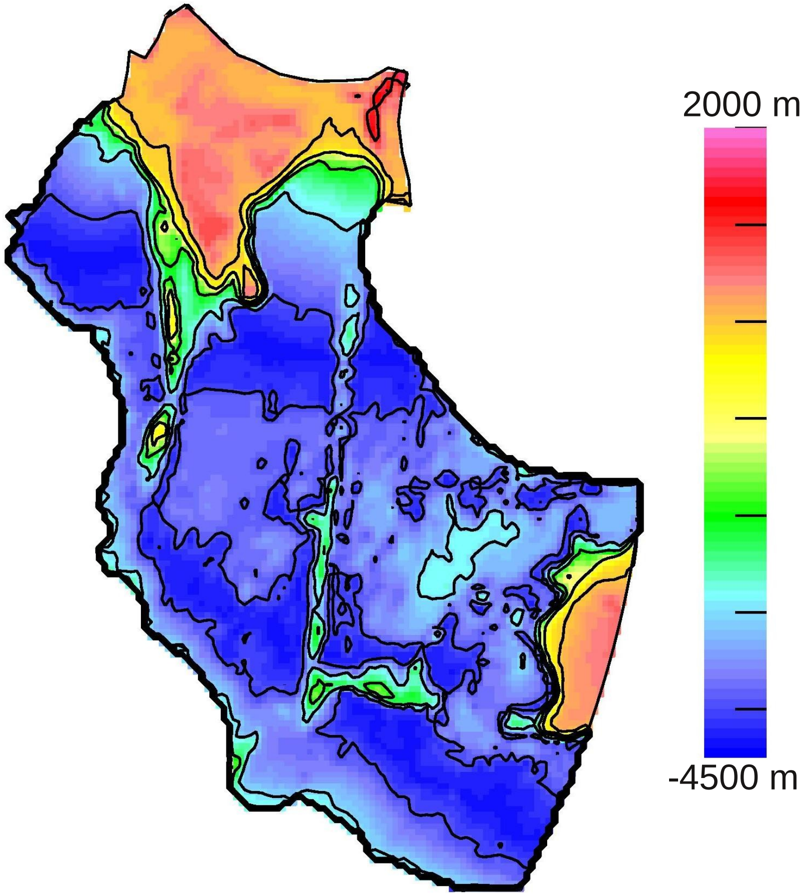
Topography

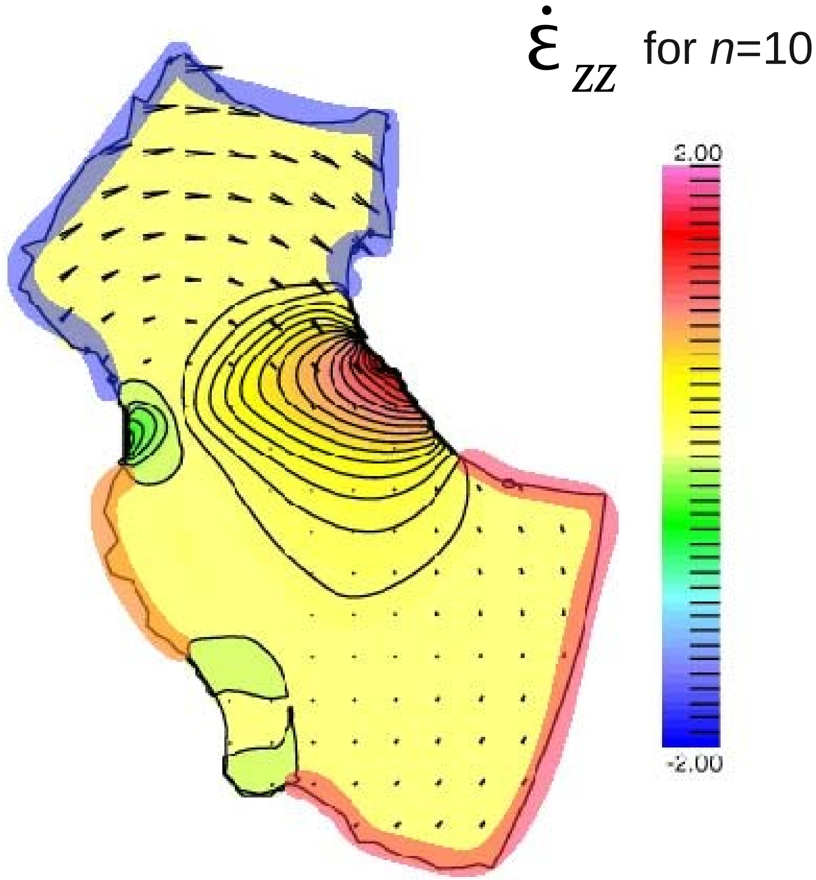
Mesh elements used for *n*=1; finer meshes used for *n*=3, 10, 30

