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Stress and strain and basic isostasy (statics)

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Advanced School on
Scaling Laws in Geophysics:
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Strain and Stress (basics)

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Strain is the term used to describe the deformation of a continuum. It describes an internal distortion (stretching, shortening or shearing) of the medium.

Any point in a deforming medium undergoes a displacement **u**, and strain occurs when **u** is not uniform, i.e. when $\nabla \boldsymbol{u} \neq 0$.

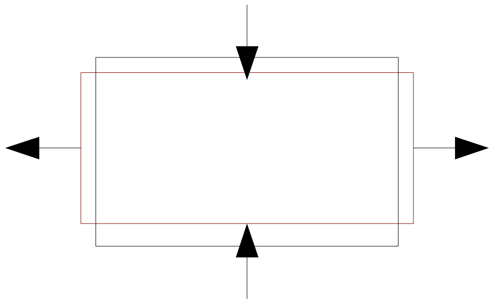
Uniform displacement or solid body rotation does not produce deformation because $\nabla \boldsymbol{u} = 0$.

u is a vector, so there are 3 components of the gradient for each of the 3 components of **u**. Thus $\nabla \boldsymbol{u}$ is a 3 x 3 tensor:

$$\nabla \mathbf{u} = \left[\frac{\partial u_i}{\partial x_j} \right] = \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right] + \frac{1}{2} \left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right] = \left[\varepsilon_{ij} + \frac{\omega_k}{2} \right]$$

The **defomation gradient tensor** thus is affected by changes in shape (strain), and changes in orientation (rotation).

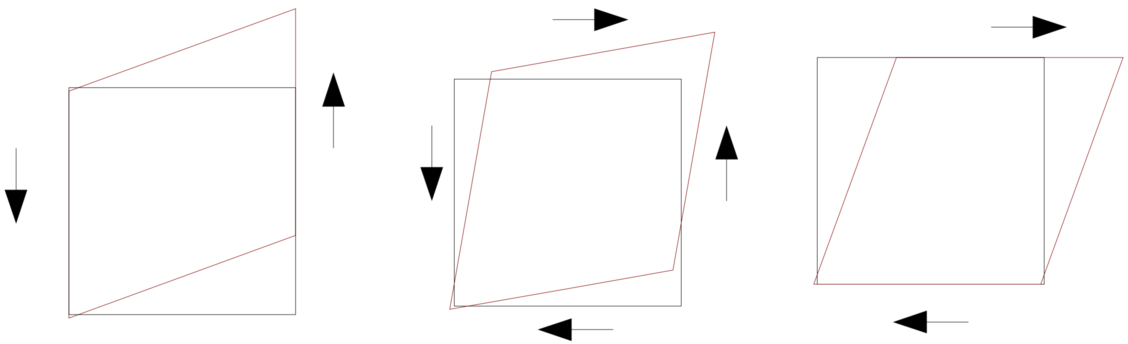
The simplest type of strain involves an extension or shortening, e.g.,



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \text{or} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}$$

This type of deformation is referred to as pure shear or co-axial deformation. Lines that are parallel to the principal strain axes remain parallel.

Simple shear of a rectangle occurs when a boundary is moved parallel to itself, but simple shear is just pure shear plus a rotation as these 3 examples show. If the shear orientation remains constant, the material rotates relative to principal axes.



$$\varepsilon_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]$$

The **Strain** tensor ε_{ij} simply describes deformation – its relation to the forces that may have produced the deformation is yet to be defined. A theory which describe only displacements and strains is referred to as a **kinematic theory**. Plate tectonics is an example of a kinematic theory.

In a **solid:** strains are small, but elastic deformation may be important (e.g. elastic strain is released in an earthquake).

In a **fluid:** strains may become very large, and often we work with strain-rates $\dot{\varepsilon}_{ij}$, the rates of change of strain, obtained from gradients of velocity rather than displacement.

