The Abdus Salam
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2240-5

Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Examples of a cooling lithosphere: both cooling plate and cooling half-space, comparison of different boundary conditions (fixed temperature and fixed heat flux).

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Heat tranfer

1) Derivation of equation
2) Features of solution
3) Space-time relationships
4) Effect of phase transformations

Fourier's law

units W/km

$$q = -k \frac{dT}{dy}$$

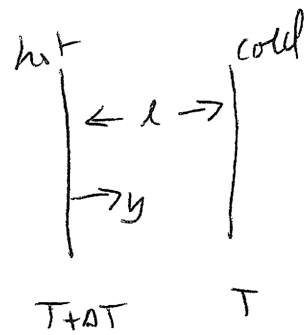
why sign?

units J/s (W)

$$= k \frac{\Delta T}{l}$$

in 3D $\underline{q} = -\underline{\underline{k}} \cdot \nabla T$

need not be a scalar



typical numbers

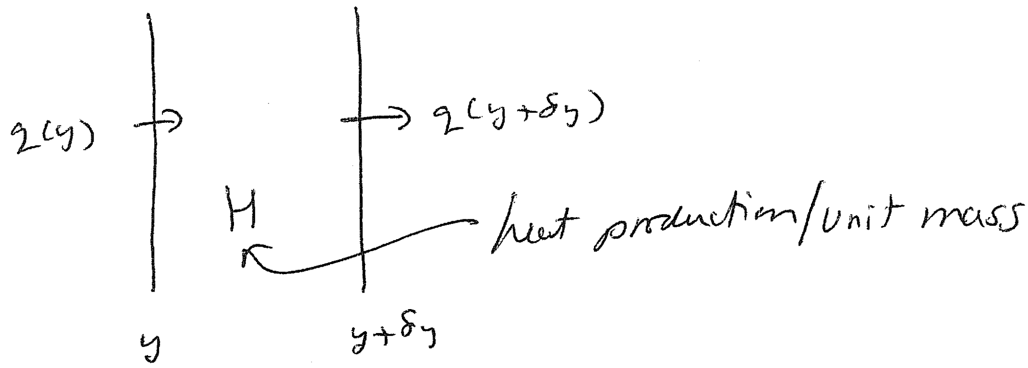
prob1

$dT/dy \sim 20-30$ K/km

$k \sim 2-3$ W/mK

$\Rightarrow q \sim 40-90$ mW/m^2

1D heat conduction equation



$$q(y+\delta y) - q(y) = \rho H \, \delta y$$

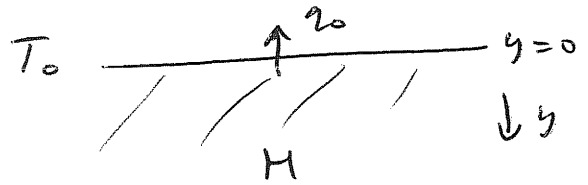
$$\underbrace{\frac{q(y+\delta y) - q(y)}{\delta y}}_{\partial q/\partial y} = \rho H$$

recall $q = -k \, dT/dy$

$$\frac{d}{dy}\left(k \frac{dT}{dy}\right) + \rho H = 0$$

if $H = 0$ $\quad d^2T/dy^2 = 0$

in 3D $\quad k\nabla^2 T + \rho H = 0$



$T(y) = ?$

$$k \frac{d^2T}{dy} = -\rho H$$

integrate $\quad k\dfrac{dT}{dy} = -\rho H y + C_1$

at $y=0$, $C_1 = -q_0$

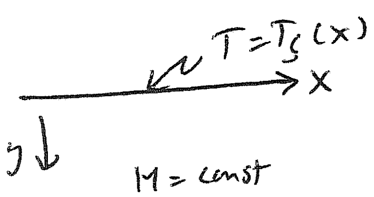
integrate again

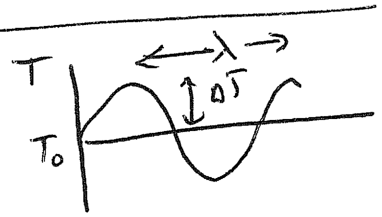
$$\frac{\rho H}{2} y^2 = -kT + q_0 y + C_2$$