Assumed-rigid outline of Indian component plate

Assumed-rigid outline of Capricorn component plate

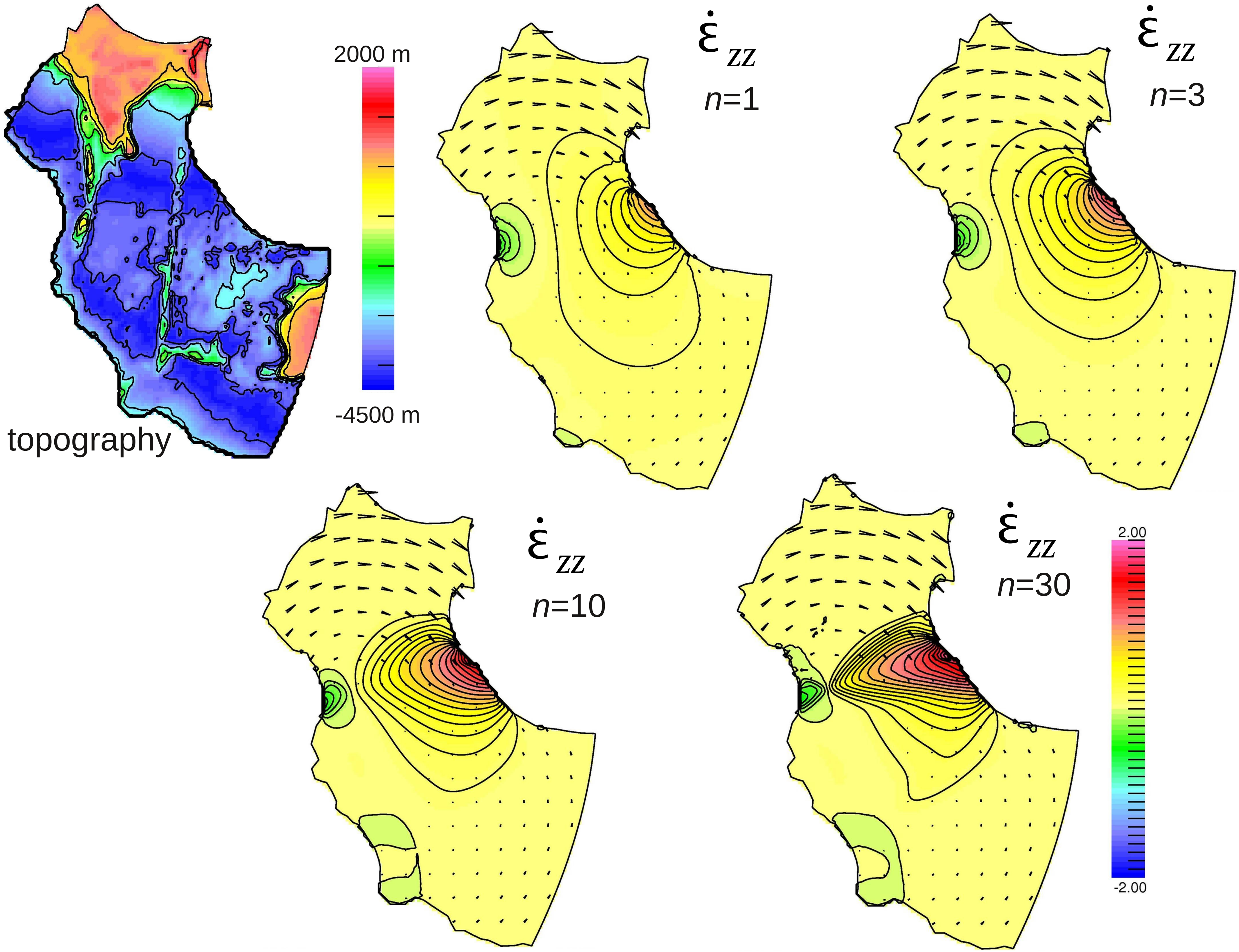
Assumed-rigid (truncated) outline of Australian component plate



