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Strokes flow and dynamic topography Stokes problem of a sinking sphere

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Sphere sinking through a viscous half-space: Dynamic topography

Morgan, W. J., Gravity anomalies and convection currents. 1. A sphere and a cylinder sinking beneath the surface of a viscous fluid, *J. Geophys. Res., 70*, 6175-6187, 1965.

Hager, B. H. (1984), Subducted slabs of lithosphere and the geoid: constraints on mantle rheology and flow, *J. Geophys. Res., 89*, 6003-6015.

Richards, M. A., and B. H. Hager (1984), Geoid anomalies in a dynamic mantle, *J. Geophys. Res., 89*, 5987-6002.

Dynamic topography

Dynamically induced flow creates spatial variations in pressure and deviatoric stress.

Such differences in stress can deflect the earth's surface.

Most convection and many other geodynamic models predict such surface deflections.

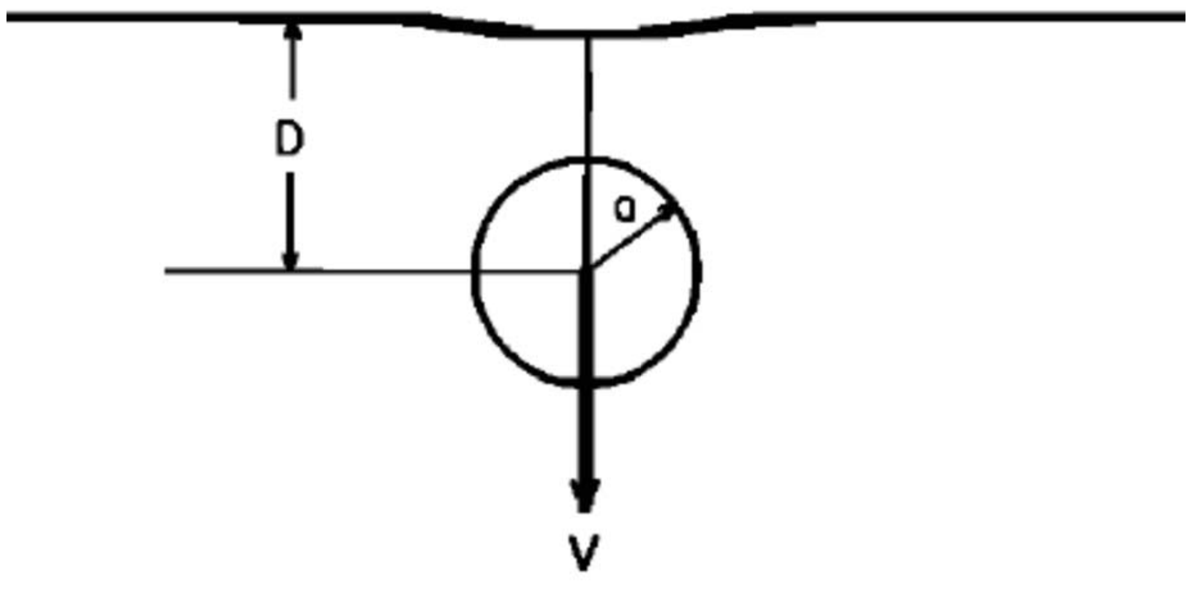
In principles, both geological and geodetic methods (gravity, geoid, GPs velocities) can test such predictions.

(This is hard!)

Sinking sphere in an infinite medium

$$V = \frac{2R^2 \Delta\rho g}{9\eta}$$

But, within a half-space, the surface should be deflected. In an infinite space, flow above the sphere is downward, and pressure is negative.



Effects of sphere

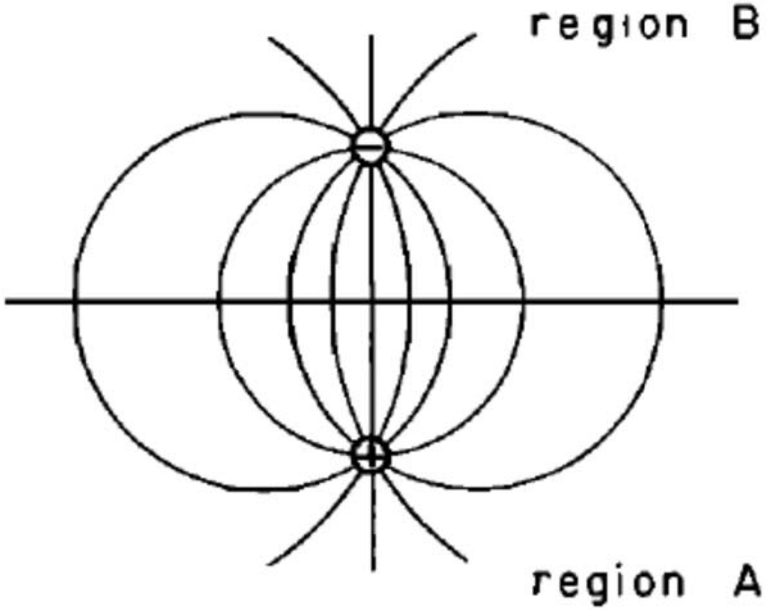
Gravitational attraction of a dense (low-density) sphere pulls (pushes) the surface down (up).

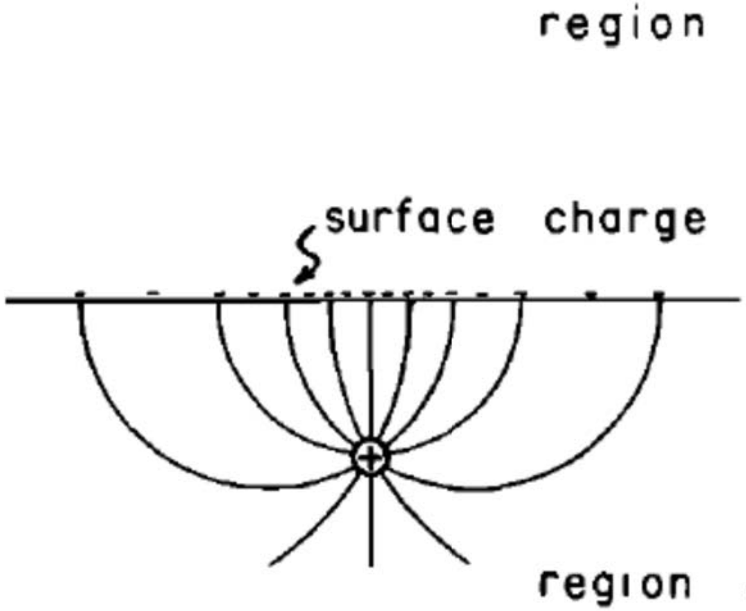
More important, a sinking (rising) sphere induces flow within the surrounding fluid, which also pulls (pushes) the surface down (up).

Morgan separates these, asking first what is the gravitational effect of the sphere on the surface.

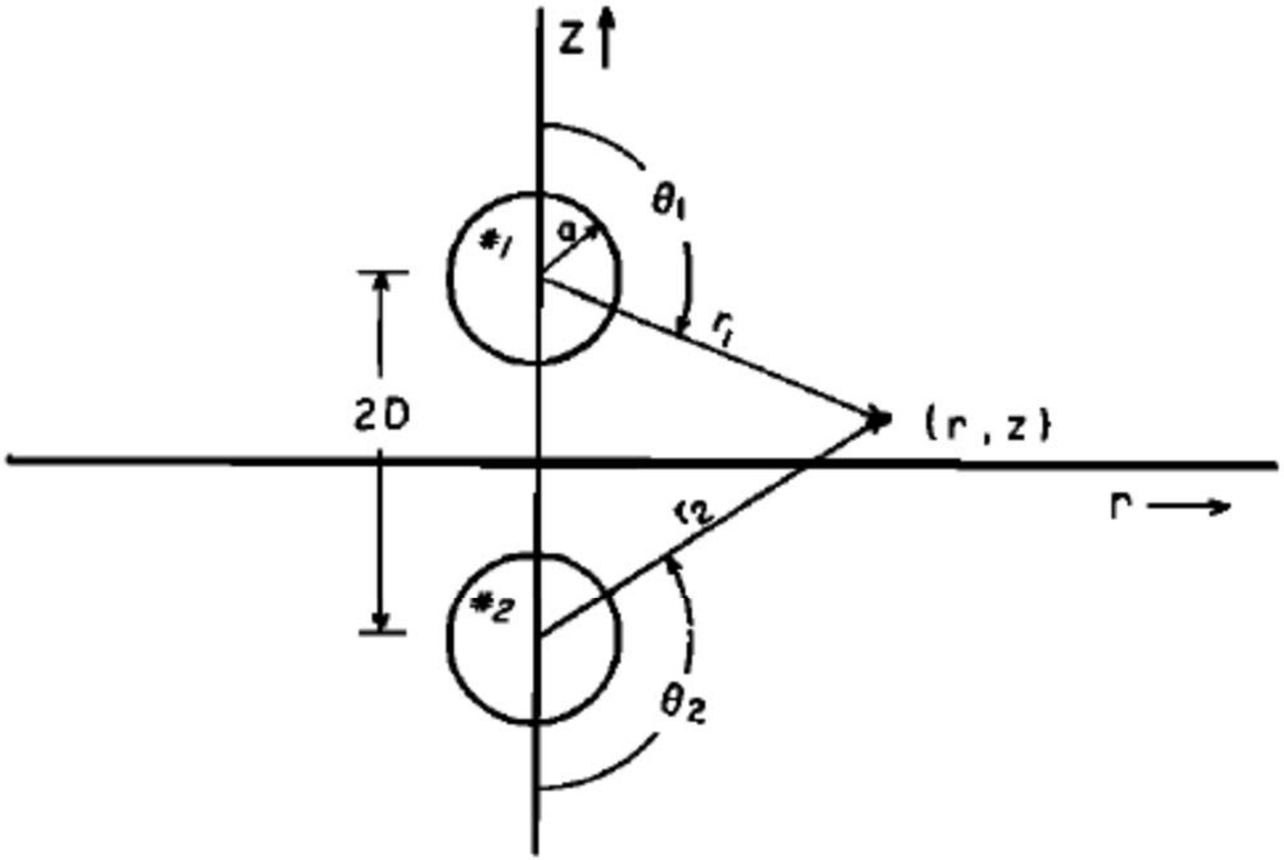
Gravitational attraction of the sphere at depth D, as measured at the surface

Morgan asks, by analogy with electrostatics, what shape must the surface take to cancel the gravitational attraction of the sphere. He does this with images.





Switch to cylindrical coordinates



Gravitational attraction of the sphere and its image, as measured at the surface

Imagine one sphere of mass M at a depth D, and a second of mass $-M$ at a height, D, above the surface.

$$g(z=0) = 2G\frac{MD}{\left(r^2 + D^2\right)^{3/2}}$$

This attraction can then be matched by a surface density (mass/area), analogous to an electric charge distribution, given by

$$\sigma = -\frac{\delta g}{4\pi G} = -\frac{2}{4\pi G}\frac{4\pi R^3 \Delta\rho}{3\left(r^2 + D^2\right)}\frac{D}{\left(r^2 + D^2\right)^{1/2}} = -\frac{2\Delta\rho R^3 D}{3\left(r^2 + D^2\right)^{3/2}}$$

Sinking sphere and its rising image sphere

Of course, with two spheres moving apart, they require that vertical (z) components of velocity at the surface vanish: $u_z(z{=}0) = 0$.

