The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2240-3

Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

23 May - 3 June, 2011

Derivation of the basic heat transfer equation and heating or cooling by diffusion

M. Manga
Univ. of California
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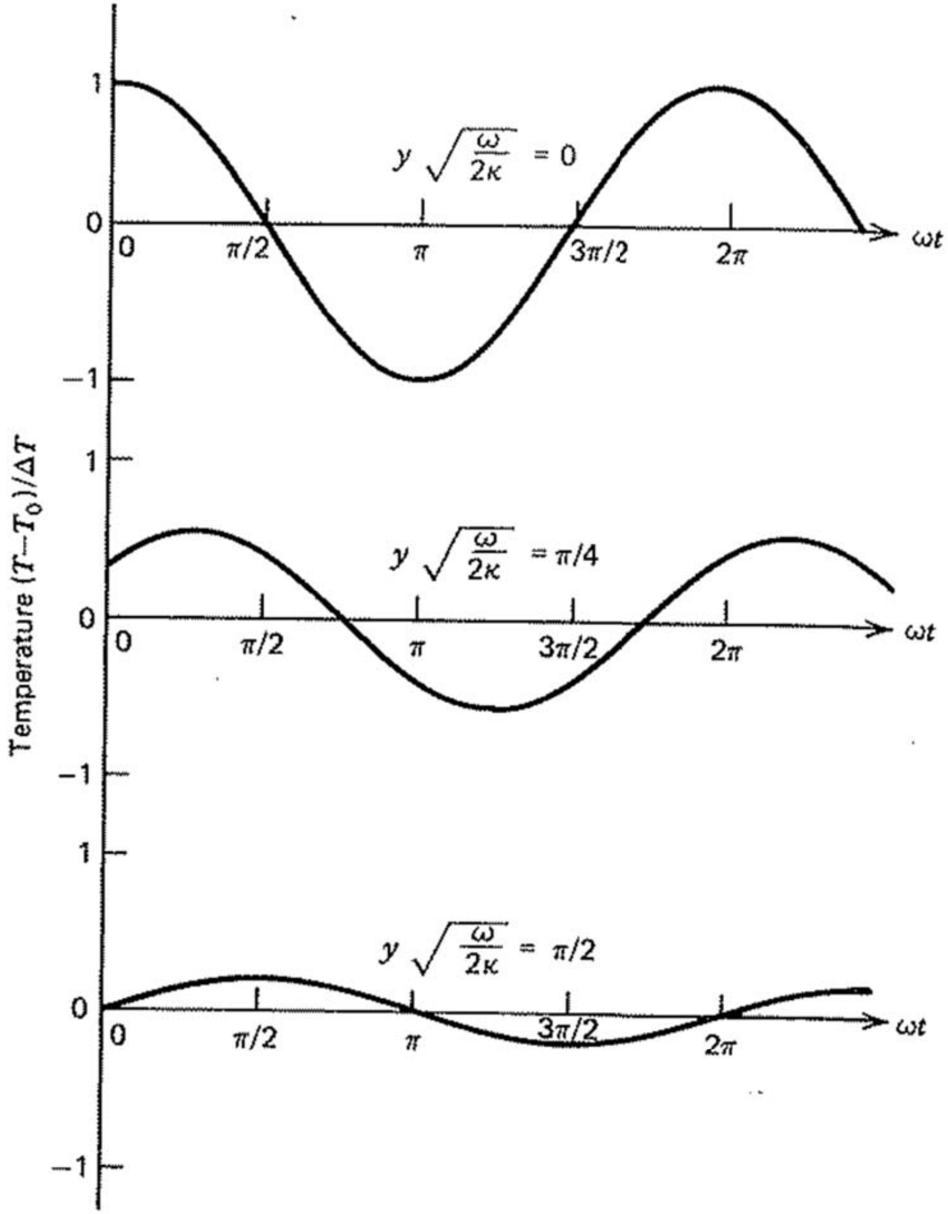
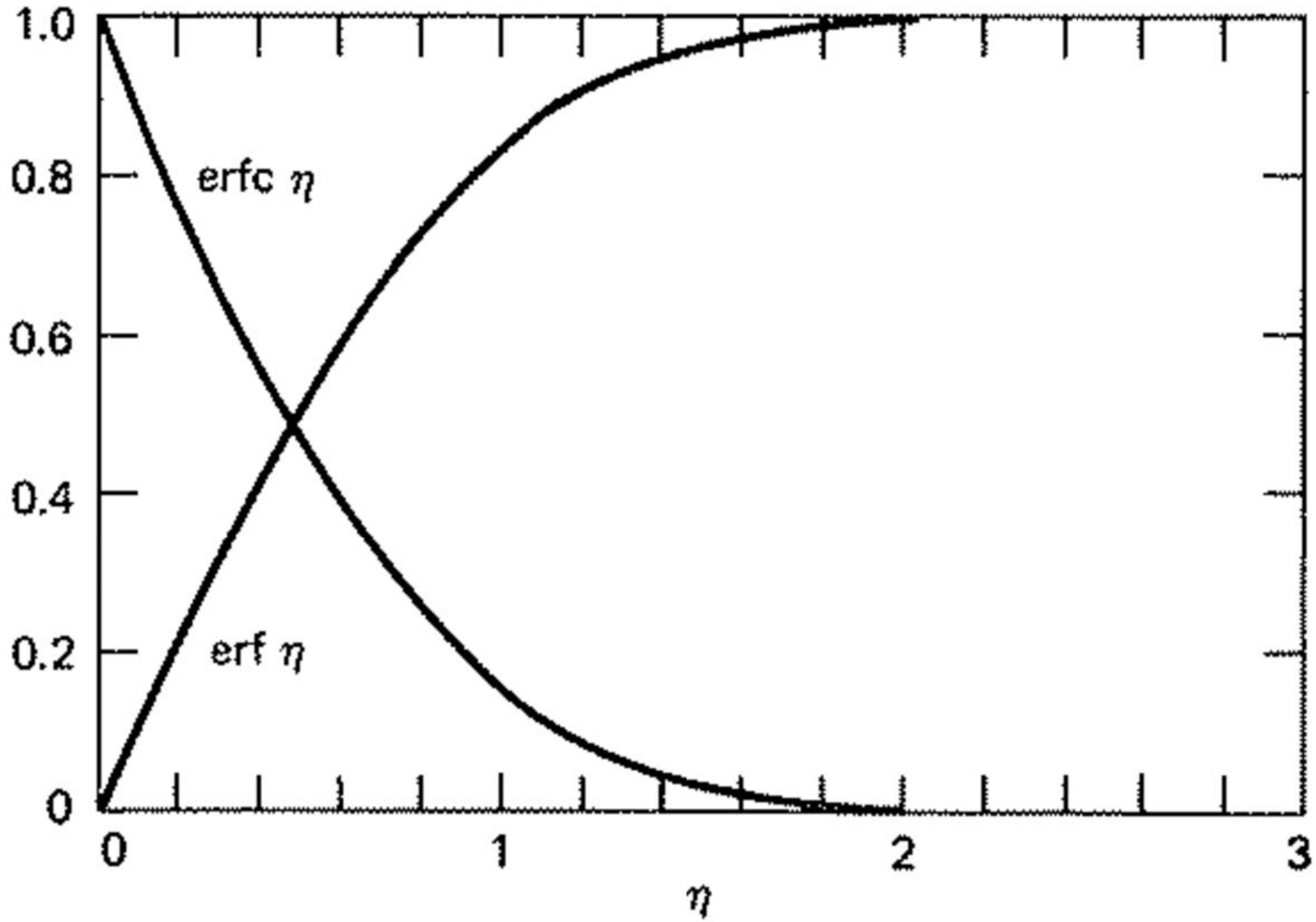
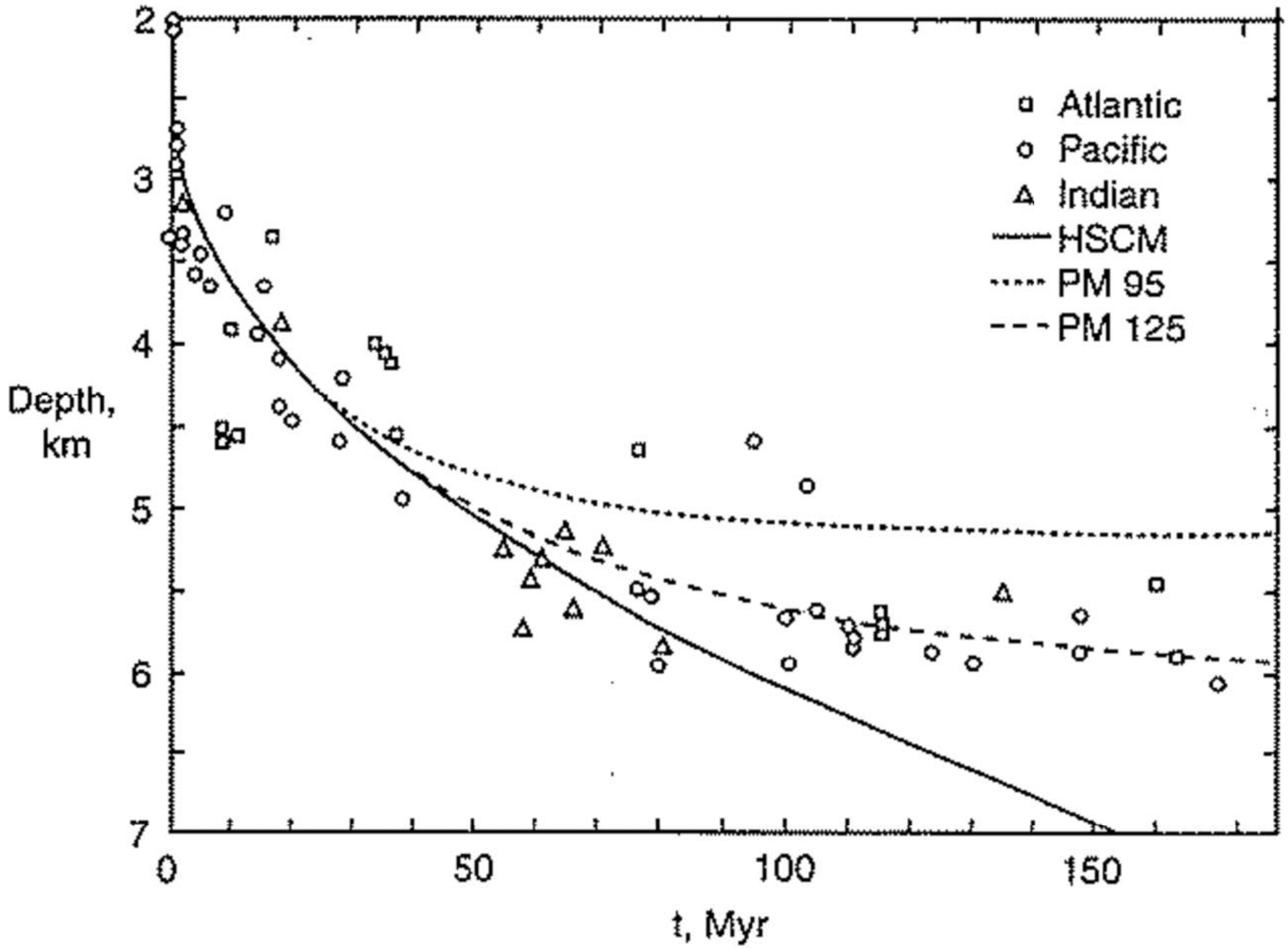
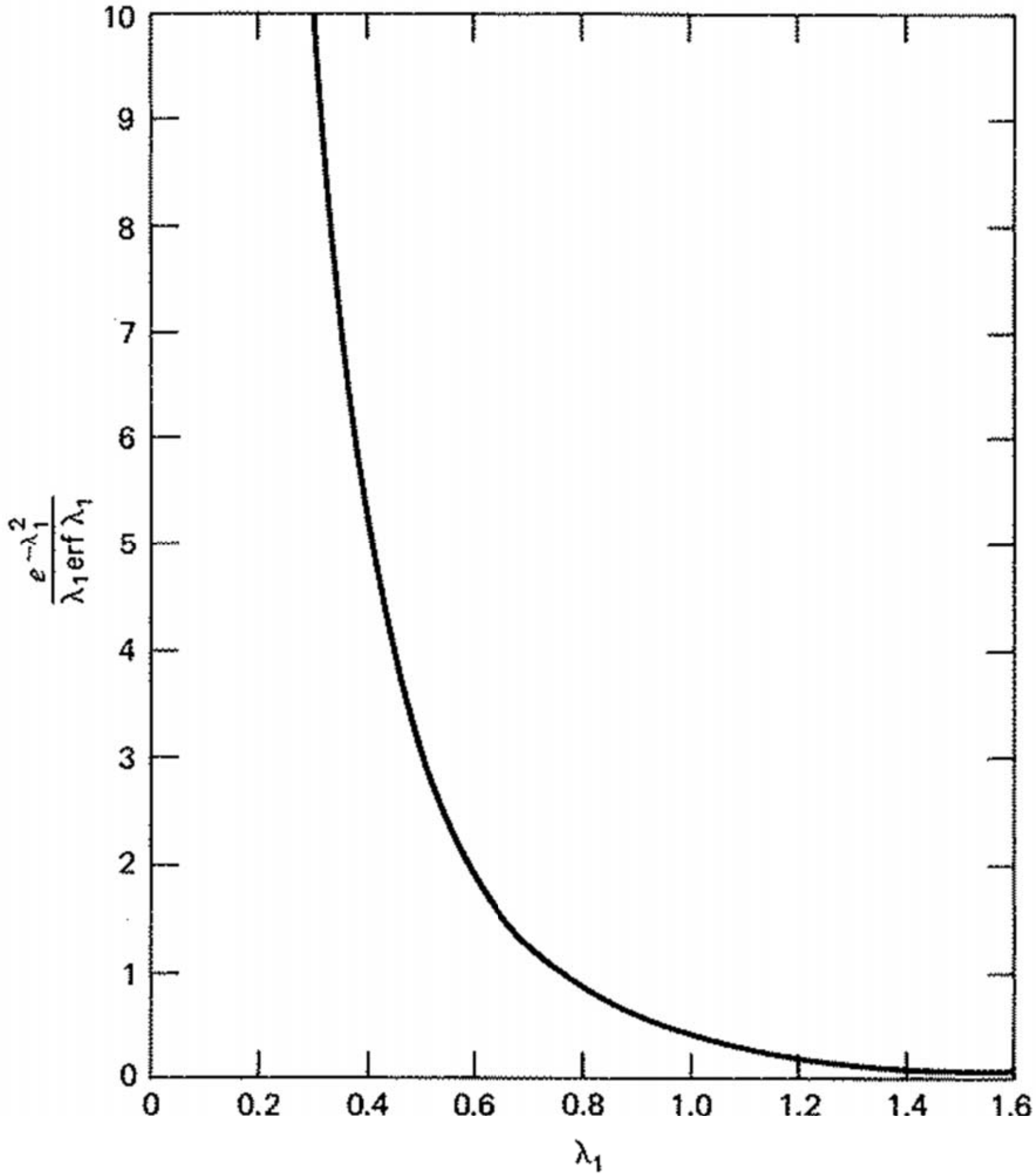


