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Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

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Convection - Part III

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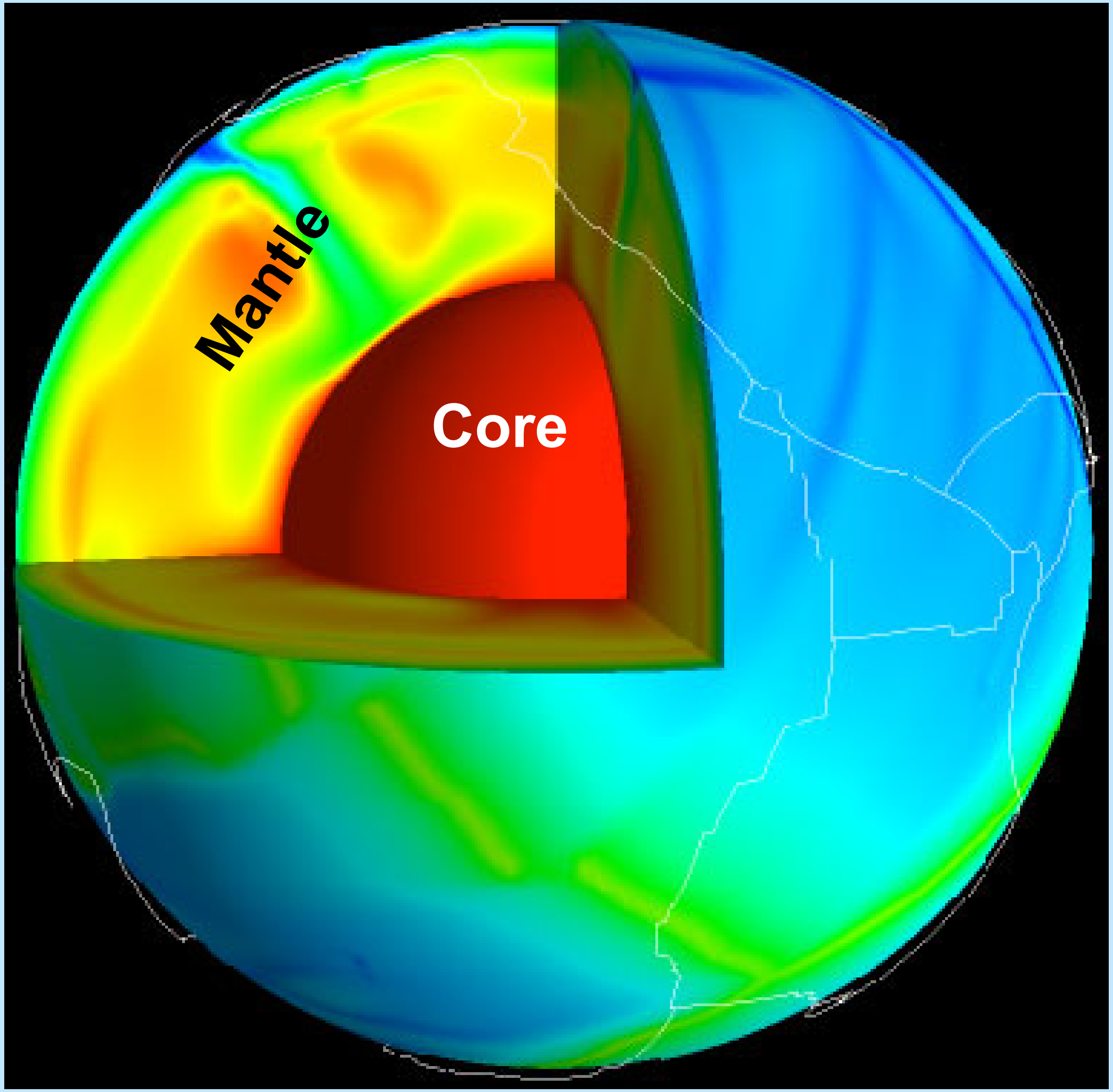
Some aspects of convection in mantle-like material

Very large Prandtl number
Free boundaries
Significant amounts of internal heat generation
Not in steady-state : cooling
Large variations of viscosity
Non-Newtonian rheology (probably)

CONVECTION DRIVEN BY INTERNAL HEAT GENERATION



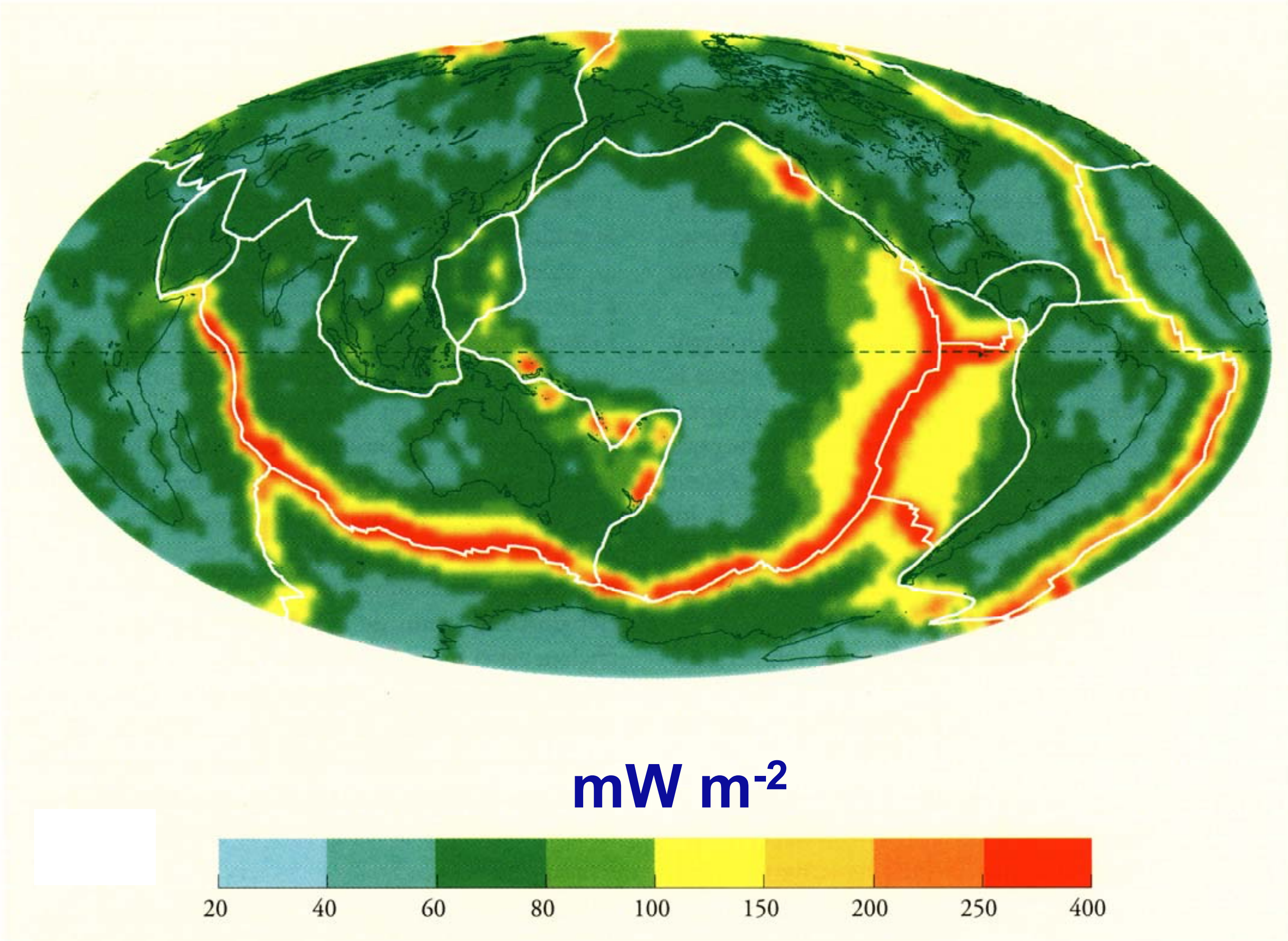
Core has no
U and Th
(K?)
but provides heat
To mantle

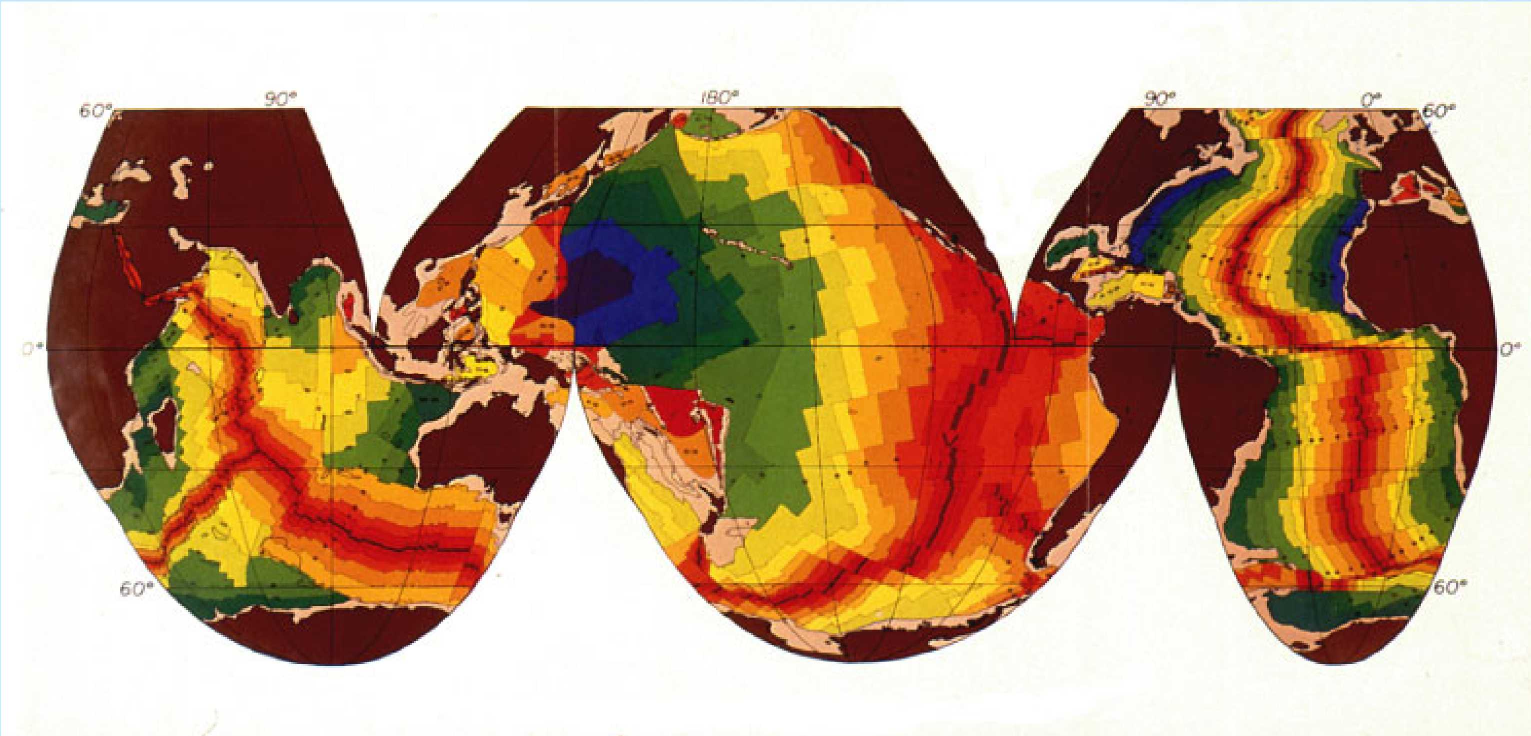
Radioactive elements

Nucleide	^{238}U	^{235}U	^{232}Th	^{40}K
$T_{1/2}$ (Ga)	4,46	0,70	14,0	1,26

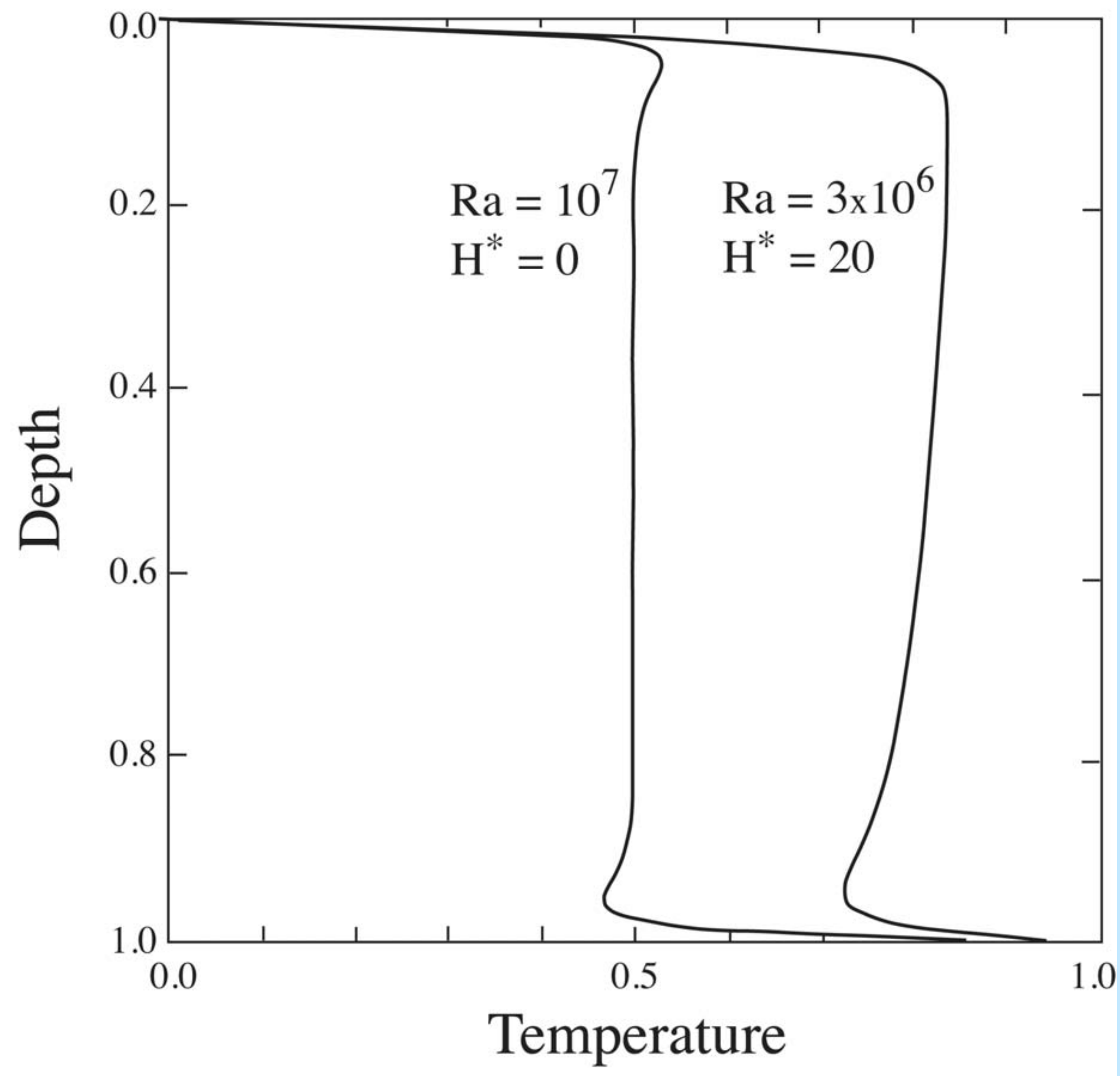
Heat production in silicate Earth (no core) $H = 5$ pW kg^{-1} (x 10^{-12} W kg^{-1}).