at $y = y_0$ $\quad T = T_0$ $\quad$ so $\quad C_2 = kT_0$

$$T = T_0 + \frac{q_0}{k} y - \frac{\rho H}{2k} y^2$$

<u>periodic variation in surface T</u>





$T_S = T_0 + \Delta T \cos(2\pi x/\lambda)$

linear so ...

$$\underbrace{\overline{\quad\quad} \; T=0, \; H = \text{const}}_{\text{just solved}} \quad + \quad \overline{\quad\quad} \; T = T_S, \; H = 0$$

$$\underbrace{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y} = 0}_{\text{Laplace's eq.}}$$

let $T(x,y) = T_0 + A(x)\,B(y) \quad (1)$ ← separation of variables

to satisfy B.C. $A(x) = \cos(2\pi x/\lambda) \quad (2)$

sub (1) + (2) into $\nabla^2 T = 0$

$$-\frac{4\pi^2}{\lambda^2}\cancel{A(x)}\,B(y) + \cancel{A(x)}\frac{d^2B}{dy^2} = 0$$

$$\frac{d^2B}{dy^2} - \frac{4\pi B}{\lambda^2} = 0$$

$$B(y) = c_1 e^{-2\pi y/\lambda} + c_2 e^{2\pi y/\lambda}$$

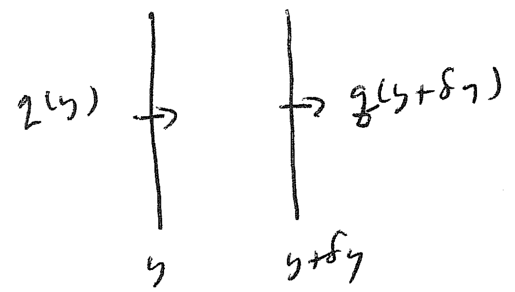
as $y \to \infty$ $B(y)$ finite so $c_2 = 0$

at $y = 0$ $T = T_0 + c_1 \cos(2\pi x/\lambda)$

$$\Rightarrow c_1 = \Delta T$$

So $$T(x,y) = T_0 + \underbrace{\Delta T\, e^{-2\pi y/\lambda}}_{\text{exponential decay of perturbation}} \cos(2\pi x/\lambda) + \frac{q_0}{k}y - \frac{\rho H}{2k}y^2$$

Unsteady heat conduction



$C \equiv$ specific heat
(energy to raise T by 1 degree)
$\sim 10^3$ J/kg

per unit area . . . .

rate of internal energy change

$$\rho C \frac{\partial T}{\partial t} \delta y$$

rate at which heat is added $\quad q(y) - q(y+\delta y) \; (+\rho H \delta y)$

$$\therefore \quad \rho C \frac{\partial T}{\partial t} = \underbrace{\frac{q(y) - q(y+\delta y)}{\delta y}}_{-\partial q/\partial y}$$

but $q = -k \, \partial T/\partial y$

$$\rho C \frac{\partial T}{\partial t} = k \, \partial^2 T/\partial y^2$$

or

$$\frac{\partial T}{\partial t} = \kappa \, \partial^2 T/\partial y^2$$

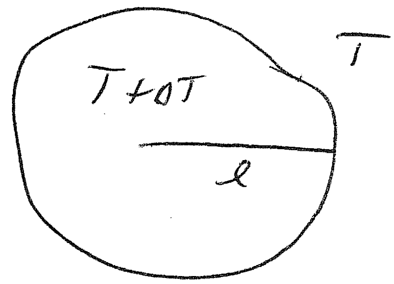
thermal diffusivity $\sim 10^{-6}$ m^2/s

in 3D $\quad \dfrac{\partial T}{\partial t} = \kappa \nabla^2 T \; (+ \rho H)$

if make heat

over what time τ does T change?