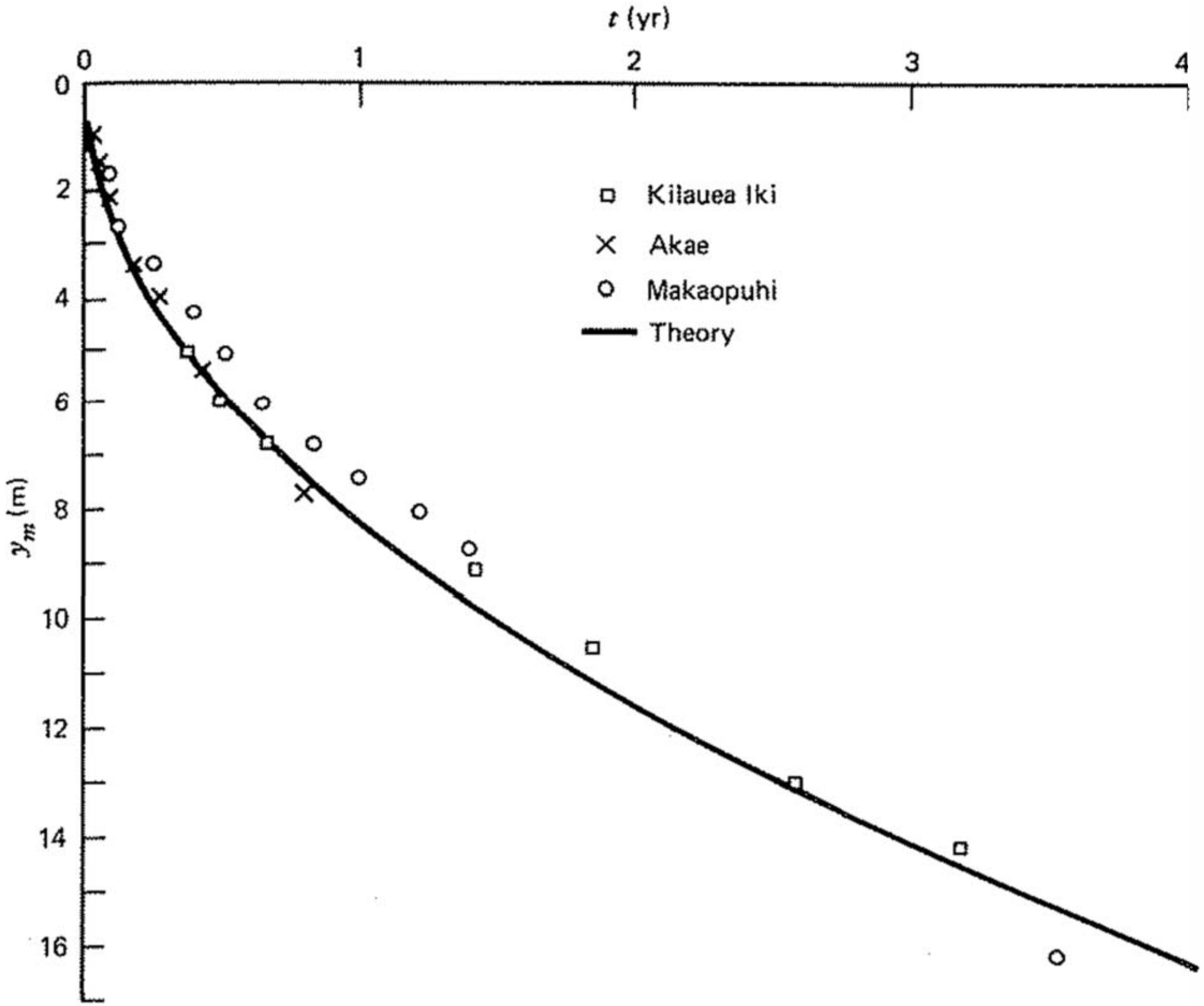
4–19 Phase shift and amplitude decay with depth of a time-periodic surface temperature variation.


1.0
0.8
0.6
0.4
0.2
0
erfc η
erf η
0
1
2
3
η


Atlantic
Pacific
Indian
HSCM
PM 95
PM 125
Depth, km
t, Myr
2
3
4
5
6
7
0
50
100
150

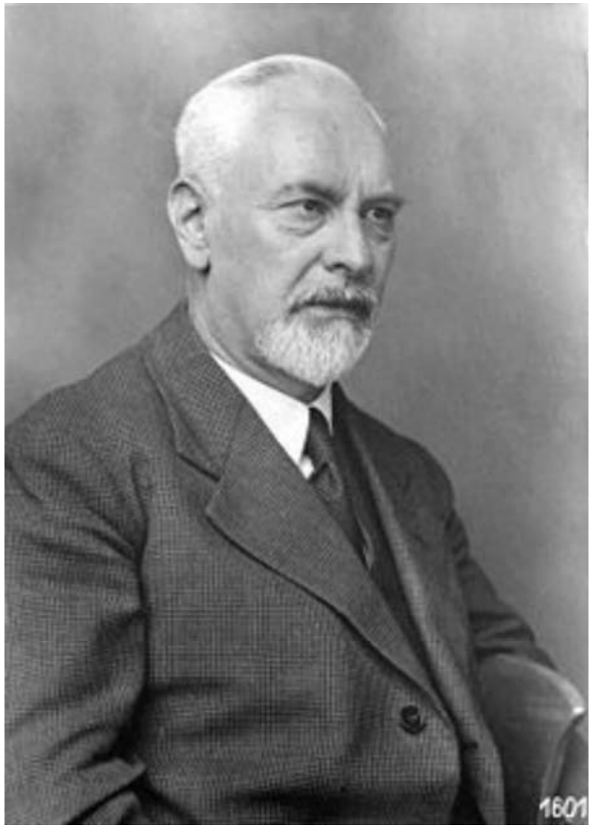


10
9
8
7
6
5
4
3
2
1
0
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
λ_1
$\frac{e^{-\lambda_1^2}}{\lambda_1 \operatorname{erf} \lambda_1}$



4–33 The thicknesses of the solidifying crusts on the lava lakes in the three pit craters Kilauea Iki (1959), Alae (1963), and Makaopuhi (1965) on the volcano Kilauea, Hawaii (Wright et al. 1976). The theoretical curve is from Equations (4–136) and (4–141).