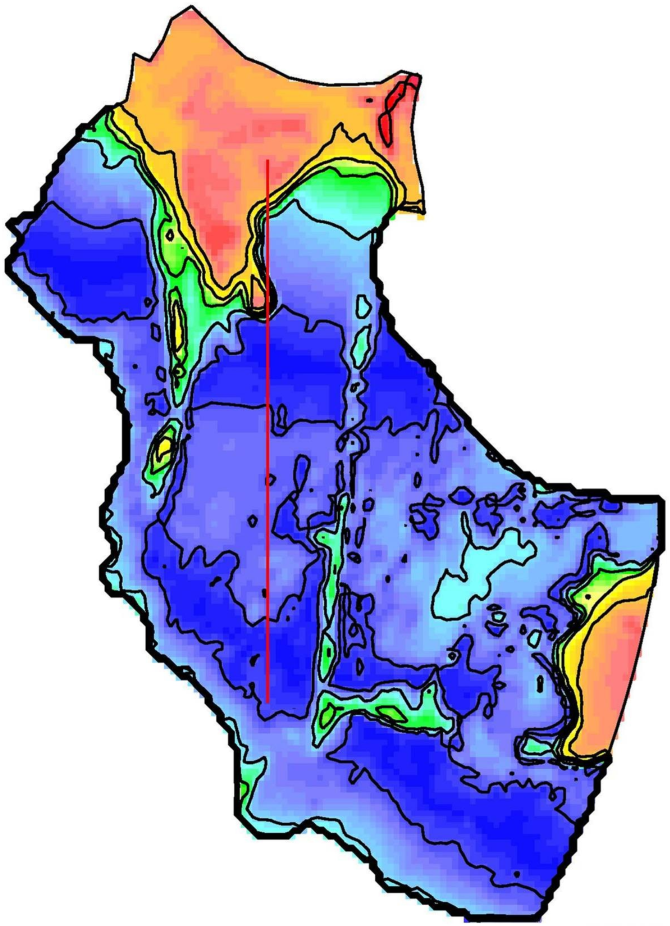


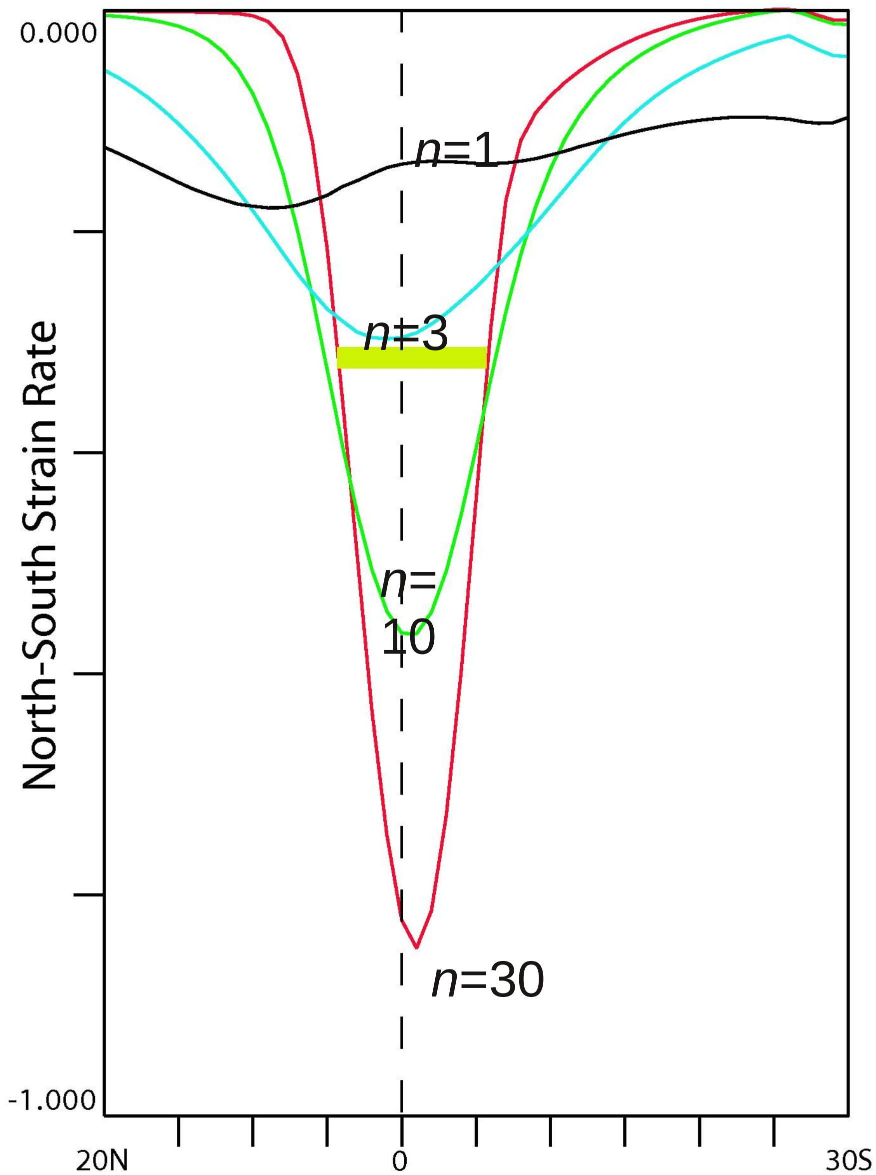
Kinematic boundary conditions: relative plate angular velocities are applied to the rigid outlines of 3 component plates (Capricorn outline arbitrarily held fixed).