Total heat generation rate 20 TW (20 x 10^{12} W)


mW m-2
20
40
60
80
100
150
200
250
400



Total flux $\Phi \approx$ 40 TW = 40 10^{12} Watts



0.0
0.2
0.4
0.6
0.8
1.0
Depth
0.0
0.5
1.0
Temperature
Ra = 10^7
H* = 0
Ra = 3x10^6
H* = 20

Two dimensionless numbers characterize convection

($Pr \rightarrow \infty$)

$$\mathrm{Ra} = \frac{\rho_o g \alpha \Delta T_c h^3}{\kappa \mu}$$

$$H^* = \frac{Hh^2}{k \Delta T}$$

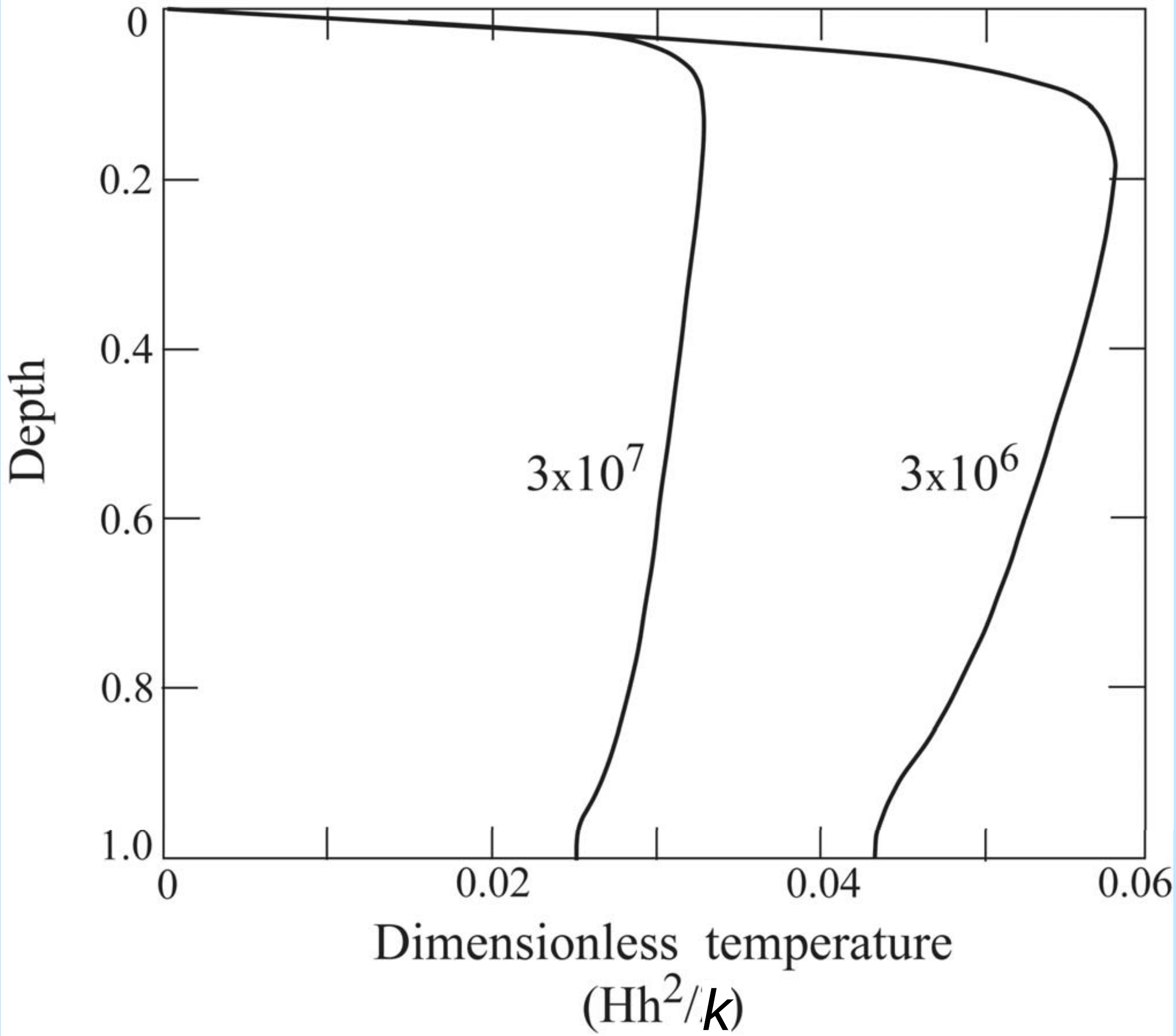
1. Pure internal heating (zero heat flux at base of fluid layer)

In all the following H stands for heat production per unit volume (what we wrote as ρH before).

Only one dimensionless numbers (infinite Pr limit)

$$\Delta T_H = \frac{Hh^2}{k},$$

$$\mathrm{Ra}_H = \frac{\rho_0 g \alpha H h^5}{k \kappa \mu}.$$



0
0.2
0.4
0.6
0.8
1.0
Depth
3x10^7
3x10^6
0
0.02
0.04
0.06
Dimensionless temperature
(Hh^2/k)

In steady-state, the heat flux out of the layer is known:
it must evacuate the total amount of heat released in the layer

$$Q = Hh,$$

The unknown is the temperature difference across the upper boundary layer, noted ΔT_i.
We seek a relationship between the dimensionless temperature and the only variable dimensionless number ($Pr \rightarrow \infty$)

$$\frac{\Delta T_i}{\frac{Hh^2}{k}} = f\,(\mathrm{Ra}_H),$$

$$0 = -\frac{d}{dz}\left(-k\frac{d\overline{T}}{dz} + \rho_o C_p \overline{w\theta}\right) + H\,,$$

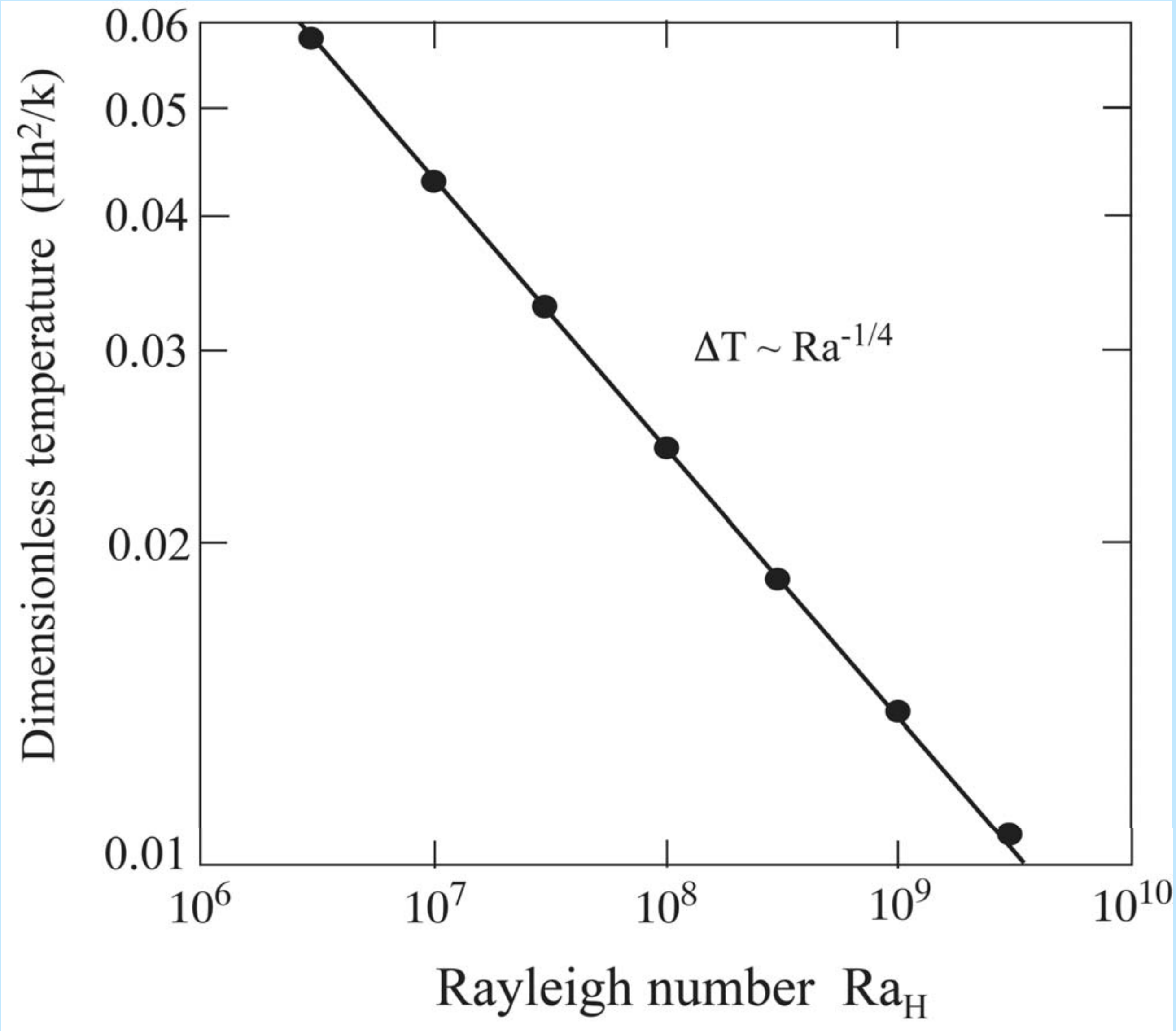
$$\begin{aligned}
\int_0^h \rho_o \alpha g \overline{w\theta} dz &= \int_0^h \frac{\alpha g}{C_p}(\rho_o C_p \overline{w\theta}) dz \\
&= \frac{\alpha g}{C_P}\int_0^h \left[k\frac{d\overline{T}}{dz} + H_z\right] dz \\
&= \frac{\alpha g}{C_P}\left(-k\,\Delta T + H\frac{h^2}{2}\right).
\end{aligned}$$

$$+\int_V \rho\alpha g w\theta\, dV - \int_V \psi\, dV = 0.$$

$$\mu\left(\frac{U}{h}\right)^2 hS \sim \frac{\alpha g}{C_P}\, H\frac{h^2}{2}S$$

$$Q \sim k\frac{\Delta T_i}{\delta}\ ,\ \delta \sim \sqrt{\frac{\kappa h}{U}},$$

$$\Delta T_i \sim \frac{Hh^2}{k}\mathrm{Ra}_H^{-1/4}.$$



0.06
0.05
0.04
0.03
0.02
0.01
Dimensionless temperature (Hh^2/k)
$\Delta T \sim Ra^{-1/4}$
10^6
10^7
10^8
10^9
10^{10}
Rayleigh number Ra_H

$$\Delta T_i \sim \frac{Hh^2}{k} \mathrm{Ra}_H^{-1/4}.$$

$$Q = Hh,$$

$$Q = C_Q k \left(\frac{\rho_o g \alpha}{\kappa \mu} \right)^{1/3} \Delta T_i^{4/3},$$

Constant for the local heat flux scaling law

Free boundaries

Heating mode	C_Q	Reference
Internally heated	0.302	Parmentier and Sotin, 2000
Mixed §	0.346	Sotin and Labrosse, 1999
Heated from below	0.378	Hansen *et al.*, 1992

† From numerical calculations in the infinite Pr limit.

§ The fluid layer is heated from below and from within.

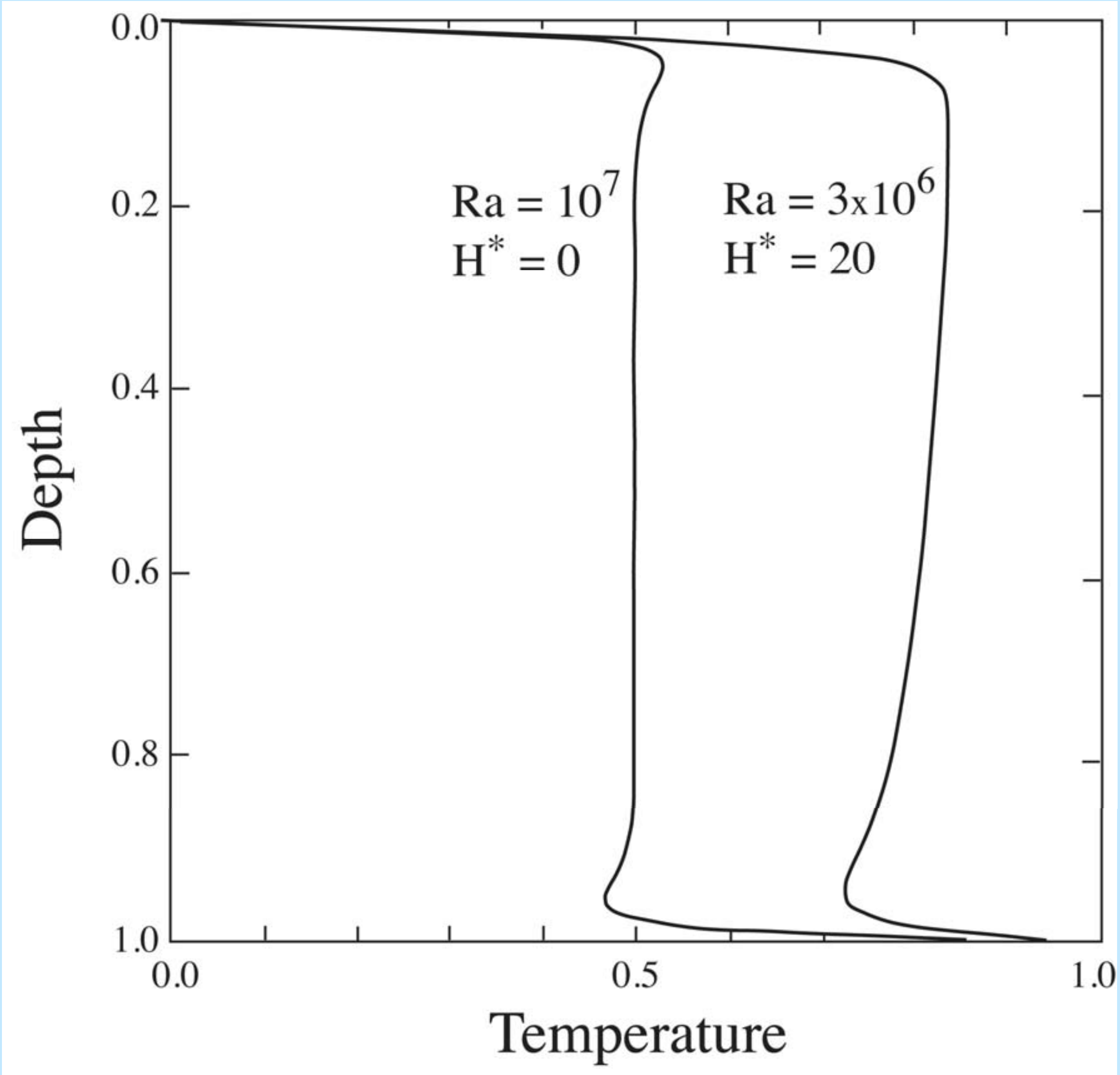
2. Layer heated from below and from within.