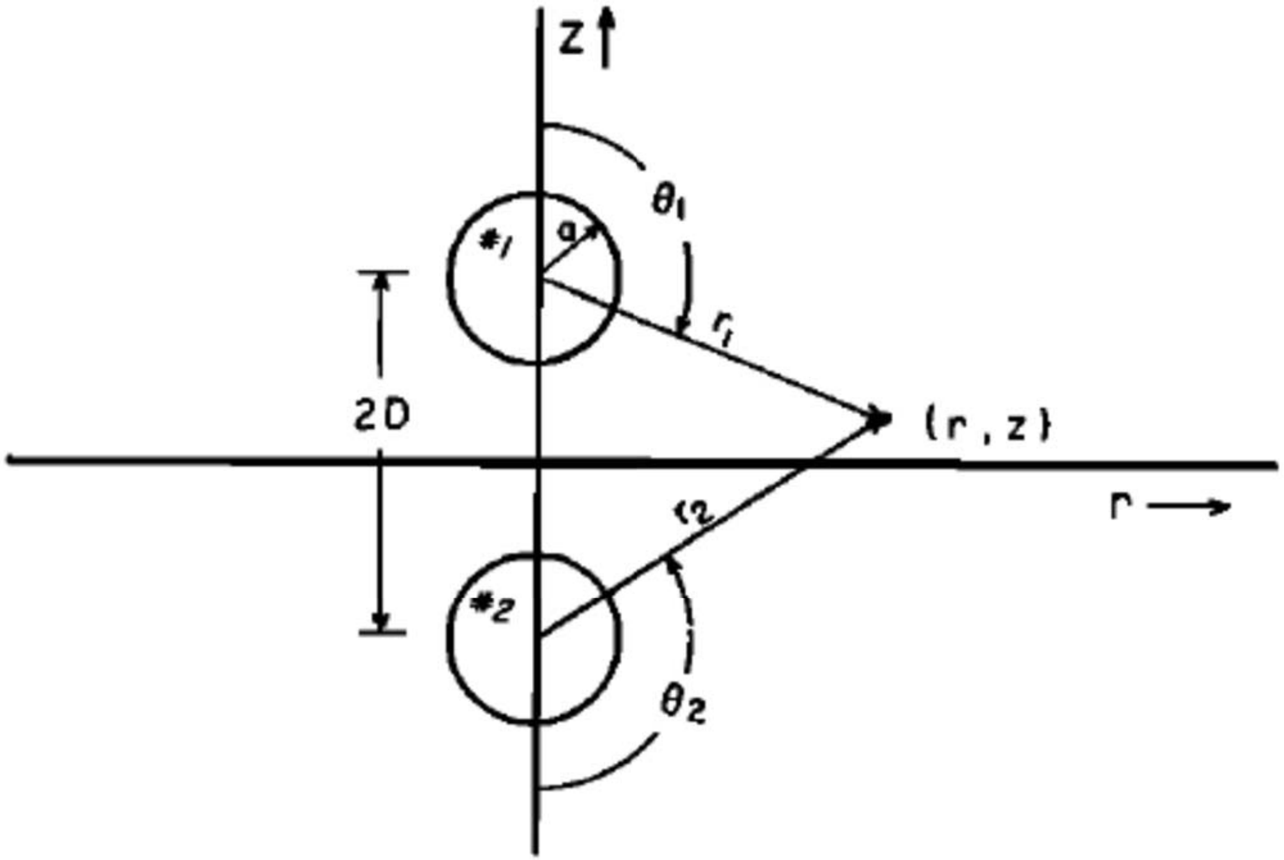
Both spheres, however, induce radial components of velocity along the boundary: $u_r(z{=}0) \neq 0$.

Effect of surface (as manifested by images)

The presence of the surface limits the flow, and hence the sphere ought not sink as rapidly as it would in an infinite space.

Morgan evaluated this, again, using images. First, an image sphere rising in the fluid above the interface affects the flow at the interface, but second, the presence of the image sphere also introduces a boundary within the (otherwise) infinite fluid that should be taken into account.

The image sphere assures that
$$u_z(z=0)=0$$

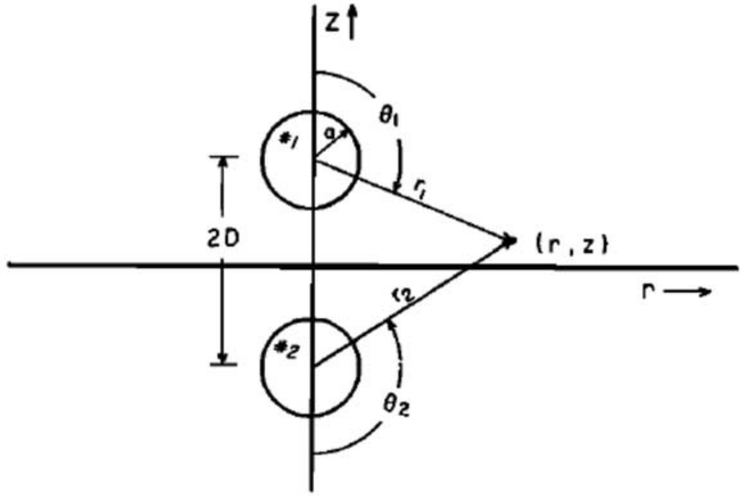


The rigid boundary of one sphere, however, affects flow induced by the other.

Morgan’s solutions

First, the vertical component, where $e = R/D$.

$$u_z = V\left(1 + \frac{3e}{4} + \frac{9e^2}{16}\ldots\right)\left\{\begin{array}{l} e\dfrac{D}{r_1} - e\dfrac{D}{r_2} \\ + \dfrac{eD}{4}\dfrac{r_1^2 - R^2}{r_1^3}\left[3\cos^2\vartheta_1 - 1\right] \\ - \dfrac{eD}{4}\dfrac{r_2^2 - R^2}{r_2^3}\left[3\cos^2\vartheta_2 - 1\right] \end{array}\right\}$$

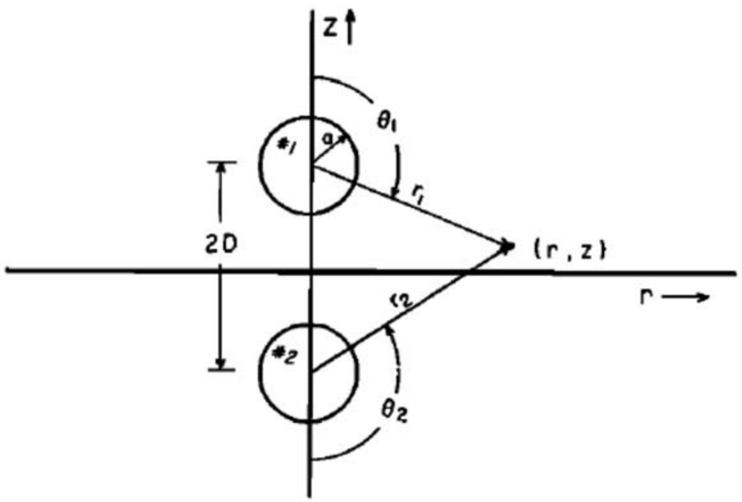


Of course, $u_z(z = 0) = 0$.

Morgan's solutions

Next, the radial component, where $e = R/D$.

$$u_r = V\left(1 + \frac{3e}{4} + \frac{9e^2}{16}\ldots\right)\left\{\begin{array}{l} \dfrac{eD}{4}\dfrac{r_1^2 - R^2}{r_1^3} 3\sin\vartheta_1\cos\vartheta_1 \\ + \dfrac{eD}{4}\dfrac{r_2^2 - R^2}{r_2^3} 3\sin\vartheta_2\cos\vartheta_2 \end{array}\right\}$$



Of course, $u_r(z=0) \neq 0$.

Morgan's solutions

Finally, the pressure, where $e = R/D$.

$$p = -\frac{3R\eta V}{2D}\left(1 + \frac{3e}{4} + \frac{9e^2}{16} + \frac{19e^3}{64}\ldots\right)$$

$$\left\{\frac{D^2}{r_1^2}\cos\vartheta_1 + \frac{D^2}{r_2^2}\cos\vartheta_2\right\}$$

$$-\frac{3e^3}{8}\left\{\frac{D^3}{r_1^3}\left[3\cos\vartheta_1 - 1\right] + \frac{D^3}{r_2^3}\left[3\cos\vartheta_2 - 1\right]\right\}$$

Drag on the sphere

With the same logic as before, the drag on the sphere becomes:

$$F_d = 6\pi R\eta V\left(1+\frac{3e}{4}+\frac{9e^2}{16}+\ldots\right)$$

Thus,

$$V = \frac{2\Delta\rho g R^2}{9\eta}\frac{1}{\left(1+\frac{3e}{4}+\frac{9e^2}{16}+\ldots\right)}$$

With $e = R/D$, this reduces to the infinite medium, when e is small, and hence $D >> R$.

Normal stress at the surface

(Note: on the surface, $z = 0$, $\cos\,\theta = D/(D^2 + r^2)^{1/2}$.)

$$\sigma_{zz} = -p + 2\eta \frac{\partial u_z}{\partial z} = \frac{2\Delta\rho g R^3 D^3}{\left(D^2 + r^2\right)^{5/2}}$$

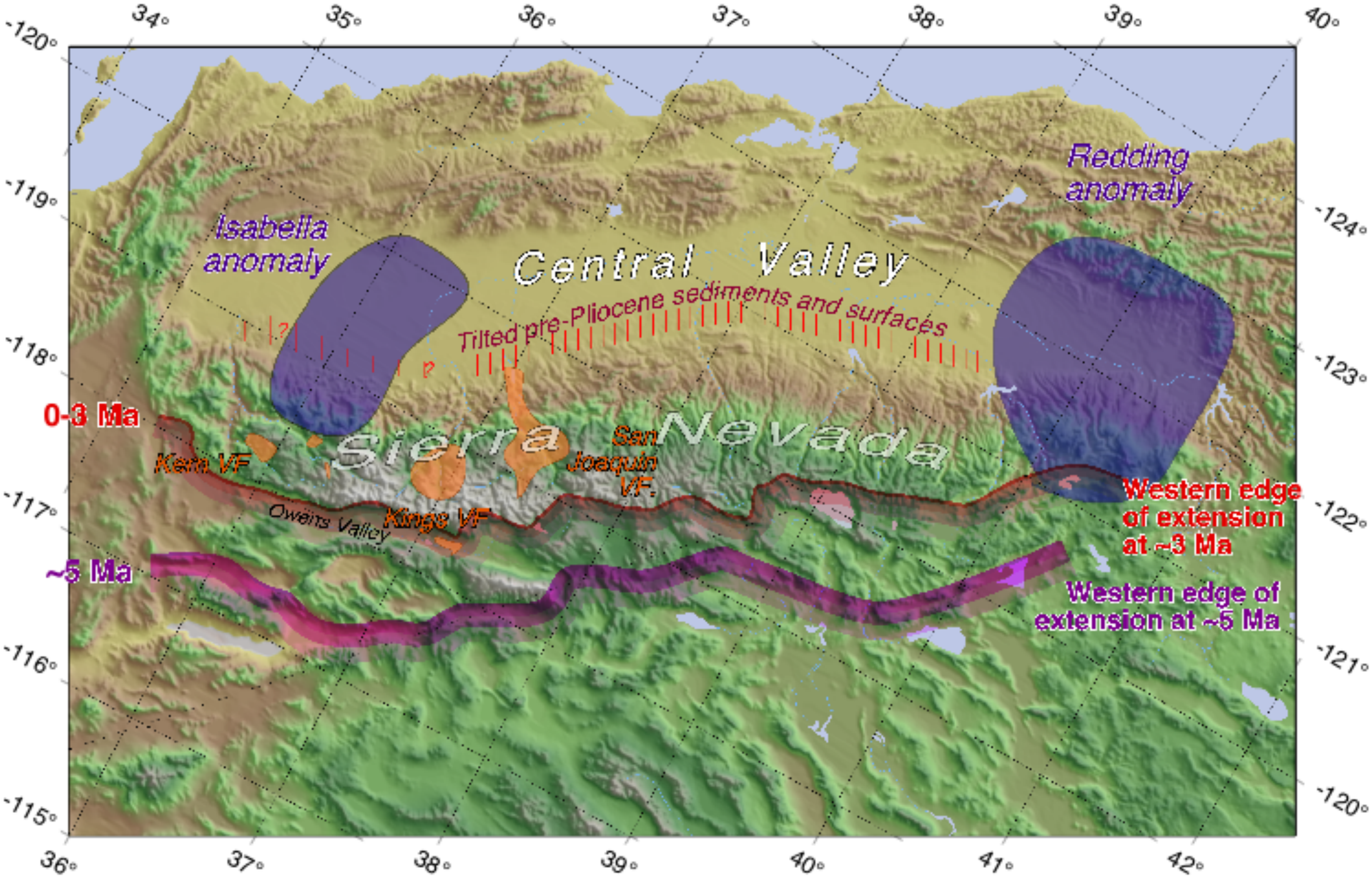
Hence, the surface is deflected downward by

