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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Derivation of basic equation for growth rate

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Advanced School on
Scaling Laws in Geophysics:
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Rayleigh-Taylor Instability

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Streamfunction

Helmholtz Theorem:

any well behaved vector field can be partitioned into a divergence-free part and a curl-free part:

$$\mathbf{u} = \nabla \Phi + \nabla \times \Psi$$

where Φ is a scalar potential and Ψ is a vector potential.

The properties of the div and curl operators ensure that the first term has zero curl and the second term has zero divergence. The additional degree of freedom (one component of Φ and three components of Ψ) is removed by requiring that:

$$\nabla \cdot \Psi = 0$$

in representing an incompressible Stoke's flow field we automatically satisfy the incompressibility condition (zero divergence) if we use only the vector potential:

$$\mathbf{u} = \nabla \times \Psi$$

Substituting this representation of the velocity field into the equation for incompressible Stokes flow (assuming low Reynolds number and constant viscosity) we get:

$$\eta \nabla^2 \left(\nabla \times \Psi \right) + \nabla\, p + \rho\, \mathbf{g} = 0$$

Now take the curl of this equation to eliminate pressure:

$$\eta \nabla^2 \left(\nabla \times \nabla \times \Psi \right) + \nabla \times \left(\nabla\, p \right) + \nabla \times \left(\rho\, \mathbf{g} \right) = 0$$

We use the identities that curl of grad is zero and.

$$\nabla \times \nabla \times \Psi = \nabla \left(\nabla \cdot \Psi \right) - \nabla^2 \Psi = -\nabla^2 \Psi$$

Thus, for incompressible Stokes flow, the vector streamfunction satisfies:

$$\eta \nabla^4 \Psi = \nabla \times (\rho \mathbf{g}) = 0$$

Taking the divergence of the stress equation gives

$$\nabla^2 p + \nabla \cdot (\rho \mathbf{g}) = 0$$

If the density and gravity are everywhere constant, then the second term in both equations is zero everywhere. In that case the flow field has a stream function that everywhere satisfies the biharmonic equation:

$$\nabla^4 \Psi = 0$$

and a pressure that everywhere satisfies the Laplace equation.

$$\nabla^2 p = 0$$

2D Streamfunction

The streamfunction formulation thus provides a simplified path to the solution of many incompressible (plane-strain) viscous flow problems. In the case of 2D flows, the vector Ψ is everwhere perpendicular to the flow, and we need only solve for its scalar amplitude:

$$\eta \nabla^4 \psi = \partial_x (\rho g) = 0$$

(*g* acting in y-direction). The velocity field is then provided by:

$$(u, v) = \left(\frac{\partial \psi}{\partial y}, \; -\frac{\partial \psi}{\partial x} \right)$$

This last expression is probably quite familiar, but the advantage of starting with the Helmholtz expression is that the potential field expressions are equally useful for 3D problems if we retain additional components of the Ψ vector.

Spheroidal and Toroidal Potential

In problems for which spherical symmetry provides a natural way to formulate the problem, we can, instead of specifying:

$$\nabla \cdot \Psi = 0$$

choose to define Ψ in terms of two other scalar potentials:

$$\Psi = \nabla \times \left(S\, \hat{\mathbf{r}} \right) + T\, \hat{\mathbf{r}}$$

defined relative to the radial vector, which of course usually defines the direction of the primary body force, and so defines a natural separation into parts of the vector field into those parts of the vector field that interact with the body force (spheroidal: *S*, Φ) and those that do not (toroidal: *T*). Thus:

$$\mathbf{u} = \nabla \Phi + \nabla \times \nabla \times \left(S\, \hat{\mathbf{r}} \right) + \nabla \times \left(T\, \hat{\mathbf{r}} \right)$$

Vorticity

We define vorticity as the curl of the velocity field:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times \nabla \times \Psi = -\nabla^2 \Psi \tag{1}$$

Vorticity is also a fundamental aspect of the flow field, describing the circulation of the flow. The Stokes equation can also be written in terms of the vorticity,

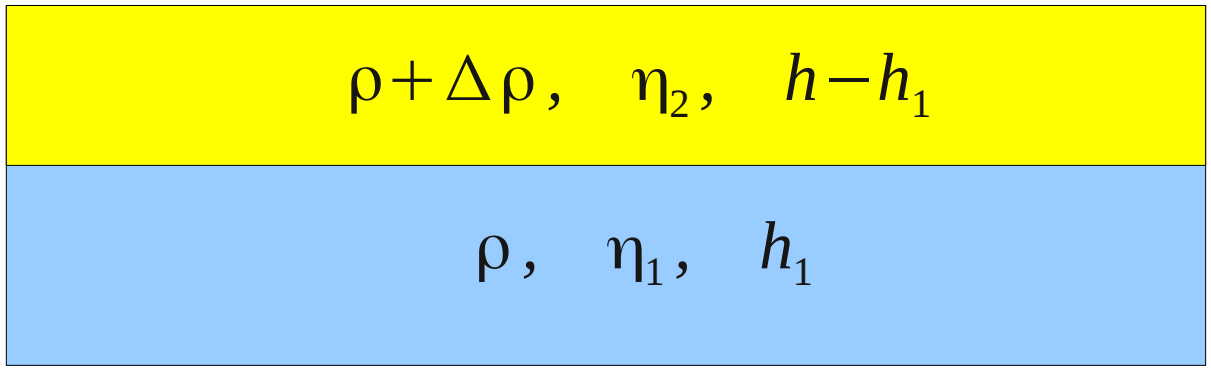
$$\eta \nabla^2 \boldsymbol{\omega} = -\nabla \times (\rho \mathbf{g}) = 0 \tag{2}$$

and in solving Stokes flow problems it is often useful to first obtain the vorticity by solving the Poisson or Laplace equation (2), and then solving equation (1).

From this pair of equations it is very clear that we need to have horizontal variations in density, or there will be no source of vorticity – nothing that drives the flow.

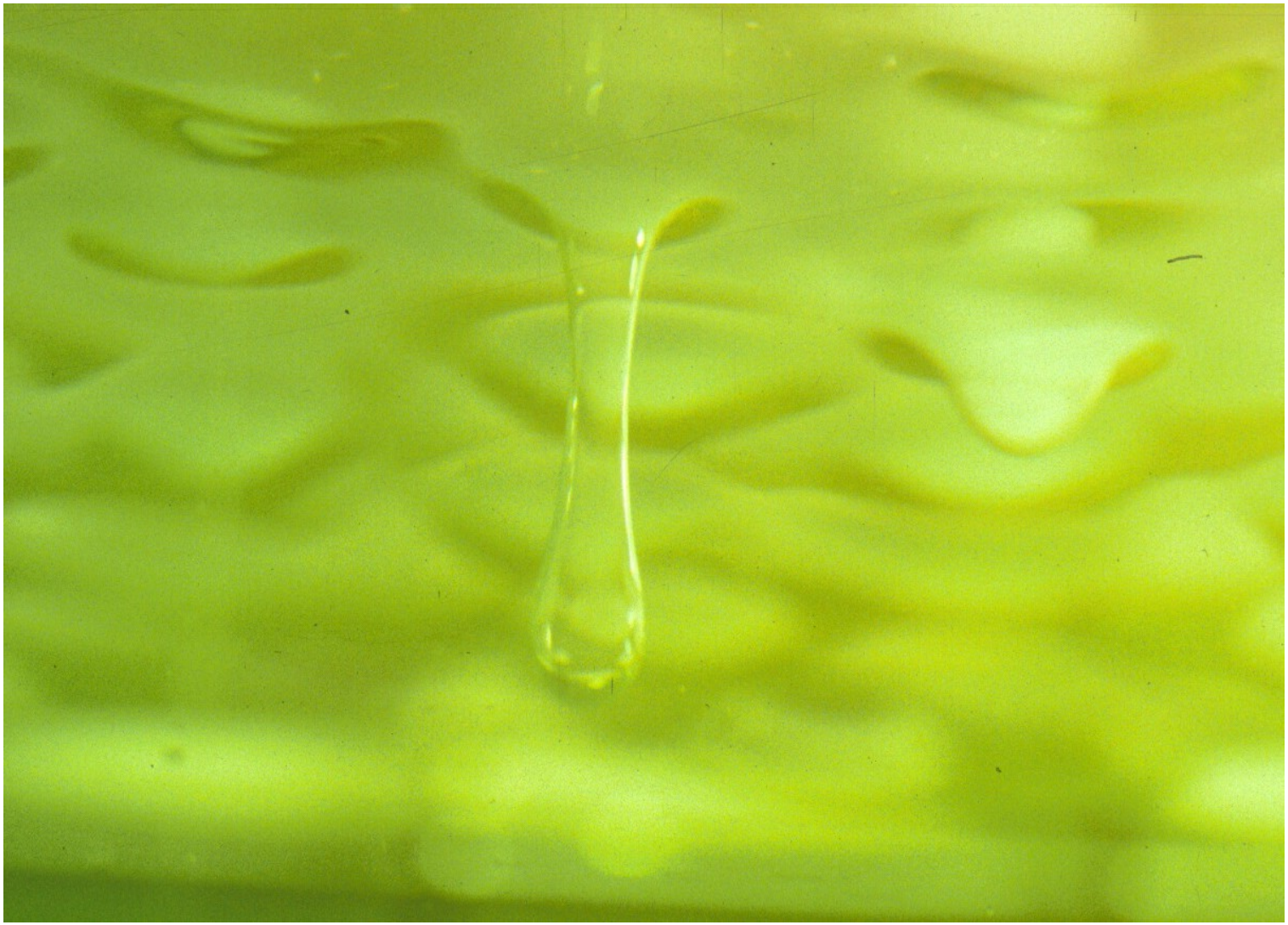
Rayleigh-Taylor Instability

The classic Rayleigh-Taylor problem is that of two layers of fluid in which the heavier fluid overlies the lighter, so the initial situation is unstable. The layers could have different thicknesses and different viscosities. The Reynolds number could be large for this problem, but I will focus on *Re* << 1



Each layer is homogeneous, so there is no source of vorticity anywhere in this problem, unless the interface is deflected from the horizontal, so we examine what happens when there is a small perturbation relative to the horizontal (with *g* acting in negative y-direction).

Rayleigh-Taylor instability of a thin layer of viscous silicone oil



Rayleigh-Taylor Instability

Suppose the unstable interface is perturbed by

$$s = s_0 \cos\left(\frac{2\pi x}{\lambda}\right) = s_0 \cos(k x)$$

then the local horizontal gradient of density at the interface drives a flow. Where the deflection is up, the flow will be upward, enhancing the upward deflection – the recipe for growth of the perturbation, or instability.

The upward flow is resisted only by the fact that energy is dissipated in deforming the viscous liquid.

A perturbation at any wavelength might grow – but some wavelengths might be more unstable than others, so in analysing this problem we would like to estimate growth rate as a function of wavelength.

Scaling Analysis

In this problem the viscous dissipation scales as the pressure gradient induced by the deflection of the boundary

$$\eta \nabla^2 u \sim \nabla p$$

In estimating the relative magnitude of these two terms, two limits are relevant:

$$\lambda > h: \qquad \eta \frac{V}{h^2} \sim \left(\frac{\Delta \rho g s}{\lambda} \right) \qquad \rightarrow \qquad \frac{1}{s}\frac{ds}{dt} \sim \left(\frac{\Delta \rho g h}{\eta} \right)\left(\frac{h}{\lambda} \right)$$

$$\lambda < h: \qquad \eta \frac{V}{\lambda^2} \sim \left(\frac{\Delta \rho g s}{\lambda} \right) \qquad \rightarrow \qquad \frac{1}{s}\frac{ds}{dt} \sim \left(\frac{\Delta \rho g h}{\eta} \right)\left(\frac{\lambda}{h} \right)$$

In either case the term on the right hand side is a growth rate and has the dimensions of inverse time. We also see that in either limit, the growth rate diminishes as the ratio h/λ either increases or decreases away from 1.

RT flow field

In each layer a streamfunction applies, and satisfies

$$\nabla^4 \psi = 0$$

If the flow has a characteristic horizontal periodicity of λ, then the solution in each layer might be expected to look like

$$\psi_i = \sin(k\,x) f_i(y)$$

where the subscript *i* indicates layer 1 or 2.
Substitute into the biharmonic equation to obtain:

$$\left[\frac{d^2}{dy^2} - k^2\right]^2 f_i(y) = 0$$

solutions are:

$$\psi_i = \left[\left(A_i + C_i ky\right)\exp(ky) + \left(B_i + D_i ky\right)\exp(-ky)\right]\sin(kx)$$

Pressure solution

The pressure field satisfies Laplace's equation, so its solution is:

$$p_i(y)=\left[E_i\exp(ky)+F_i\exp(-ky)\right]\cos(kx)$$

The pressure and stream function solutions are linked by the Navier-Stokes equation, so we need expressions for *u*, *v*, and their derivatives:

$$u_i=k\left[\left(A_i+C_i+C_i\,ky\right)\exp(ky)-\left(B_i-D_i+D_i\,ky\right)\exp(-ky)\right]\sin(kx)$$

$$v_i=-k\left[\left(A_i+C_i\,ky\right)\exp(ky)+\left(B_i+D_i ky\right)\exp(-ky)\right]\cos(kx)$$

By substituting the solutions into:

$$\eta\nabla^2\mathbf{u}+\nabla p=0$$

we can express the pressure coefficients in terms of the streamfunction coefficients.

Boundary conditions

The other 8 integration constants needed to define the solution are obtained from boundary and continuity conditions, using $v = ds/dt = qs$ to describe the growth of boundary deflection *s* at rate *q*.
Upper and lower boundaries rigid:

$$\psi_2 = \partial\psi_2 = 0 \quad \text{on} \quad y = 1$$
$$\psi_1 = \partial\psi_1 = 0 \quad \text{on} \quad y = 0$$

Velocity continuous on the interface:

$$\psi_2 = \psi_1 \quad \text{and} \quad \partial\psi_2 = \partial\psi_1 \quad \text{on} \quad y = h_1$$

Traction continuous on the interface:

$$\sigma_{xz,1} = \sigma_{xz,2} \quad \text{and} \quad \sigma_{zz,1} = \sigma_{zz,2} - \Delta\rho\, g\, v/q$$

where normal stress in the bottom layer is decreased by an upward deflection of the interface of amount s.