Two dimensionless numbers (Pr $\longrightarrow \infty$)

$$\mathrm{Ra} = \frac{\rho_o g \alpha \Delta T_c h^3}{\kappa \mu}$$

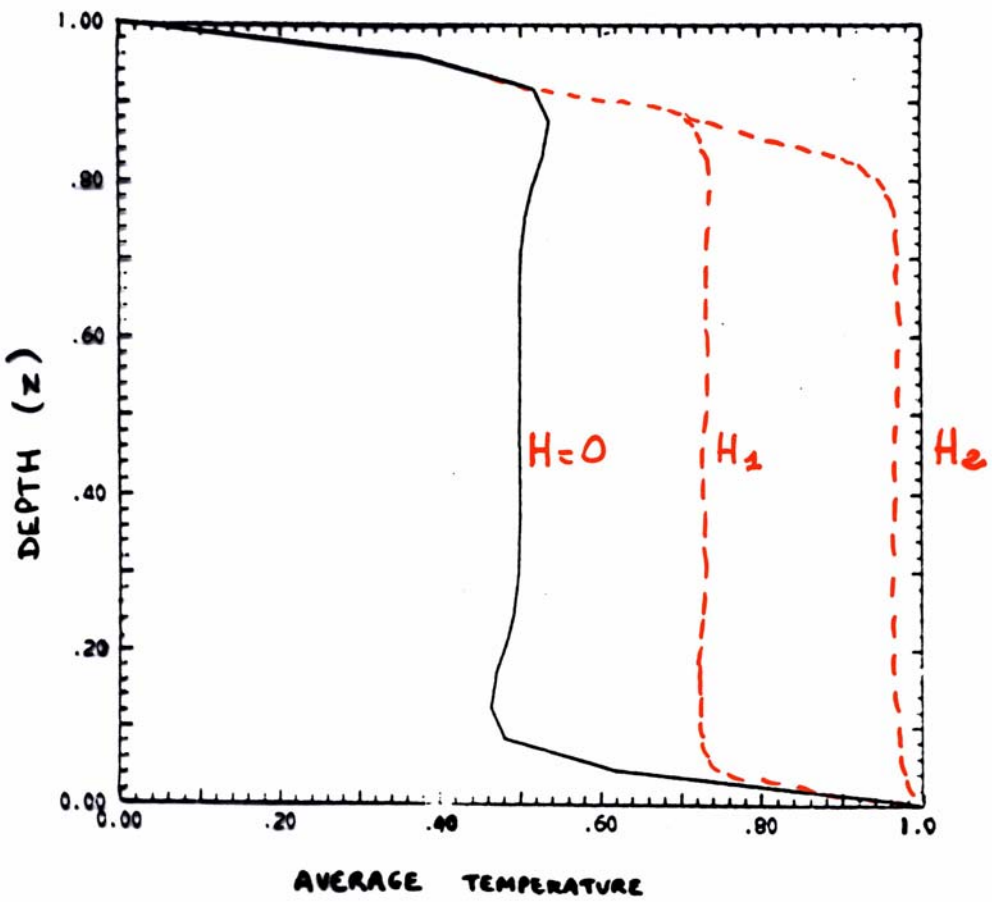
$$H^* = \frac{H h^2}{k \Delta T}$$

$$\Theta = \Delta T_i / \Delta T$$



0.0
0.2
0.4
0.6
0.8
1.0
Depth
0.0
0.5
1.0
Temperature
Ra = 10^7
H* = 0
Ra = 3x10^6
H* = 20

$Ra = 10^6$

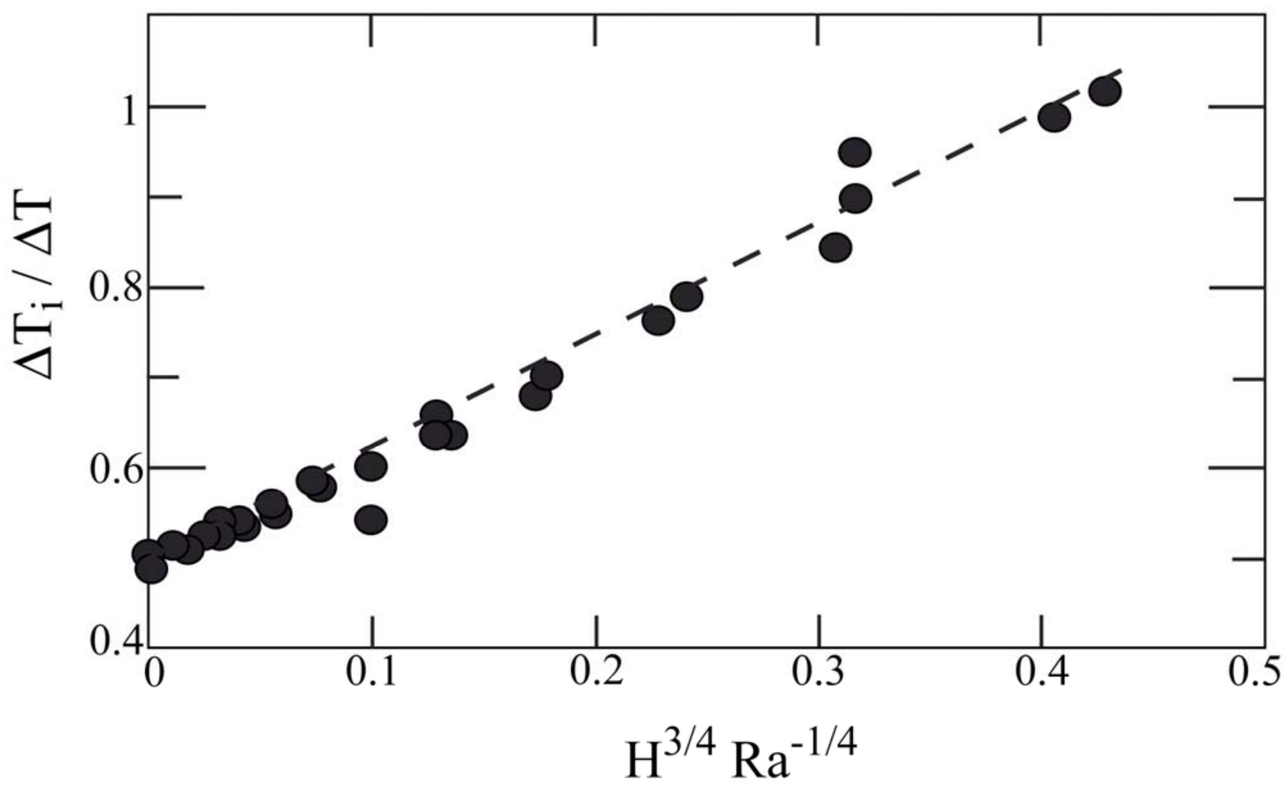

1.00
.80
.60
.40
.20
0.00
DEPTH (z)
0.00
.20
.40
.60
.80
1.0
AVERAGE TEMPERATURE
H=0
H1
H2

$$\Delta T_i = \frac{\Delta T}{2} + C^* \frac{Hh^2}{k} \left(\frac{\rho g \alpha H h^5}{\lambda \kappa \mu} \right)^{-1/4},$$

$$\Delta T_i = \frac{\Delta T}{2} + C^* \Delta T H^{*3/4} \mathrm{Ra}^{-1/4}$$

$$\Theta = \frac{1}{2} + C^* H^{*3/4} \mathrm{Ra}^{-1/4}.$$

$$C^* = 1.24$$



1
0.8
0.6
0.4
0
0.1
0.2
0.3
0.4
0.5
ΔT_i / ΔT
H^3/4 Ra^-1/4

$$Q = C_Q k \left(\frac{\rho_o g \alpha}{\kappa \mu} \right)^{1/3} \Delta T_i^{4/3},$$

$$\mathrm{Nu} = \frac{Q_T}{k \Delta T / h} = 0.346 \mathrm{Ra}^{1/3} \Theta^{4/3},$$

Constant for the local heat flux scaling law

Free boundaries

Heating mode	C_Q	Reference
Internally heated	0.302	Parmentier and Sotin, 2000
Mixed §	0.346	Sotin and Labrosse, 1999
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† From numerical calculations in the infinite Pr limit.

§ The fluid layer is heated from below and from within.

Temperature-dependent viscosity

$$\mu = \mu_o \exp\left(-\frac{T - T_o}{\Delta T_R}\right),$$

Add one dimensionless number

viscosity ratio $\mu_o/\mu(T_o + \Delta T)$

or

temperature ratio $\Delta T/\Delta T_R$

Arrhenius temperature dependence ?

$$\mu = a \exp\left(\frac{E + pV}{RT}\right),$$

$$\mu \approx \mu_i \exp\left[-\frac{(E + pV)(T - T_i)}{RT_i^2}\right],$$

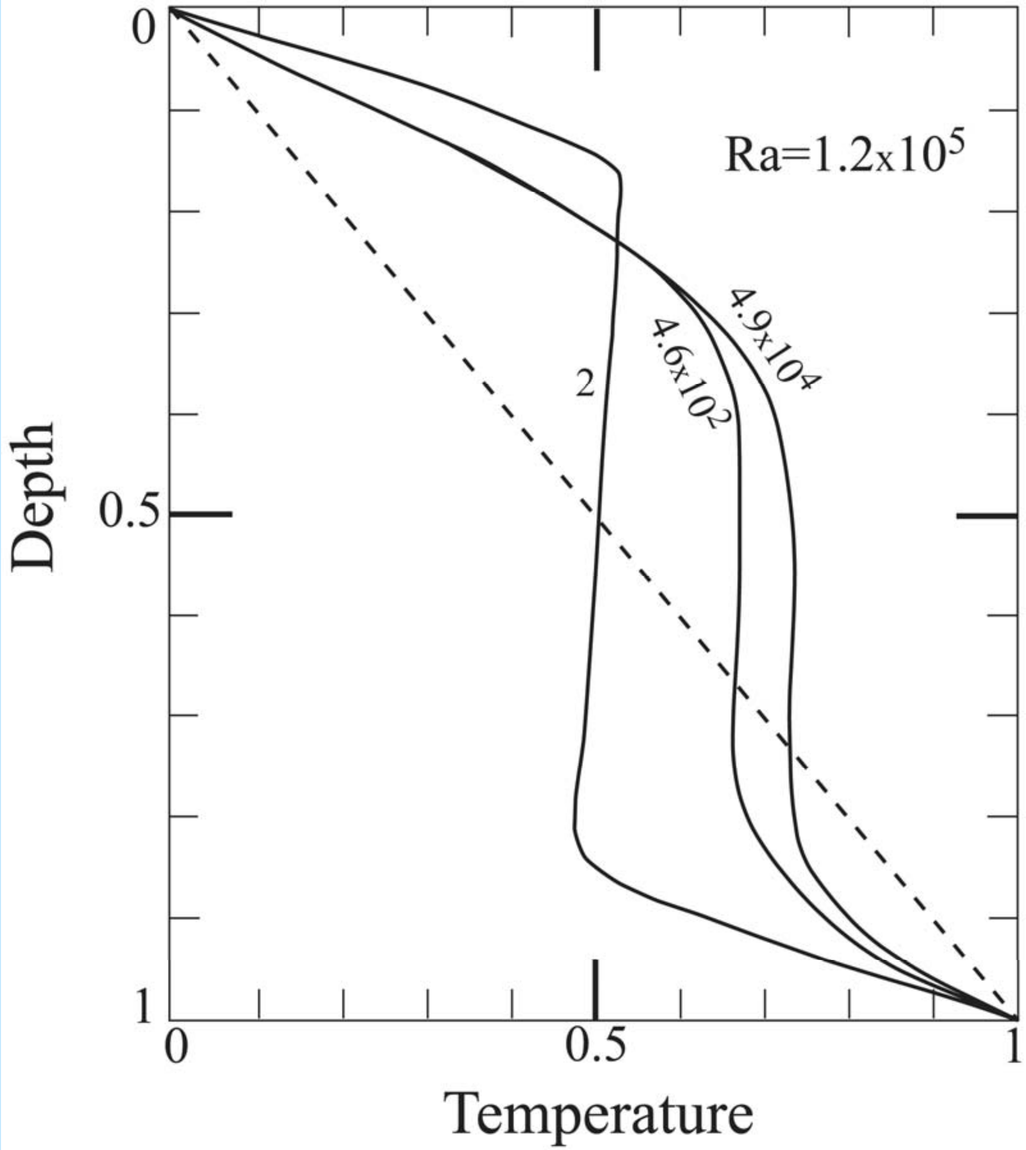
$$\boxed{\Delta T_R = \frac{RT_i^2}{E + pV}.}$$

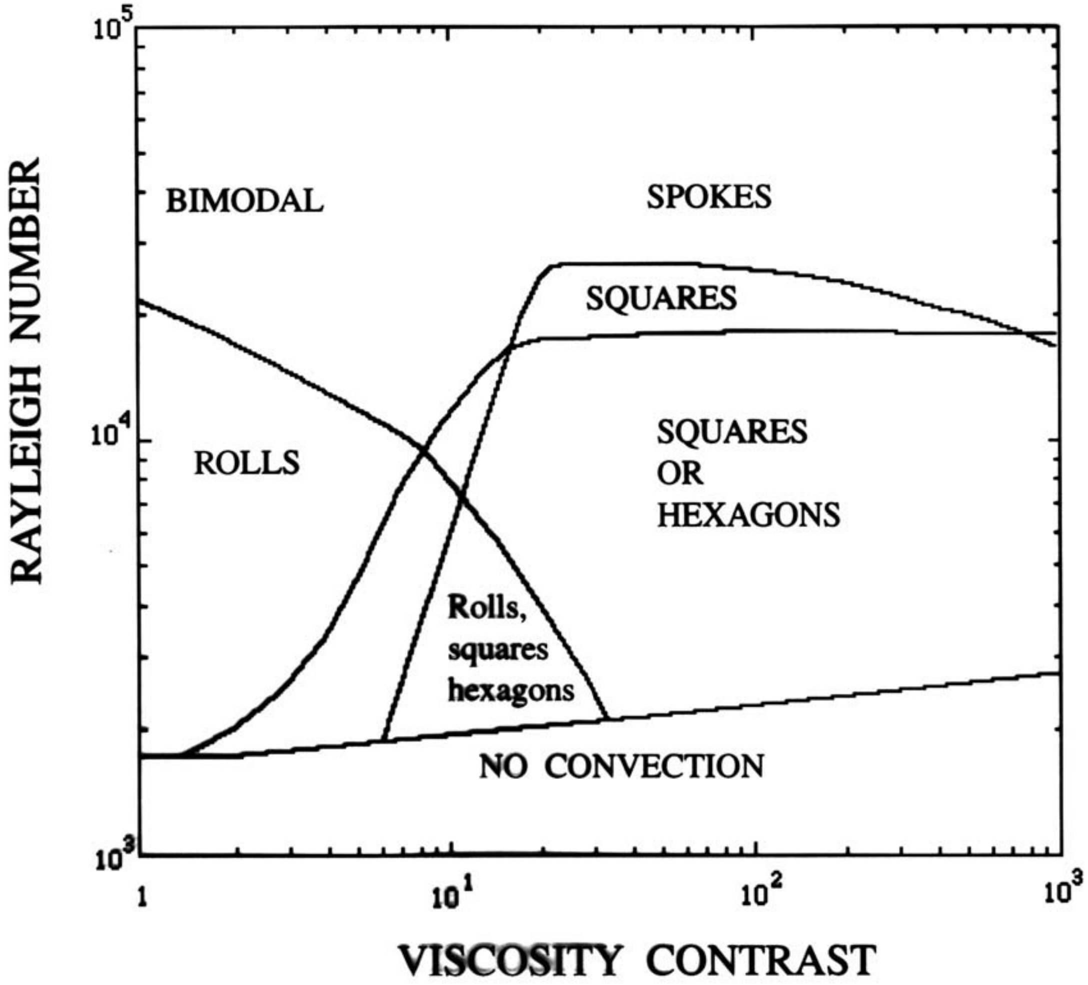
Rheological temperature scale for mantle rheologies †

Creep regime	E (kJ mole^{-1})	V (cm^3 mole^{-1})	ΔT_R (K) §
Dry diffusion	261	6	92
Wet diffusion	387	25	62
Dry dislocation	610	13	39
Wet dislocation	523	4	46

† Representative values from (Korenaga and Karato, 2008).