Prandtl (1905)



On the motion of fluids with very little friction

"Uber Flussigkeitsbewegung bei sehr keliner reibung"

Proceedings on the Third International Mathematics Congress

Euler's equations (1755)

- True basis of continuum mechanics was Newton's second law applied to infinitessimally small elements in a fluid

- Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- Conservation of momentum

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + \nabla p = 0$$

- No viscosity (no internal friction)

D’Alembert’s paradox

“Thus I do not see, I admit, how one can satisfactorily explain by theory the resistance of fluids. On the contrary, it seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance: a singular paradox which I leave to future geometers for elucidation”.

D’Alembert’s memoirs (1768)

Navier-Stokes equations

- Conservation of mass
  (incompressible, that is, constant density)

$$\nabla \cdot \mathbf{v} = 0$$

- Conservation of momentum

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\substack{\text{Unsteady}\\ \text{acceleration}}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\substack{\text{Convective}\\ \text{acceleration}}}\Big)}^{\text{Inertia}} = \underbrace{-\nabla p}_{\substack{\text{Pressure}\\ \text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}} + \underbrace{\mathbf{f}}_{\substack{\text{Other}\\ \text{forces}}}$$

As far as I can see, there is today no reason not to regard the hydrodynamic equations of Navier and Stokes as the exact expression of the laws that rule the motion of real fluids.
Helmholtz (1873)

Momentum (velocity) boundary layer

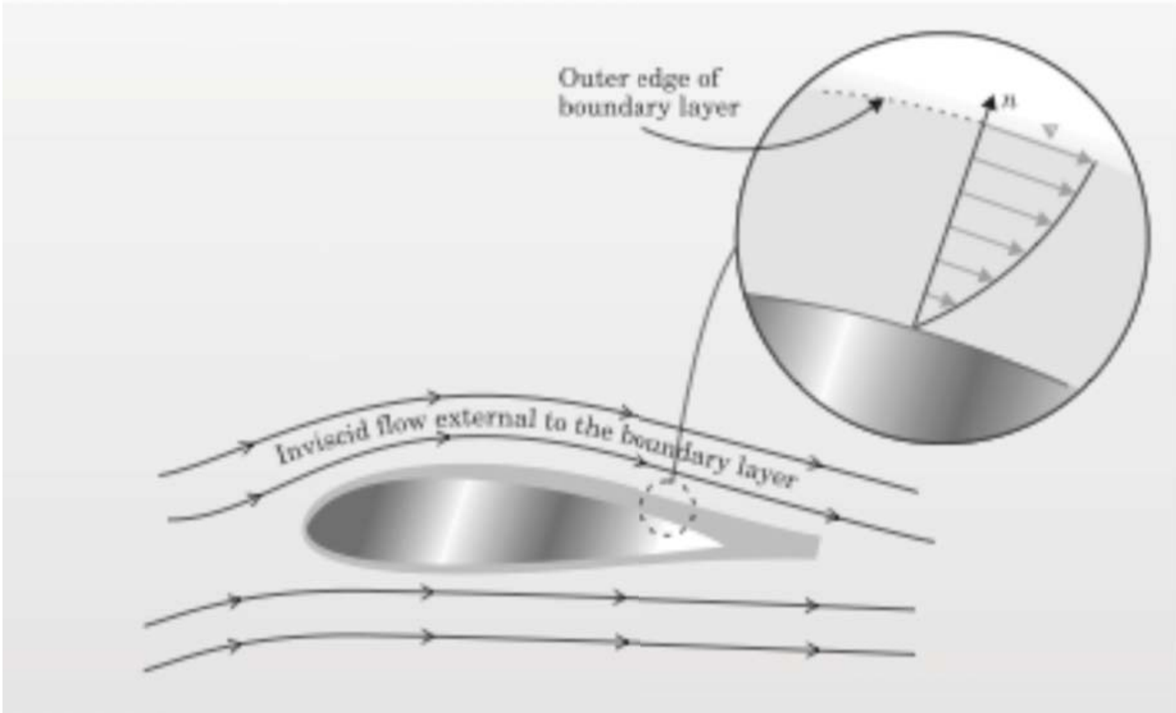


Figure 2. A fluid flow may be viewed as comprising two parts. In a thin boundary layer (blue) adjacent to the surface, the effects of friction are dominant. Outside the boundary layer, the flow is inviscid. The blowup of the boundary layer shows how the flow velocity **v** changes, as a function of the normal distance *n*, from zero at the surface to the full inviscid-flow value at the outer edge.

From Anderson, Physics Today (2005)

Prandtl (1905): boundary layers

“A very satisfactory explanation of the physical process in the boundary layer (grenzschicht) between a fluid and a solid body could be obtained by the hypothesis of an adhesion of the fluid to the walls, that is, by the hypothesis of a zero relative velocity between fluid and wall. If the viscosity was very small and the fluid path along the wall not too long, the fluid velocity ought to resume its normal value at a very short distance from the wall. In the thin *transition layer* (ubergangsschist) however, the sharp changes of velocity, even with small coefficient of friction, produce marked results.