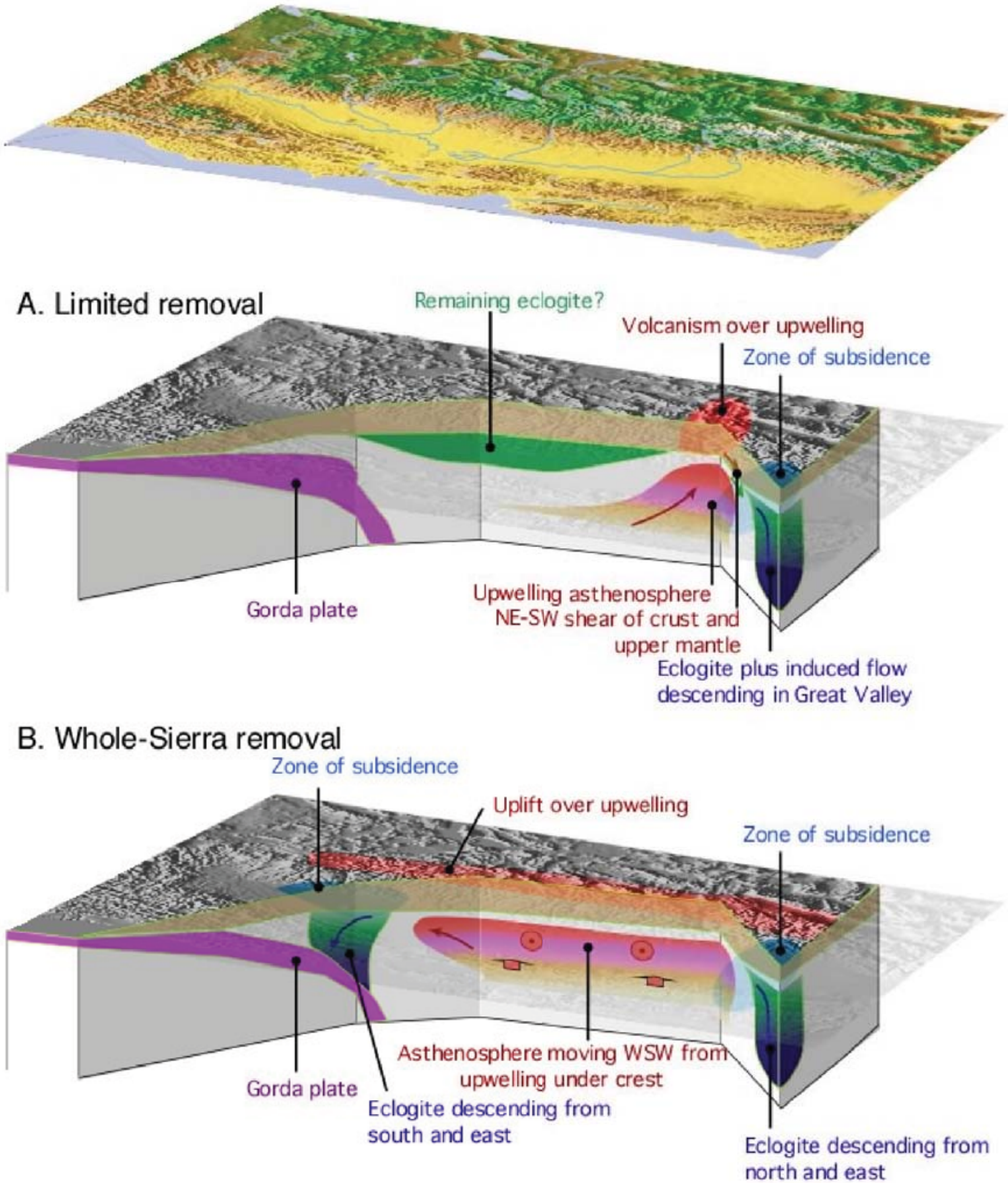
$$\Delta h = \frac{\sigma_{zz}}{\rho_m g} = \frac{2\Delta\rho R^3 D^3}{\rho_m \left(D^2 + r^2\right)^{5/2}}$$

Some special qualities

1. Both the normal stress and the deflection of the surface are independent of viscosity (for this case of constant viscosity).
2. The radius of the sphere enters only in the volume of the sphere; what matters is the mass of the sphere.
3. Gravity measured at the surface (gravity anomalies) will consist of two parts: one due to the gravitational attraction of the mass at depth, and the other due to the deflection of the surface.

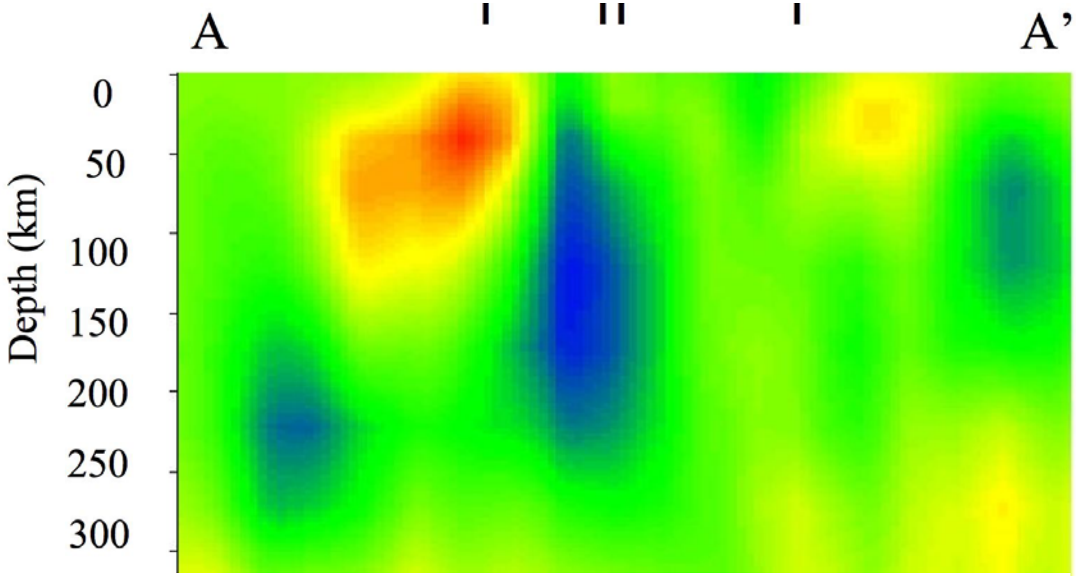
Sierra Nevada (Craig Jones et al.)