T_i = 1700 K, p = 6 GPa


Ra=1.2x10^5
4.9x10^4
4.6x10^2
2
Depth
0
0.5
1
0
0.5
1
Temperature

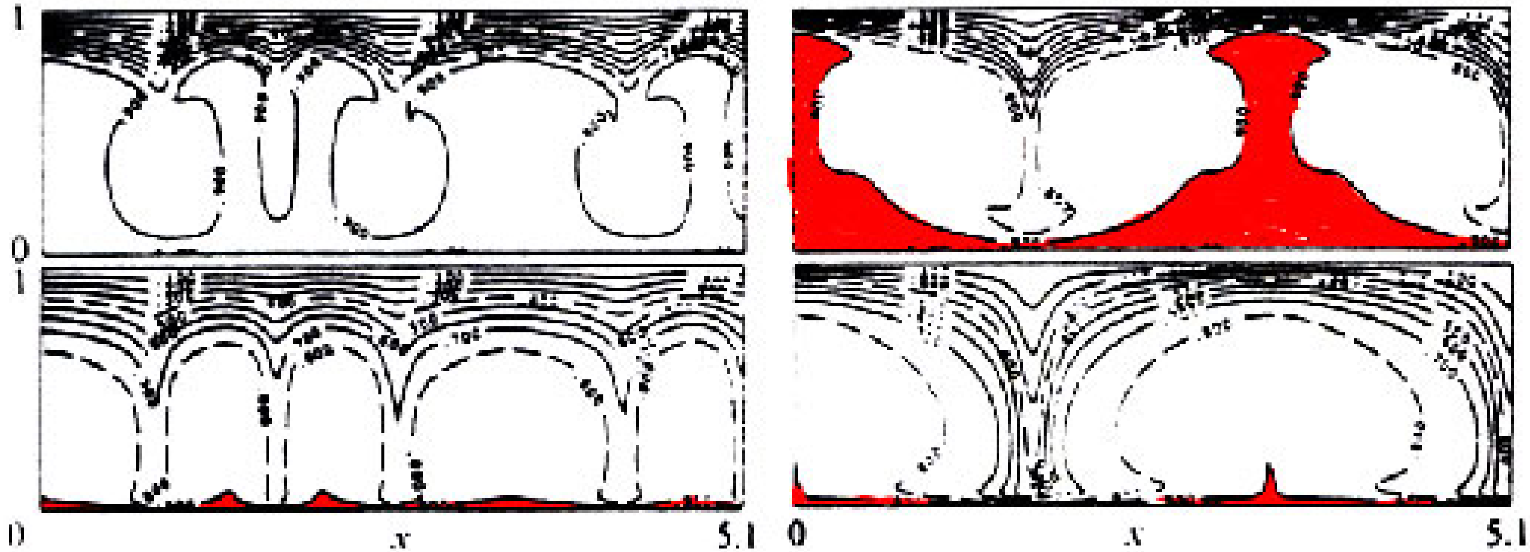


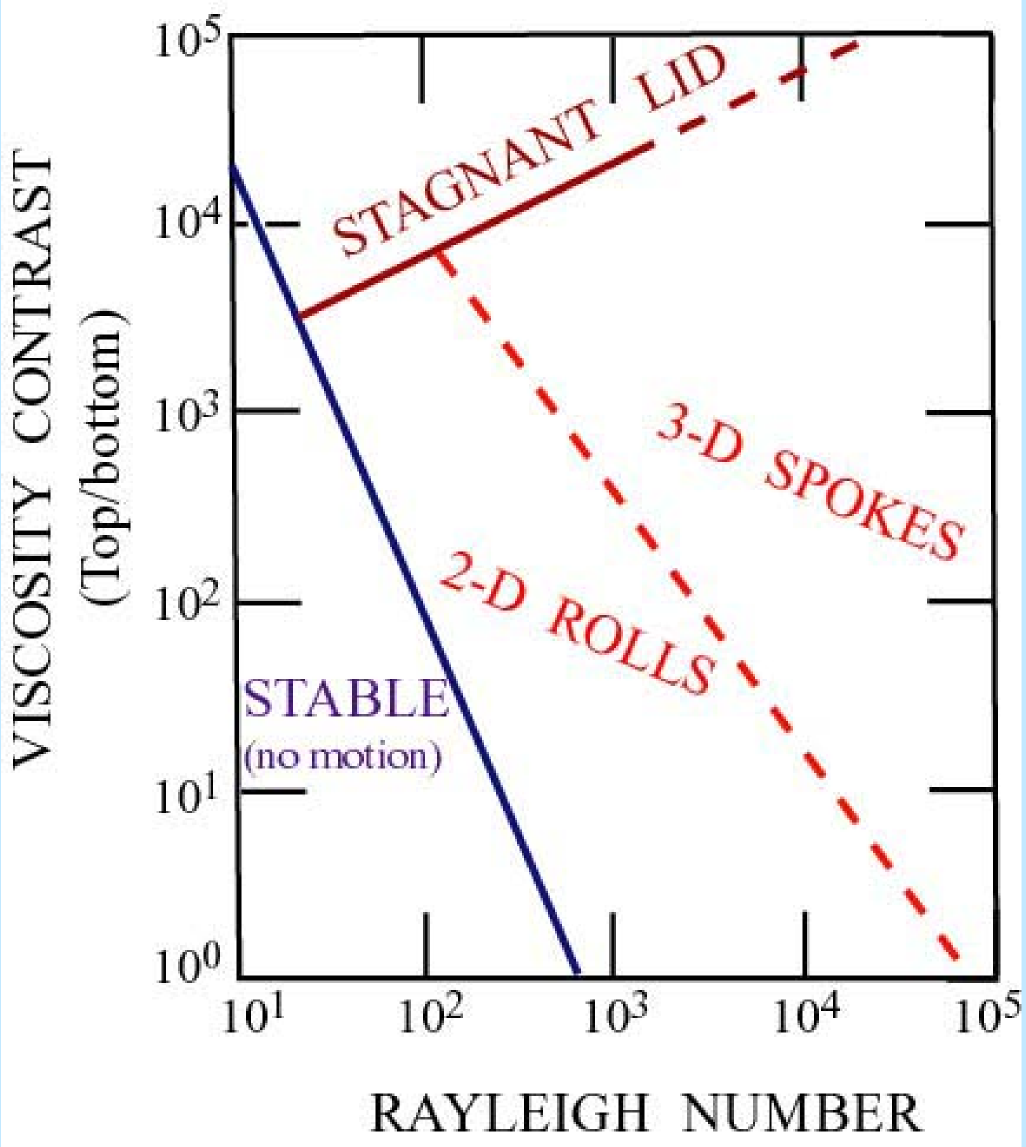
RAYLEIGH NUMBER
VISCOSITY CONTRAST
10^5
10^4
10^3
1
10^1
10^2
10^3
BIMODAL
SPOKES
SQUARES
ROLLS
SQUARES
OR
HEXAGONS
Rolls,
squares
hexagons
NO CONVECTION

Ra = 3.2 10^6

Viscosity ratio

$r = 3.2 \times 10^4$ $r = 3.2 \times 10^3$



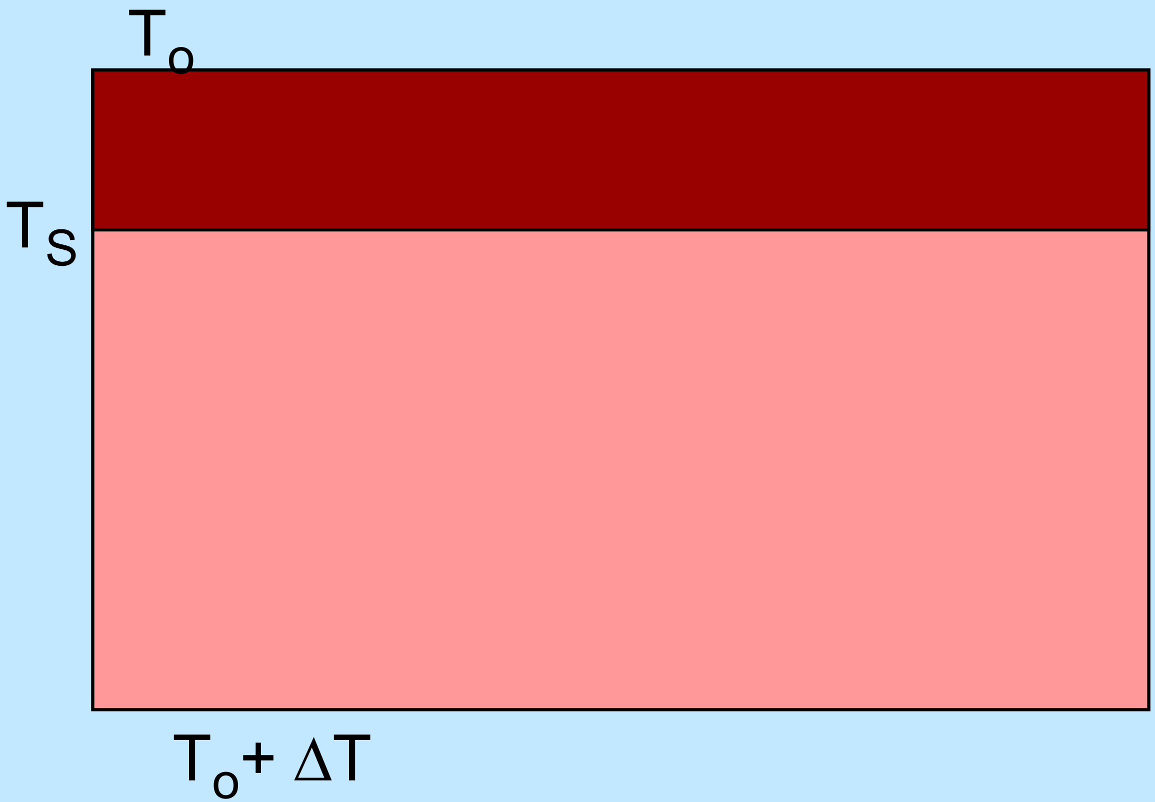


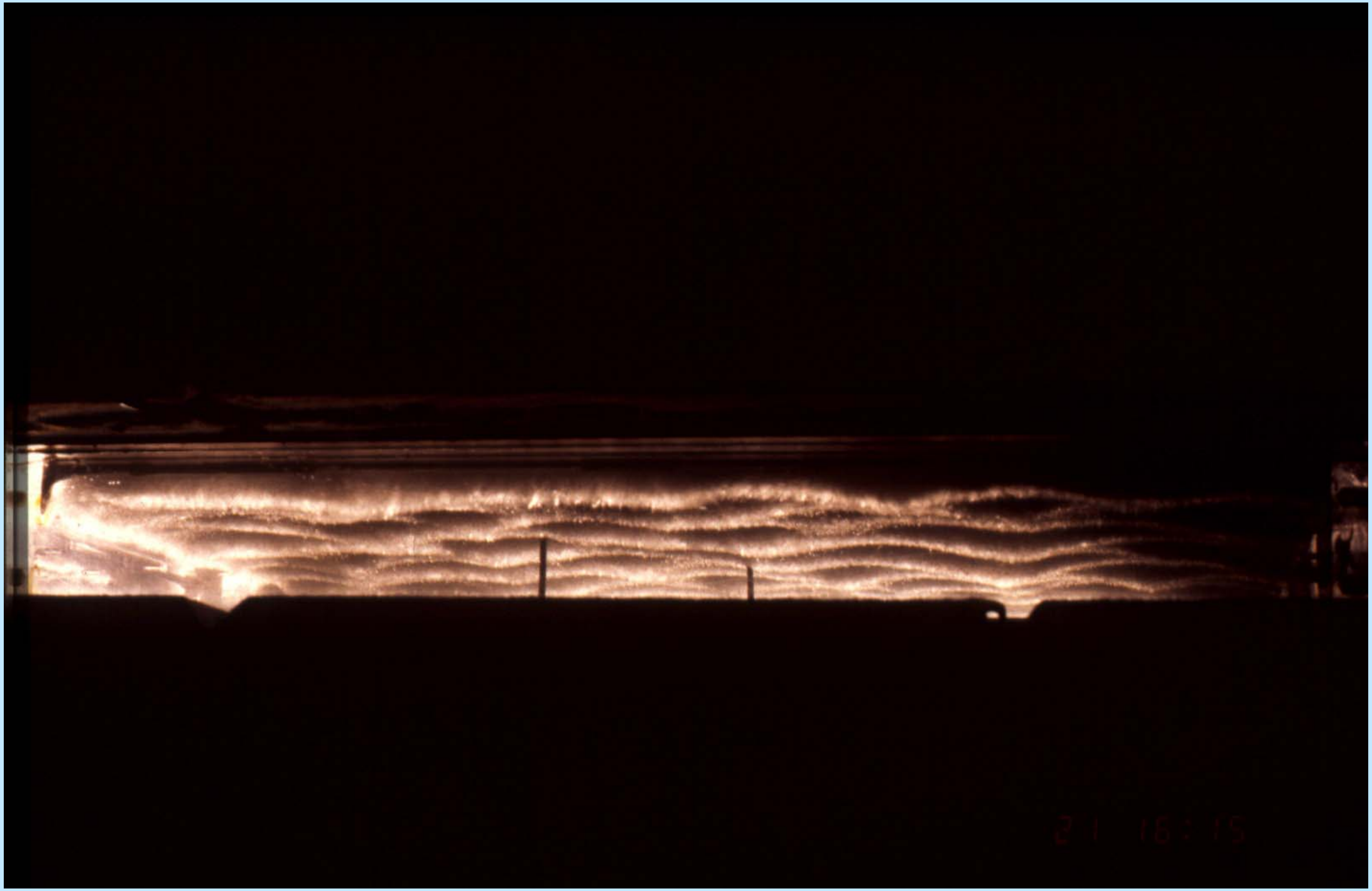
VISCOSITY CONTRAST
(Top/bottom)
RAYLEIGH NUMBER
STAGNANT LID
3-D SPOKES
2-D ROLLS
STABLE
(no motion)
10^0
10^1
10^2
10^3
10^4
10^5