Velocity of outline smoothly interpolated between rigid outlines.



2000 m
-4500 m
topography
$\dot{\varepsilon}_{zz}$
n=1
$\dot{\varepsilon}_{zz}$
n=3
$\dot{\varepsilon}_{zz}$
n=10
$\dot{\varepsilon}_{zz}$
n=30
2.00
-2.00





green reference bar is 10° long

N-S extent of deformation along 81.5°E: ≈10°

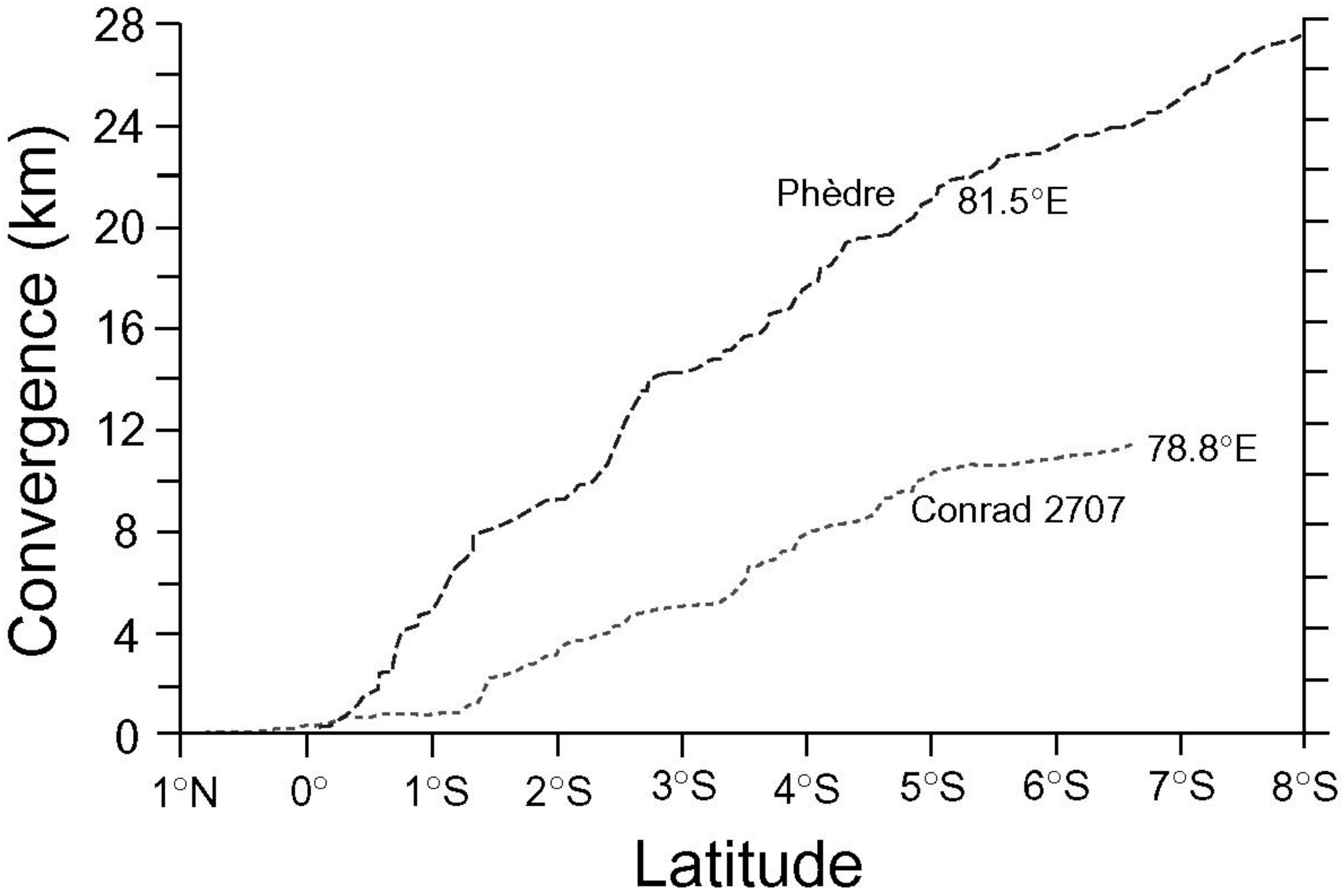
n < 30 → too wide.