$$\underbrace{\frac{\partial T}{\partial t}} = \kappa \underbrace{\frac{\partial^2 T}{\partial y^2}}$$



$$O\left(\frac{\Delta T}{\tau}\right) \sim O\left(\kappa \frac{\Delta T}{l^2}\right)$$

$\Rightarrow \quad \tau \sim L^2/\kappa$

time to cook a chicken (1 kg) is 1 hour

time to cook a turkey (10 kg)?

$$t \propto L^2 \qquad L \propto M^{1/3}$$

$$t \propto M^{2/3}$$

$$\frac{t_1}{t_2} = \left(\frac{M_1}{M_2}\right)^{2/3}$$

$$t_{\text{turkey}} = (10)^{2/3} \sim 4.64 \text{ hours}$$

prob 2+3

(7)

periodic in time changes of surface T

$T = T_0 + \Delta T \cos(\omega t)$ $\qquad \dfrac{\partial T}{\partial t} = \kappa \dfrac{\partial^2 T}{\partial y^2}$

again use separation of variables

$$T(y,t) = T_0 + A(y)\,B(t)$$

$$= T_0 + \underbrace{A_1(y)\cos\omega t}_{\text{in phase}} + \underbrace{A_2(y)\sin\omega t}_{\text{out of phase}}$$

sub. into $\partial T/\partial t = \kappa\,\partial^2 T/\partial y^2$, apply B.C.

$$T = T_0 + \underbrace{\Delta T\, e^{-y\sqrt{\omega/2\kappa}}}_{\text{amplitude}} \cos\underbrace{\left(\omega t - y\sqrt{\omega/2\kappa}\right)}_{\text{phase}}$$

amplitude decreases exponentially by factor of e at

$$y_e = \left(\frac{2\kappa}{\omega}\right)^{1/2}$$

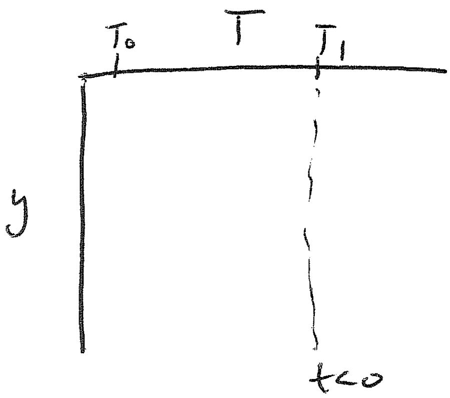
if $\omega = 1/\text{day}$, $y_e = 17\,\text{cm}$

$\omega = 1/\text{year}$, $y_e = 3.2\,\text{m}$

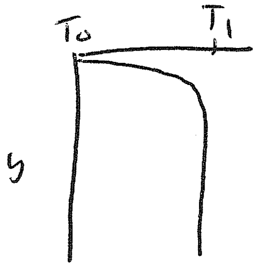
$\omega = \dfrac{1}{10^4\,\text{years}}$, $y_e = ?$ (320 m)

prob 4

Cooling of a half-space



at $t=0$ change T at $y=0$ from T_1 to T_0



$t>0$ $\quad \dfrac{\partial T}{\partial t} = \kappa \dfrac{\partial^2 T}{\partial y^2}$ (*)

$q(t)$?