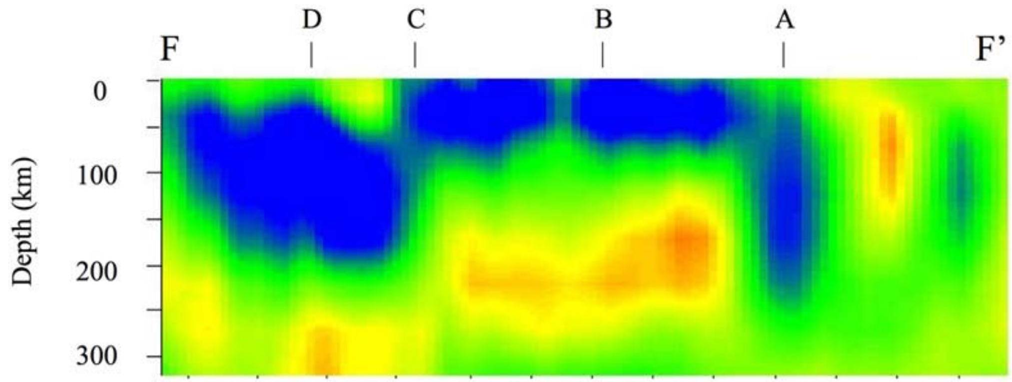


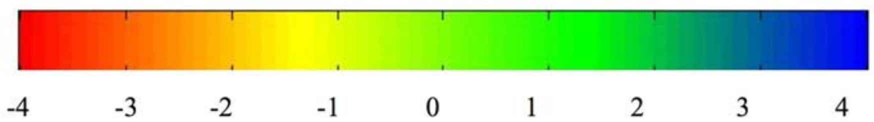


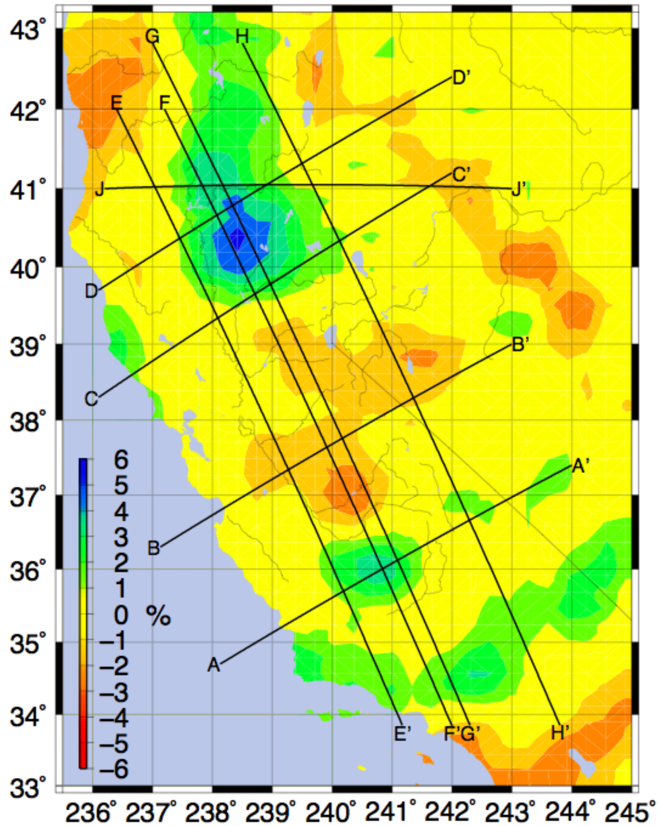
Drawn
by
Craig
Jones

Sierra Nevada Tomographic cross-sections

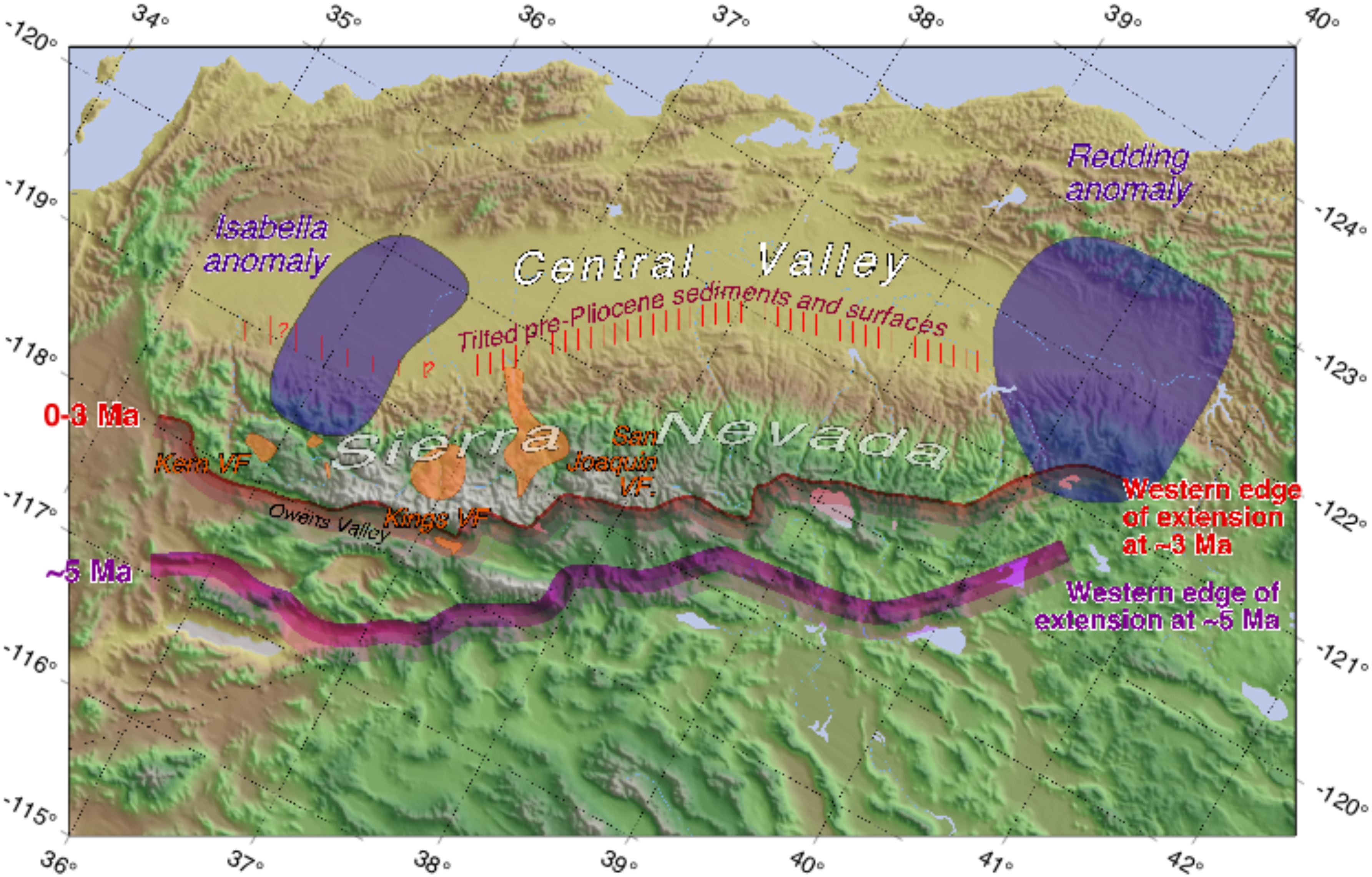




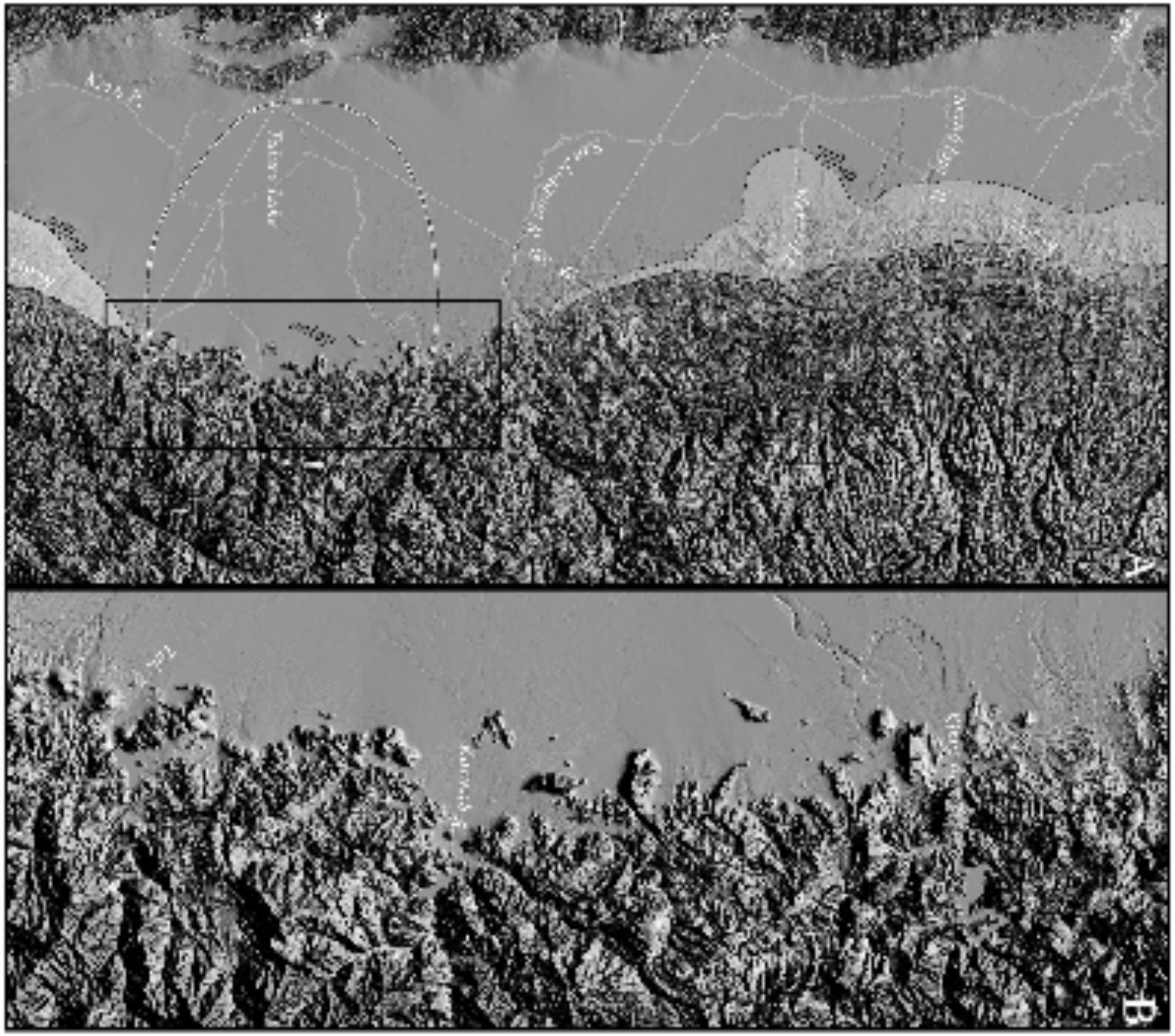




Sierra Nevada (Craig Jones et al.)



Submergence of the Southeast San Joaquin Valley and drowning of topography by thick sediment



Saleeby and Foster [2004]

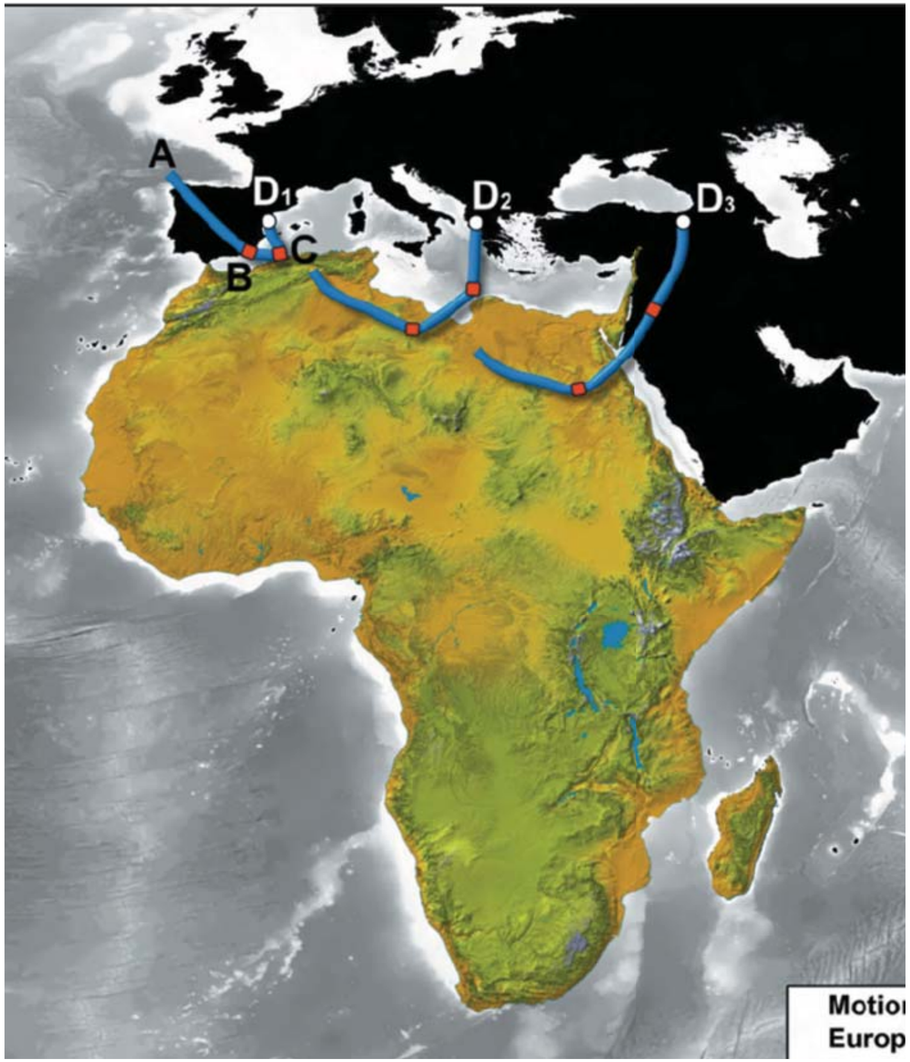
Sierra Nevada: Isabella Anomaly

Suggestion of sinking blob of cold material: Isabella anomaly with high P-wave speeds.

Another suggestion: drowning of topography, as if the surface is pulled down.

What is the mass anomaly associated with the Isabella anomaly , and could it pull the surface down dynamically, as some believe?

This is an open question.