n=30 : closest

Modeled deformation zone offset to N relative to observed (≈0°S to ≈10°S).

Modeled deformation zone more symmetric than observed

N-S seismic profiles used to estimate convergence

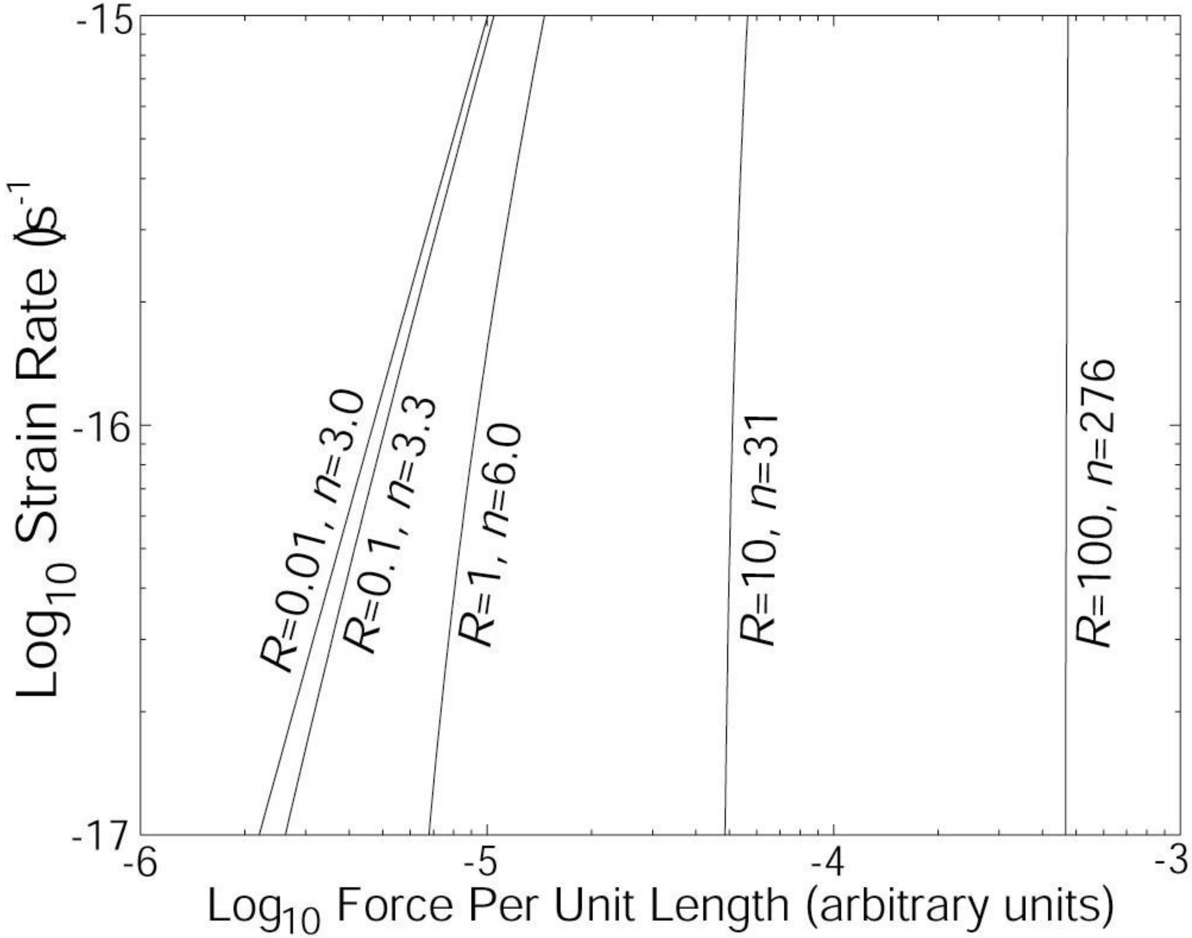


After Van Orman et al 1995

Strain-rate versus stress for composite model of brittle layer above ductile creeping layer