The strain tensor is a 3x3 matrix which satisfies an eigenvalue equation:

$$\boldsymbol{\varepsilon} = \mathbf{V} \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \mathbf{V}^T$$

The columns of the eigenvector matrix **V** define the 3 principal directions of the strain. The eigenvalue ε_{ii} defines the principal strain occurring in that orientation. The principal strains are independent of the original choice of coordinate system

The quantity $$\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \nabla \cdot \mathbf{u}$$

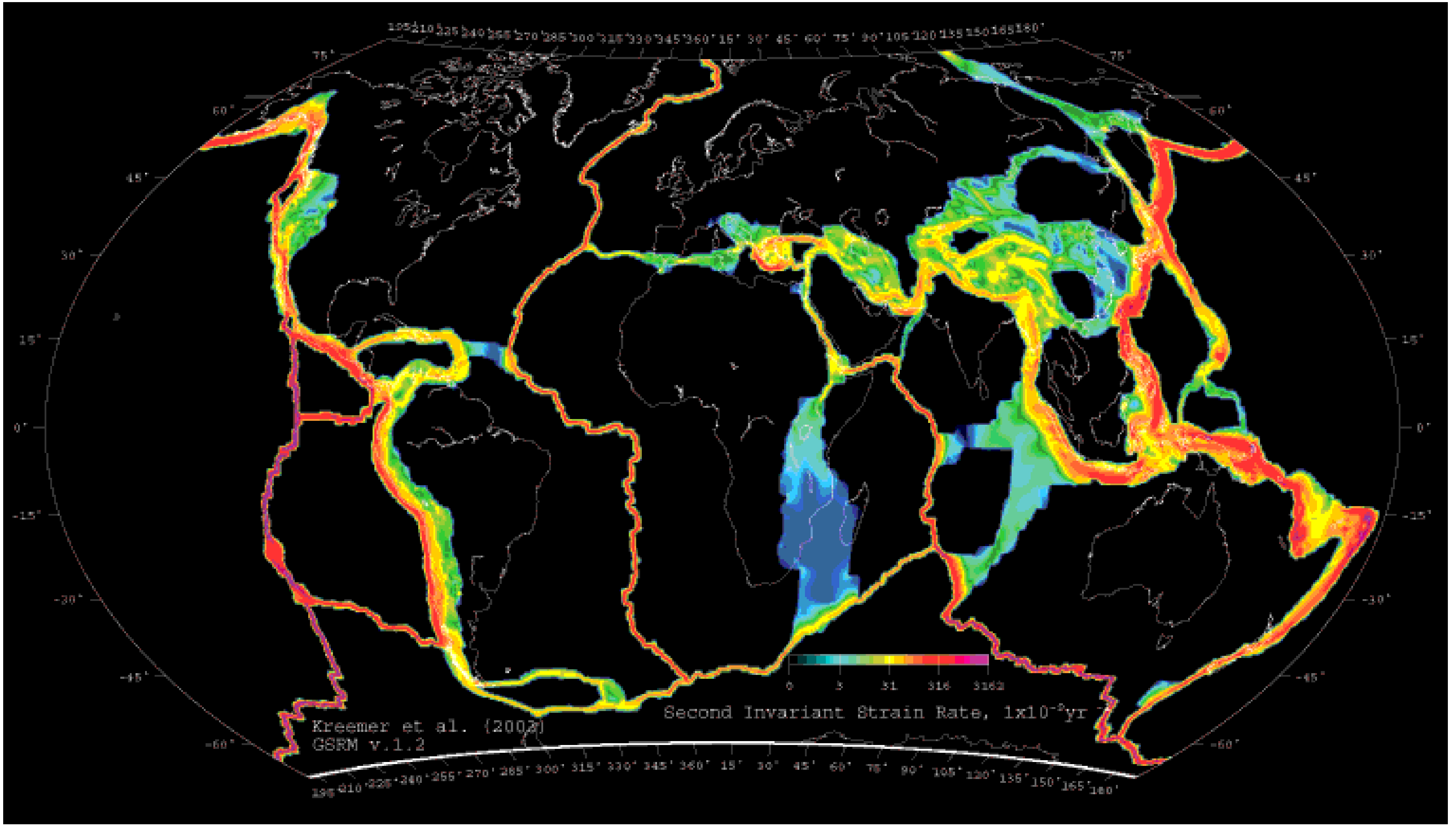
is one of the invariants of the strain tensor and it represents the volumetric strain. Often for earth deformation problems $\theta \approx 0$