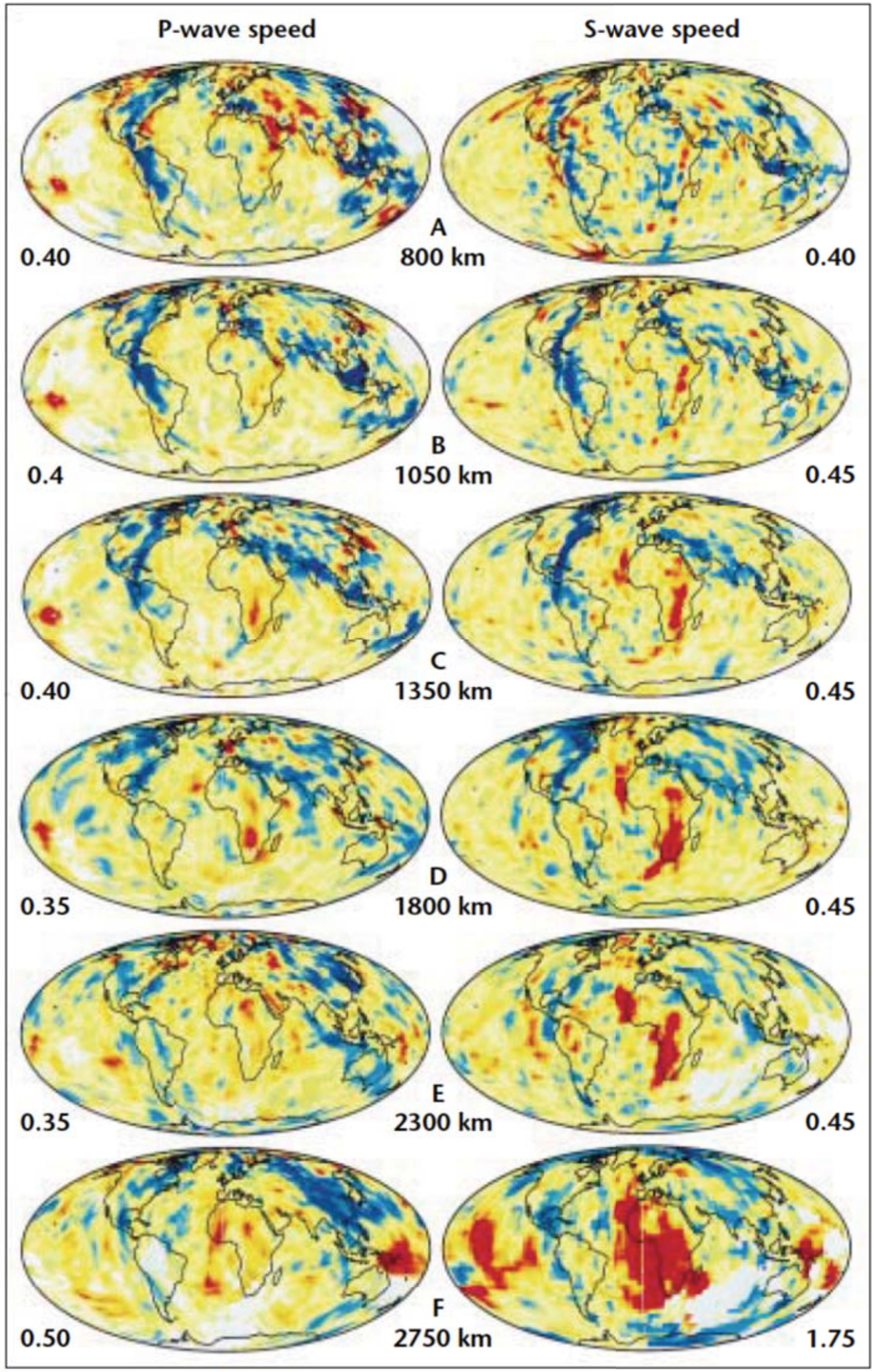


[*de Wit* 2007]

Consider Southern Africa

A region ~ 2000 km in dimension stands ~1 km above sea level.

Is this an example of dynamic topography?

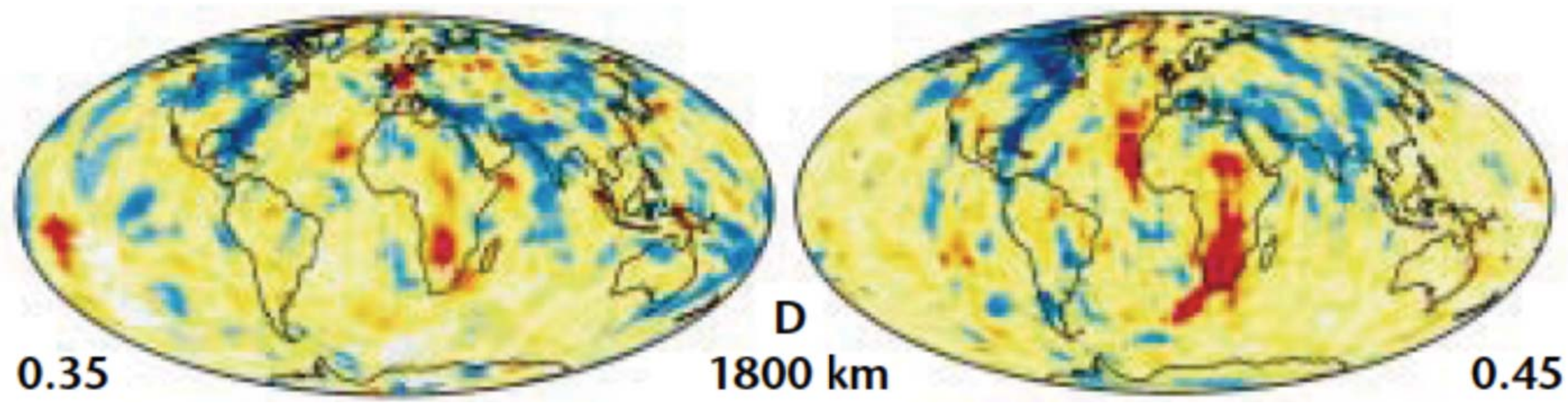


Some do think that high southern Africa results from dynamic topography

- Low P- and S-wave speeds in the lower mantle beneath southern Africa suggests that low-density material is rising.

[*Grand et al*. 1997]

Blow-up of tomographic images from 1800 km [*Grand et al*. 1997]

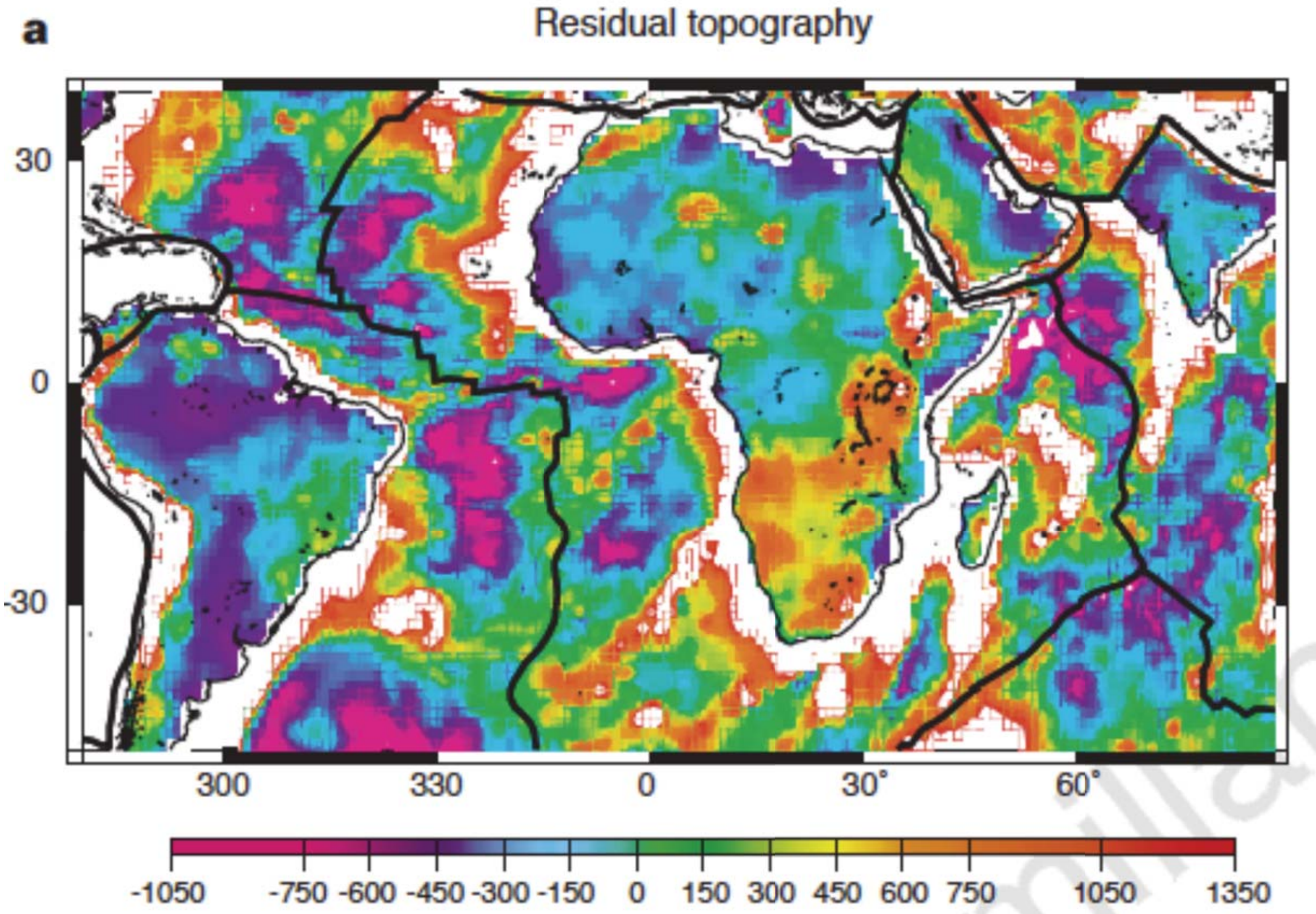


Low P- and S-wave speeds in the lower mantle beneath southern Africa suggests that low-density material is rising.

Red areas show regions with speeds ~0.35-0.45% lower than average.

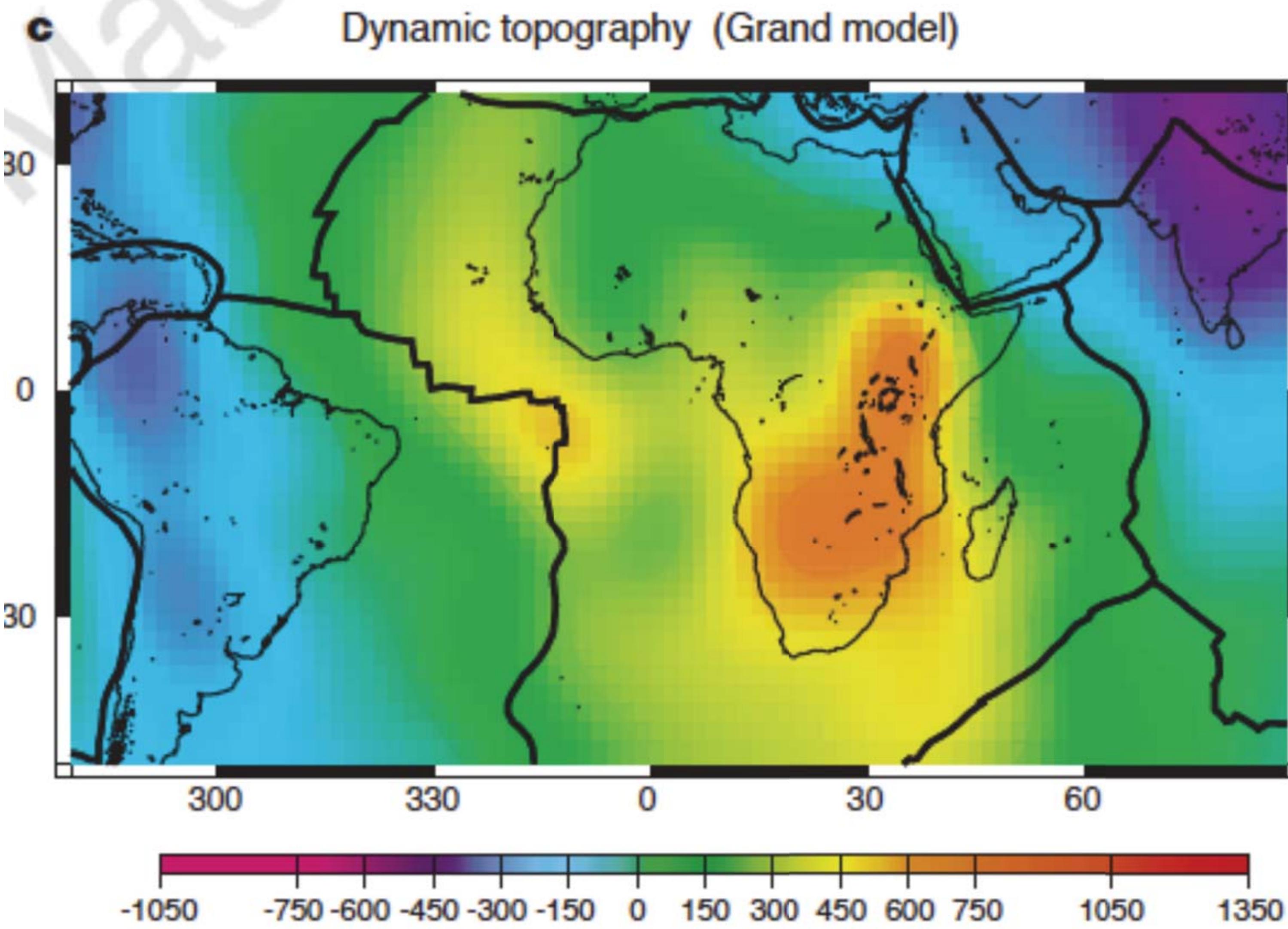
Therefore, P waves travel with speeds ~0.05 km/s slower than average, and S waves by 0.03 km/s slower than average.