$T(y,t)$?

let $\theta = (T - T_1)/(T_0 - T_1)$ so $0 < \theta < 1$

$$T = (T_0 - T_1)\theta + T_1$$

sub into (*)

$$\frac{\partial \theta}{\partial t} = \kappa \frac{\partial^2 \theta}{\partial y^2}$$

I.C. $\theta(y,0) = 0$

B.C. $\begin{cases} \theta(0,t) = 1 \\ \theta(\infty,t) = 0 \end{cases}$

recall that κ, t and length were related

hypothesize that θ depends on y, t being related

$$\eta = \frac{y}{2\sqrt{\kappa t}}$$

$$\frac{\partial\theta}{\partial t} = \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial t} = \frac{\partial\theta}{\partial\eta}\left(-\frac{1}{4}\overset{2\eta}{\frac{y}{\sqrt{\kappa t}}}\frac{1}{t}\right) = -\frac{\eta}{2t}\frac{d\theta}{d\eta}$$

$$\frac{\partial\theta}{\partial y} = \frac{\partial\theta}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} = \frac{1}{2\sqrt{\kappa t}}\frac{\partial\theta}{\partial\eta}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{1}{2\sqrt{\kappa t}}\frac{\partial^2\theta}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial y} = \frac{1}{4\kappa t}\frac{\partial^2\theta}{\partial\eta^2}$$

$$\therefore \underbrace{-2\eta\frac{d\theta}{d\eta} = \frac{d^2\theta}{d\eta^2}}_{\substack{\text{ODE} \\ \text{(not PDE)}}}$$

B.C. $t=0 \Rightarrow \eta\to\infty$ $\quad \theta(\eta=\infty)=0$

$y\to\infty \Rightarrow \eta\to\infty$ $\quad \theta(\eta=0)=1$

$y=0 \Rightarrow \eta=0$

2 B.C. for 2nd order ODE

let $f = d\theta/d\eta \Rightarrow -2\eta f = df/d\eta$

integrate $-\eta^2 = \ln f + c$

$$f = c_1 e^{-\eta^2} = d\theta/d\eta$$

integrate again

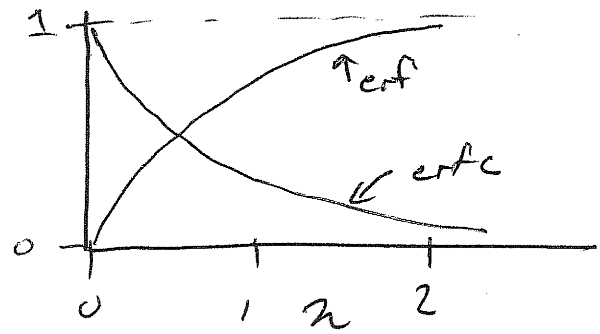
$$\theta = c_1\int_0^\eta e^{-\xi'^2}d\xi' + c_2$$

$\theta(0)=1 \Rightarrow c_2 = 1$

$\theta(\infty)=0$ so $\theta = c_1\underbrace{\int_0^\infty e^{-\xi^2}d\xi}_{\sqrt{\pi}/2} + 1 = 0 \Rightarrow c_1 = -2/\sqrt{\pi}$

$$\theta = \frac{T - T_1}{T_0 - T_1} = 1 - \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi^2} d\xi}_{\mathrm{erf}(\eta)}$$

$$= \mathrm{erfc}(\eta)$$



$= y/2\sqrt{\kappa t}$] y, t related!!

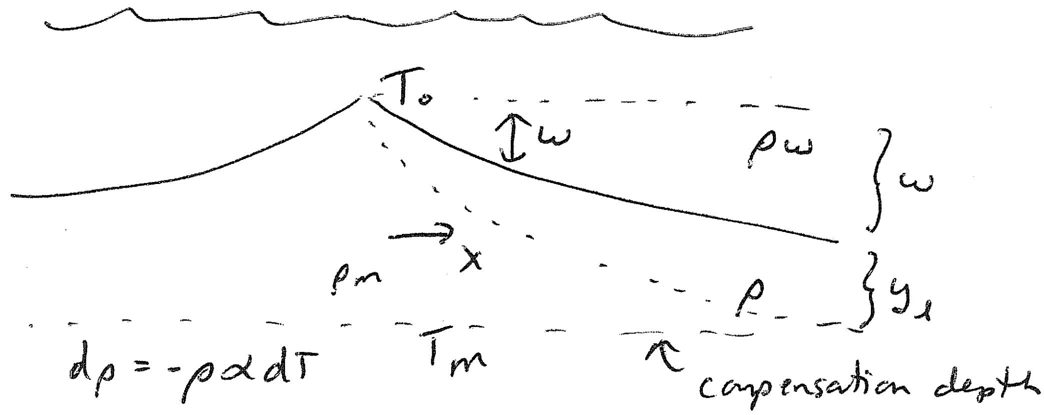
heat flow?