Determination of Growth Rate

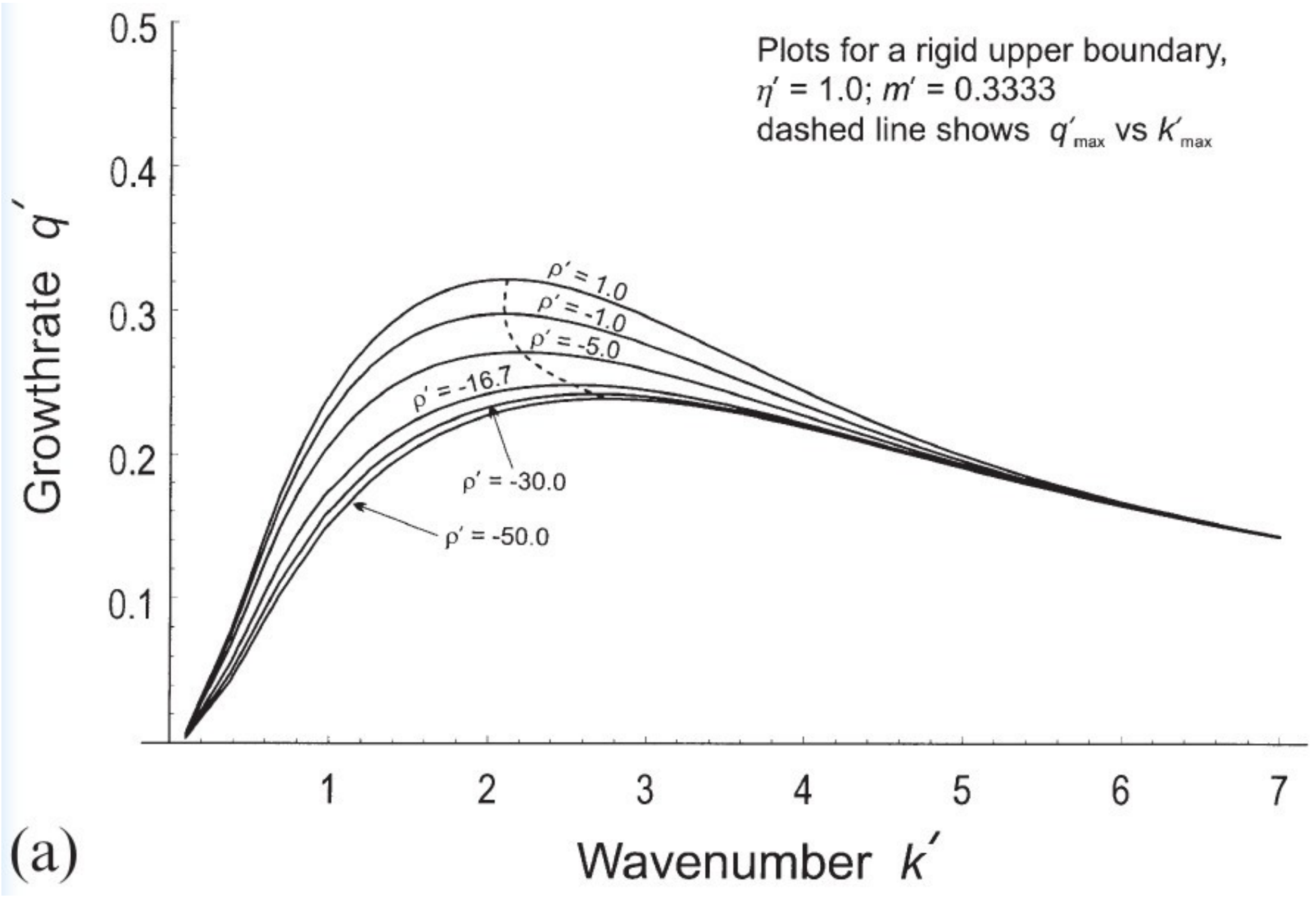
The set of simultaneous equations that constrains the 8 unknown integration constants represents a homogeneous matrix equation of the form: $\mathbf{K}\,\mathbf{x}=0$

For a non-trivial solution to exist, we require det (**K**) = 0

This condition provides a constraint on the growth rate *q* in terms of the other physical variables, and it allows us then to determine the set of physical constants that define the solution, save for one unconstrained amplitude coefficient.

We may want to add additional layers, e.g., a low density crust above a higher density mantle layer. The implicit equation for *q* obtained from the determinant of the matrix will have as many solutions as there are distinct interfaces.

Growth Rate dependence on wavenumber



Lithospheric Gravitational Instability

Removal of the lower part of the lithosphere by viscous creep flow

Often associated with crustal convergence

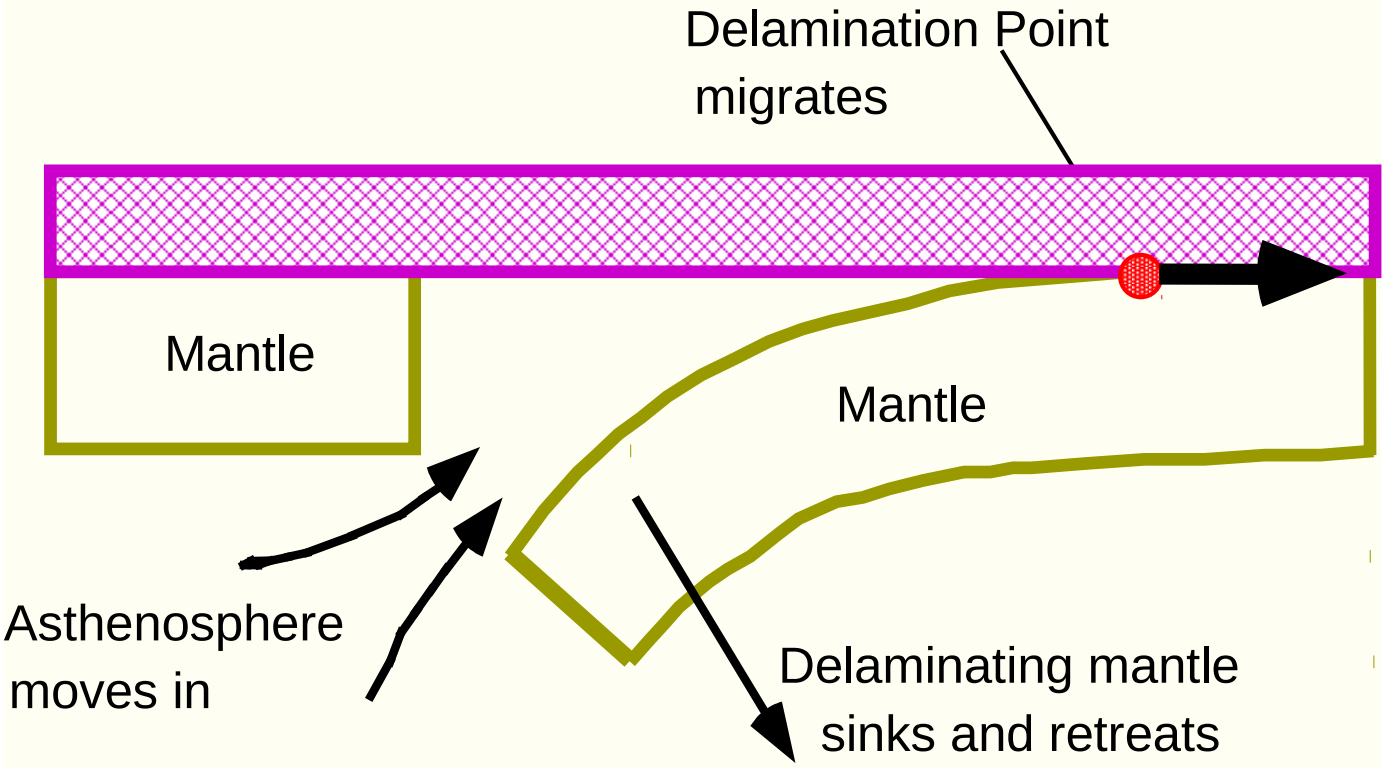
Cold mantle lithosphere is replaced by hot asthenosphere (downwelling and upwelling)

Magmatism: volcanics and plutonism associated with extension

Mantle downwelling can be directly detected by seismic methods (in principle)

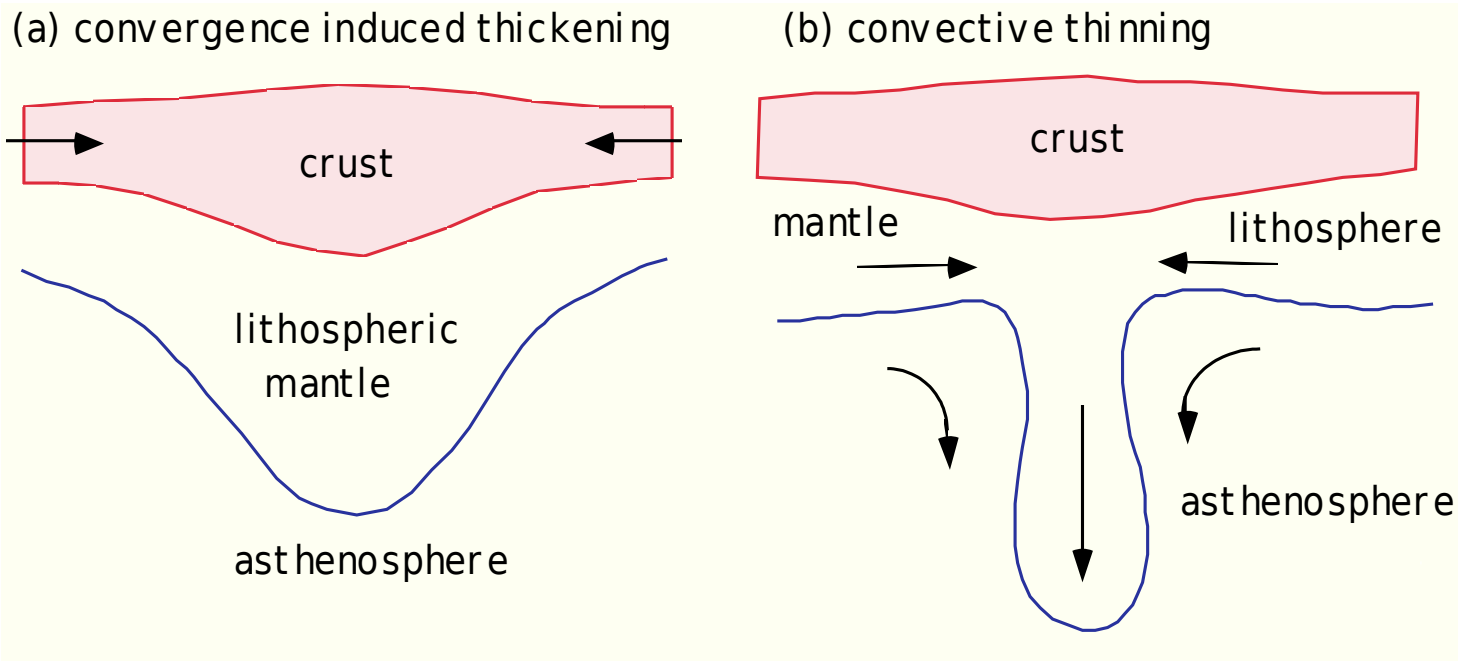
Instability Mechanisms - Delamination

Delamination is the progressive peeling away of the mantle and its replacement by asthenosphere. The delamination point migrates across the thinned region (Bird, 1979).



The Convective Thinning Mechanism

Convective thinning occurs as the classic Rayleigh-Taylor instability, with the dense layer descending in viscous drips. Growth is driven by negative buoyancy but perhaps triggered by imposed convergence.



Delamination or Convective Thinning ?

If delamination occurs, the entire thickness of mantle lithosphere could be replaced, with the layer peeling off as a coherent unit.

Delamination would cause a “magmatic front” of lithospheric heating to migrate progressively across the affected region.

Convective thinning may result in stationary or migrating down-welling and progressive expansion of the thinned region, even while convergence continues.

Convective “drips” could occur as sheets or plumes.

Crustal Growth: Magmatism

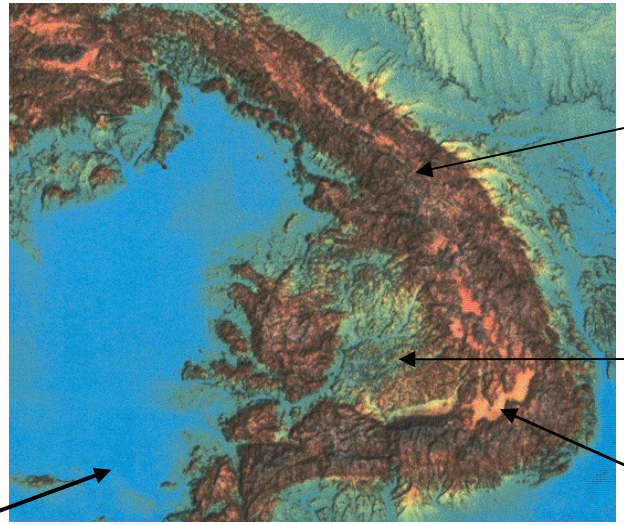
- upwelling of asthenosphere generates basaltic melts, producing alkaline volcanism distributed across the extended crust

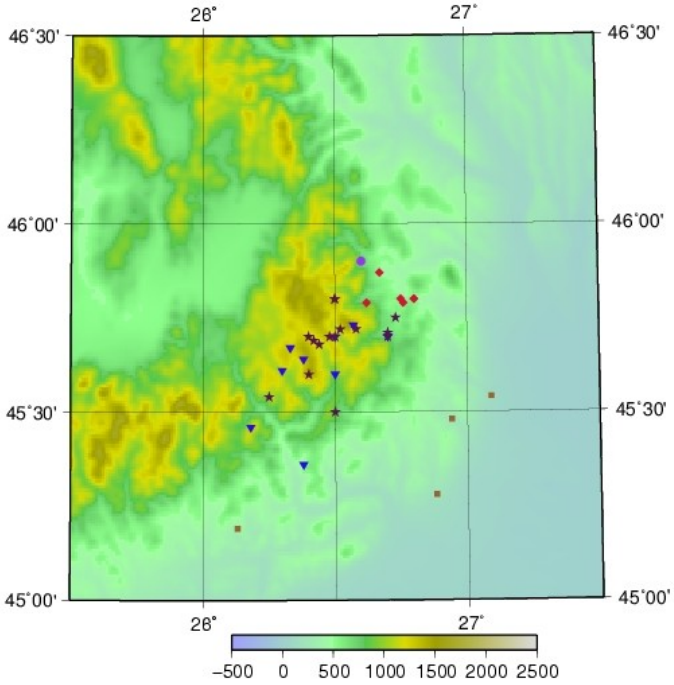
- juxtaposition of lower crust with attenuated mantle lithosphere produces melting of lower crust and silicic volcanism and plutonism

- mantle downwelling is apparently associated with high-K andesites – maybe related to re-melting of metasomatized mantle lithosphere.