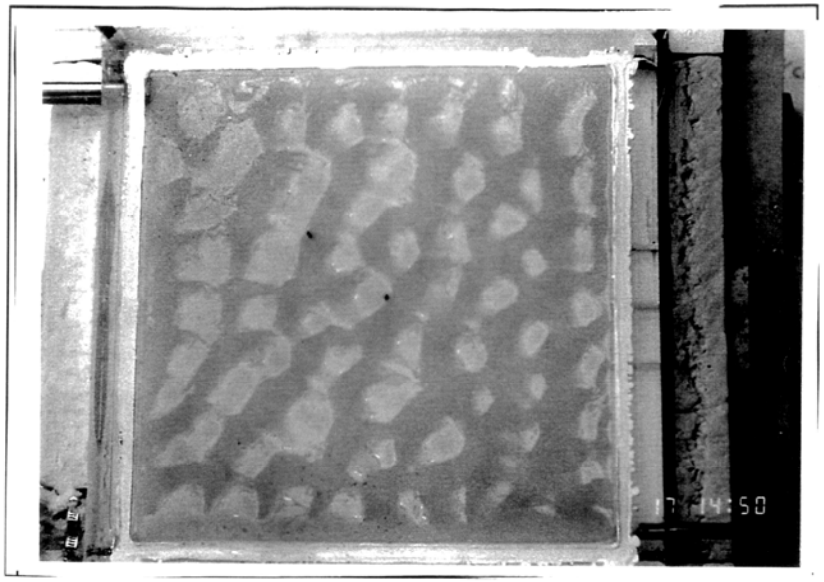
EXPERIMENTS WITH GLYCEROL

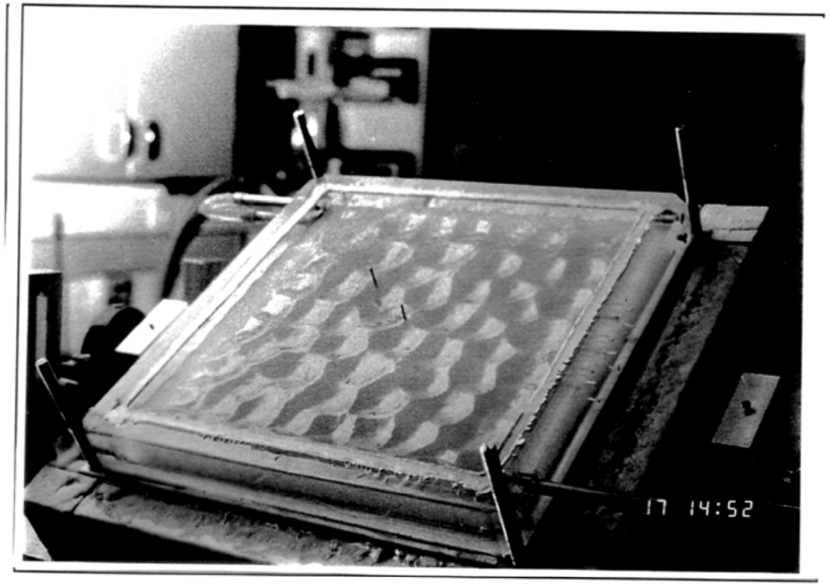
Solidifies at 18° C

High Prandl number ($Pr = 9x10^3$)






17 14:50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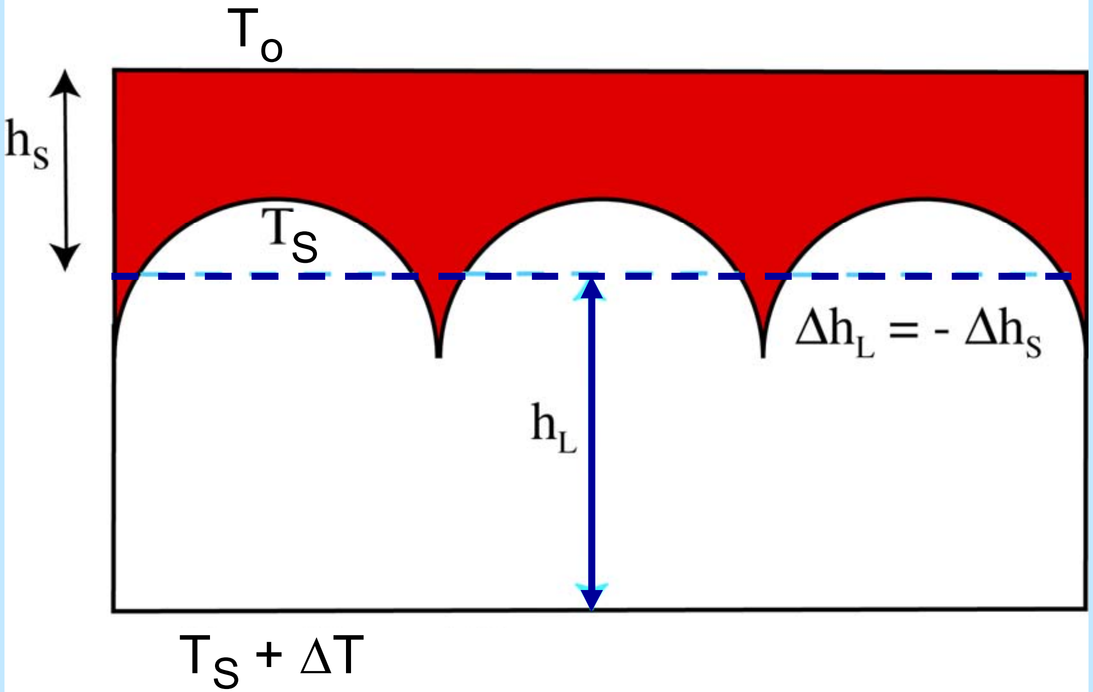
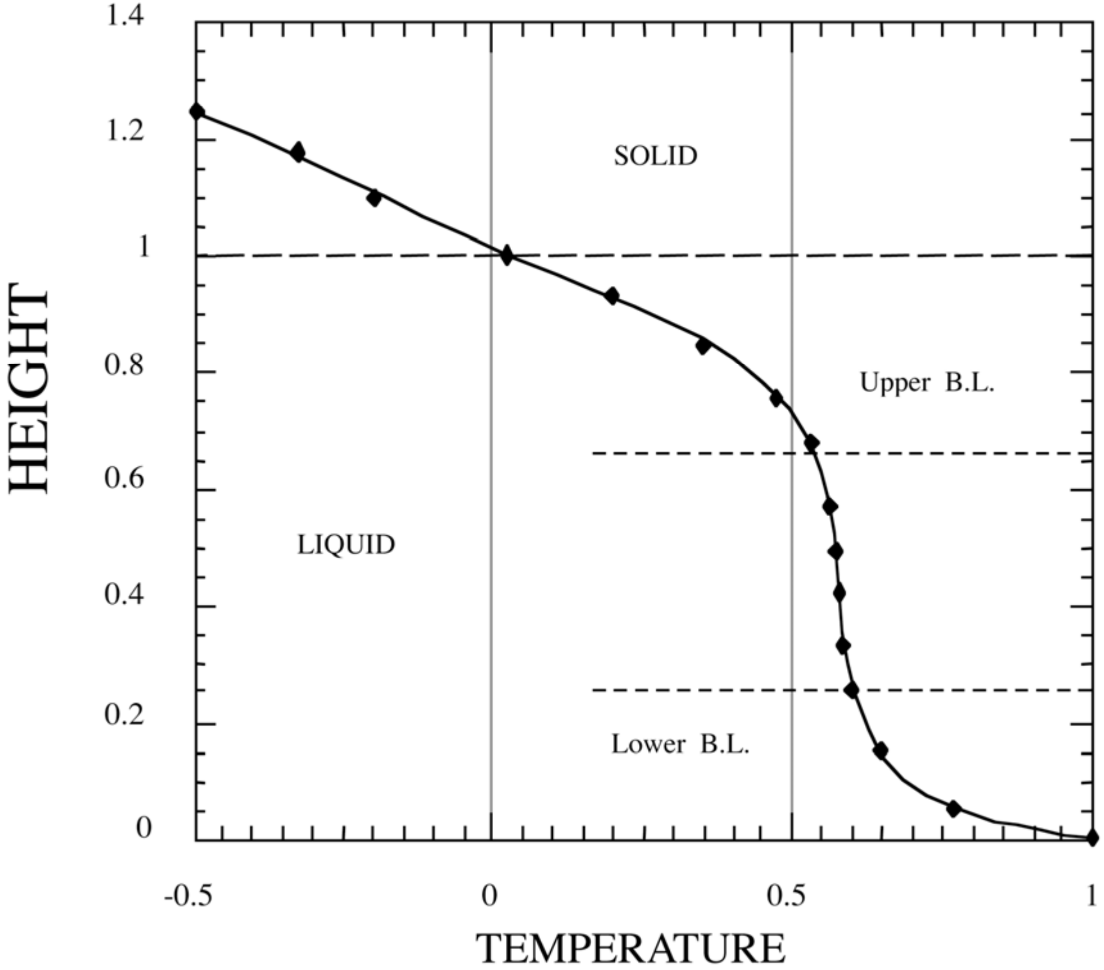


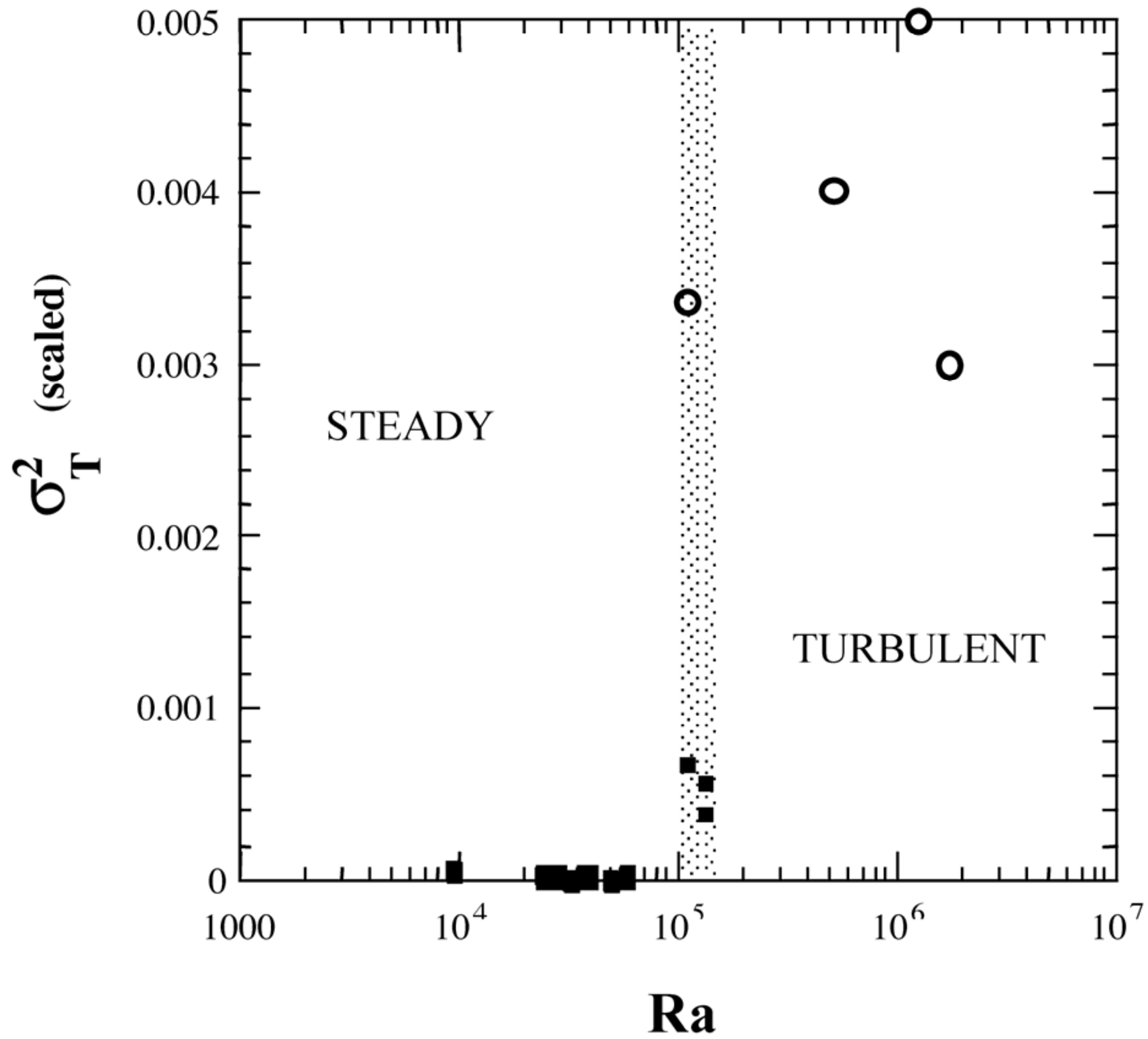
Numbers which define the experiments

$$Ra = \frac{g\,\alpha\,\Delta T\, h_L^{\,3}}{\kappa\,\nu} \qquad Pr = \nu / \kappa$$