The second invariant of the strain tensor is also often a useful indicator of the “magnitude” of strain:

$$E = \sqrt{\sum_{ij} \varepsilon_{ij} \varepsilon_{ij}}$$

In formulating any description of the physical world involving tensor quantities, we use invariants to ensure that the results are independent of the choice of coordinate system.

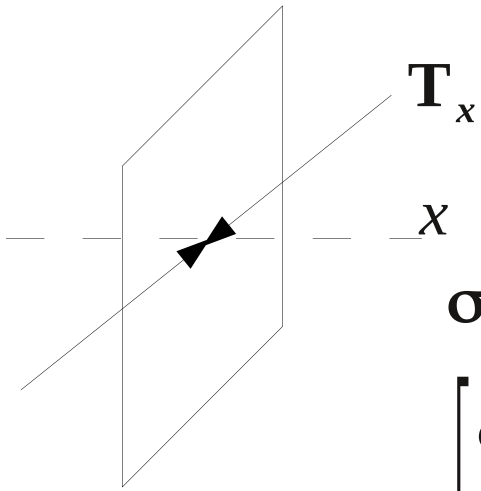
A global map of plate strain (quantified by 2nd invariant of strain-rate tensor), produce by: The Global Strain-rate Map Project: http://gsrm.unavco.org/intro/



Stress is the term used to describe the internal forces in a continuum. Like strain it is a tensor quantity.

Consider the plane $x = 0$ passing through a continuous medium which is in local force equilibrium (not accelerating). At any point on that plane a **traction** (force per unit area) on one side of the plane is balanced by an equal and opposite traction on the opposite side.

There are 3 orthogonal planes at any point, so 3 independent traction vectors are needed.



$$\boldsymbol{\sigma} = \left[\mathbf{T}_x, \mathbf{T}_y, \mathbf{T}_z\right]$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$

Each of the 3 traction vectors is specified by 3 components, so the internal state of force at that point requires 9 components: the **stress tensor**. The 3 columns of the stress tensor are the 3 traction vectors. Torque equilibrium requires a symmetric stress tensor.

We can find the traction acting on any plane defined by the unit normal vector $\hat{\mathbf{n}}$, by contracting the tensor with that vector:

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$

In particular, we can find 3 orthogonal planes for which the traction vectors are normal to those planes. This operation is equivalent to diagonalizing the stress tensor by rotation of the coordinate axes:

$$\boldsymbol{\sigma} = \mathbf{V} \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \mathbf{V}^T$$

to obtain the principal stress components acting in the orientation of the **principal stress axes** (defined by the eigenvectors in **V**).

The invariant $$p = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

is defined as pressure. For a fluid in which shear forces are not sustained, we have: $p = \sigma_{11} = \sigma_{22} = \sigma_{33}$

Pressure may cause volumetric strain (compaction), but in general does not produce shape-changing deformation, so it is useful to separate out the pressure to define the **deviatoric stress** tensor:

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} - p & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} - p & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - p \end{bmatrix}$$

Deviatoric stresses cause shape-changing deformation.

A **constitutive relation** defines how strain changes under the effect of stress. They are usually based on empirical measurement. Examples are:

Elasticity: $\sigma_{ij} = \lambda \theta \delta_{ij} + 2 \mu \varepsilon_{ij}$

Newtonian viscosity: $\tau_{ij} = 2 \eta \dot{\varepsilon}_{ij}$

For linear isotropic relations principal strain and stress axes coincide.