Southern Africa is high



[*Lithgow-Bertelloni and Silver* 198]

Dynamic flow supports high Southern Africa?



[*Lithgow-Bertelloni and Silver* 198]

Surface deflection: Some numbers

$$\Delta h = \frac{2\Delta\rho R^3 D^3}{\rho_m \left(D^2 + r^2\right)^{5/2}}$$

First, to make it easy, let's look at $r = 0$.

$$\Delta h = 2\frac{\Delta\rho}{\rho_m}\frac{R^3}{D^2}$$

Suppose that Δh = 1 km, R = 1000 km,
D = 2000 km, and ρ = 3.3 x 10^3 kg/m^3.
What must $\Delta\rho$ be?

$\Delta\rho$ = 6.6 kg/m^3

What Temperature anomaly does $\Delta\rho$ = 6.6 kg/m^3 require?

$$\Delta\rho = \rho\alpha\Delta T$$

$r = 5 \text{ x } 10^3 \text{ kg/m}^3$

$\alpha = 2 \text{ x } 10^{-5} \text{ °C}^{-1}$

Thus,

$\Delta T = 80°\text{C}.$

This is a sensible value.

Does this match gravity anomalies?

Two mass anomalies contribute to the gravity anomaly: the mass of the sphere and the dynamic topography itself.

$$\Delta g = -G\frac{\Delta M}{D^2} = -G\frac{4\pi R^3 \Delta\rho}{3D^2}$$

With G = 6.67 x 10^{-11} N m^2/kg^2, Δg = -46 mgal.

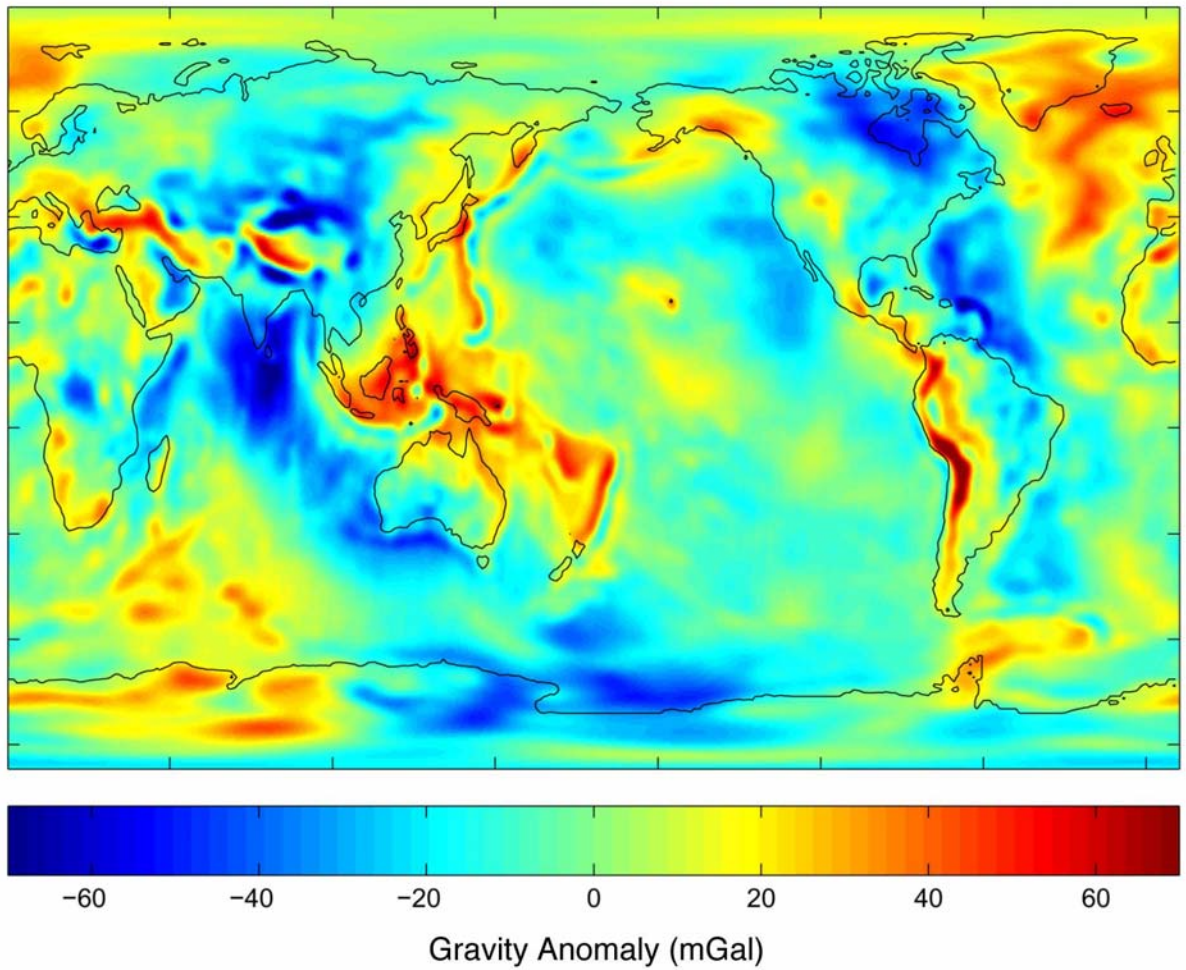
For the surface deformation:

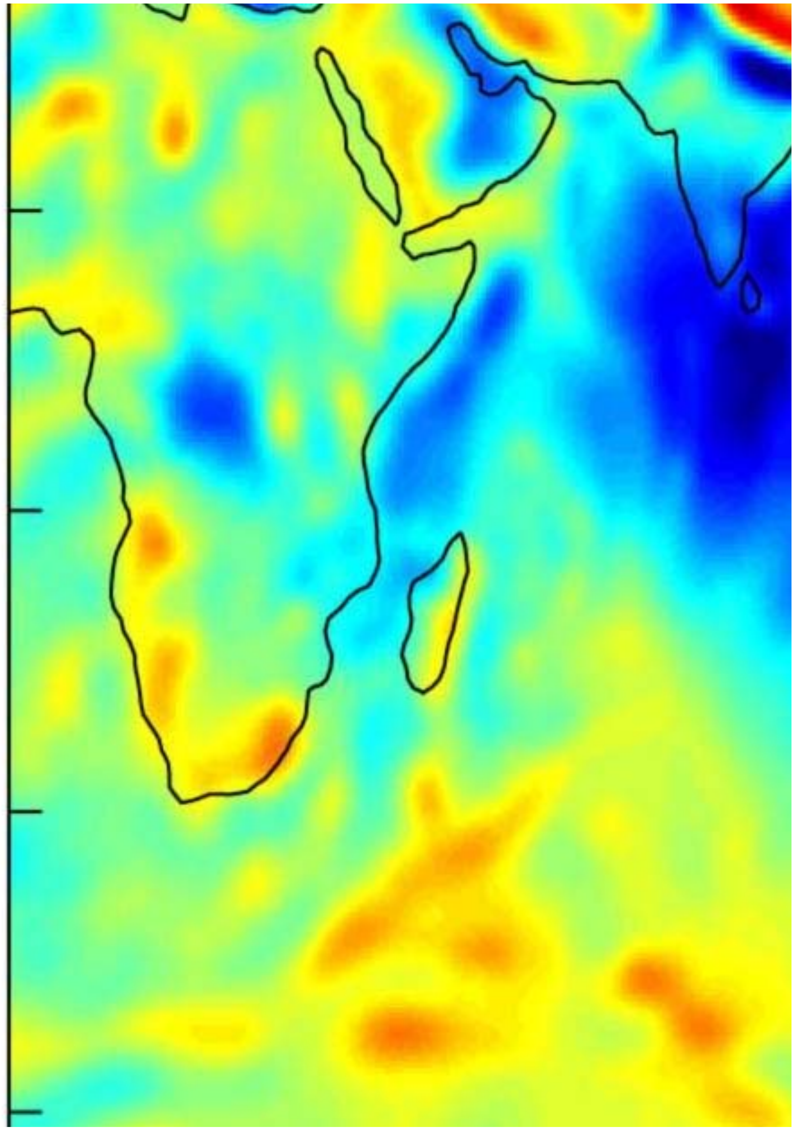
$$\Delta g = 2\pi G \rho_m \Delta h$$

Δg = 138 mgal.

So the total should be Δg = (138 – 46 =) 92 mgal.

Gravity anomalies over Africa (from the *GRACE* mission)





http://www.zonu.com/fullsize-en/2009-11-19-11208/Gravity-anomalies-in-the-world.html

A high southern Africa: Dynamic topography or not?

A reasonable density anomaly in the lower mantle, qualitatively consistent with tomographic images, can account for 1 km of excess topography.

The temperature anomaly associated with the density anomaly is reasonable.

The geoid (allegedly) is matched by the convection model.

Dynamic topography predicts a gravity anomaly that is too large!

Assumptions

Rising sphere?

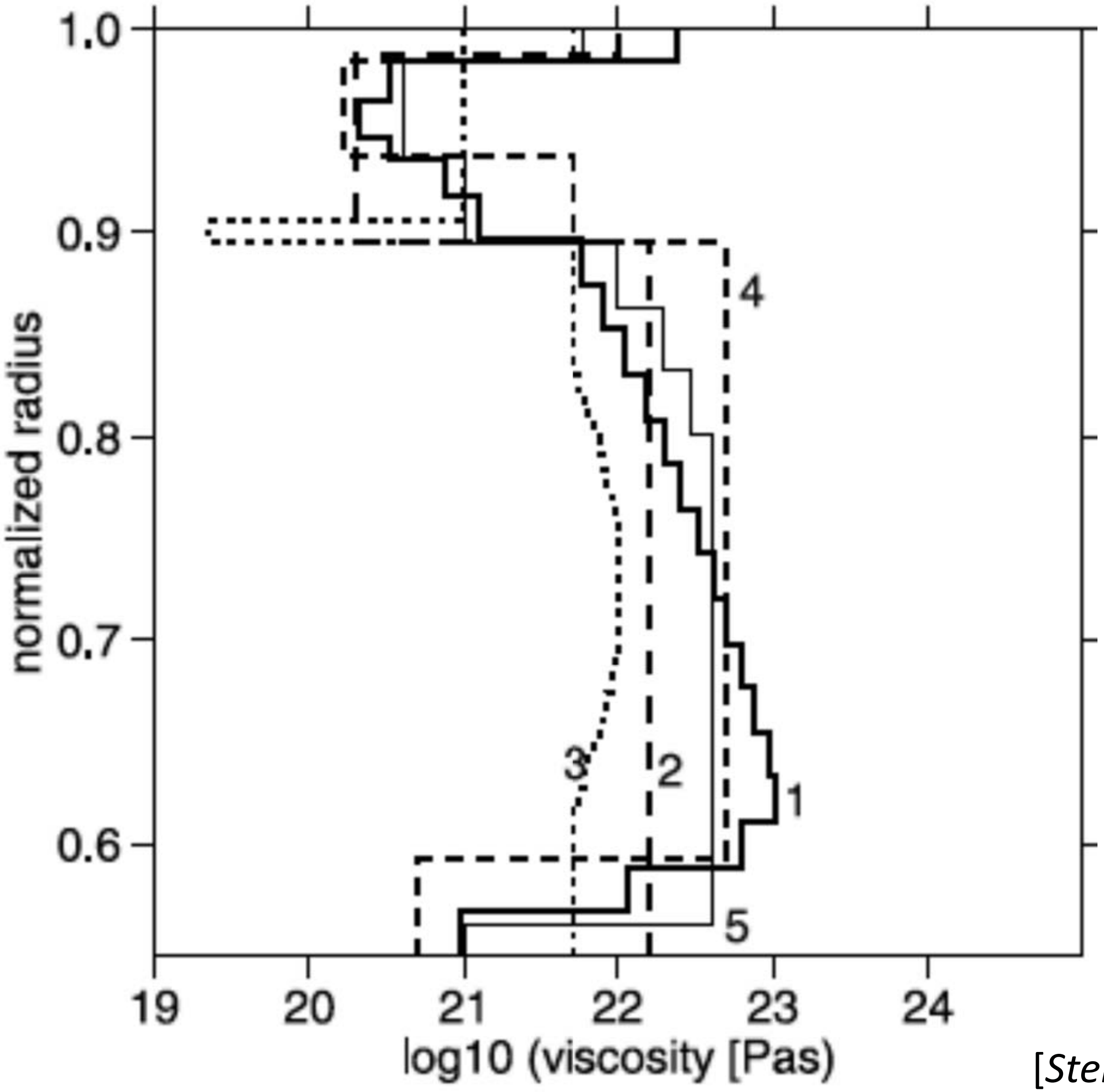
Not a realistic distribution of mass.