Vrancea – the SE Carpathians

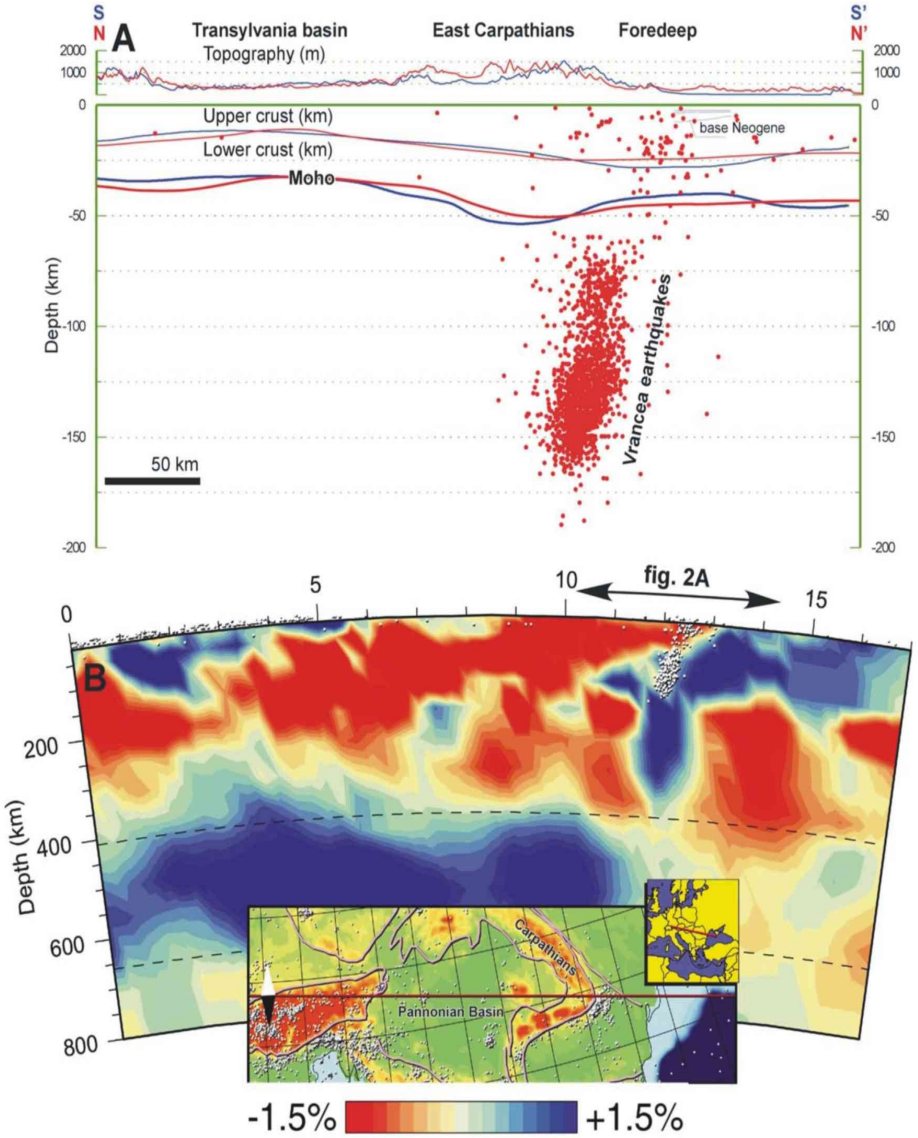
- The Vrancea region in the SE Carpathians is characterised by a cluster of strong seismicity to depths of about 200 km
- Typical models for Vrancea are based on subduction of the oceanic lithosphere
- Or does the present seismicity result from an instability of continental mantle lithosphere ?





- The large earthquake hypocentres fall within a vertical rectangular block, approximately 80 km long, 30 km wide, 200 km deep

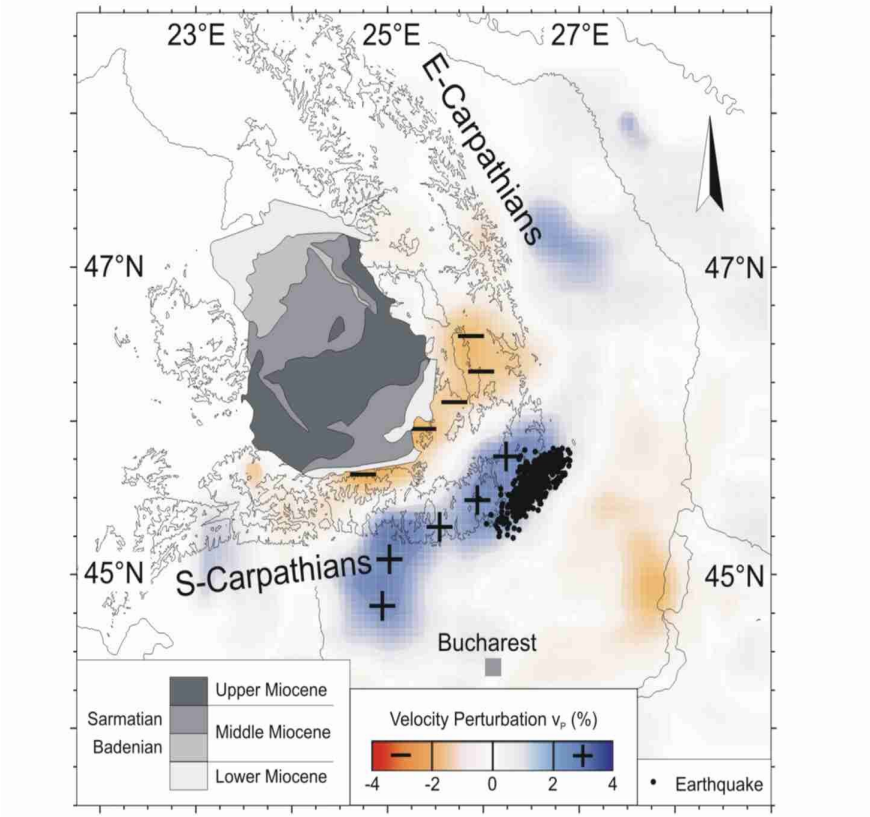
Mantle Downwelling beneath the SE Carpathians



Wortel and Spakman, Science, 2000.

Cloetingh et al., Earth Planet. Sci. Lett., 2004.

Image from: B.Sperner et al., Tectonophysics 382 (2004) 51-84.



Miocene sediment distribution and seismic P-velocity perturbation in the depth range 110-150 km beneath the SE Carpathians (from the Calixto experiment; Martin et al., 1991).

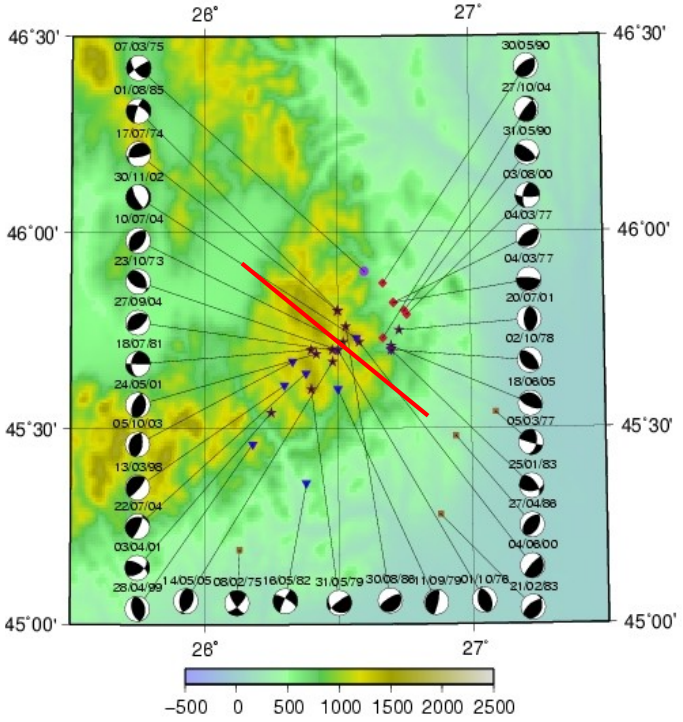
Strain rates Determined from the Seismicity in Vrancea

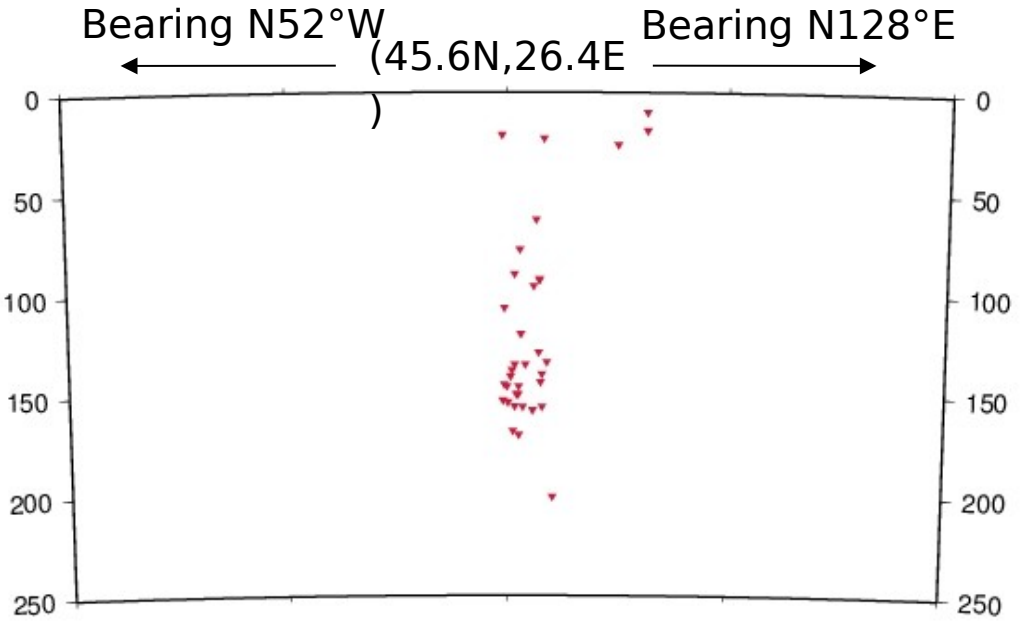
- 31 earthquake moment tensors (1967-2007; with magnitude > 5.3) used to determine the strain-rate tensor with Kostrov formula:

$$\dot{\varepsilon}_{ij} = \frac{1}{2\mu\, Vt} \sum_{k=1}^{N} M_{ij}^{(k)}$$

$M_{ij}^{(k)}$ = seismic moment tensor $\mu = 7.4 \times 10^{10}\, N/m^2$

$V = 1.2 \times 10^{14}\, m^3$ = seismogenic volume, t = 40 yr





- NW-SE cross section:

Lorinczi and Houseman, Tectonophysics (2009)

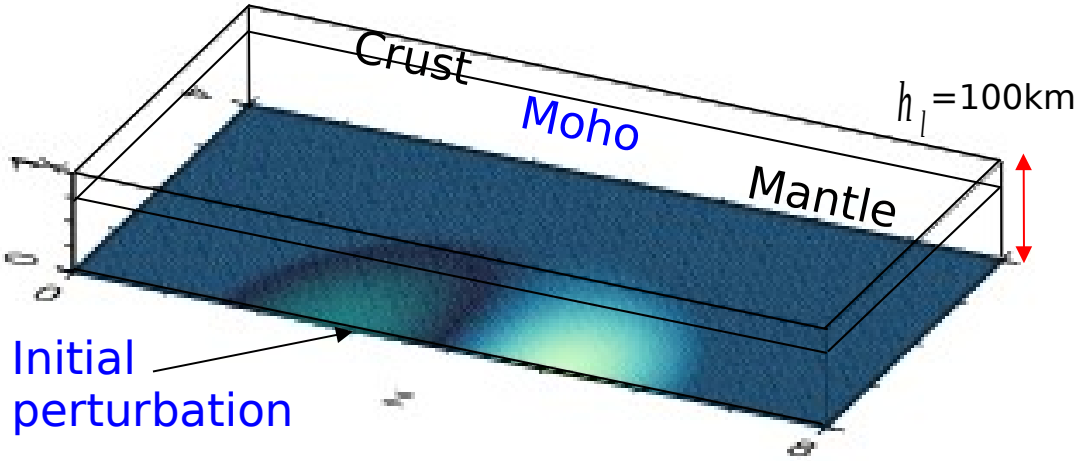
Seismogenic Strain-rates

Depth	(s^{-1})	$\dot{\varepsilon}_1$	$\dot{\varepsilon}_2$	$\dot{\varepsilon}_3$
0-50 km	x 10^{-17}	-0.28	0.00	0.28
50-100 km	x 10^{-13}	-0.10	-0.01	0.11
100-150 km	x 10^{-14}	-0.34	-0.03	0.37
150-200 km	x 10^{-16}	-0.09	-0.03	0.12

- Eigenvectors: the extension axis is near vertical and the compression axis in the range 50-100 km is NW-SE; at deeper levels the P axis rotates to a more E-W direction
- Maximum vertical extension rates on the order of 10^{-14} s^{-1} (35% per Myr) in the depth range 50-100 km: such high strain rates could not be maintained for a very long time.
- At 200 km vertical velocity is 23 mm/yr.

Continuum Deformation Model of Lithospheric Downwelling

- Viscous flow of the lithospheric mantle permits the development of local mantle downwelling, in the form of a Rayleigh-Taylor instability. Deformation field solved numerically using Lagrangean-frame finite element method (program *oregano*).
- A low-density crust on top of a dense lithospheric mantle on top of a less dense asthenosphere is unstable once perturbed. Initial perturbation to depth of LAB here uses a first order Bessel function ($m = 1$).



- Conservation of momentum in a very viscous medium

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = -\rho g_i \quad i = 1,2,3$$

- pressure and deviatoric stress:

$$\sigma_{ij} = p\delta_{ij} + \tau_{ij}$$

- Viscous constitutive relation:

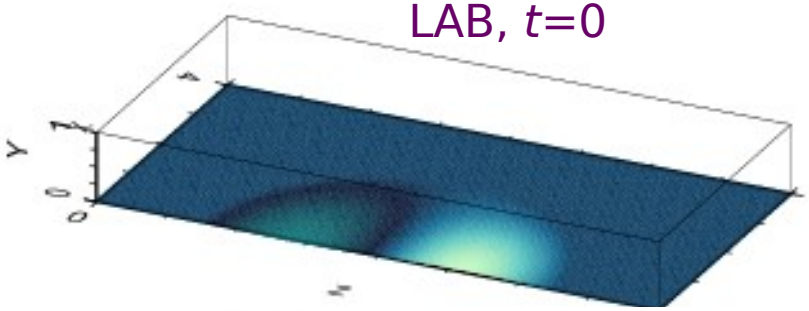
$$\tau_{ij} = B\ \dot{E}^{(1/n-1)}\ \dot{\varepsilon}_{ij}$$

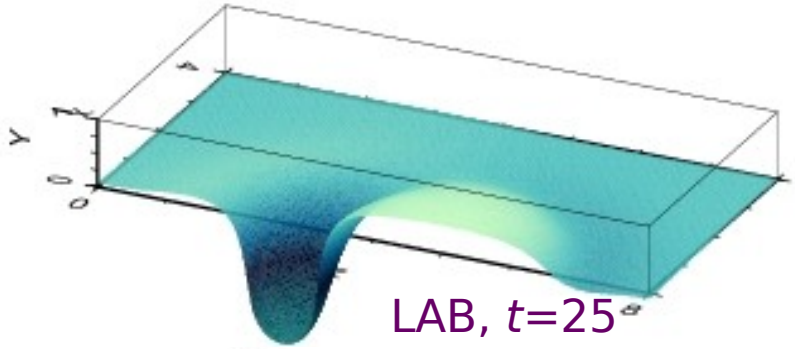
$$\dot{E} = \sqrt{\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}}$$ 2nd invariant of strain-rate

- Mass conservation: $\nabla \cdot u = \sum_{i=1}^{3} \dot{\varepsilon}_{ii} = 0$

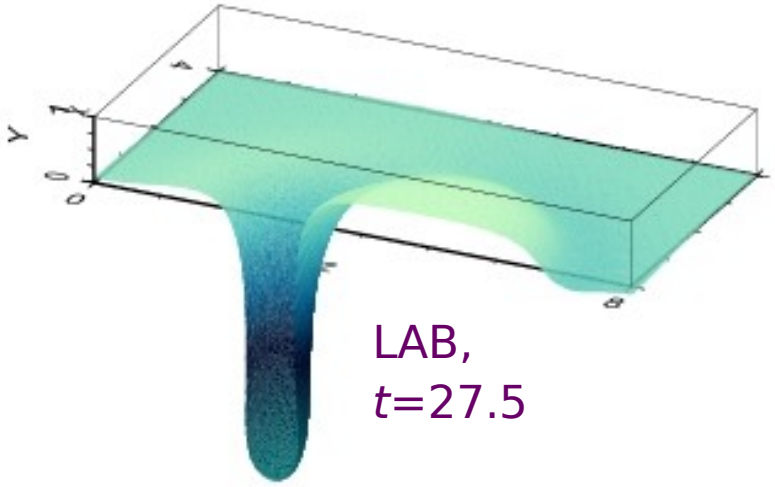
- Mantle downwelling and thickening beneath lithosphere that is initially thicker (Vrancea) is balanced by upwelling and lithospheric thinning in the adjacent area where the lithosphere is initially thinner (Transylvanian Basin).