$$R = \frac{\text{strength of upper lithosphere}}{\text{strength of lower lithosphere at ref } \dot{e}}$$

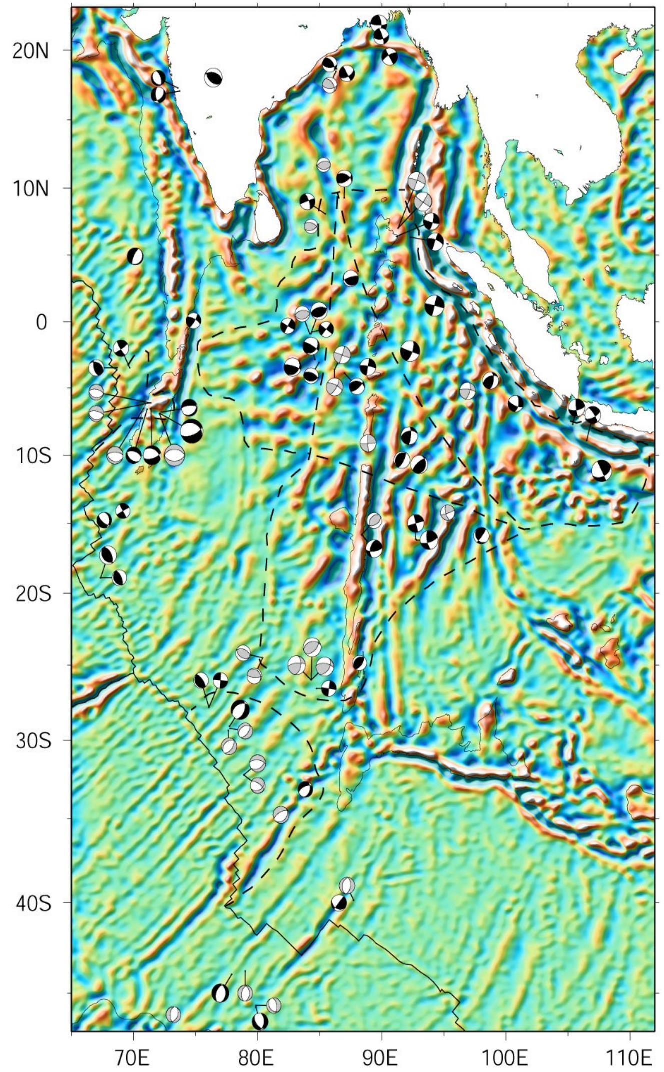
ref $\dot{e} = 10^{-16}\ s^{-1}$
($0.3\%\ Myr^{-1}$)

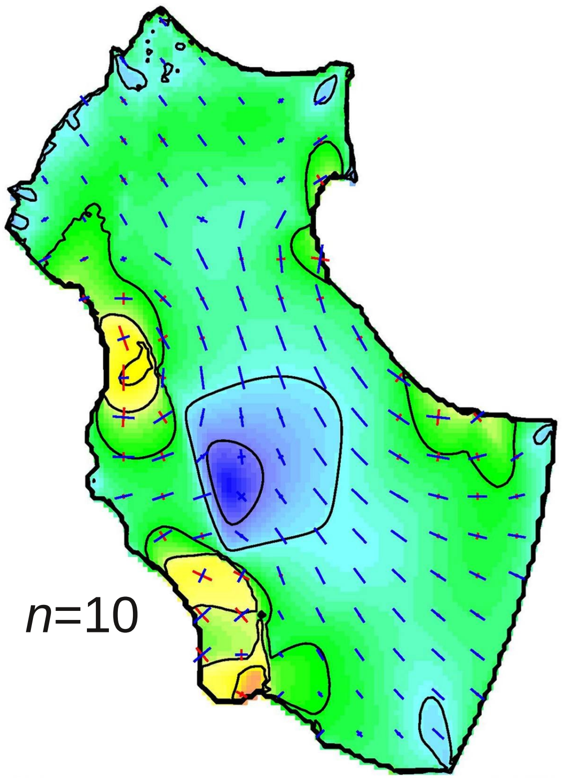


The slope of the curve gives the exponent for the power-law model triaxial strain-rate tensor is decomposed into two equivalent double-couple moment tensors, e.g.,