Flat earth?

Not so good for a mass anomaly at 2000 km depth.

Constant viscosity in the mantle?

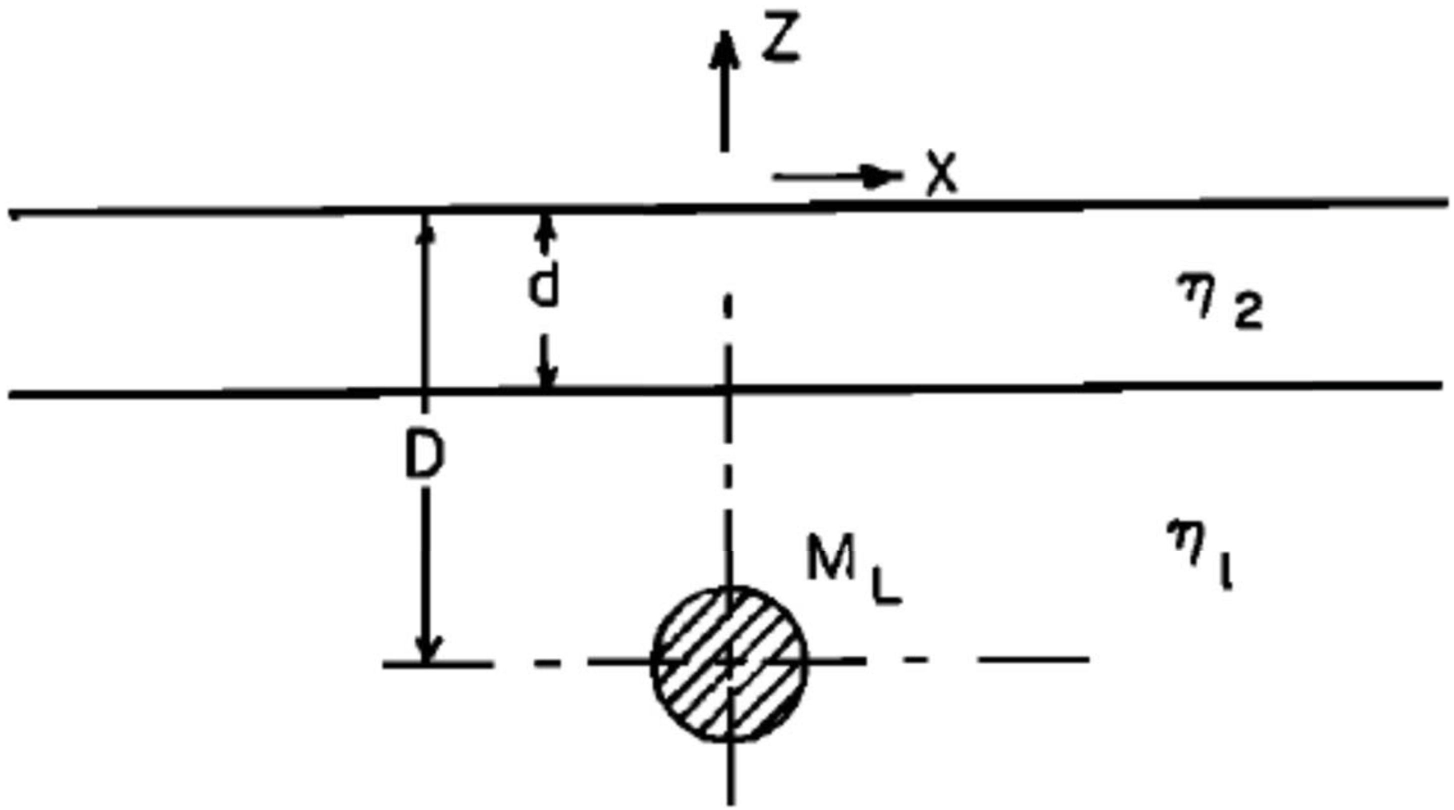
Patently false!



Viscosity Structure of the earth

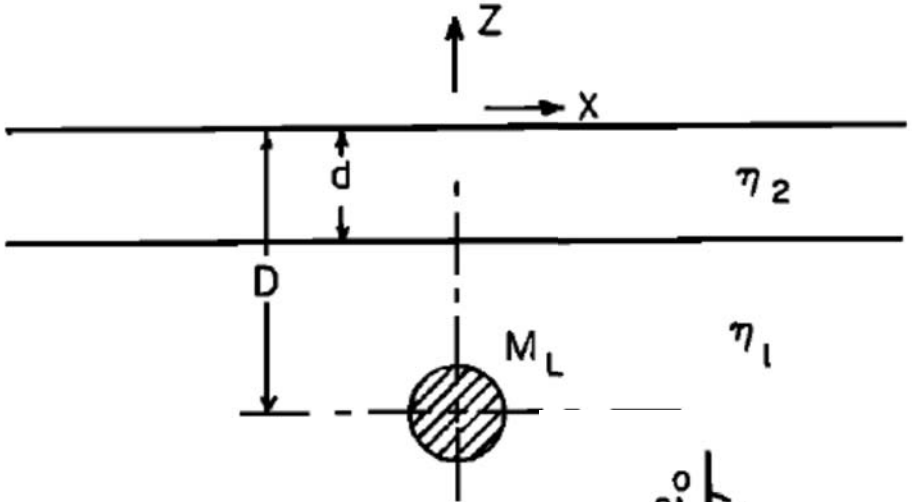
[*Steinberger and Holme* 2002]

Two-layered structure (*Morgan* 1965)



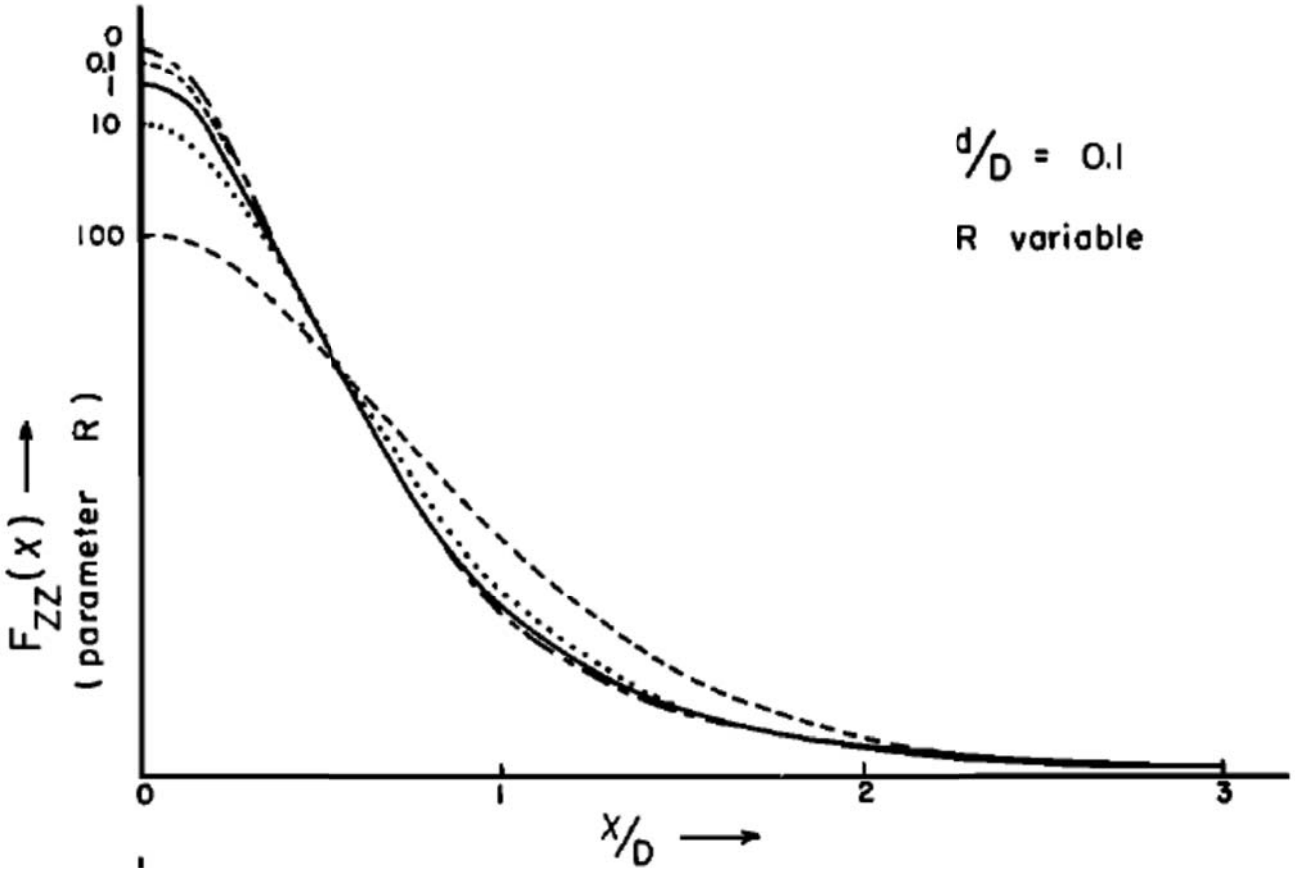
This must be solved numerically.