LAB, t=0



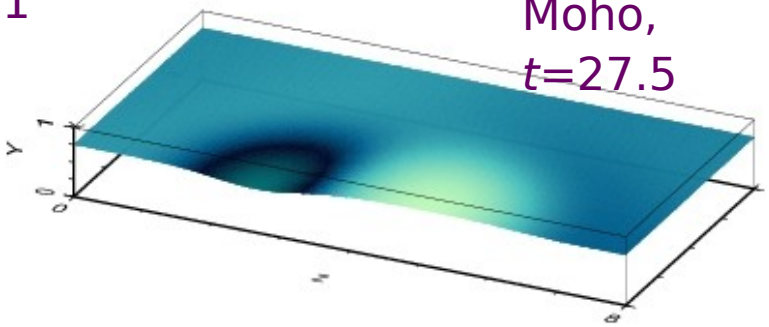


LAB, t=25



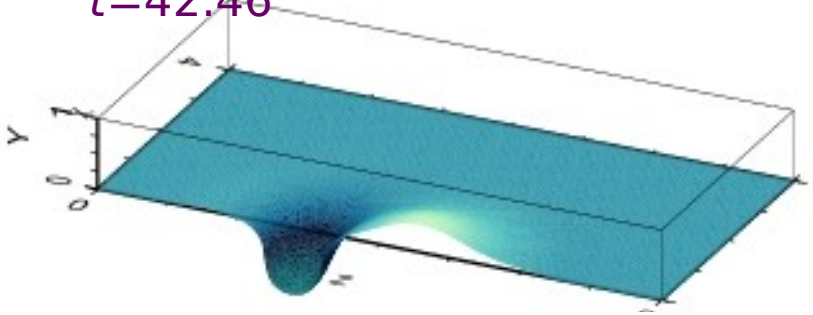
LAB, t=27.5

Newtonian, n = 1



Moho, t=27.5

non-Newtonian, n = 3, t=42.46



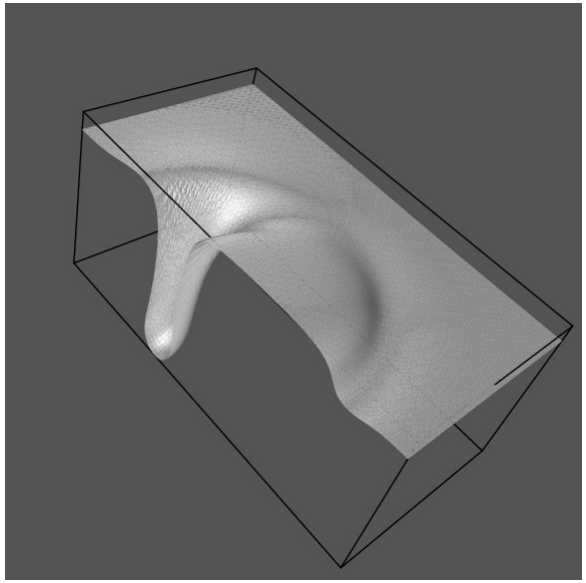
Lorinczi and Houseman, Tectonophysics (2009)

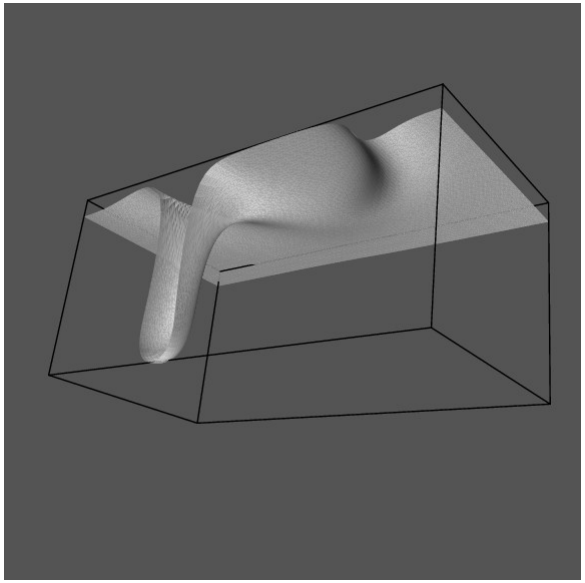
LAB viewed from different perspectives beneath the lithosphere, after growth of RT instability

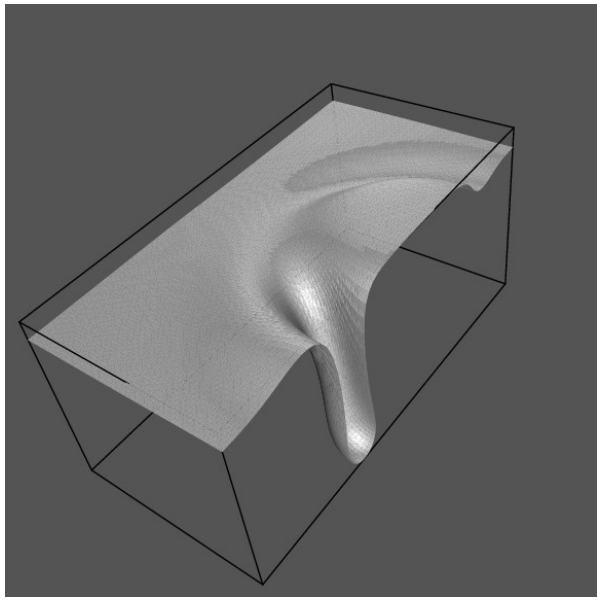
RT instability has a preferred spatial wavenumber (for maximum growth rate) but any function that satisfies

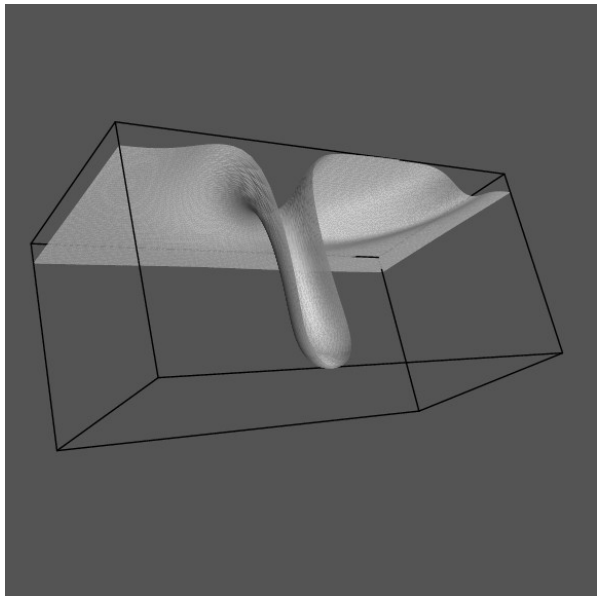
$$\nabla_H^2 f(x,y) = -k^2 f(x,y)$$

will initially grow at the same rate, including rolls, hexagons, and solutions to the Bessel function Here illustrated is the growth of an instability described by the $m = 1$ Bessel function



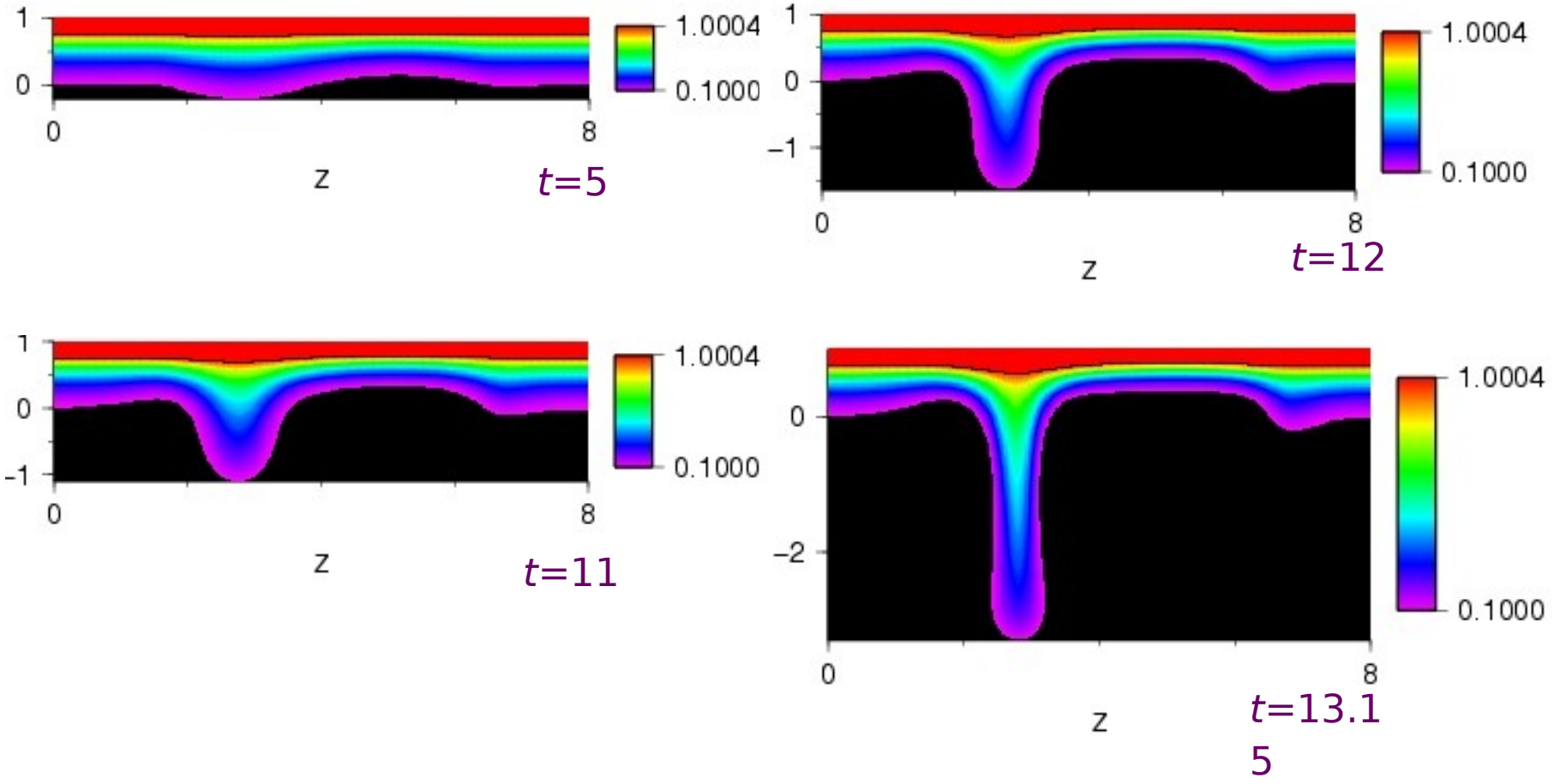






Growth of Instability with Stratified Viscosity

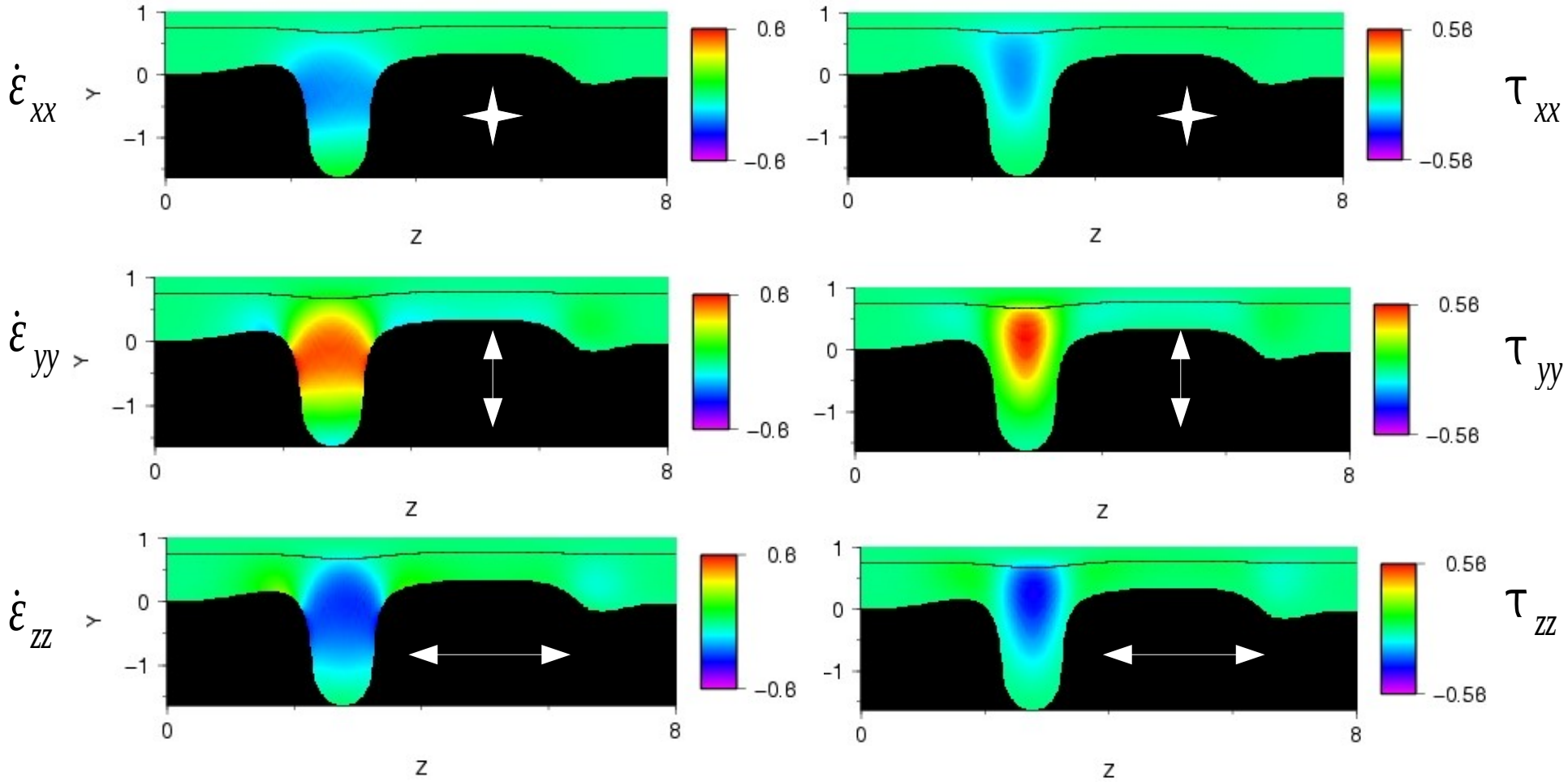
section along the symmetry plane
viscosity increases by factor of 10 from LAB to Moho