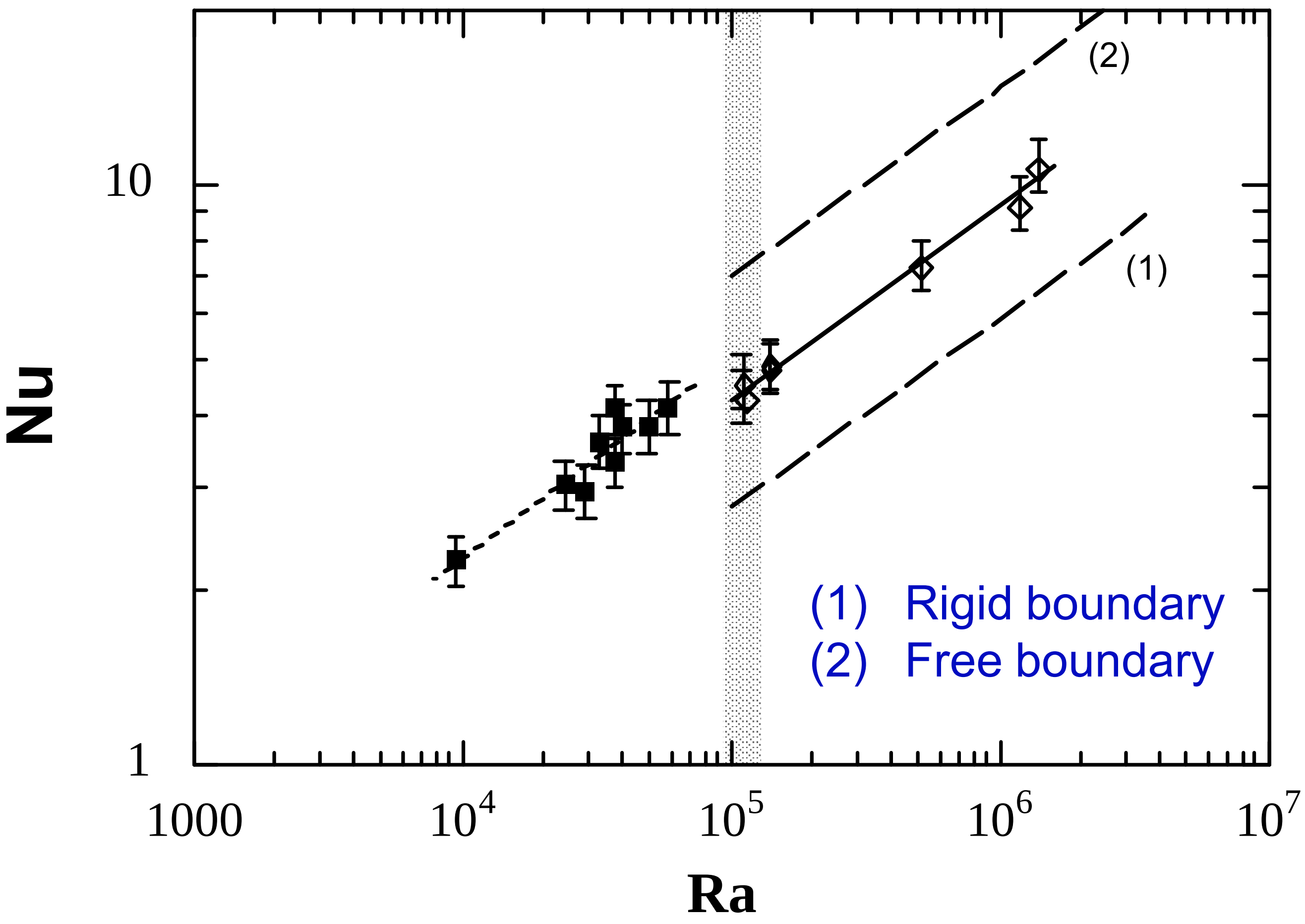
$$A = h_S / h_L$$



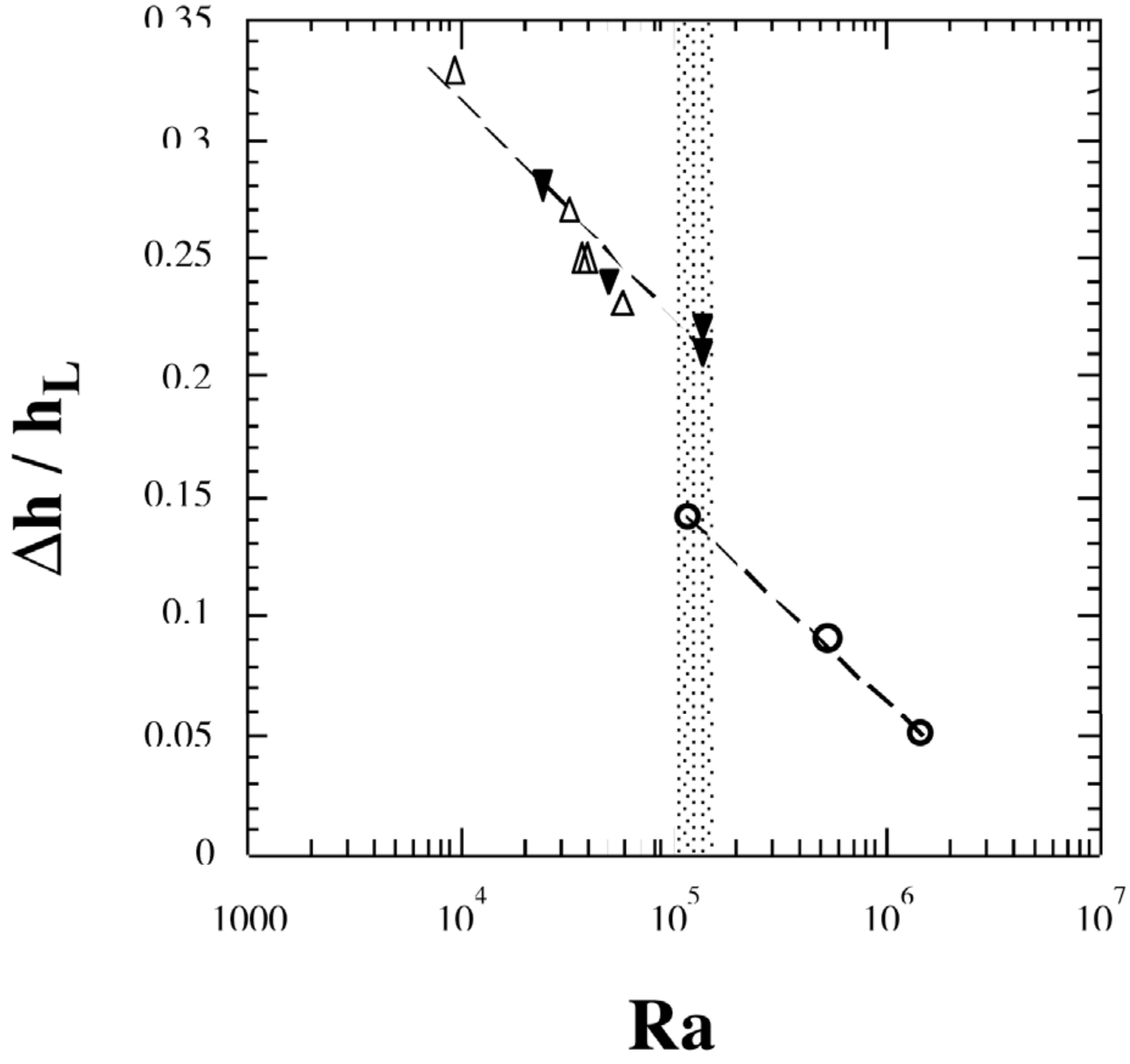
1.4
1.2
1
0.8
0.6
0.4
0.2
0
HEIGHT
SOLID
LIQUID
Upper B.L.
Lower B.L.
-0.5
0
0.5
1
TEMPERATURE



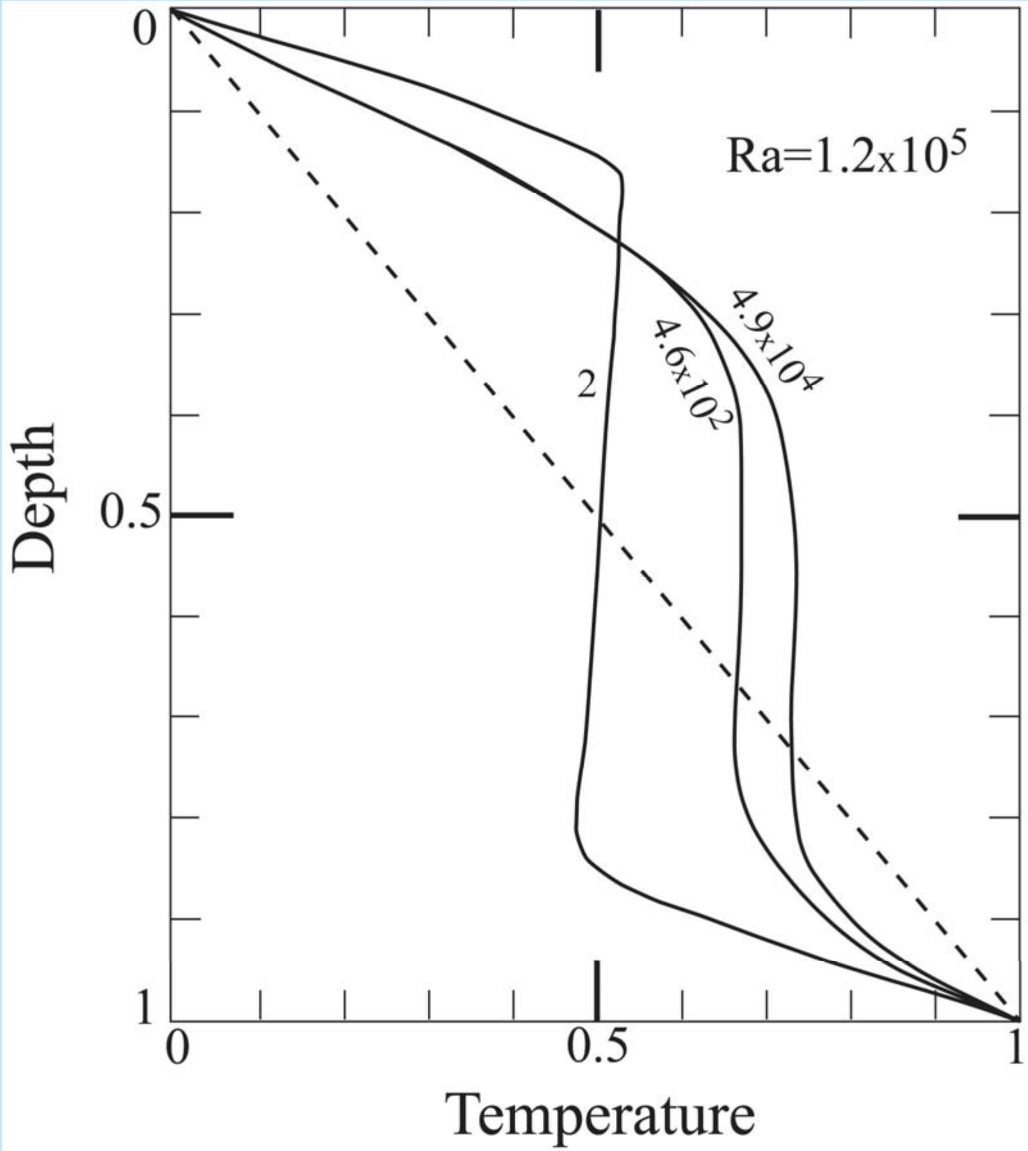
0.005
0.004
0.003
0.002
0.001
0
1000
10^4
10^5
10^6
10^7
STEADY
TURBULENT
σ_T^2 (scaled)
Ra



Nu
Ra
10
1
1000
10^4
10^5
10^6
10^7
(1)
(2)
(1) Rigid boundary
(2) Free boundary



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Δh / h_L
1000
10^4
10^5
10^6
10^7
Ra



0
0.5
1
Depth
0.5
Temperature
Ra=1.2x10^5
2
4.6x10^2
4.9x10^4

1. Moderate viscosity variations

10^1 < viscosity ratio < $\approx 10^3$

Two different thermal boundary layers

δ_T and δ_B, such that $\delta_T >> \delta_B$

ΔT_T and ΔT_B such that $\Delta T_T > \Delta T_B$
and such that $\Delta T_T + \Delta T_B = \Delta T$

Different viscosity values

μ_o at the top

μ_i in the interior (away from the boundary layers)

μ_B at the base.

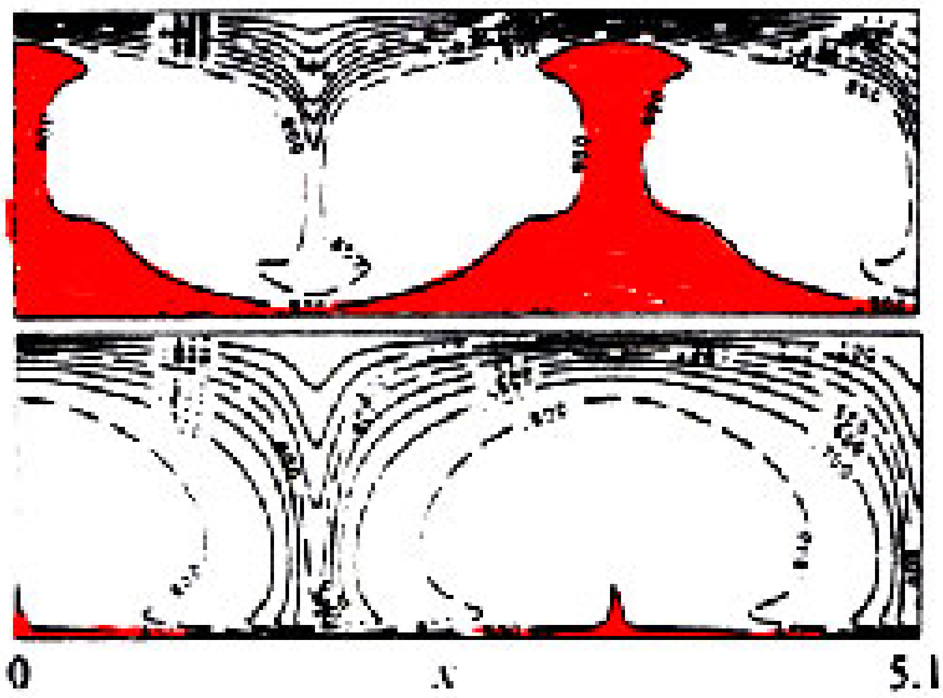
Because δ_B is thin, $\mu_i \approx \mu_B$.

$$Q = k\frac{\Delta T_T}{\delta_T} = k\frac{\Delta T_B}{\delta_B},$$

$$\Delta T_T = \Delta T\,\frac{\delta_T}{\delta_T + \delta_B}$$

$$\delta_T \sim \sqrt{\frac{\kappa h}{U_T}}.$$

$r = 3.2 \times 10^3$



In the cold upper boundary layer, dissipation is associated with the bending of a viscous layer as it is entrained into a downwelling current. For bending along a circular trajectory, the strain rate is U_T/δ_T. Dissipation is achieved in the small circular quadrant sector where bending occurs, $\delta V \approx \delta_T^2 h$. Thus,

$$\epsilon_{U,T} \sim \mu_o \left(\frac{U_T}{\delta_T}\right)^2 \delta_T^2 h.$$

Dissipation in the hot interior is achieved with velocity gradients that are distributed through the layer, implying that,

$$\epsilon_{U,B} \sim \mu_i \left(\frac{U_B}{h}\right)^2 h^3,$$

$$\mu_o \left(\frac{U_T}{\delta_T} \right)^2 \delta_T^2 h \sim \mu_i \left(\frac{U_B}{h} \right)^2 h^3 \sim \frac{\alpha g Q}{C_P} h^3.$$

$$\frac{\epsilon_{U,T}}{\epsilon_{U,B}} \sim \frac{\mu_o U_T^2}{\mu_i U_B^2} \sim \frac{\mu_o \Delta T_B^4}{\mu_i \Delta T_T^4},$$

$$\frac{\Delta T_T}{\Delta T_B} \sim \left(\frac{\mu_o}{\mu_i} \right)^{1/4}, \; \frac{U_T}{U_B} \sim \left(\frac{\mu_o}{\mu_i} \right)^{-1/2}, \; \frac{\delta_T}{\delta_B} \sim \left(\frac{\mu_o}{\mu_i} \right)^{1/4}.$$

$$\Delta T_T \sim \Delta T \frac{(\mu_o/\mu_i)^{1/4}}{1+(\mu_o/\mu_i)^{1/4}}$$

$$\mathrm{Nu} \sim \mathrm{Ra}_o^{1/3} \frac{(\mu_o/\mu_i)^{1/3}}{\left[1+(\mu_o/\mu_i)^{1/4}\right]^{4/3}},$$

where the Rayleigh number Ra_o has been calculated with the viscosity at the top.

$$\mathrm{Ra}_o = \frac{\rho_o g \alpha \Delta T h^3}{\kappa \mu_o}.$$

$$\delta_T \sim h Ra_o^{-1/3} \left[\frac{(\mu_o/\mu_i)^{1/4}}{\left[1+(\mu_o/\mu_i)^{1/4}\right)} \right]^{4/3}.$$