translated in Ackroyd et al. 2001

Within boundary layer, balance inertia and viscous stresses

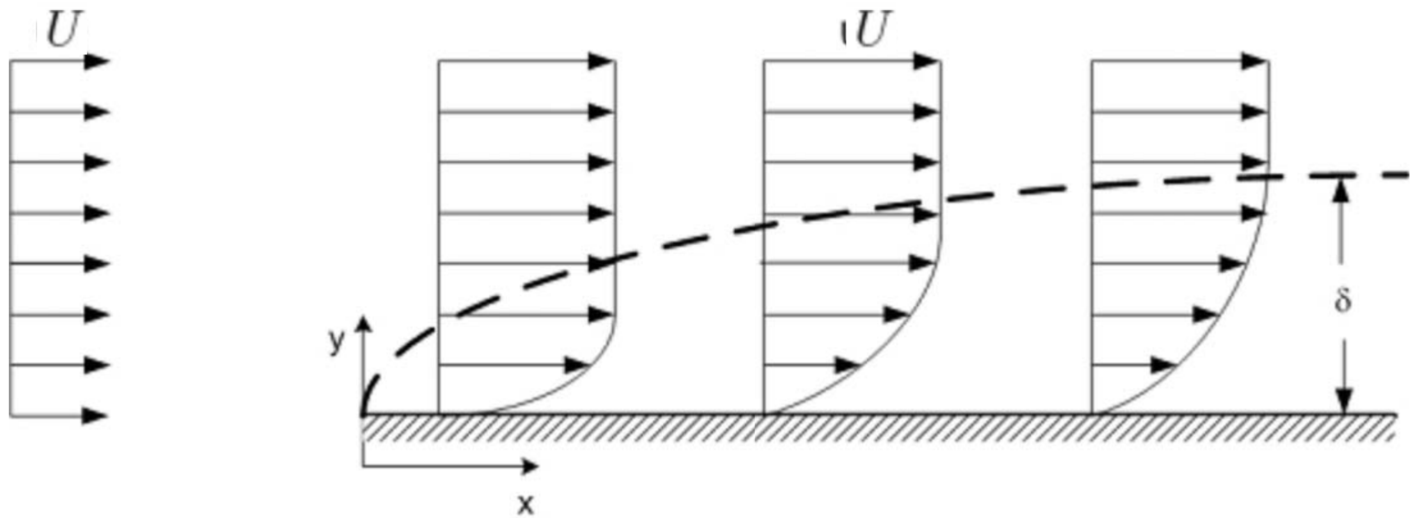
$$\frac{U^2}{L} \sim \nu \frac{U}{\delta^2} \tag{1}$$

The steady boundary layer equations reduce to

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{2}$$

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = \nu \frac{\partial^2 u_x}{\partial y^2} \tag{3}$$

with boundary conditions $u_x = u_y = 0$ at $y = 0$.



From (1) we expect $\delta(x) \sim \sqrt{\nu x/U}$ and thus introduce a similarity variable (explicitly relating x and y)

$$\eta = \frac{y}{\delta} = y\sqrt{U/\nu x} \tag{4}$$

and expect that

$$u_x = U f(\eta) \tag{5}$$

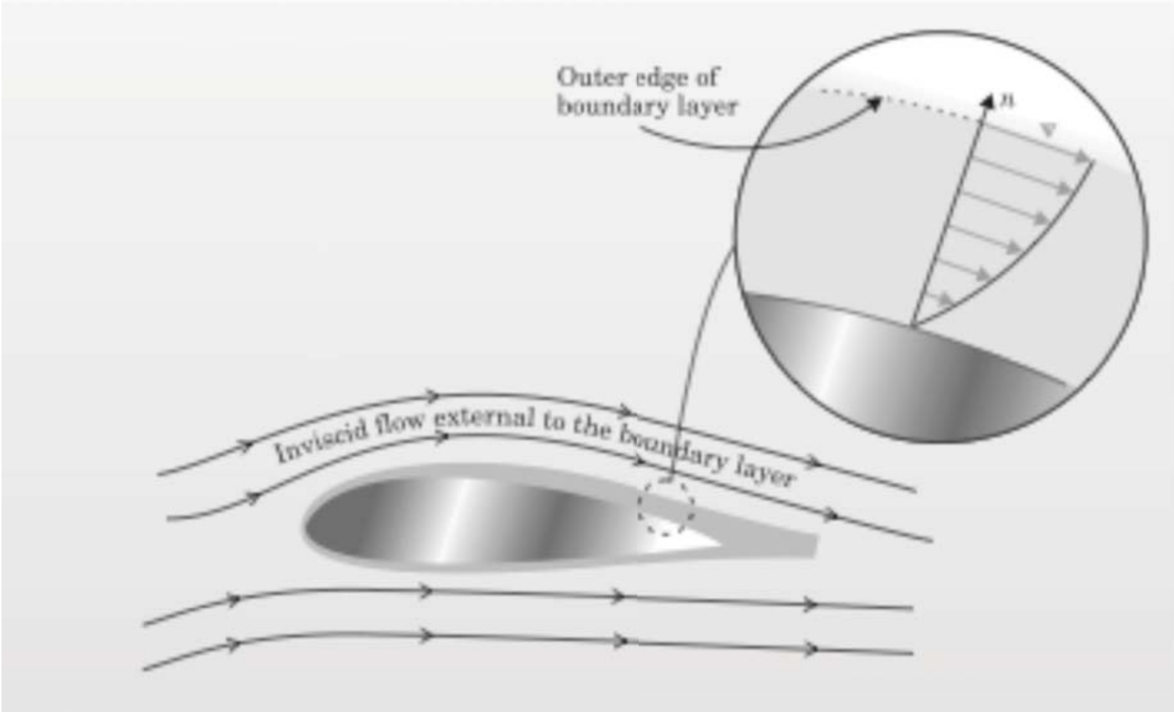
with new boundary conditions $f = 0$ and $\eta = 0$, and $f = f' \rightarrow 1$ as $\eta \rightarrow \infty$.

Replacing (5) into (1)-(2) we obtain a nonlinear ODE

$$f''' + \frac{1}{2} f f'' = 0 \tag{6}$$

$$\text{shear stress} = \mu \frac{\partial u_x}{\partial y} = 0.332\rho\sqrt{\nu U^3/x} \tag{7}$$

Momentum (velocity) boundary layer



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A more general definition

- A transitional area between two distinct regions with different physical properties

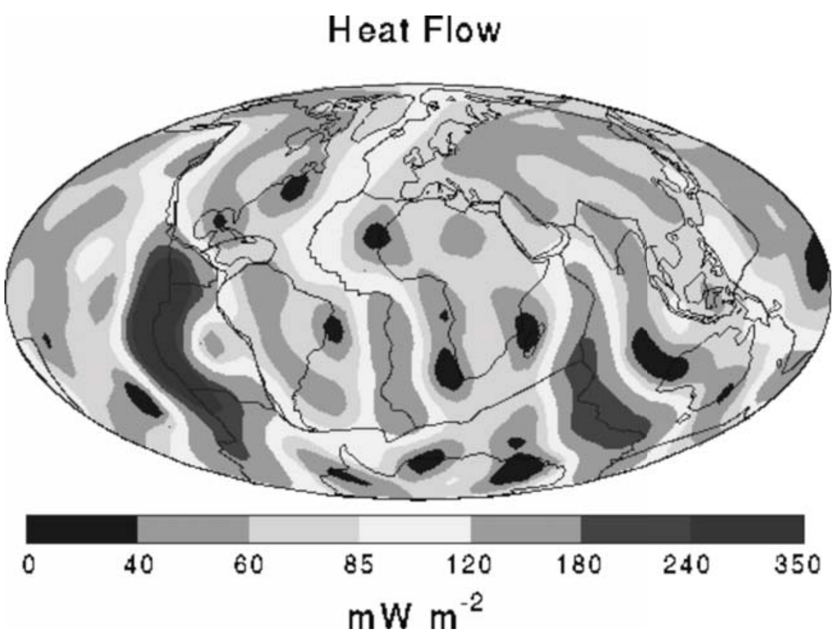
In Earth's mantle . . .