High-stress Core in Stratified Viscosity Experiment

strain-rates

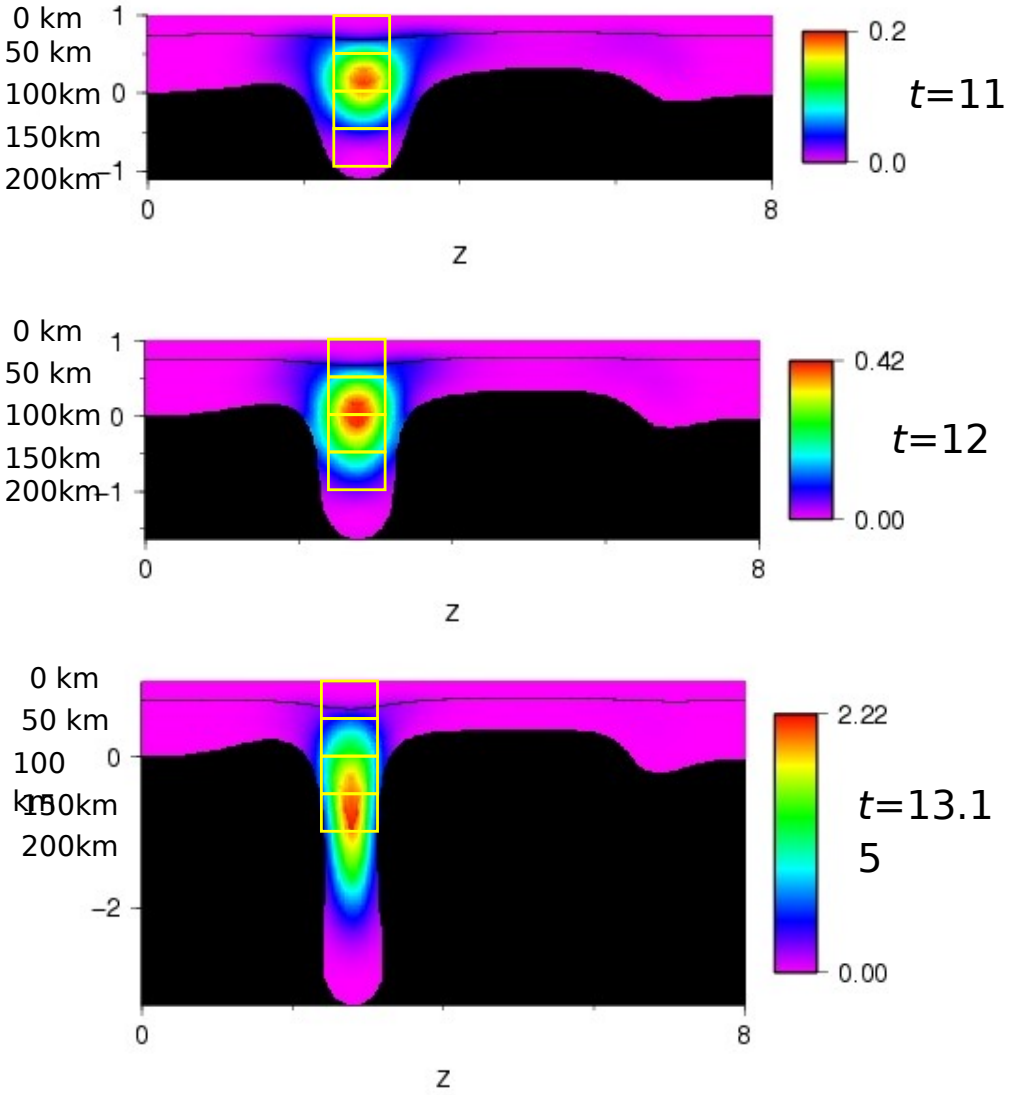
deviatoric stress

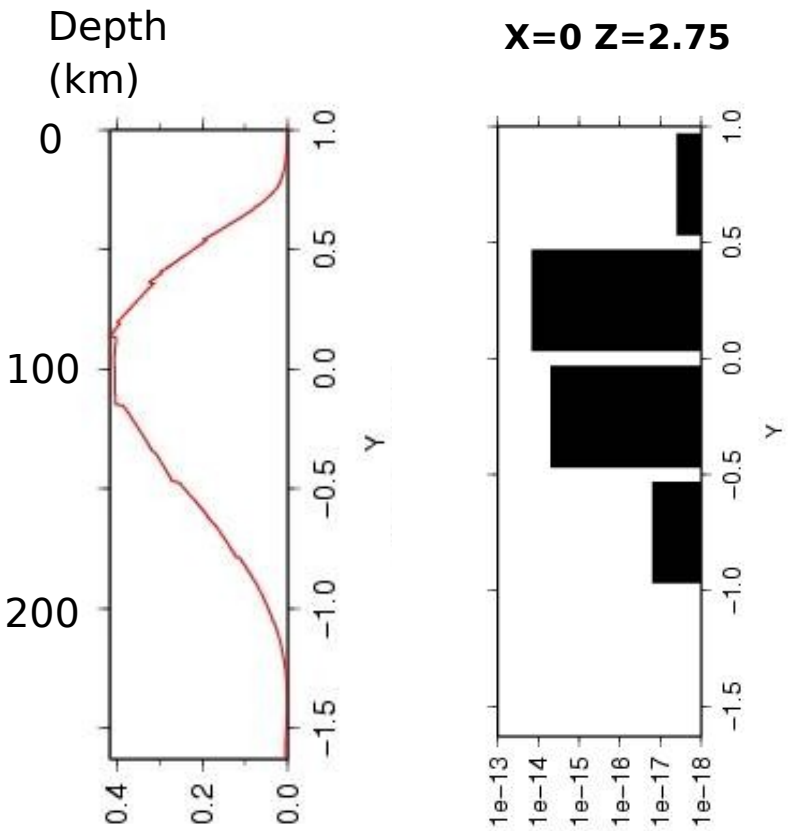


t=12

- The mechanical work done in deforming the viscous material (thermal dissipation) may be the best measure to compare with seismic strain-rates:
- t=12 (dimensionless time ~ 5 or 10 Myr) best matches the present state of Vrancea

$$\Phi = \sum_{1 \le i,j \le 3} \tau_{ij} \dot{\varepsilon}_{ij}$$



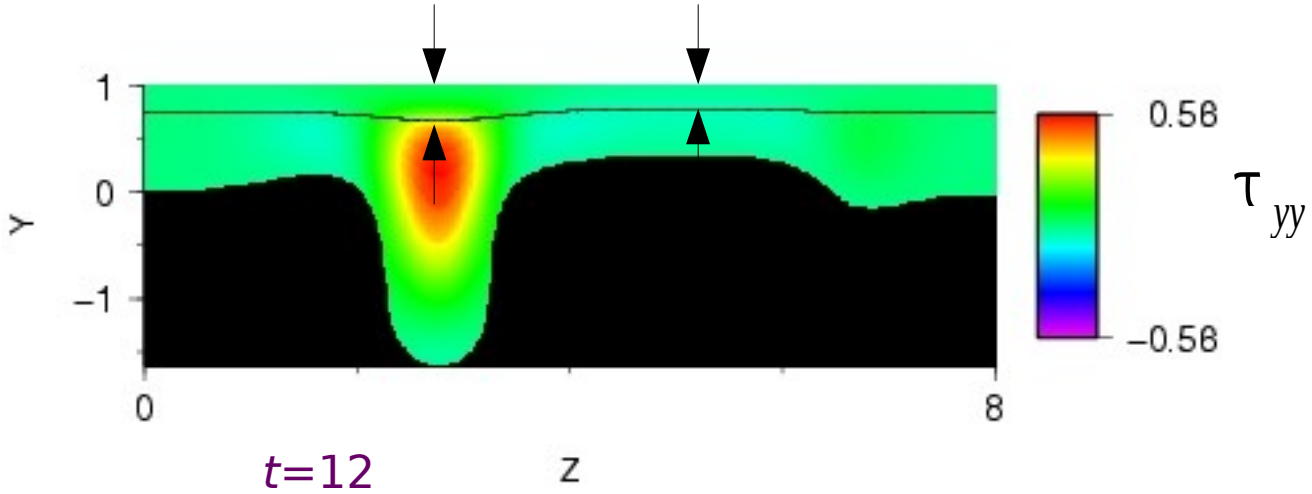


- Depth profile of seismic strain rate intensity (right) matches profile of the thermal dissipation (left) with a peak thermal dissipation core at around 100 km for t = 12.

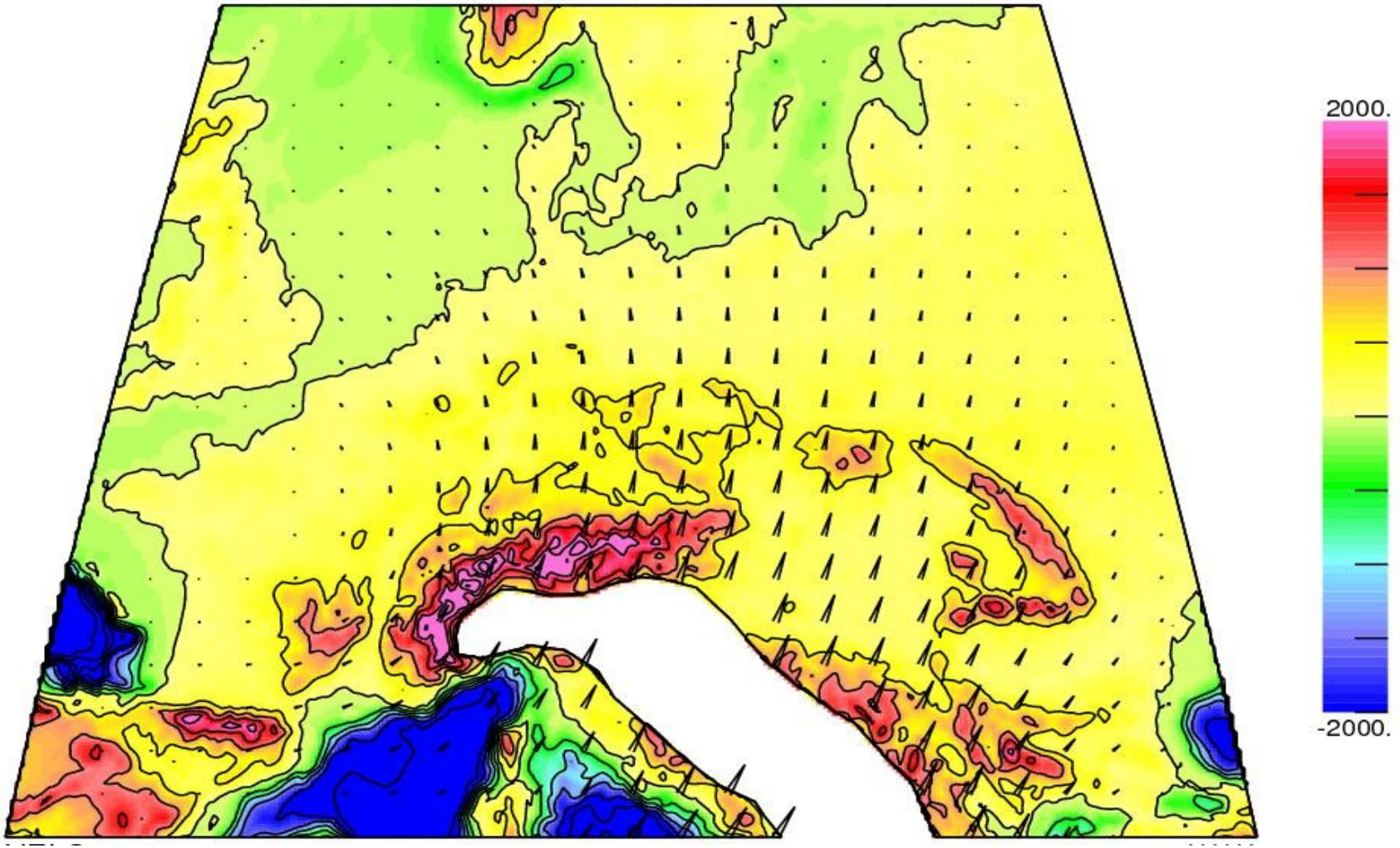
Lorinczi and Houseman, Tectonophysics (2009)

Seismically determined estimates of crustal thickening (1.2 ~ 45/37.5 km) and crustal thinning factors (1.07 ~ 37.5/35 km) can be compared with calculations, to constrain rheological parameters.

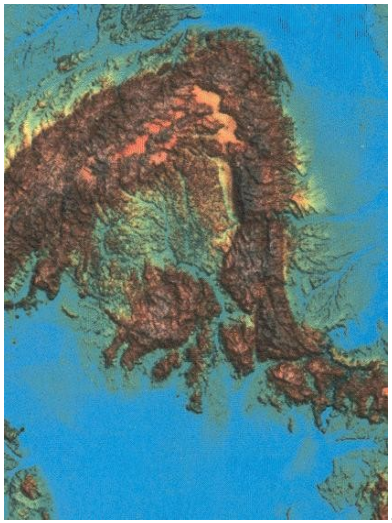
Mantle η	0.333 to 1			0.1 to 1			0.033 to 1		
Crustal η	0.5	1	2	0.5	1	2	0.5	1	2
Thickness	1.35	1.28	1.22	1.26	1.22	1.17	1.18	1.14	1.11
Ext. factor	1.07	1.07	1.06	1.07	1.06	1.05	1.06	1.05	1.05



Thin viscous sheet calculation (*basil*) showing the convergence imposed by the boundary push from the Adriatic block against the European margin (using the rotation pole of Battaglia et al., 2004). Displacement rate vectors for a constant viscosity (n=1) thin viscous sheet (longest arrow ~ 5 mm/yr), here superimposed on topography (m).