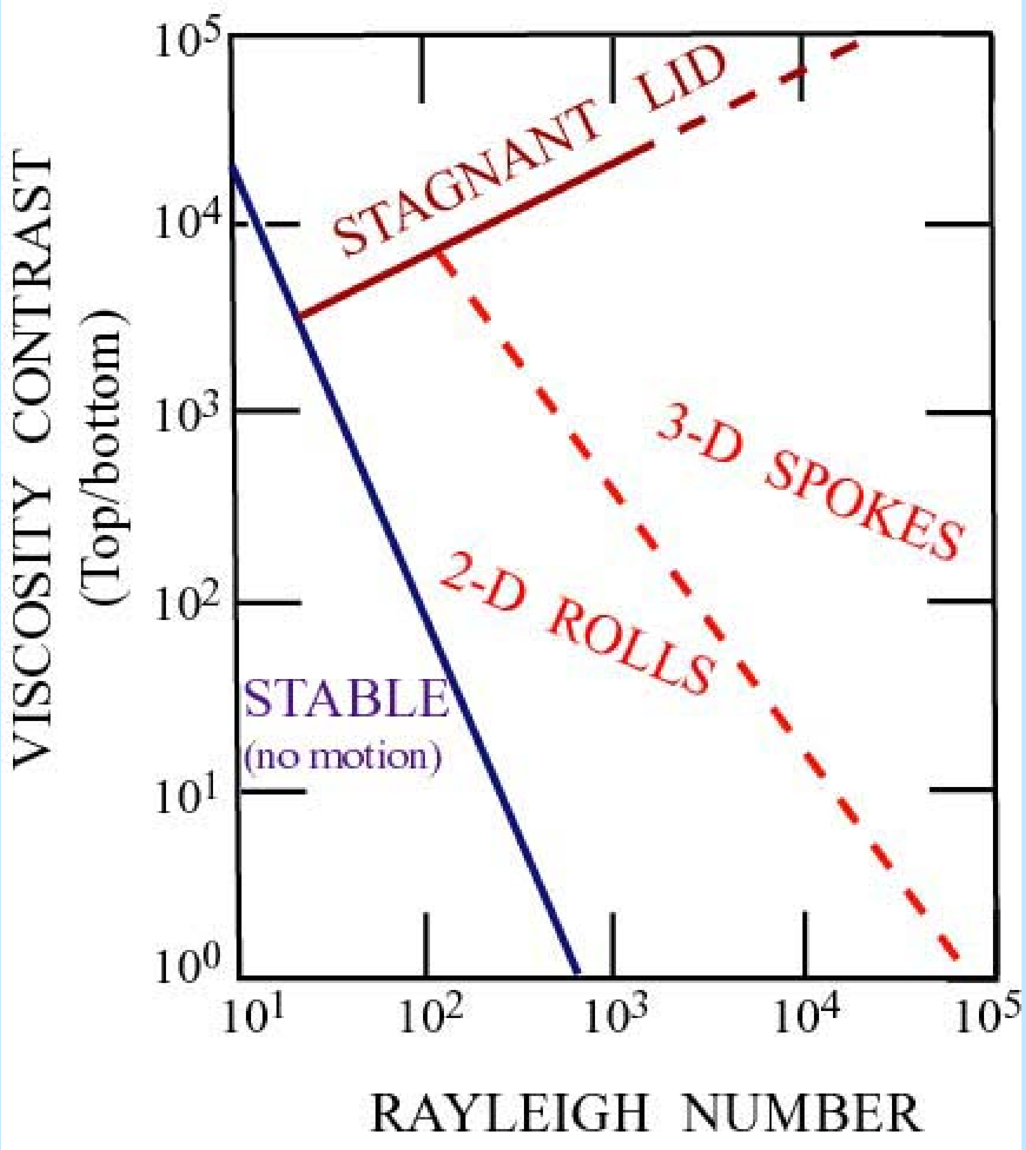
2.Large viscosity variations

viscosity ratio > ≈10^3

Upper part of the fluid remains stagnant.
Temperature scale for convecting interior is no longer ΔT, but ΔT_R.

Heat flux scale is

$$Q \sim Q_R = k\left(\frac{\rho_o g \alpha}{\kappa \mu_i}\right)^{1/3} \Delta T_R^{4/3},$$



VISCOSITY CONTRAST
(Top/bottom)
10^5
10^4
10^3
10^2
10^1
10^0
STAGNANT LID
3-D SPOKES
2-D ROLLS
STABLE
(no motion)
10^1
10^2
10^3
10^4
10^5
RAYLEIGH NUMBER

As argued above, $\Delta T_T \gg \Delta T_B$ because the upper thermal boundary layer is made of a thick stagnant lid and a thin unstable sub-layer. Also, $\Delta T_B \sim \Delta T_R$.

$$\mathrm{Nu} \sim \mathrm{Ra}_i^{1/3} \left(\frac{\Delta T_R}{\Delta T} \right)^{4/3} \ , \ \delta_B \sim h \mathrm{Ra}_i^{-1/3} \left(\frac{\Delta T_R}{\Delta T} \right)^{-1/3},$$

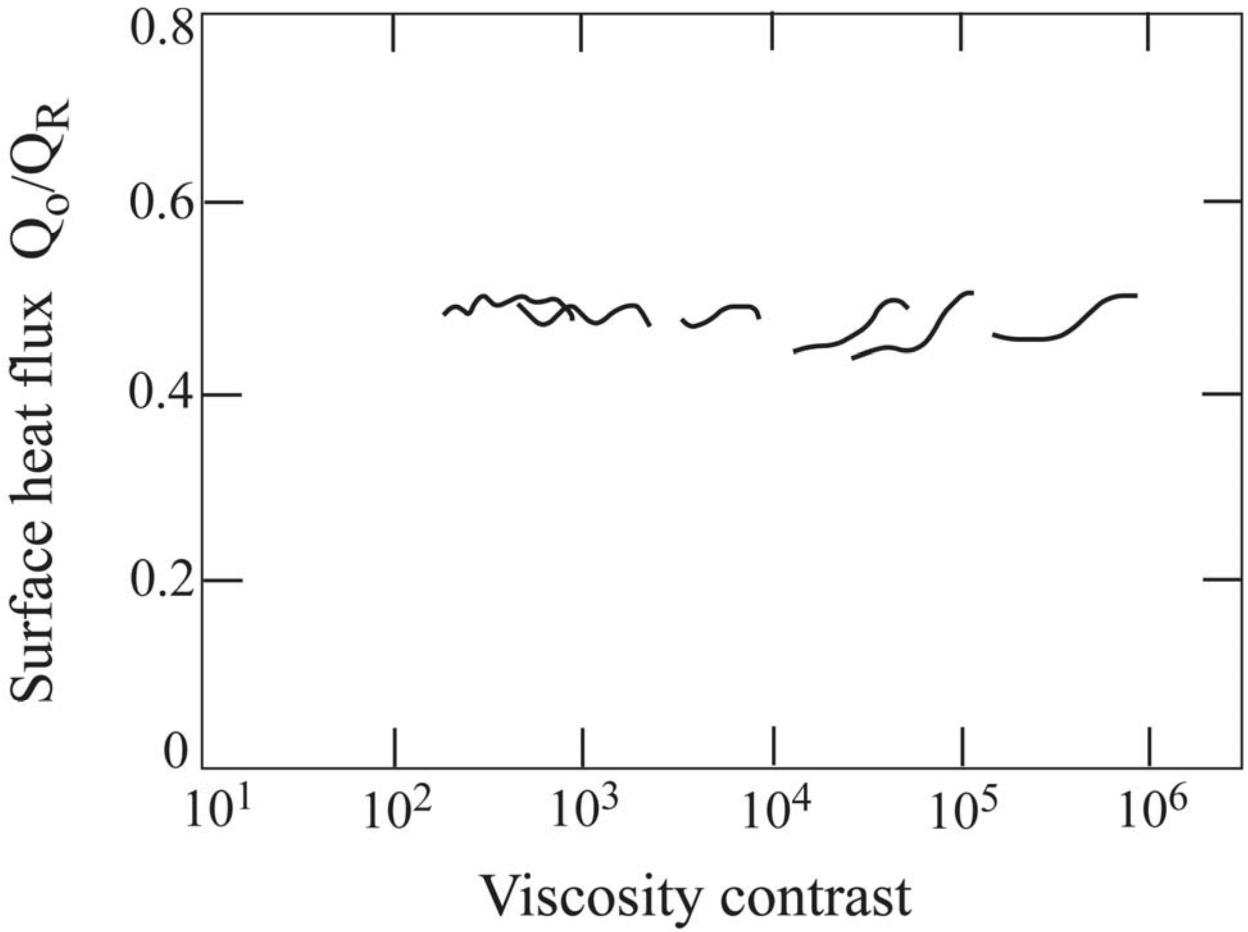
where Ra_i is a Rayleigh number calculated with the interior viscosity μ_i,

$$\mathrm{Ra}_i = \frac{\rho_o g \alpha \Delta T h^3}{\kappa \mu_i}.$$

A scaling for the thickness of the upper boundary layer can only be obtained in the limit of $\Delta T \gg \Delta T_R$. In this case, $\Delta T_T \sim \Delta T$:

$$\begin{aligned} \delta_T &\sim h \mathrm{Ra}_i^{-1/3} \left(\frac{\Delta T}{\Delta T_R} \right)^{4/3} \\ &\sim \delta_B \frac{\Delta T}{\Delta T_R}. \end{aligned}$$

$$Q \sim Q_R = k\left(\frac{\rho_o g \alpha}{\kappa \mu_i}\right)^{1/3} \Delta T_R^{4/3},$$



Non-Newtonian rheology

$$\dot{e} = b^{-1}\sigma^{n}\exp(-H/RT),$$

$$\sigma_2 = \sqrt{(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})/2}.$$

$$\dot{e}_{ij} = \frac{1}{2b}\sigma_2^{n-1}\exp\left(\frac{T-T_o}{\Delta T_R}\right)\sigma_{ij}.$$

Apparent viscosity

$$\mu = \frac{\sigma_{ij}}{2\dot{e}_{ij}} = \frac{b}{\sigma_2^{n-1}}\exp\left(-\frac{T-T_o}{\Delta T_R}\right) = \frac{b^{1/n}}{\dot{e}_2^{\frac{n-1}{n}}}\exp\left(-\frac{T-T_o}{n\Delta T_R}\right),$$

Apparent viscosity

$$\mu = \frac{\sigma_{ij}}{2\dot{e}_{ij}} = \frac{b}{\sigma_2^{n-1}} \exp\left(-\frac{T - T_o}{\Delta T_R}\right) = \frac{b^{1/n}}{\dot{e}_2^{\frac{n-1}{n}}} \exp\left(-\frac{T - T_o}{n\Delta T_R}\right),$$

$\mu(\sigma)$: which scale for magnitude of convective stresses ?

Add one dimensionless number : **n**

$$\mu = \frac{\sigma_{ij}}{2\dot{e}_{ij}} = \frac{b}{\sigma_2^{n-1}} \exp\left(-\frac{T-T_o}{\Delta T_R}\right) = \frac{b^{1/n}}{\dot{e}_2^{\frac{n-1}{n}}} \exp\left(-\frac{T-T_o}{n\Delta T_R}\right),$$

Proper temperature scale ?

ΔT_R or ($n\ \Delta T_R$) ?

What is the stress-scale for convection ?

$$\dot{e} \sim U/h$$

Use the dissipation equation

$$\sigma \frac{U}{h} hS \sim \frac{\alpha g}{C_p} QhS,$$

Add boundary layer scalings

$$Q \sim k \Delta T/\delta$$

$$\delta \sim \sqrt{\kappa h/U},$$

$$\boxed{\sigma \sim \rho_o g \alpha \Delta T \delta.}$$

1. Moderate viscosity variations

10^1 < viscosity ratio < ≈10^3

Vicosity scale for the fluid interior :

$$\mu_i = \frac{b^{1/n}}{(U/h)^{\frac{n-1}{n}}} \exp\left(-\frac{T_i - T_o}{n\Delta T_R}\right).$$

Use the dissipation equation again

$$\sigma\dot{e}h^3 \sim \frac{b^{1/n}}{(U/h)^{\frac{n-1}{n}}} \exp\left(-\frac{T_i - T_o}{n\Delta T_R}\right)\left(\frac{U}{h}\right)^2 h^3 \sim \frac{\alpha g}{C_p} Q h^3.$$

Add the boundary layer scalings

$$\delta \sim h\mathrm{Ra}_n^{-\frac{n}{n+2}}, \ \mathrm{Nu} \sim \mathrm{Ra}_n^{\frac{n}{n+2}}, \ U \sim \frac{\kappa}{h}\mathrm{Ra}_n^{\frac{2n}{n+2}}$$

where Ran is a “modified” Rayleigh number

$$\mathrm{Ra}_n = \frac{\rho_o \alpha g \Delta T h^{\frac{n+2}{n}}}{\kappa^{1/n} b^{1/n} \exp\left(-\frac{T_i - T_o}{n\Delta T_R}\right)}.$$

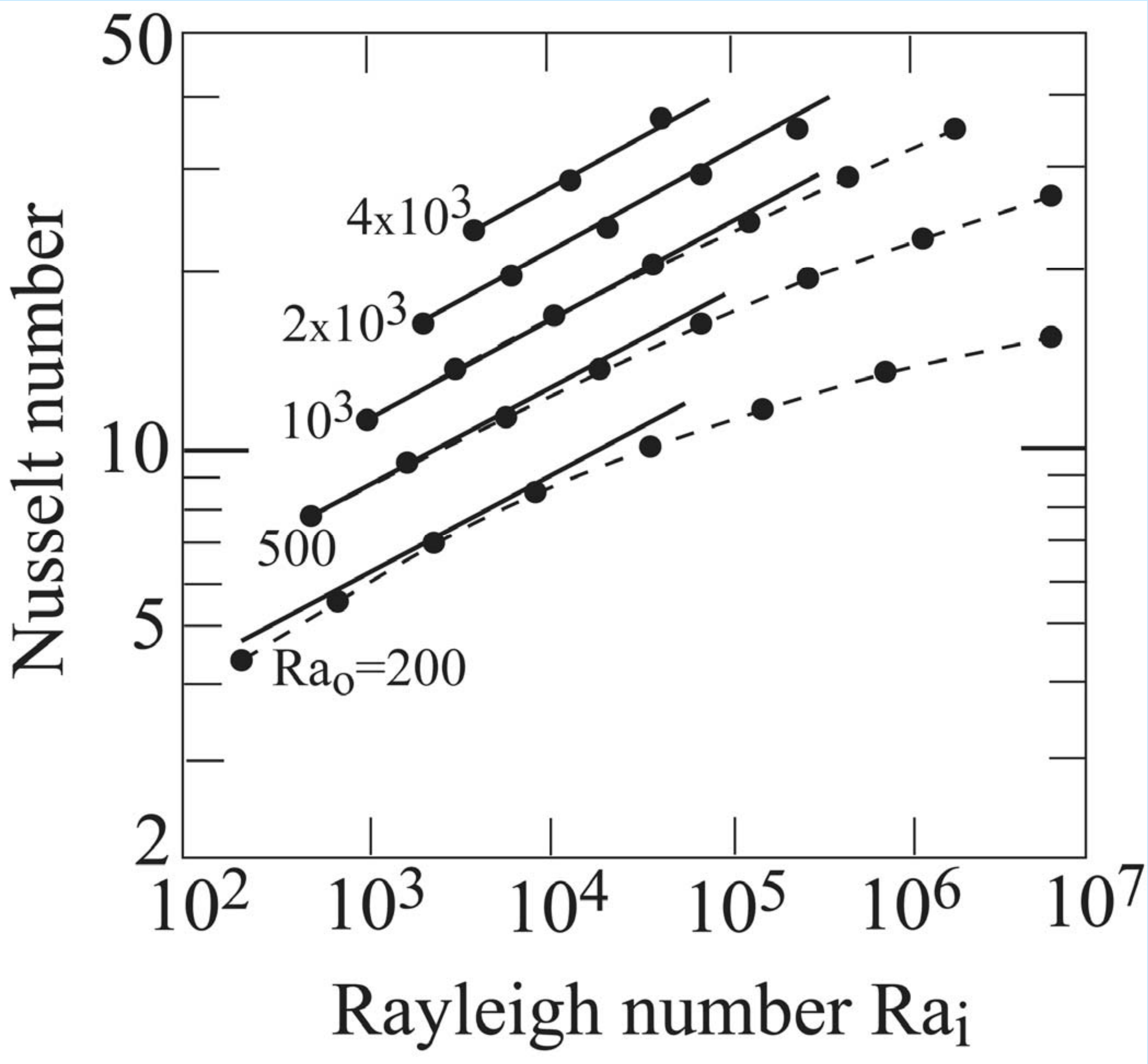
These results involve the two new dimensionless numbers
n and $\Delta \mathbf{T_R}$