- Thermal boundary layers
- Compositional boundary layers
- Rheological boundary layers
- Electrochemical boundary layer

- No momentum boundary layer

Earth's upper thermal boundary layer: Oceanic plates

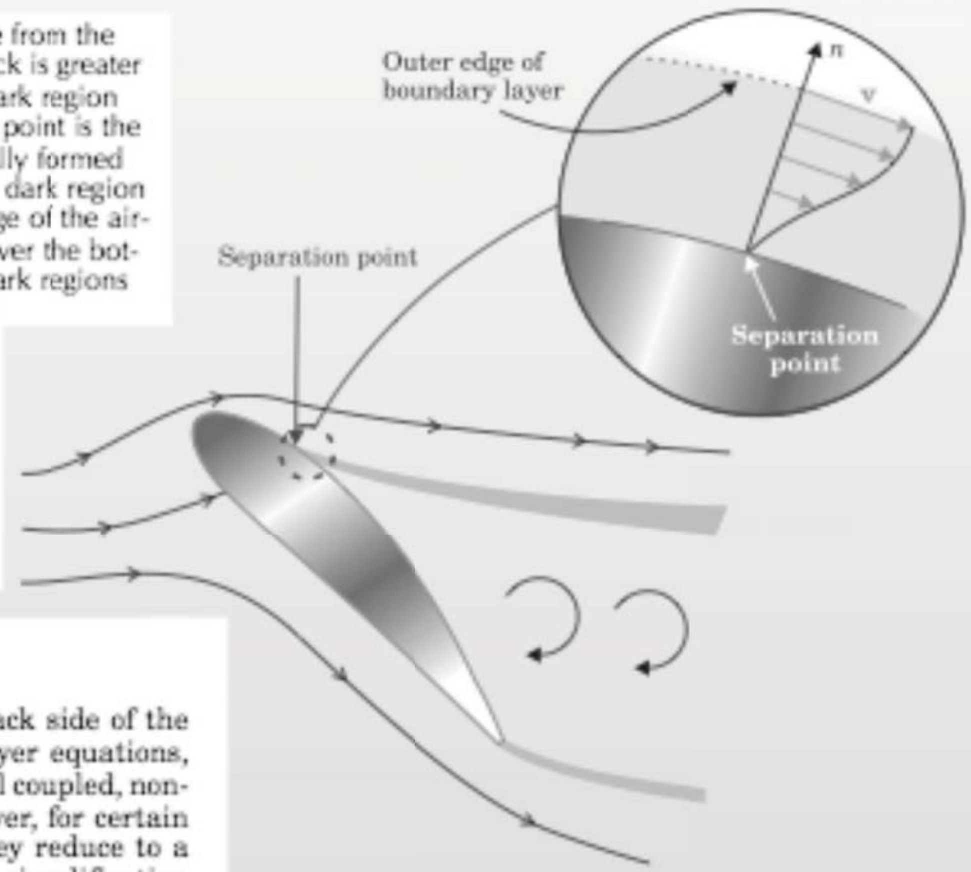
Prediction: depth ~ age$^{1/2}$, heat flow ~ age$^{-1/2}$



Degree 12
Pollack et al., *Rev Geophys* (1993)

Prandtl (1905): Separation

Figure 3. The boundary layer can separate from the top surface of an airfoil if the angle of attack is greater than the so-called stall angle. The upper dark region that trails downstream from the separation point is the remnant of the boundary layer that originally formed on the top surface of the airfoil. The lower dark region that trails downstream from the trailing edge of the airfoil is the remnant of the boundary layer over the bottom surface. When separated, these two dark regions are called shear layers, and they form the upper and lower boundaries of the separated flow region. Between the shear layers is a dead-air region. Due to the considerable flow separation illustrated here, the lift of the airfoil is dramatically reduced—the airfoil is stalled. The blowup shows the flow's velocity profile above the separation point.



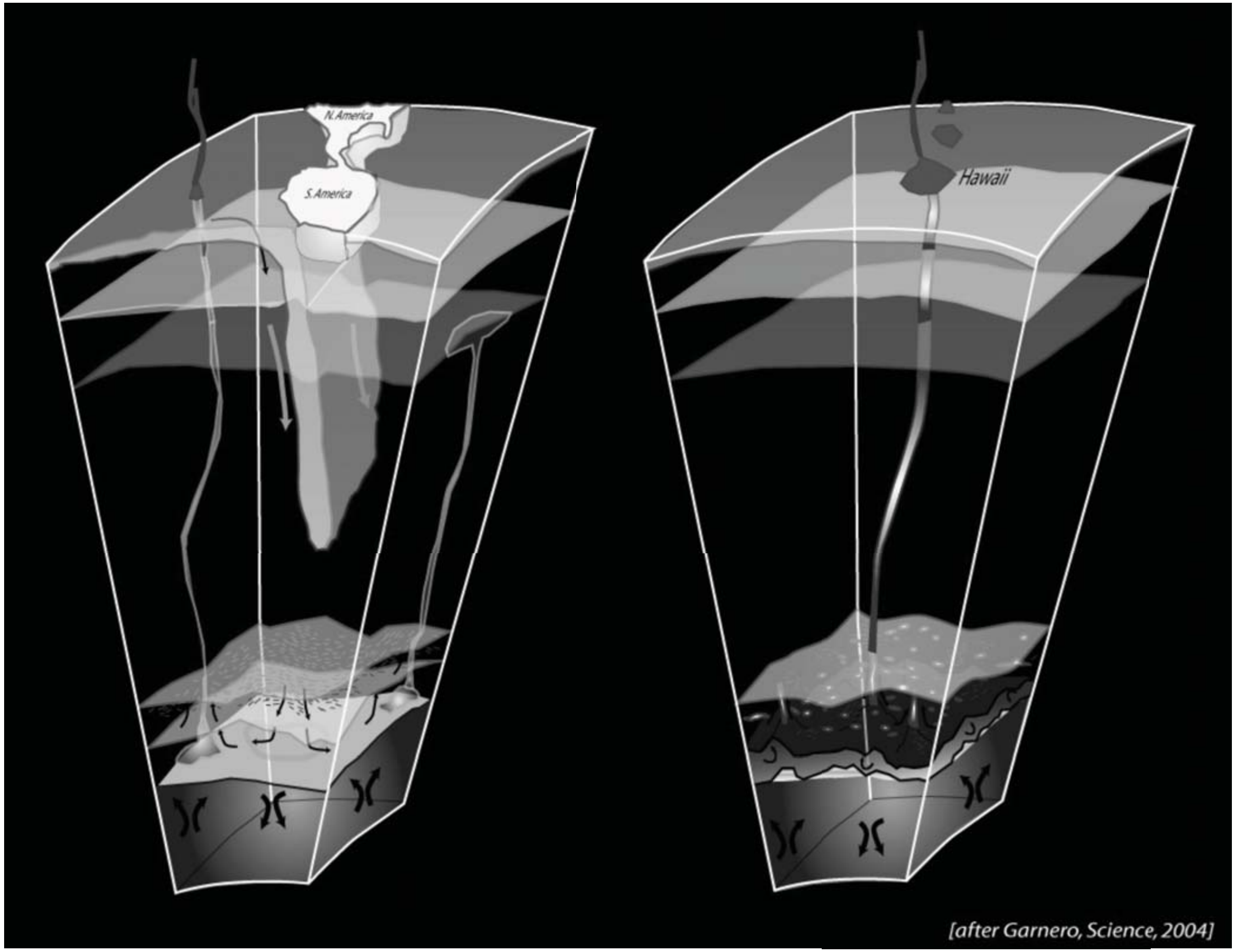
lution gave the separation points on the back side of the cylinder. As noted earlier, the boundary-layer equations, though simpler than Navier–Stokes, are still coupled, nonlinear partial differential equations. However, for certain types of pressure gradients in the flow, they reduce to a single ordinary differential equation. That simplification