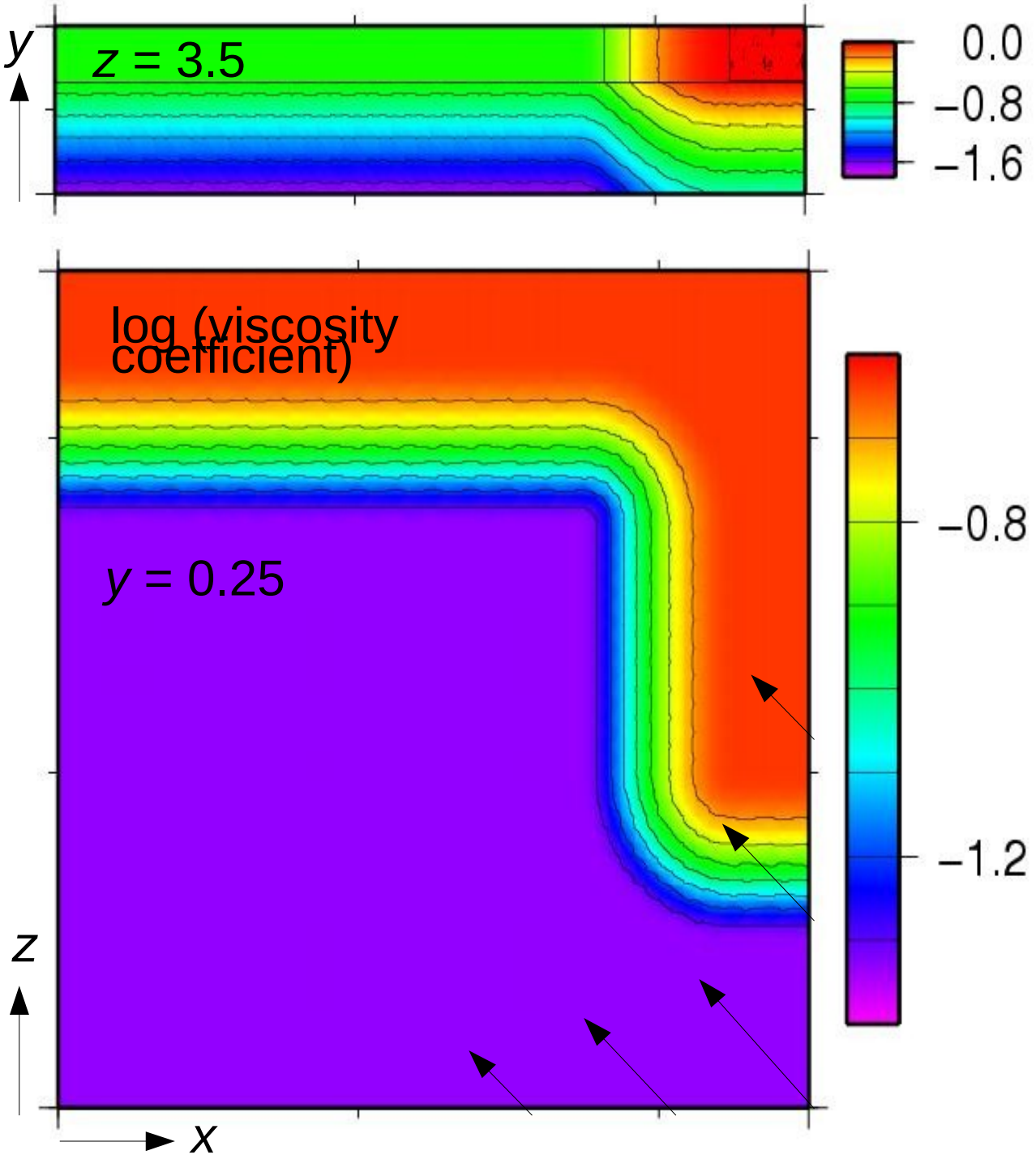


Suppose instead that the red region representing East European platform and Moesian block is 5 x more viscous than the blue region representing Pannonian Basin. Viscosity decreases by a factor of 10 across mantle

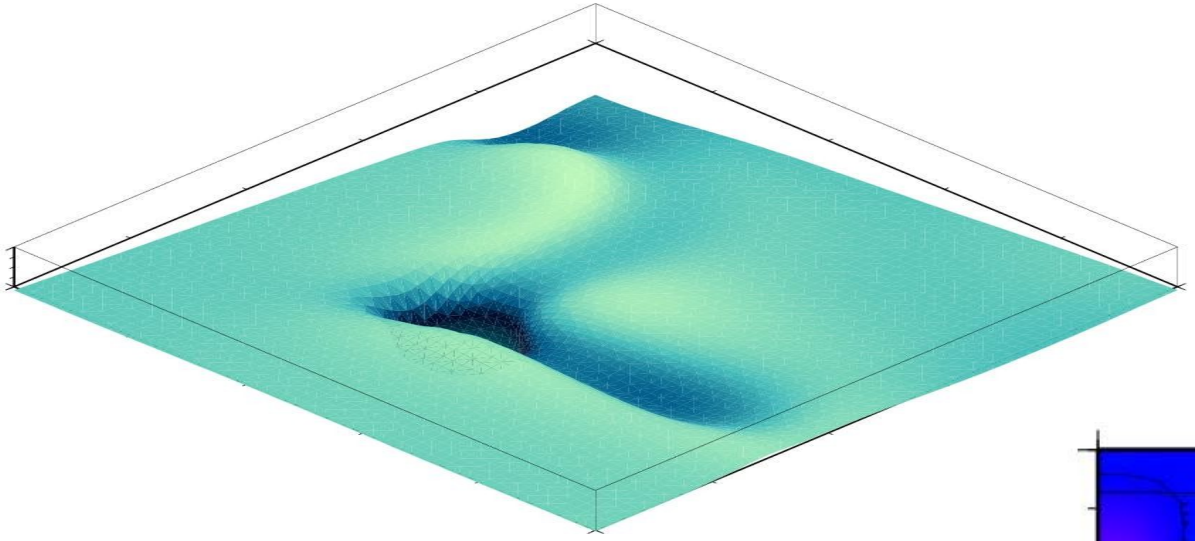


north

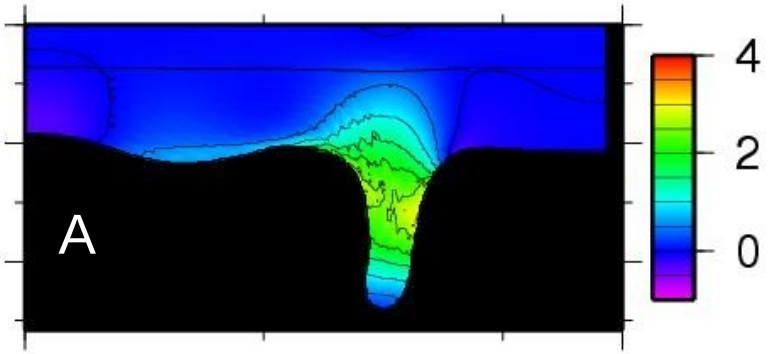
push from Adria



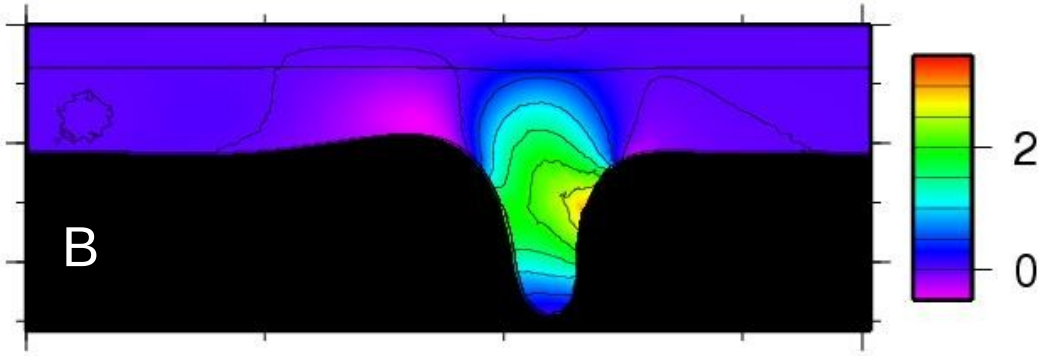
tg6n1dv10, *t* = 5.9125

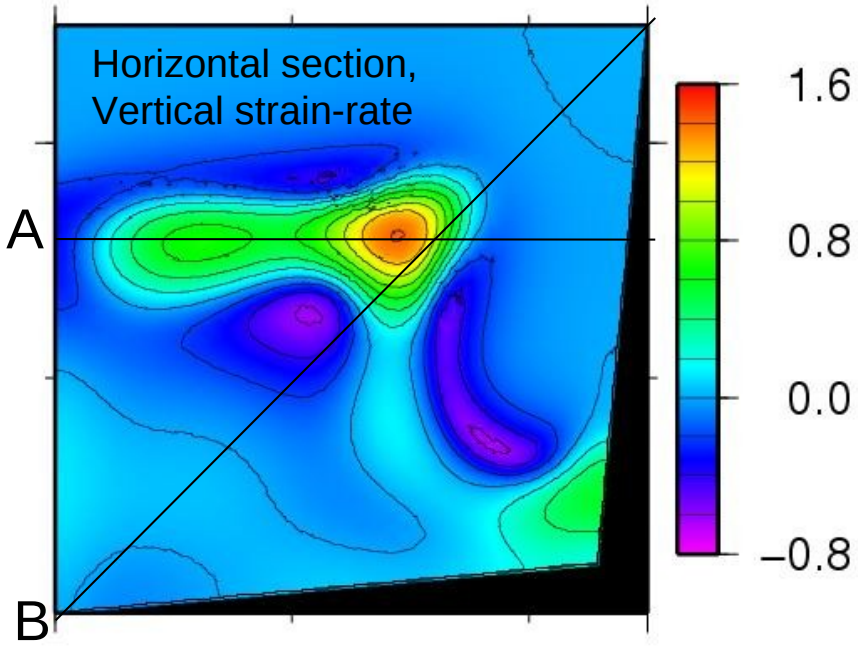


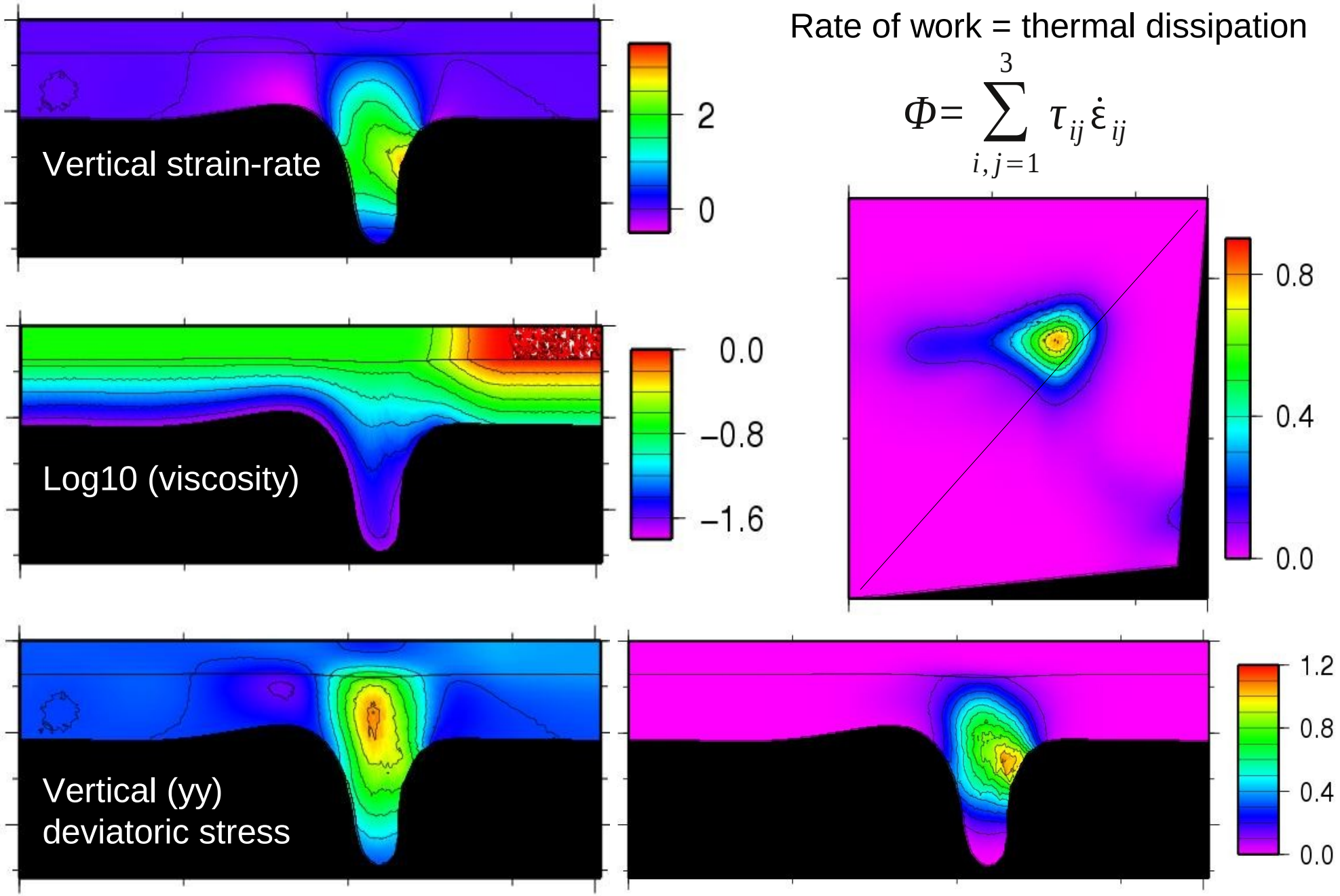
3D-perspective of the base of the layer



Vertical strain-rate maps show stretching of mantle downwelling naturally triggered by the focusing of deformation in the corner between the two strong regions of East European platform and Moesian block.