$$q = -k\frac{\partial T}{\partial y} = \frac{k\,(T_0 - T_1)}{\sqrt{\pi \kappa t}} \qquad q \propto t^{-1/2}$$

$$\Bigg(k\frac{\partial T}{\partial y} = (T_0 - T_1)\, \partial\theta/\partial \underset{2\eta\sqrt{\kappa t}}{y}$$

$$= \frac{k(T_0 - T_1)}{2\sqrt{\kappa t}} \underbrace{\frac{\partial \theta}{\partial \eta}}_{c_1 e^{-\eta^2}} \Bigg)$$

Depth of oceans



$d\rho = -\rho \alpha \, dT$

what is $w(x)$?

at ridge

$$\rho_m (w + y_l) = \int_0^{y_l} \rho^{(y)} \, dy + w \rho_w$$

$$w(\rho_w - \rho_m) + \int_0^{y_l} \underbrace{(\rho^{(y)} - \rho_m)}_{\rho_m \alpha (T_m - T)} \, dy = 0$$

we just solved for this

$$(T_m - T_0) \rho_m \alpha \int_0^{y_l} \operatorname{erfc}\left(\frac{y}{2\sqrt{\kappa t}}\right) dy$$

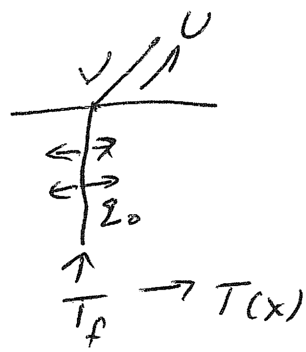
$$\therefore \; w = \frac{\rho_m \alpha (T_m - T_0)}{\rho_m - \rho_0} \int_0^{y_l} \operatorname{erfc}\left(y / 2\sqrt{\kappa t}\right) dy$$

if $y_l \to \infty$ $\quad \int \to 1/\sqrt{\pi}$

let $\eta = y / 2\sqrt{\kappa t}$ $\quad d\eta/dy = 1/2\sqrt{\kappa t}$

$$w(x) = \frac{\rho_m \alpha (T_m - T_0) \, 2\sqrt{\kappa t}}{(\rho_m - \rho_0) \sqrt{\pi}}$$

heating of faults



work = F · d

rate of work = F · U

heat flow = τU

$$q_0 = U\tau/2$$

What is T_f ?

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t}\right) = \kappa \frac{\partial^3 T}{\partial x^3}$$

$$\frac{\partial q}{\partial t} = \kappa \frac{\partial^2 q}{\partial x^2}$$ with B.C. $q(x=0) = \frac{U\tau}{2}$

$q(x \to \infty) = 0$

solution $q = q_0 \operatorname{erfc} \eta$

$$\frac{\partial T}{\partial y} = -\frac{q_0}{k} \operatorname{erfc} \eta$$

$$T(x) = T_0 - \frac{q_0}{k}\int_\infty^y \operatorname{erfc}(\xi/2\sqrt{\kappa t})\, d\xi$$

$$= T_0 + \frac{2q_0}{k}\sqrt{\kappa t}\int_\eta^\infty \operatorname{erfc} \eta' \, d\eta'$$

if $\eta \to 0$ $(y=0)$ $\int \to \frac{1}{\sqrt{\pi}}$

$$T(x=0) = T_0 + \frac{2q_0}{k}\sqrt{\frac{\kappa t}{\pi}}$$

prob 6

Phase changes (solidification)

Latent heat (L): heat liberated by solidification of 1 kg

2 complications: Where is boundary? Accounting for L?