Prandtl (1905): separation

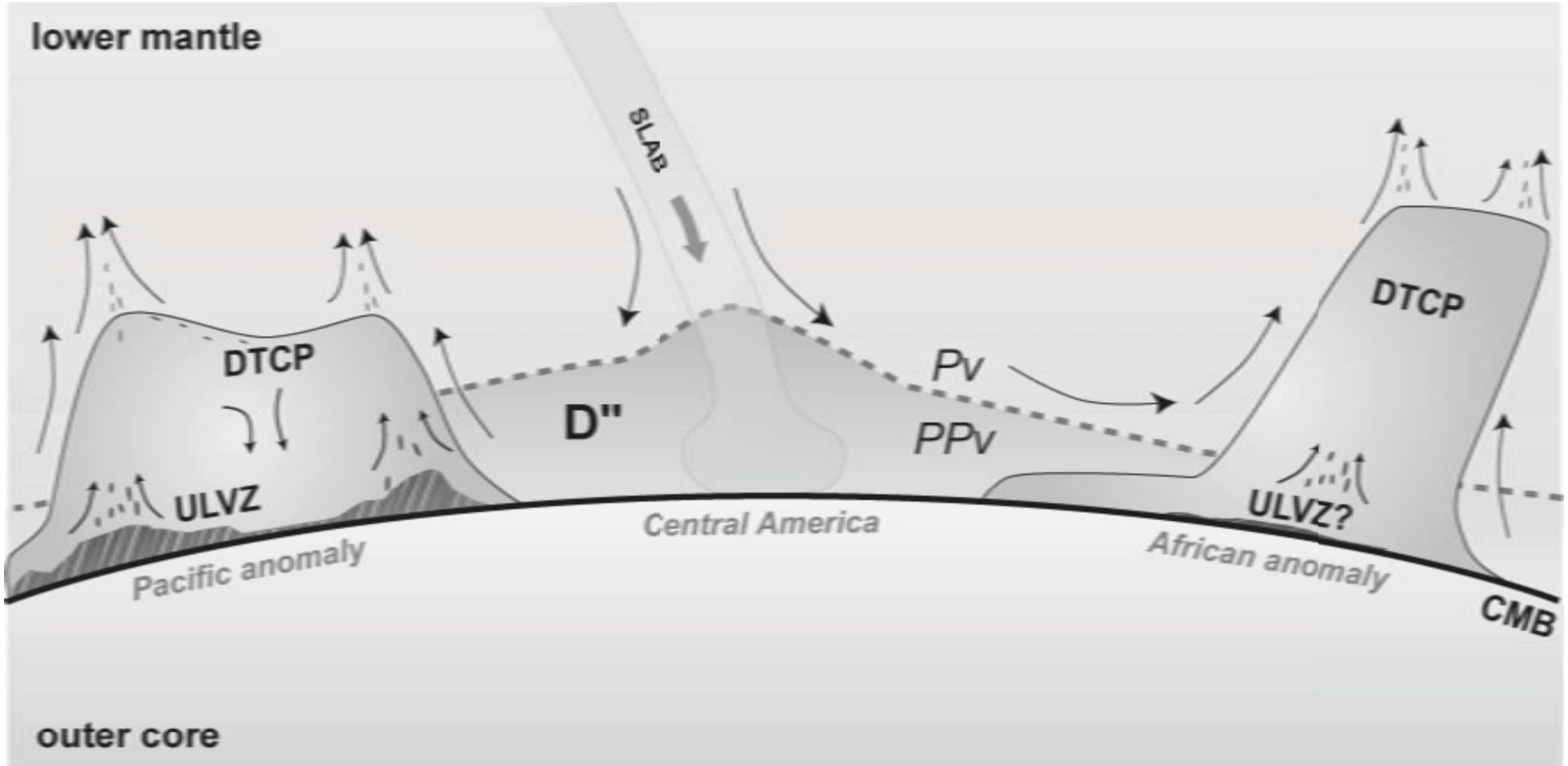
While dealing with a flow, the latter divides into two parts interacting on each other; on one side we have the "free fluid" with is dealt with as if it were frictionless, according to the Helmholtz vortex theorems, and on the other side the transition layers near the solid walls. *The motion of these layers is regulated by the free fluid, but they for their part give to the free motion its characteristic feature by the emmission of vortex sheets.*