In order to evaluate the impact of each number independently

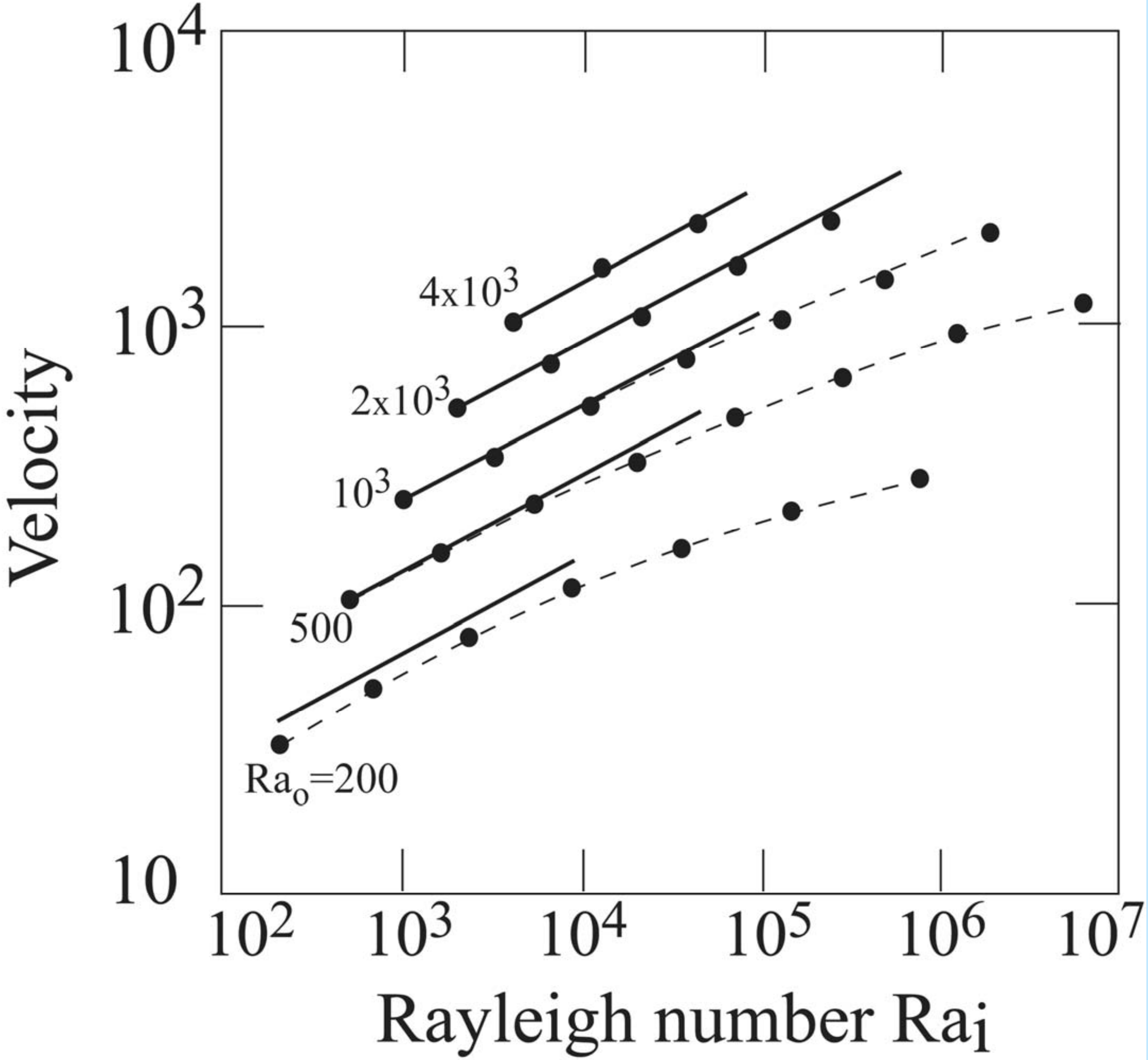
$$\mathrm{Ra}_o = \frac{\rho_o \alpha g \Delta T h^{(n+2)/n}}{\kappa^{1/n} b^{1/n}}$$

$$\mathrm{Ra}_i = \frac{\rho_o \alpha g \Delta T h^{(n+2)/n}}{\kappa^{1/n} b^{1/n} \exp\left(-\frac{T_i - T_o}{\Delta T_R}\right)} = \mathrm{Ra}_o \exp\left(\frac{T_i - T_o}{\Delta T_R}\right).$$

$$\mathrm{Nu} \sim \mathrm{Ra}_o^{\frac{n-1}{n+2}} \mathrm{Ra}_i^{\frac{1}{n+2}}.$$



50
10
5
2
Nusselt number
4x10^3
2x10^3
10^3
500
Ra_o=200
10^2
10^3
10^4
10^5
10^6
10^7
Rayleigh number Ra_i



Velocity
10^4
10^3
10^2
10
$4x10^3$
$2x10^3$
10^3
500
Ra_o=200
10^2
10^3
10^4
10^5
10^6
10^7
Rayleigh number Ra_i

The modified Rayleigh number that enters these scaling laws, Ra_n can be written in terms of a reference viscosity μ_n:

$$\mathrm{Ra}_n = \frac{\rho_o g \alpha \Delta T h^3}{\kappa \mu_n},$$

$$\text{where} \quad \mu_n = \frac{b^{1/n}}{(\kappa/h^2)^{\frac{n-1}{n}}} \exp\left(-\frac{T_i - T_o}{n \Delta T_R}\right).$$

Heat flux

$$Q \sim k\left(\frac{\rho_o g\alpha}{\kappa^{1/n} b^{1/n} \exp\left(-\frac{T_i - T_o}{n\Delta T_R}\right)}\right)^{\frac{n}{n+2}} \Delta T^{\frac{2(n+1)}{n+2}}.$$

This can be rewritten in the familiar form

$$Q \sim k\left(\frac{\rho_o g\alpha}{\kappa\mu_i}\right)^{1/3} \Delta T^{4/3},$$

where μ_i is the "effective" viscosity for convection

$$\mu_i = \frac{\left[b^{1/n} \exp\left(-\frac{T_i - T_o}{n\Delta T_R}\right)\right]^{\frac{3n}{n+2}}}{\kappa^{\frac{n-1}{n+2}} \left(\rho_o g\alpha\Delta T\right)^{\frac{2(n-1)}{n+2}}}.$$

Add boundary layer scalings

$$Q \sim \lambda \Delta T / \delta \text{ and } U \sim \kappa h / \delta^2$$

Explicit expression for convective stress scale

$$\begin{aligned}
\sigma &\sim \rho_o g \alpha \Delta T \delta \\
&\sim \mu_i \frac{U}{h} \\
&\sim \left[b\kappa \left(\rho_o g \alpha \Delta T\right)^2 \exp\left(-\frac{T_i - T_o}{\Delta T_R}\right)\right]^{\frac{1}{n+2}} .
\end{aligned}$$

Substituting for σ into rheological equation leads to viscosity μ_i.

2.Large viscosity variations viscosity ratio > ≈10^3

$$Q = C(n)\,k \frac{(\rho_o g \alpha)^{\frac{n}{n+2}}}{\left[\kappa b \exp\left(-\frac{T_i - T_o}{\Delta T_R}\right)\right]^{\frac{1}{n+2}}} \Delta T_R^{\frac{2(n+1)}{n+2}},$$

n	$C(n)$ ‡
1	0.528
2	0.755
3	0.971

Numerical calculations by Solomatov & Moresi (2000)

Temperature difference across the unstable boundary layer

(from Solomatov & Moresi, 2000)

n	$\Delta T_\delta / \Delta T_R$
1	2.4
2	3.6
3	4.8

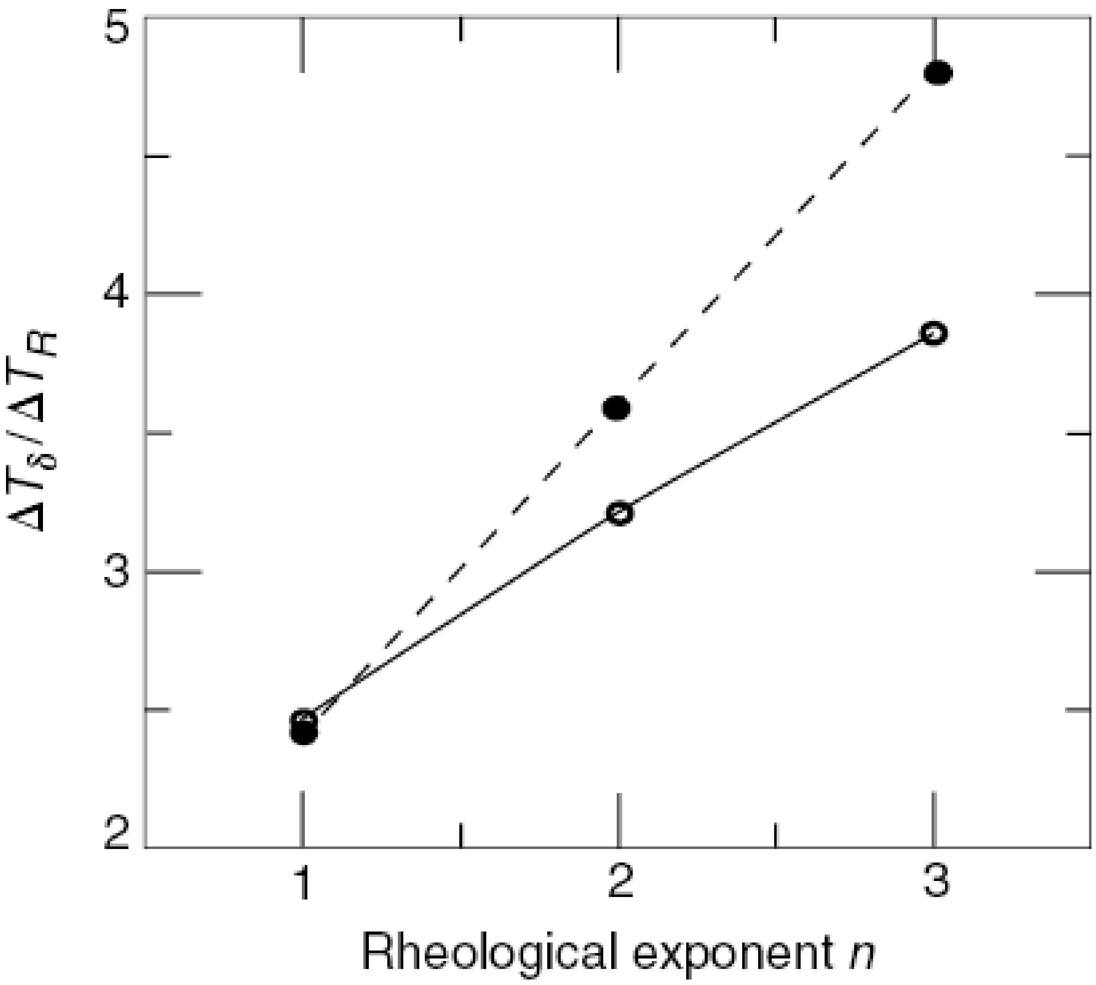
Method: from independent determination
of boundary layer thickness
using the vertical velocity profile

Another way of determining ΔT_δ

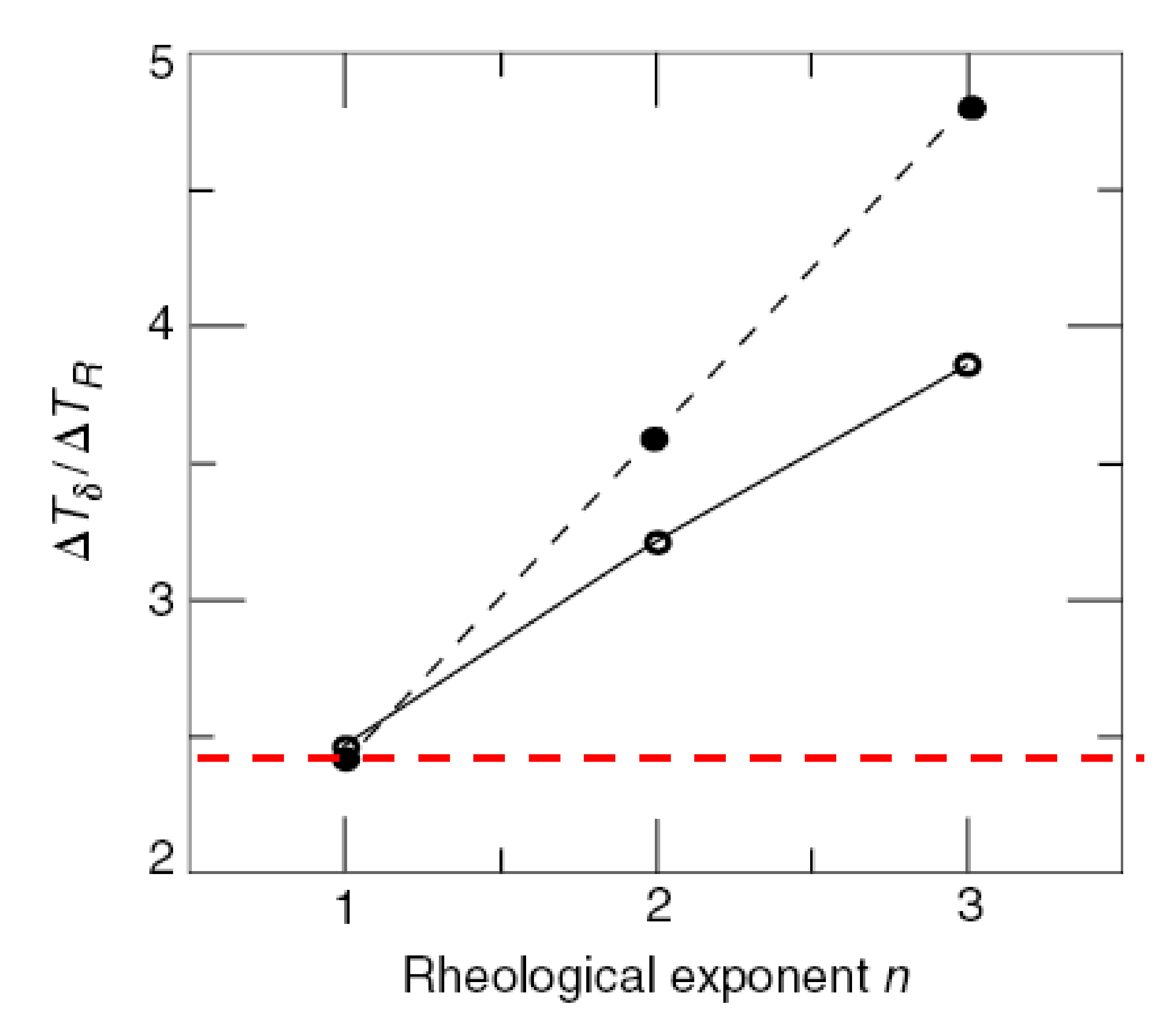
$$Q = C_o \lambda \left(\frac{\rho_o g \alpha}{\kappa \mu_i} \right)^{1/3} \Delta T_\delta^{4/3},$$

where C_o is the value determined for constant viscosity fluids. In this case:

$$\Delta T_\delta = \left(\frac{C(n)}{C_o} \right)^{3/4} \Delta T_R.$$



5
4
3
2
$\Delta T_\delta / \Delta T_R$
1
2
3
Rheological exponent n



Value from laboratory experiments

Rheological temperature scale for mantle rheologies †

Creep regime	E (kJ $mole^{-1}$)	V (cm^3 $mole^{-1}$)	ΔT_R (K) §
Dry diffusion	261	6	92
Wet diffusion	387	25	62
Dry dislocation	610	13	39
Wet dislocation	523	4	46

† Representative values from (Korenaga and Karato, 2008).

T_i = 1700 K, p = 6 GPa