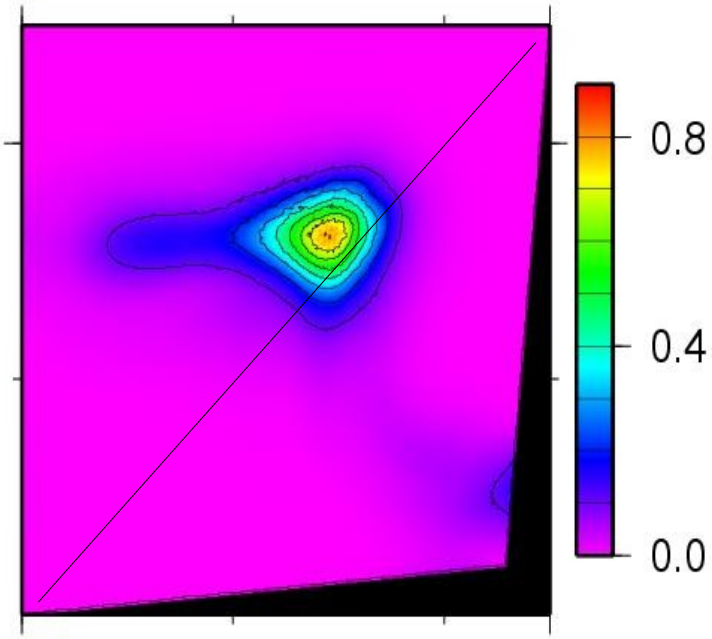


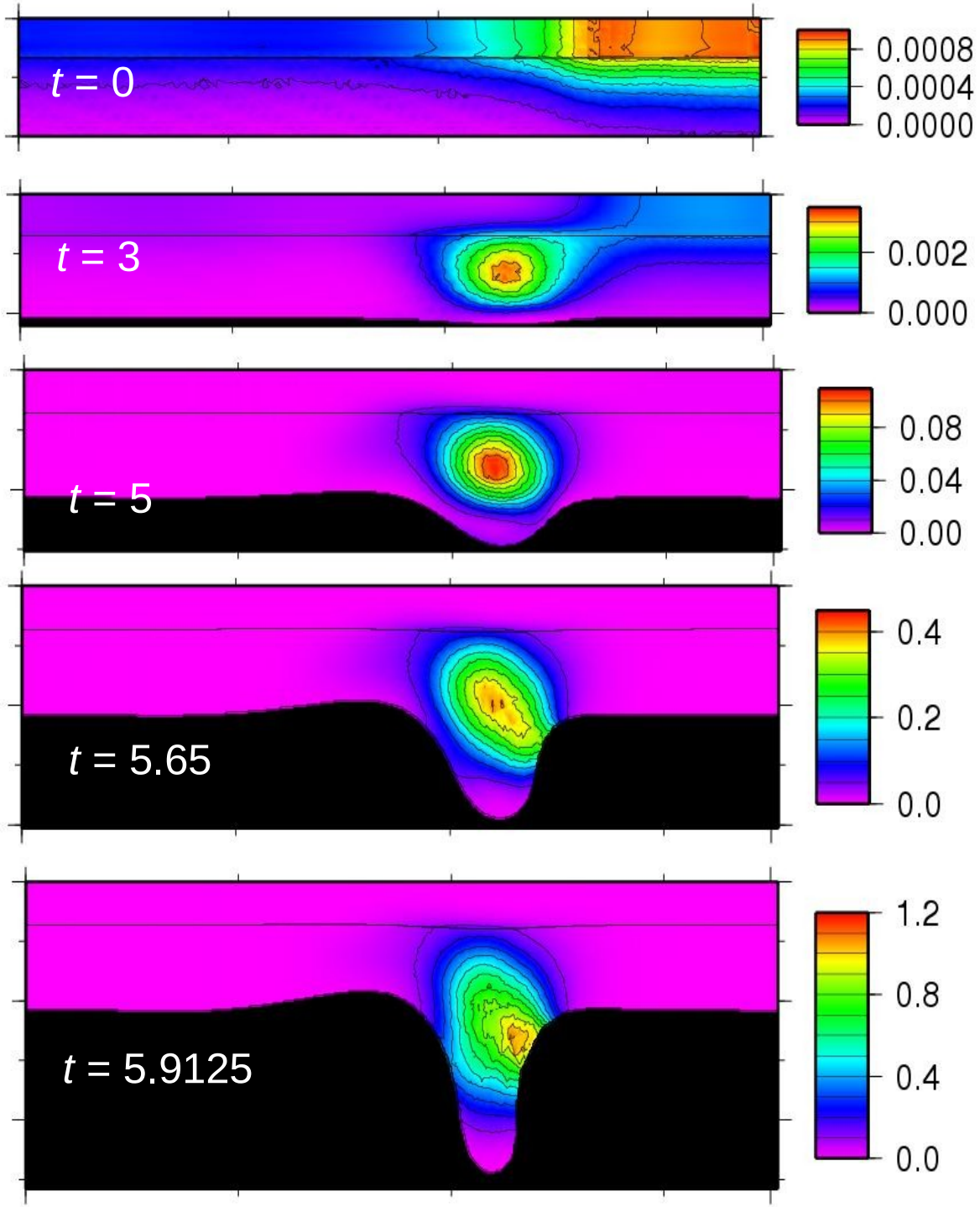

Vertical strain-rate
2
0
Log10 (viscosity)
0.0
−0.8
−1.6
Rate of work = thermal dissipation
$\Phi = \sum_{i,j=1}^{3} \tau_{ij} \dot{\varepsilon}_{ij}$
0.8
0.4
0.0
Vertical (yy)
deviatoric stress
1.2
0.8
0.4
0.0
tg6n1dv10, t = 5.9125

$$\Phi = \sum_{i,j=1}^{3} \tau_{ij} \dot{\varepsilon}_{ij}$$

Evolution of the rate of work during time as the mantle drip develops.

Note the increase in amplitude with time by over 3 orders of magnitude

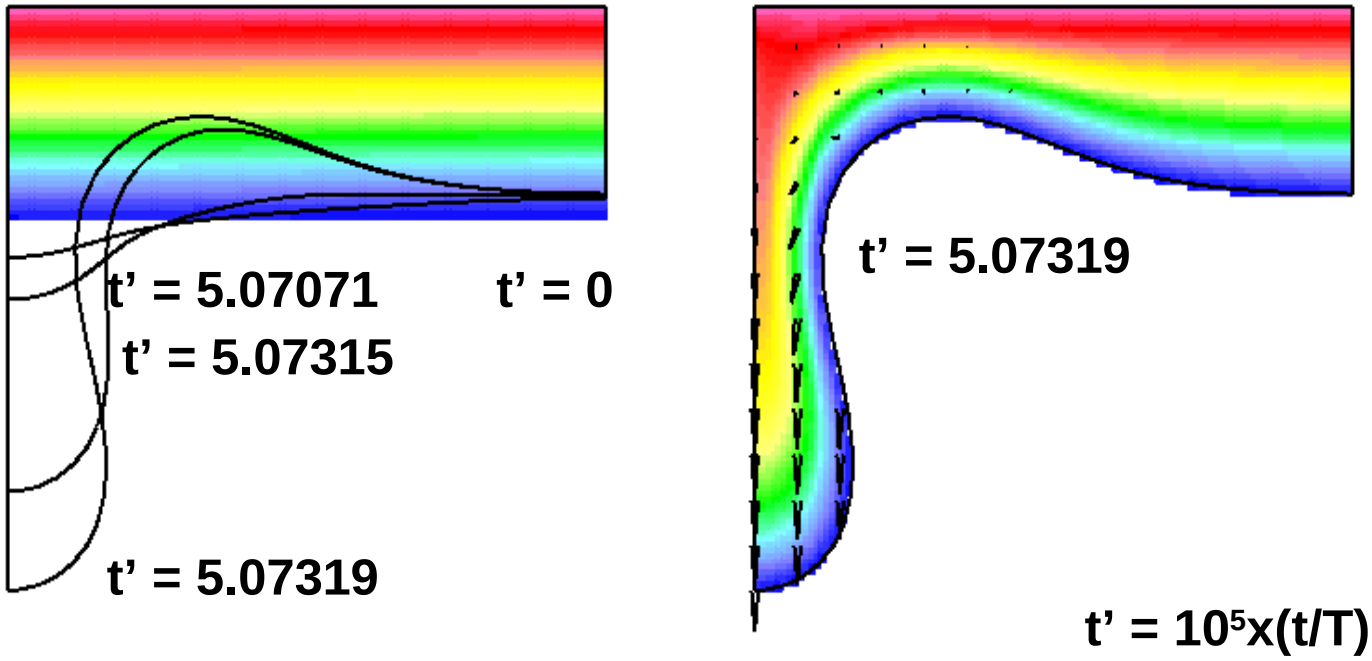




Instability of a non-Newtonian layer

For olivine, strain-rate is proportional to the 3rd power of stress. Initial growth is very slow, but accelerates rapidly as a large blob of mantle drops off.

2D, n=3, λ’ = 6, constant strength coefficient, density (colour coded) decreases linearly with depth



Rayleigh-Taylor Instability

Consider 2 layers of fluid, one above the other, the upper layer is more dense by $\Delta\rho$

Suppose the interface initially level, is perturbed by deflection of the interface – in the form of a harmonic function like:

$$\Delta h \sin(2\pi x/\lambda) \qquad (1)$$

At the interface, we create a horizontal gradient of density which drives a flow field described by the poloidal potential:

$$\eta \nabla^4 \Psi \sim g \Delta \rho \Delta h \frac{2\pi}{\lambda} \cos\left(\frac{2\pi x}{\lambda}\right) \qquad (2)$$

That flow field amplifies the initial deflection, and if viscosity is constant, then the rate at which Δh increases is proportional to Δh. This is the condition for exponential growth:

$$\Delta h = D h_0 \exp\left(t/\tau\right) \quad \text{with time constant} \quad \tau = \frac{2\eta}{g \Delta \rho h} \qquad (3)$$

Rayleigh-Taylor analysis: for small deflection, velocity *w* is proportional to displacement :

$$dZ/dt = qZ \quad \text{and} \quad dw/dt = qw \tag{4}$$

Growth rate *q* depends on viscosity of unstable layer η, density contrast $\Delta\rho$, and wavenumber

$$q = \left(gh\Delta\rho/2\eta\right) f\left(kh\right) \tag{5}$$

For Newtonian viscosity, growth is exponential. For a non-Newtonian material, viscosity depends on the rate of displacement

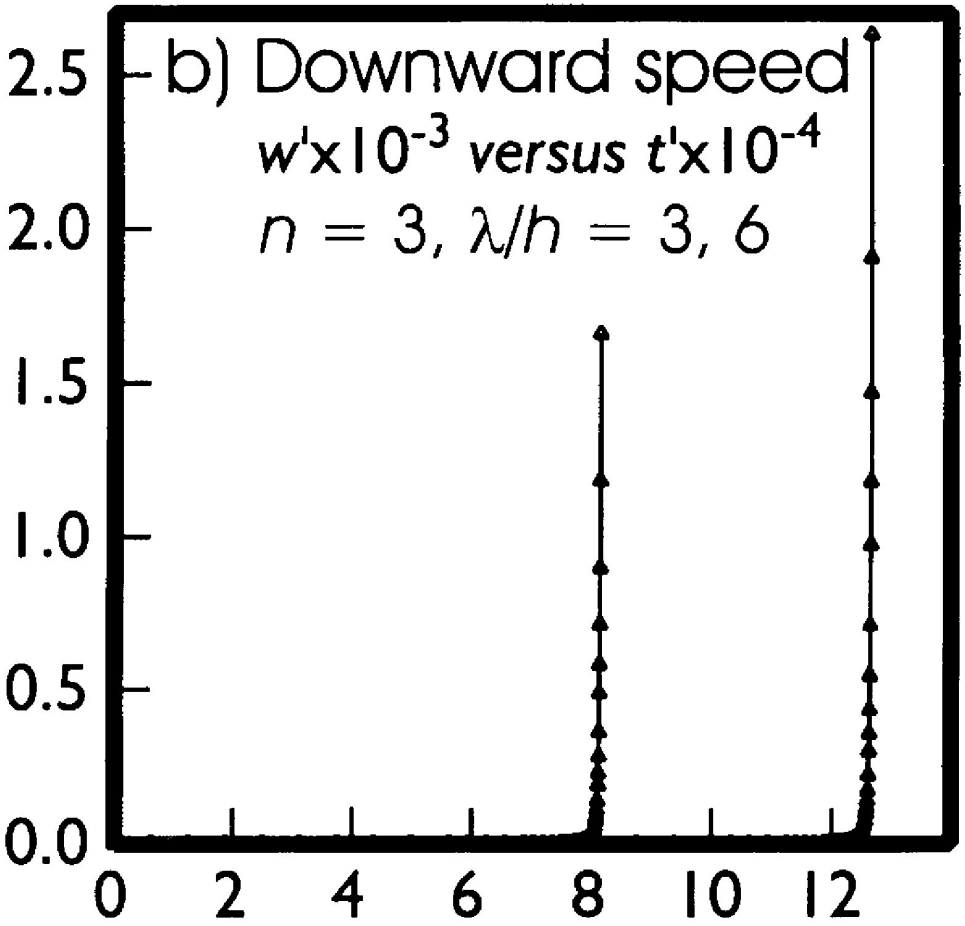
$$\eta = \left(B/2\right)\dot{E}^{1-n/n} \approx \left(B/2\right)\left(w/h\right)^{1-n/n} \tag{6}$$

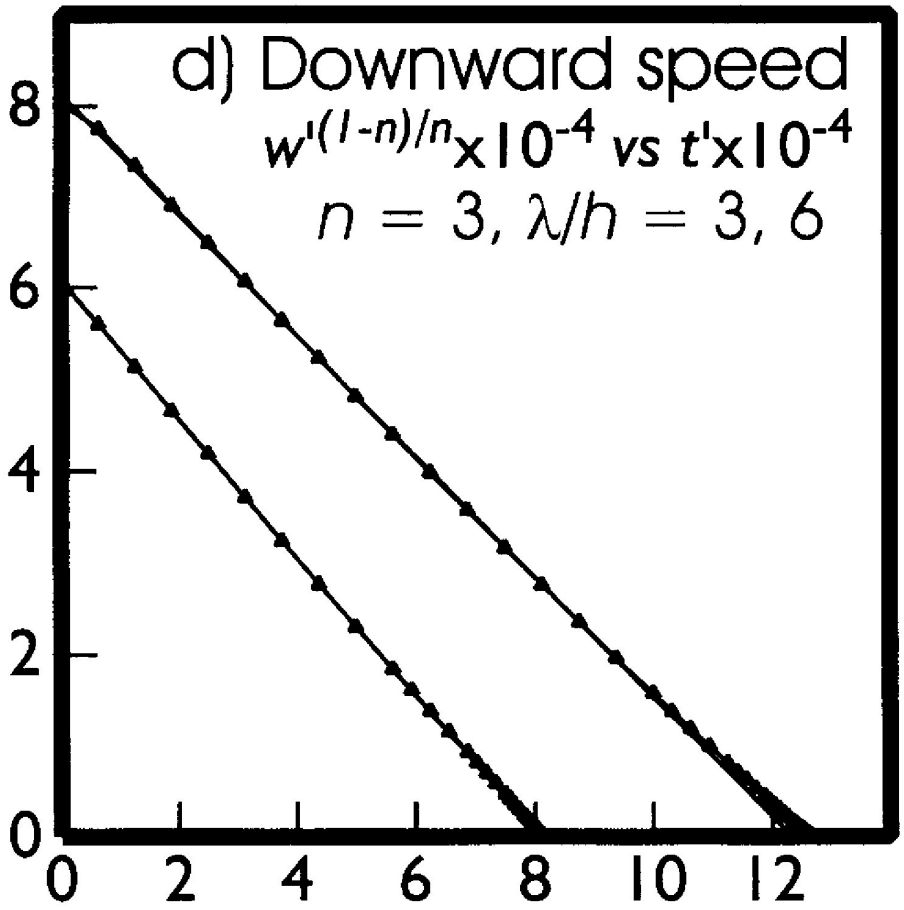
Substitute (6) and (5) into (4) and solve for :

$$w = \left[C\left(kh, n\right)\left(\frac{n-1}{n}\right)\frac{g\Delta\rho}{B}h^{1/n}\left(t_b - t\right)\right]^{n/(1-n)} \tag{7}$$

For n = 3 speed grows as: $w' \propto \left(t_b' - t' \right)^{-3/2}$

The speed of downwelling demonstrates a dramatic transition from initially negligible growth rates.