translated in Ackroyd et al. 2001

Subduction, plumes are detached boundary layers



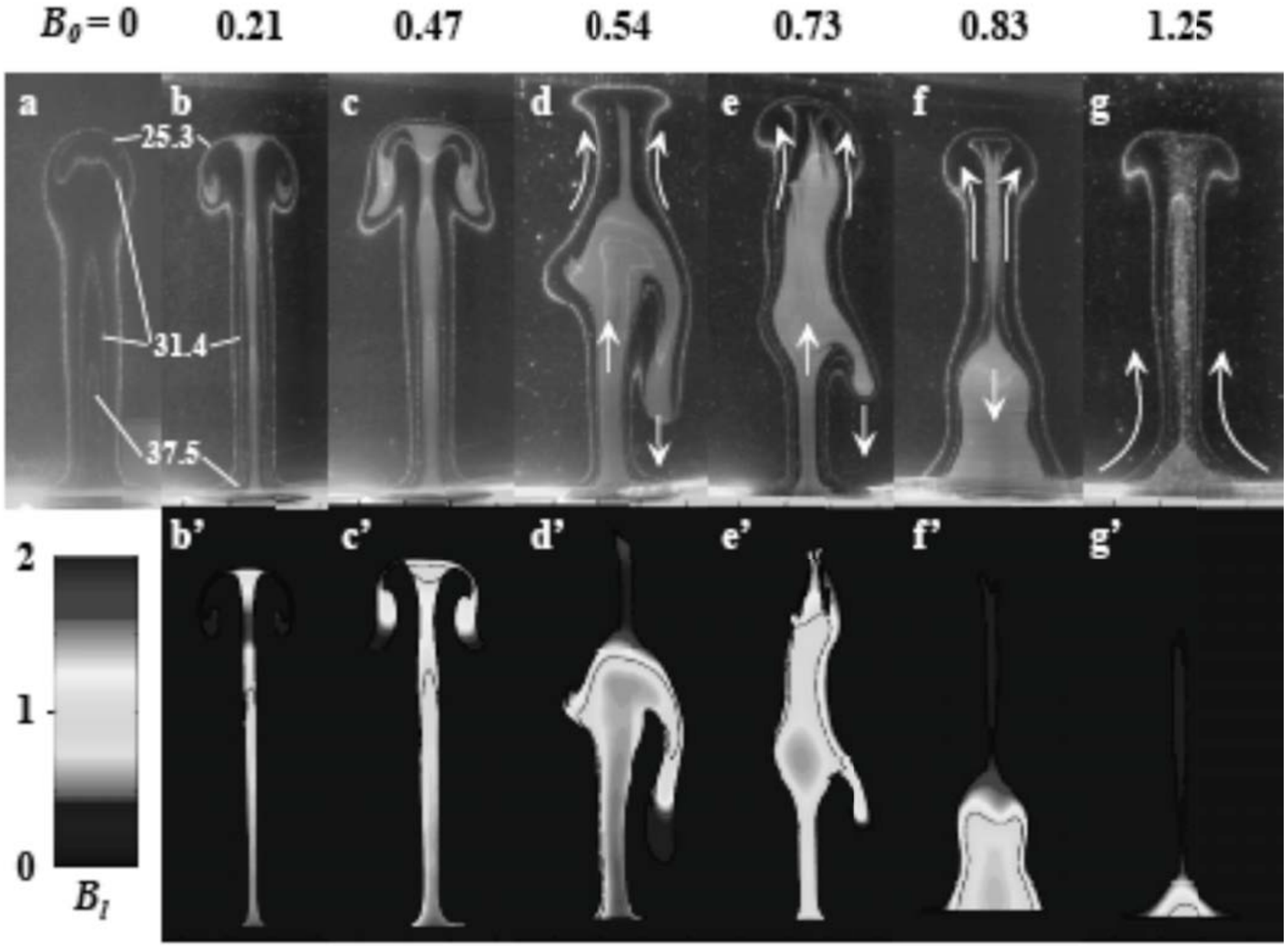
Complications 1:
Boundary layers interact with each other



[Garnero et al. [2006]

BLs:Composition, phase transitions, melting, thermal, rheology

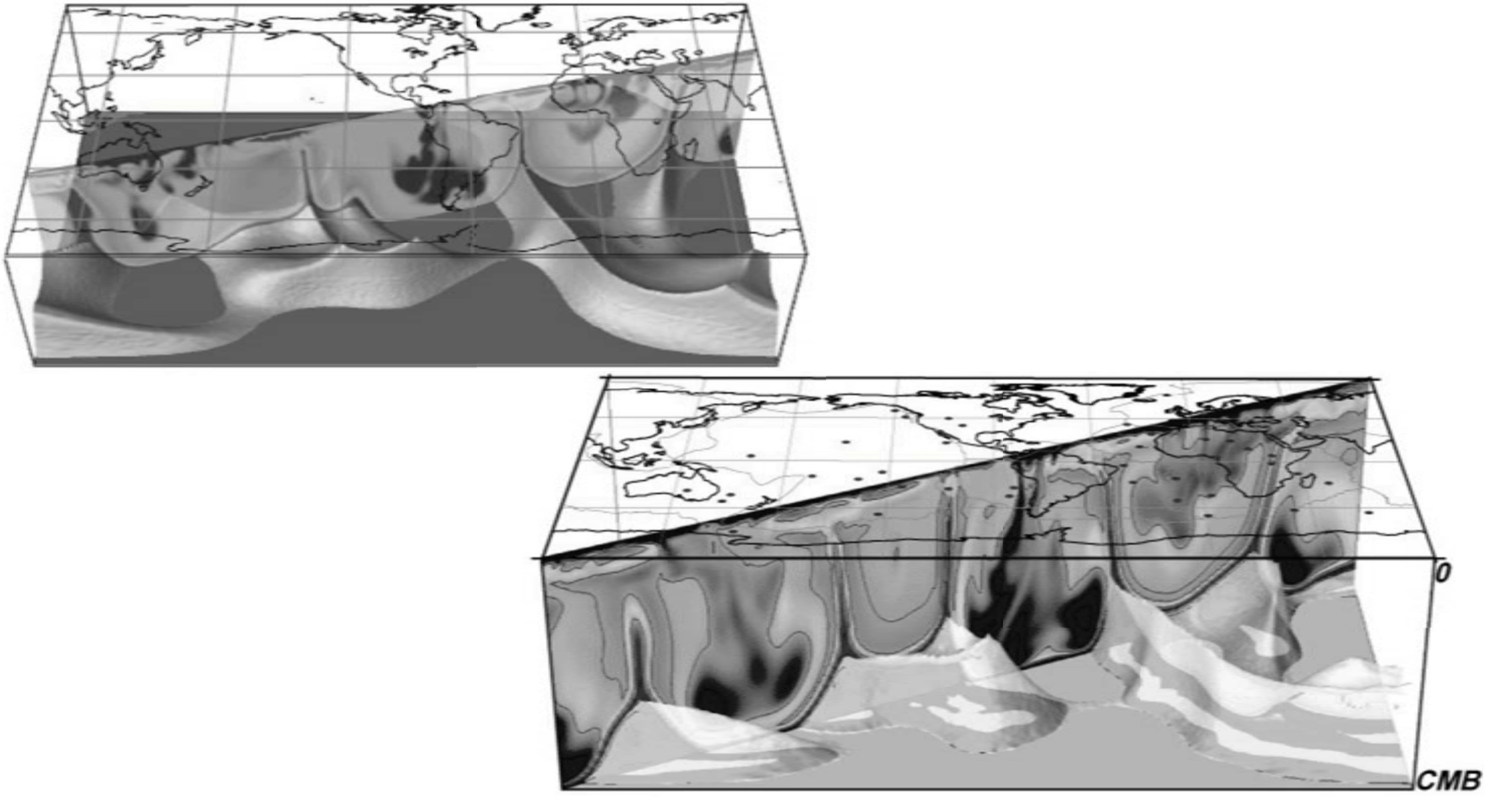
Detached boundary layers



Kumagai et al., *GRL* in review

See also Lin and van Keken, *Nature* 2005

Complication 2:
All boundary layers interact



McNamara and Zhong, *Nature* 2005

More information

- Dryden (1955) Fifty years of boundary-layer theory and experiment, *Science*
- Anderson, J.D. (2005) Ludwig Prandtl's boundary layer, *Physics Today*
- Darrigol, O. (2005) Worlds of Fluid, a history of hydrodynamics from the Bernoullis to Prandtl, Oxford Univ Press

Summary

Definition: A transitional area between two distinct regions with different physical properties

- Thermal boundary layers
- Compositional boundary layers
- Rheological boundary layers
- Electrochemical boundary layer
- ALL interact