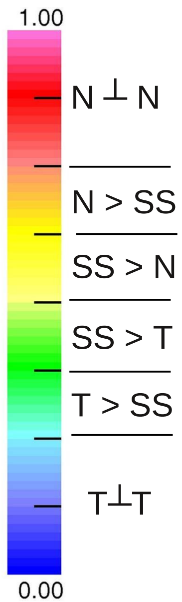
$$\begin{bmatrix} \dot{\varepsilon}_{11} & 0 & 0 \\ 0 & \dot{\varepsilon}_{22} & 0 \\ 0 & 0 & \dot{\varepsilon}_{33} \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon}_{11} & 0 & 0 \\ 0 & -\dot{\varepsilon}_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\dot{\varepsilon}_{33} & 0 \\ 0 & 0 & \dot{\varepsilon}_{33} \end{bmatrix}$$

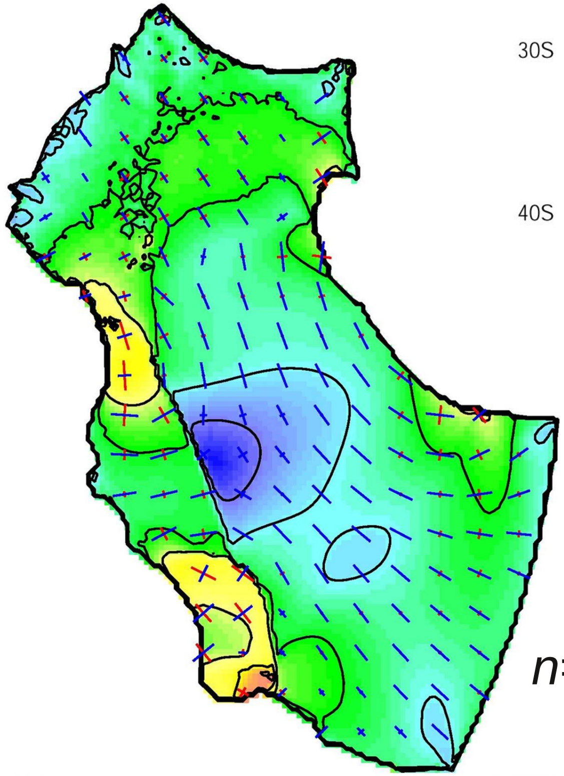
minimum energy decomposition defines the types of earthquake mechanism, SS, N, T



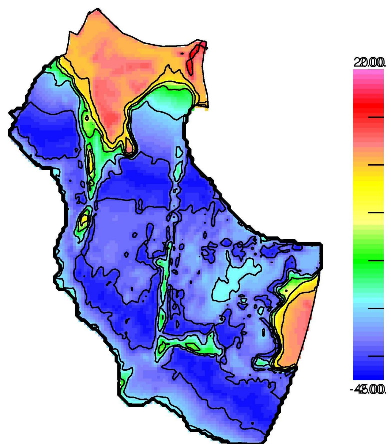


n=10





n=30



Conclusions

- Thin viscous sheet models in which the lithosphere is treated as a coherent unit deforming under the influence of boundary tractions and internal buoyancy can explain many observations of lithospheric deformation in the continents.
- The continental lithosphere responds to stress with a combination of deformation mechanisms that produce an effective non-Newtonian constitutive law.
- Spatial variations in lithospheric strength cause strain localization, which is enhanced by the non-Newtonian behaviour.
- Decoupling of crust and mantle layers caused by a low-viscosity lower crust may be important at the smaller scale, but large scale translation of crust relative to mantle is difficult unless the lower crust is very weak.
- 3D calculations allow these other processes to be incorporated more accurately, but only at the expense of a considerable increase in complexity and parameter space.

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NB: list incomplete