Integrating again, the deflection also varies as a negative power of (t_b - t), the time to downwelling.

$$Z = h\left(C\frac{hg\,\Delta\rho}{nB}\right)^{\frac{n}{1-n}}\left[(n-1)(t_b-t)\right]^{\frac{1}{1-n}} \tag{8}$$

Time to downwelling is a negative power of the amplitude of initial perturbation Z_o. *C(kh, n)* is a constant of order 1.

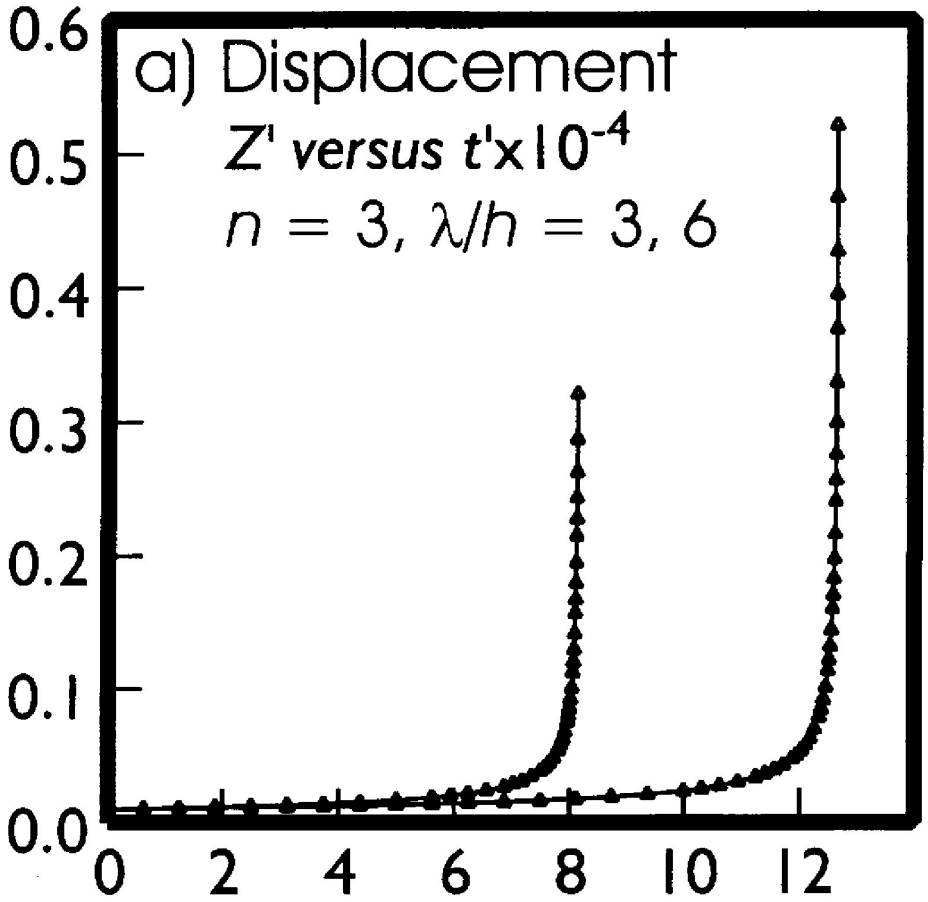
$$t_b = \frac{1}{(n-1)}\left(\frac{B}{hg\,\Delta\rho}\right)^n\left(\frac{n}{C}\right)^n\left(\frac{Z_0}{h}\right)^{1-n} \tag{9}$$

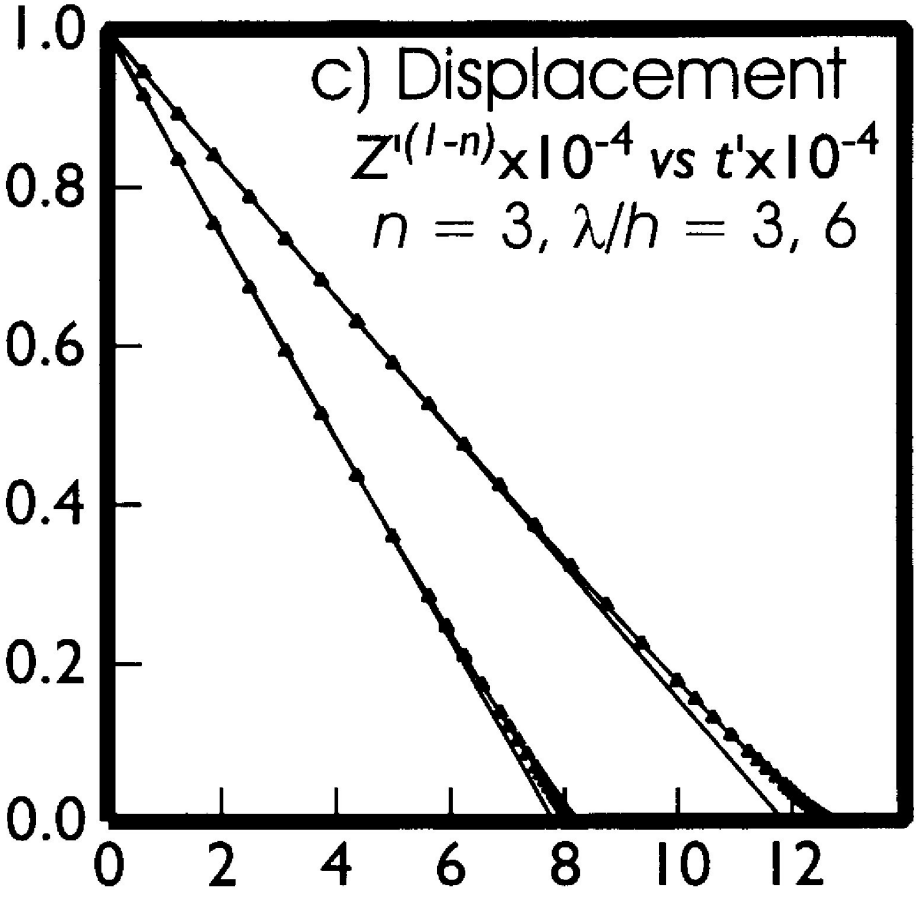
Growth rate varies inversely as the time to downwelling

$$q_w = \left[\left(\frac{n-1}{n}\right)(t_b-t)\right]^{-1}; \quad q_Z = \frac{q_w}{n} \tag{10}$$

For n = 3 displacement grows as: $Z' \propto \left(t_b' - t' \right)^{-1/2}$

slow initial growth, increasing rapidly towards instability time t_b.





Thermal Diffusion Stabilizes

If density differences are due to temperature, thermal diffusion causes harmonic disturbances to decay with rate $\propto -k^2\kappa$ (k = wavenumber, κ = diffusivity).

Short wavelengths are more effectively suppressed than long wavelengths.

The growth rate $\boldsymbol{q}$ of the Rayleigh-Taylor instability varies as $1/(t_b-t)$ $\boldsymbol{q}$ increases with time from $q_0 \propto Z_0^{(n-1)}$.

If $(q-k^2\kappa) < 0$ at $t = 0$ initial perturbation will decay. Convective instability occurs when $(q_0-k^2\kappa) > 0$.

Therefore small perturbations decay, whereas large perturbations grow catastrophically.

Convective Stability Criterion

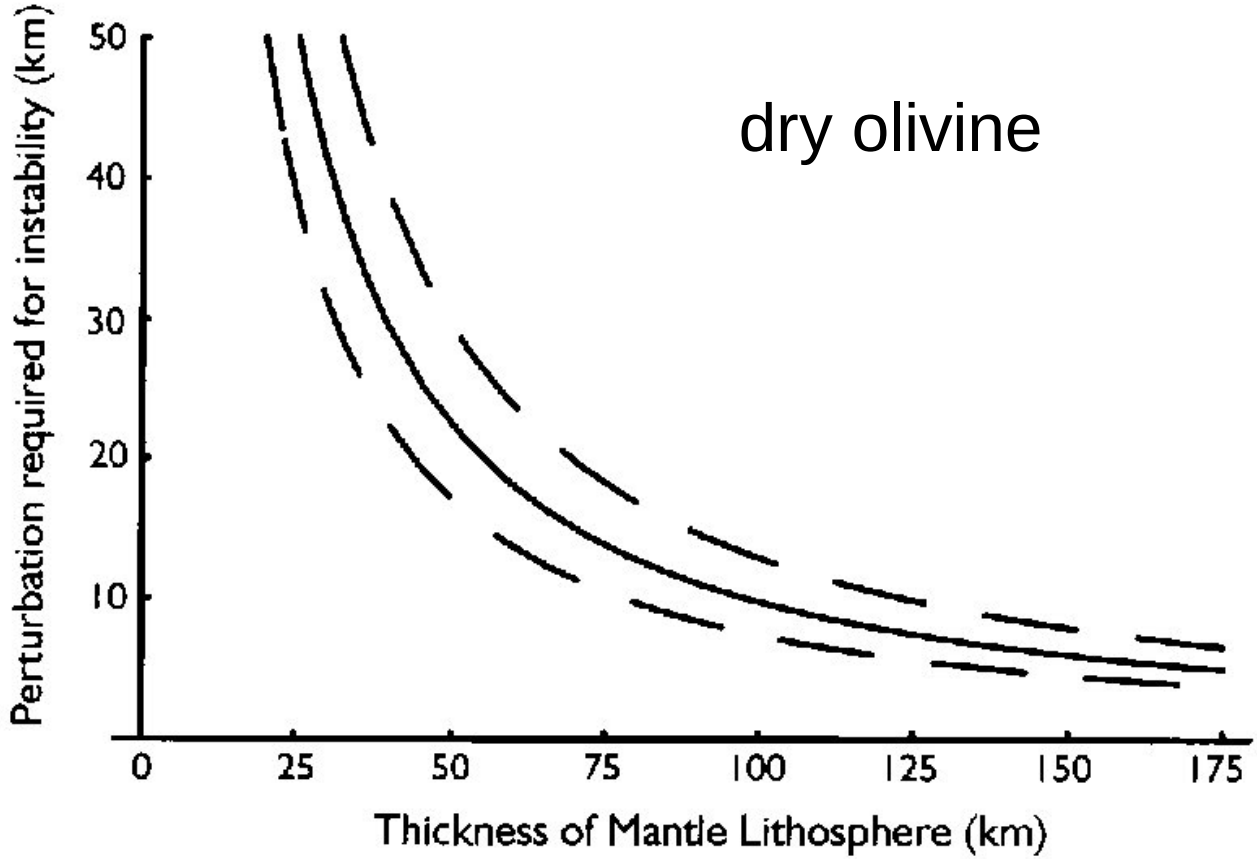
Careful scaling of the stability equations (Conrad and Molnar, 1999), combined with empirical estimates of the rheological relations for olivine (Karato et al., 1986), provide a critical Rayleigh number, Ra_{nc}

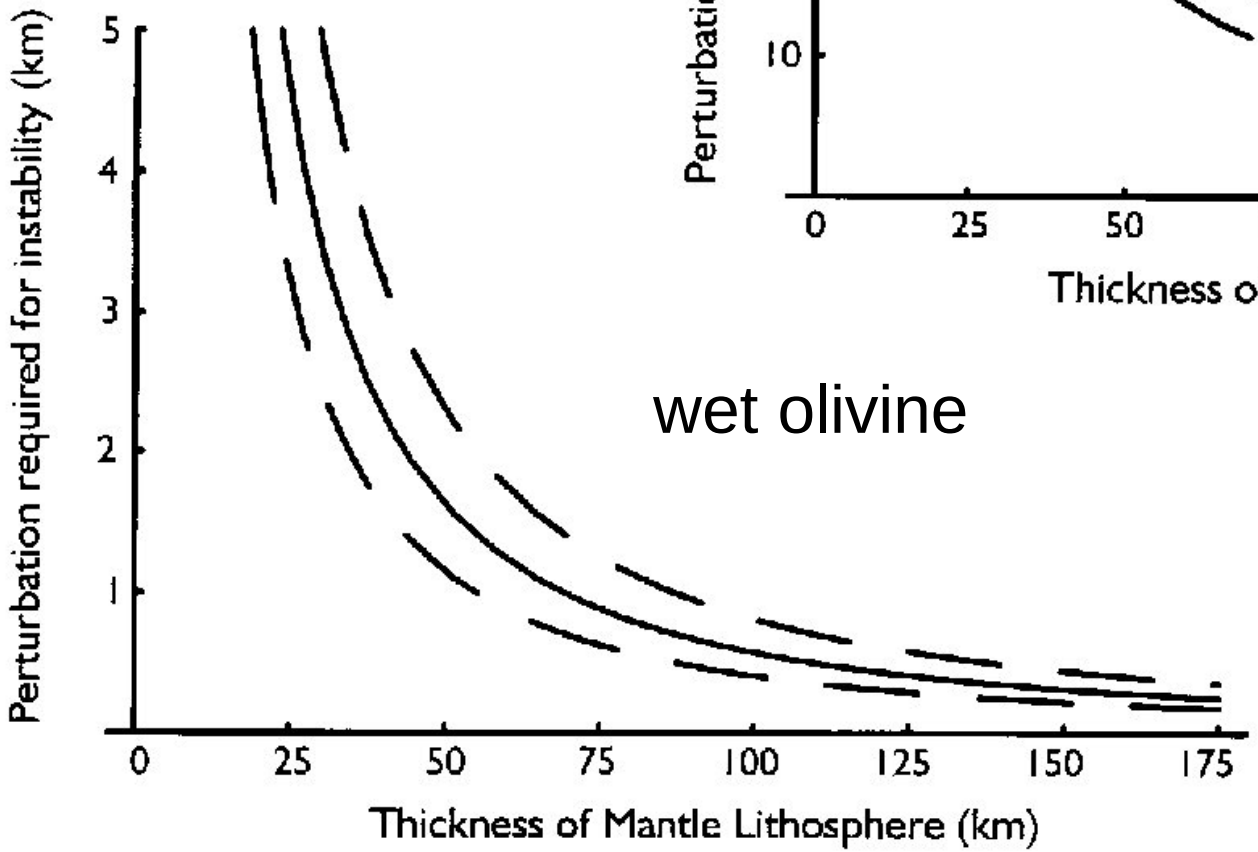
$$Ra_n = \left(\frac{g\Delta\rho h}{nB}\right)^n \left(\frac{h^2}{\kappa}\right)\left(\frac{Z_0}{h}\right)^{n-1} > Ra_{nc}$$

For non-Newtonian viscosity, a sufficiently large amplitude perturbation is required before buoyancy can overcome diffusion.

The ratio Z_0/h is a measure of the lithospheric thickening that is required to destabilise the layer.

Perturbation in lithospheric thickness required for instability to develop under dry or wet olivine law





Houseman and Molnar (2001)