Stefan (1891) problem

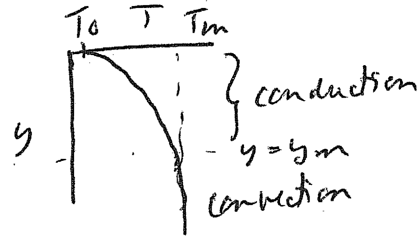
------------ $y=0$

solid

- - - - - - - - $y = y_m(t)$

liquid T_m (isothermal)

$\downarrow$ $\boxed{\dfrac{dy_m(t)}{dt}\ ? \quad y_m(t)\ ?}$



again $\theta = (T - T_0)/(T_m - T_0)$

$\eta = y/2\sqrt{\kappa t}$

define $\eta_m = y_m/2\sqrt{\kappa t} = \lambda_1$ (Answer already!)

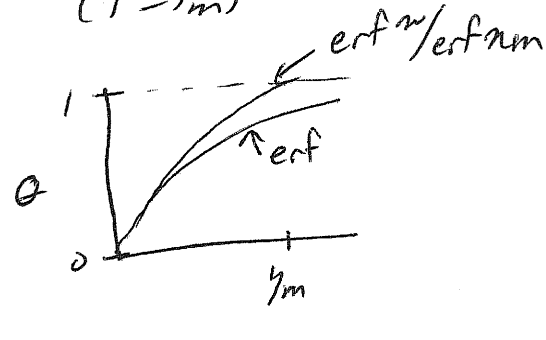
$\boxed{\text{what is } \lambda_1\ ?}$

$$2\eta \frac{d\theta}{d\eta} = \frac{d^2\theta}{d\eta^2} \quad \text{from before}$$

B.C. $\theta(\eta = 0) = 0$ $\quad (T = 0)$

$\theta(\eta = \eta_m = \lambda_1) = 1$ $\quad (T = T_m)$

$$\theta = \frac{\operatorname{erf} \eta}{\operatorname{erf}(\eta_m)}$$



prob5

to get λ_1,

in time δt, boundary moves down $\frac{dy_m}{dt}\delta t$

solidifying $\rho\left(\frac{dy_m}{dt}\right)\delta t$ /area

releasing heat $L\rho(dy_m/dt)\,\delta t$/area

heat must be conducted $-k\,\partial T/\partial y\big|_{y=y_m}\delta t$

$$\therefore\quad -k\left.\frac{\partial T}{\partial y}\right|_{y=y_m} = \rho L\frac{dy_m}{dt}$$

since $y_m = 2\lambda_1\sqrt{\kappa t}$

$$\frac{dy_m}{dt} = \lambda_1\sqrt{\kappa/t}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=y_m} = -\frac{T_m-T_0}{2\sqrt{\kappa t}}\,\frac{2}{\sqrt{\pi}}\,e^{-\lambda_1^2}\,\frac{1}{\operatorname{erf}\lambda_1}$$

$$\frac{k(T_m-T_0)}{\cancel{2}\sqrt{\kappa\cancel{t}\,\pi}}\,\frac{e^{-\lambda_1^2}}{\operatorname{erf}\lambda_1} = \rho L\lambda_1\sqrt{\frac{\kappa}{\cancel{t}}}$$

$$\underbrace{\frac{e^{-\lambda_1^2}}{\lambda_1\operatorname{erf}\lambda_1} = \frac{\cancel{\rho}L\kappa}{\cancel{\kappa}\rho c\,(T_m-T_0)}}_{\text{plot}}$$

known - get λ_1