Effect of viscosity ratio

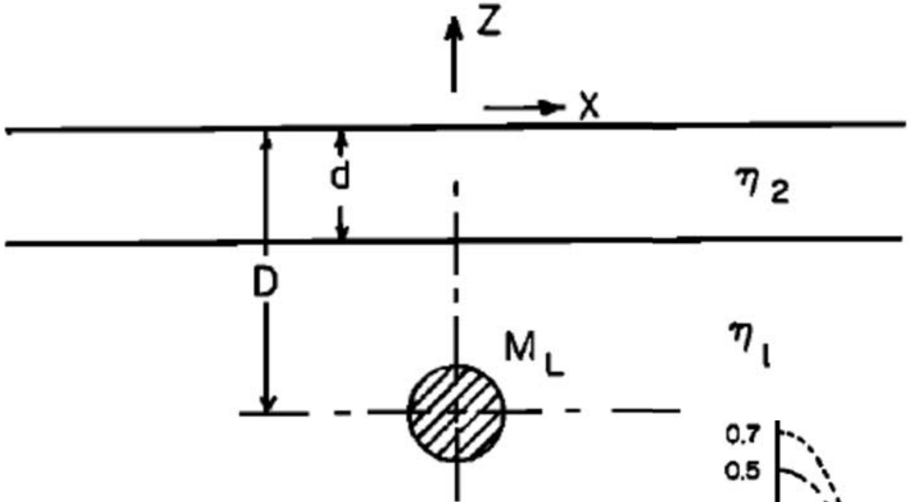


$$R = \frac{\eta_1}{\eta_2}$$

$$= \frac{\eta_{\text{Lower layer}}}{\eta_{\text{Upper layer}}}$$

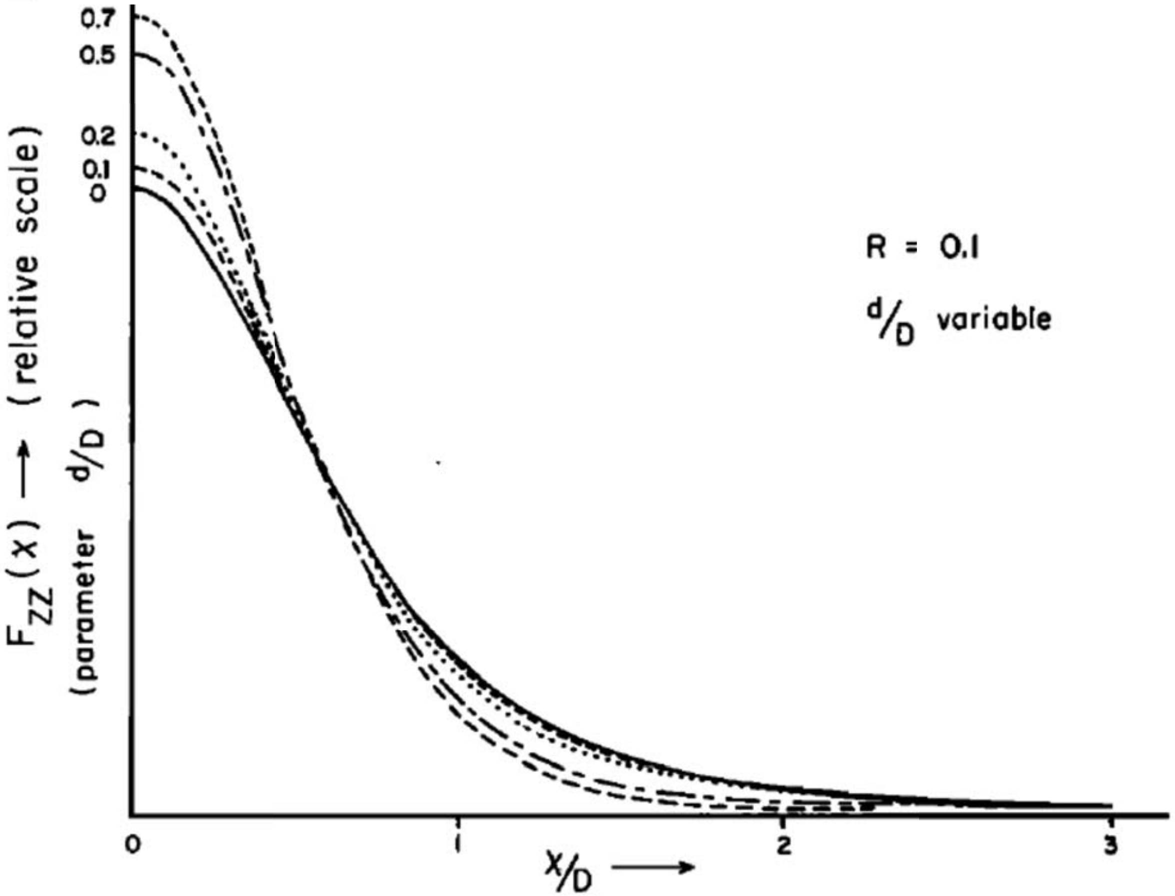


Effect of layer thickness

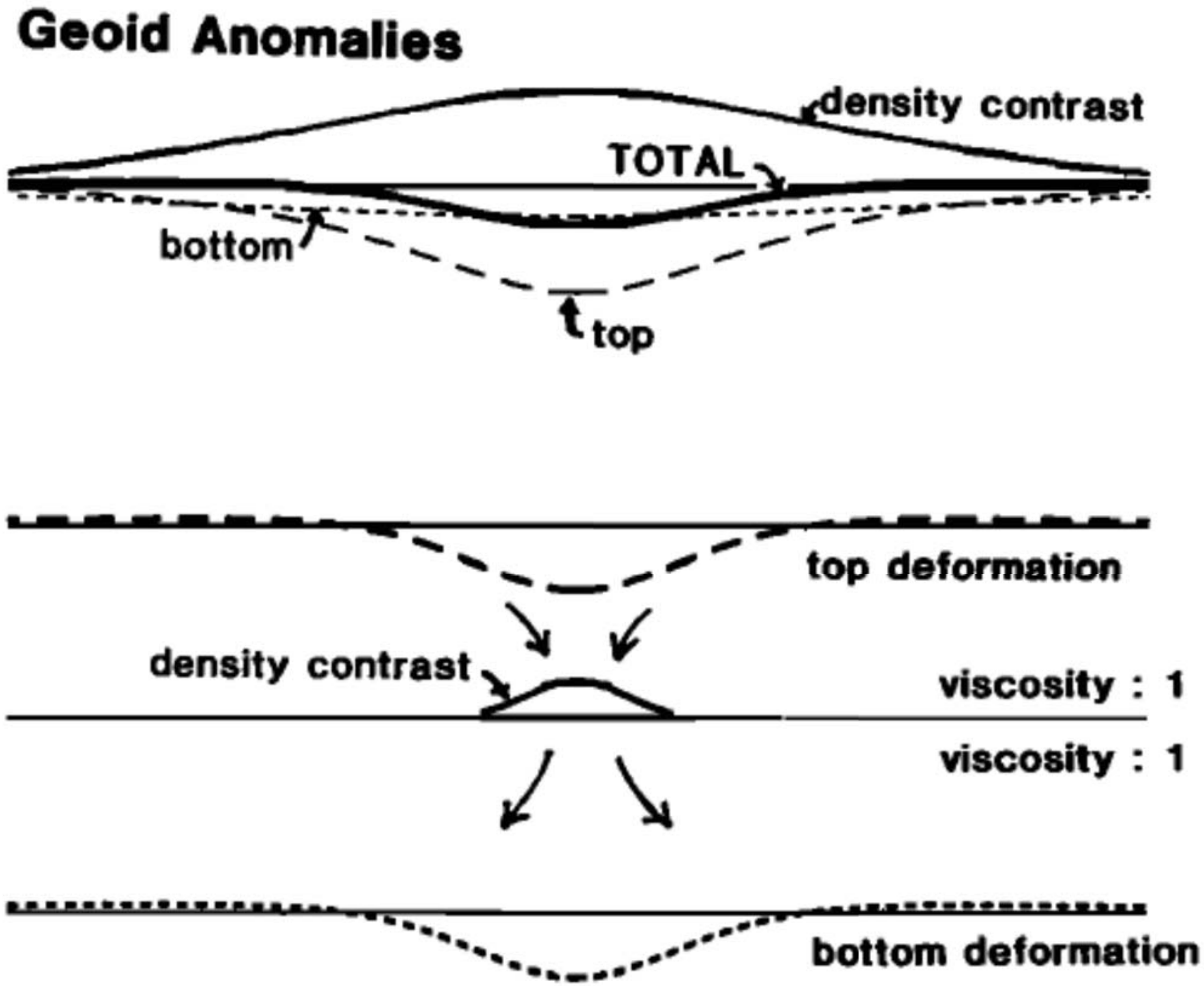


$$R = \frac{\eta_1}{\eta_2}$$

$$= \frac{\eta_{\text{Lower layer}}}{\eta_{\text{Upper layer}}}$$

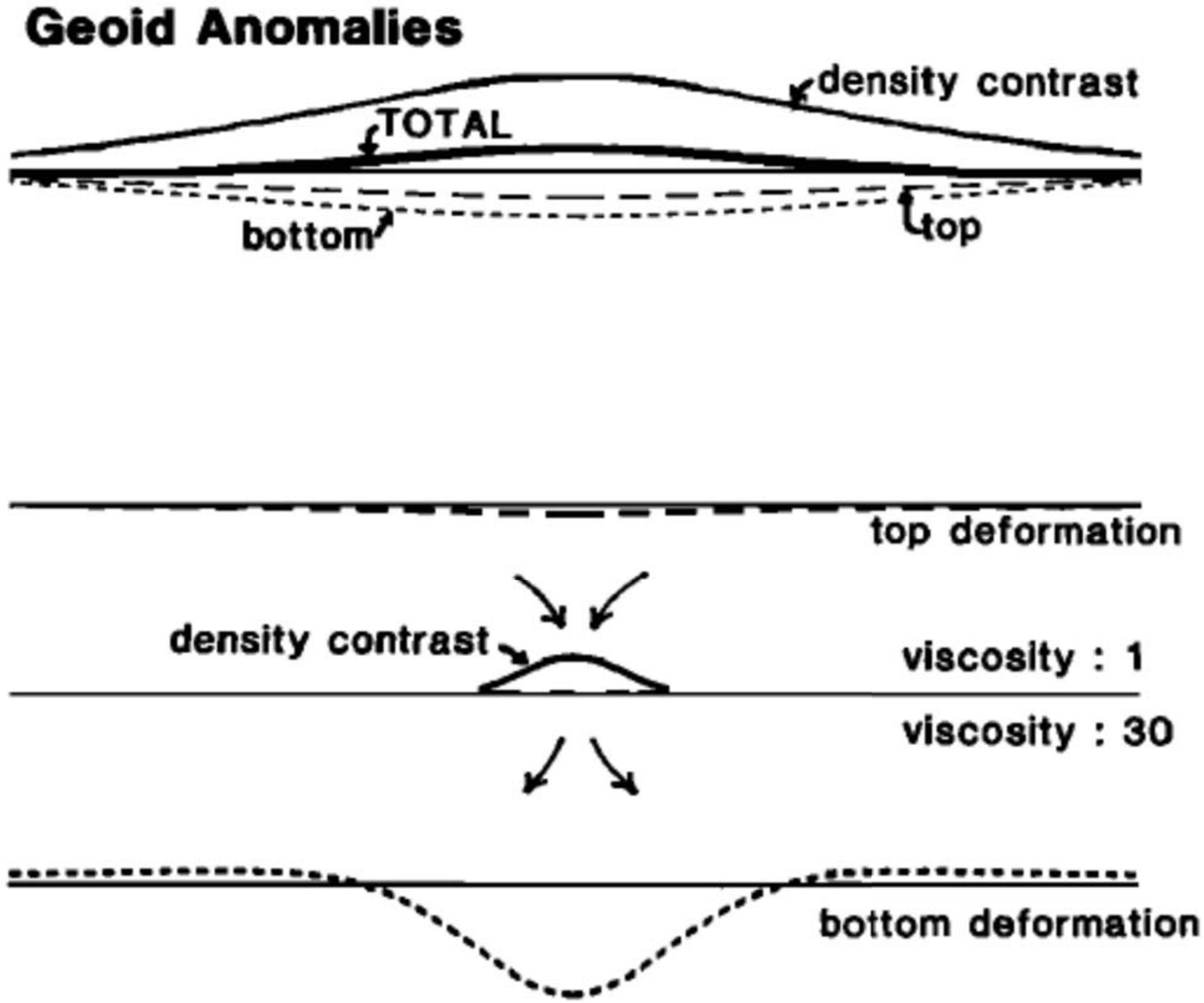


Dynamic topography in a spherical earth (no layering)



[Hager 1984]

Dynamic topography in a layered spherical earth



[Hager 1984]

Summary

Dynamic topography must exist in a dynamic earth.

Dynamic topography is a fashionable line of pursuit today.

Hager and Richards's work, exploiting the geoid, demonstrated that viscosity in the mantle must increase with depth, and that convection must penetrate the transition zone in the upper mantle.

Most workers have subsequently exploited the geoid, not gravity anomalies, to test ideas of dynamic topography.

Gravity anomalies place limits on the magnitude of dynamic topography. (*I think most overestimate it*.)