The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA International Atomic Energy Agency

2240-2

Advanced School on Scaling Laws in Geophysics: Mechanical and Thermal Processes in Geodynamics

23 May - 3 June, 2011

Strokes equation, equation of continuity, and constitutive laws (both Newtonian, and non-Newtonian with underlying physics)

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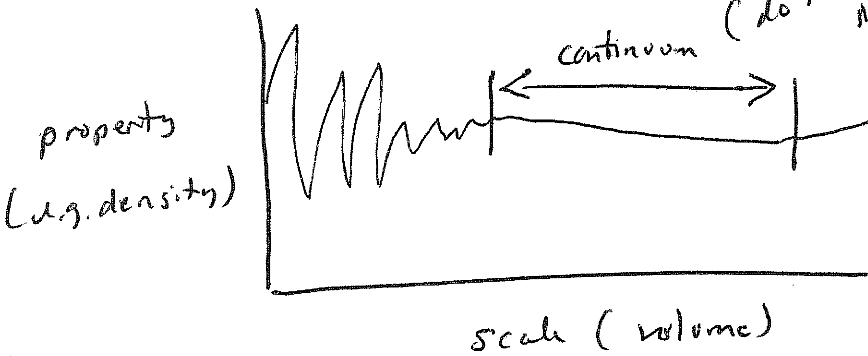
Continuum hypothesis



goal: derive equations for fluid motion (and assumptions made)

notation

a , $\underline{a}$, $\underline{\underline{a}}$

scalar, vector, 2nd rank tensor

$\underline{U} = U_i$ (or $U_i \underline{e}_i$ ← unit vector in i direction)

vector calculus review

$$\underline{a} \cdot \underline{b} = a_i \overbrace{\underline{e}_i \cdot b_j \underline{e}_j}^{\delta_{ij}} \qquad \delta_{ij} = \begin{cases} 0 & i=j \\ 1 & i \neq j \end{cases}$$

$$= a_i b_i$$

$$= \underbrace{\sum_{i=1}^{3} a_i b_i}_{\text{Einstein summation convention}} = a_x b_x + a_y b_y + a_z b_z$$

$$\underline{\underline{I}} = \delta_{ij} \underline{e}_i \underline{e}_j$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \phi \end{pmatrix}$$

$\underline{a} \wedge \underline{b} = a_i \underline{e}_i \wedge b_j \underline{e}_j$

$= a_i b_j \underline{e}_i \wedge \underline{e}_j$

$\underline{e}_i \wedge \underline{e}_j = \epsilon_{ijk} \underline{e}_k$ ← new direction ⊥ to other 2!

$$\epsilon_{ijk} = \begin{cases} 0 & \text{any 2 of } ijk \text{ equal} \\ +1 & \text{cyclic } 123 \quad 231 \quad 312 \\ -1 & \text{anticyclic } 213 \quad 321 \quad 132 \end{cases}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x_i} \underline{e}_i$$

$$\nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x_i} \overbrace{\underline{e}_i \cdot v_j \underline{e}_j}^{1} = \frac{\partial v_j}{\partial x_i} \delta_{ij} = \frac{\partial v_i}{\partial x_i}$$

$$= v_{i,i}$$

$$\nabla \wedge \underline{v} = \begin{vmatrix} \underline{e}_x & \underline{e}_y & \underline{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, -\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

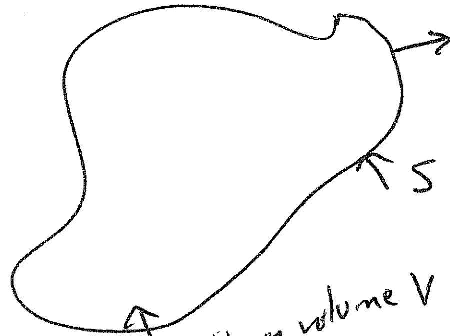
$$= \frac{\partial}{\partial x_i} \underline{e}_i \wedge v_j \underline{e}_j = \frac{\partial v_j}{\partial x_i} \epsilon_{ijk} \underline{e}_k = v_{j,i} \epsilon_{ijk} \underline{e}_k$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$\nabla^2 a = \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla^2 \underline{v} = \frac{\partial}{\partial x_k} \overbrace{\underline{e}_k \cdot \frac{\partial v_j}{\partial x_i} \underline{e}_i}^{\delta_{ik}} \underline{e}_j = \frac{\partial^2 v_j}{\partial x_i^2} \underline{e}_j$$

Conservation of mass



Mass flux at each point on S

$$\underbrace{\rho}_{\text{not necessarily constant}} \underline{u}\cdot\underline{n} \quad \text{mass flux}$$

$$\underbrace{\int_S \rho \underline{u}\cdot\underline{n}\, ds}_{\text{net flow of mass out of } V} = \underset{\text{why?}}{-} \underbrace{\int_V \frac{\partial \rho}{\partial t}\, dv}_{\text{change of mass in } V}$$

Use divergence theorem

$$\int_S \rho \underline{u}\cdot\underline{n}\, ds = \int_V \nabla\cdot(\rho \underline{u})\, dv$$

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) \right] dv = 0$$

since V is arbitrary

$$\left.\frac{\partial \rho}{\partial t} + \nabla\cdot\rho u = 0 \right] \text{ continuity equation}$$

if $\rho =$ constant

$$\underbrace{\nabla \cdot \underline{v} = 0}_{\text{incompressible fluid}} \qquad \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \right)$$

Material derivation – derivative moving with fluid $\underline{u}$

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial x} + u \cdot \nabla \phi$$

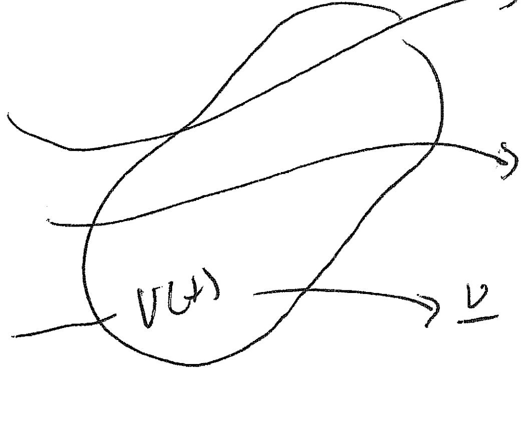
ϕ varies in $\underline{x}, t$

$$d\phi = \left.\frac{\partial \phi}{\partial t}\right|_x dt + \left.\frac{\partial \phi}{\partial x}\right|_t dx$$

$$\frac{d\phi}{dt} = \frac{\partial \phi}{dt} \cdot \frac{dt}{dt} + \frac{\partial \phi}{\partial x} \cdot \underbrace{\frac{dx}{dt}}_{v}$$

generalize

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \underline{u} \cdot \nabla \phi$$



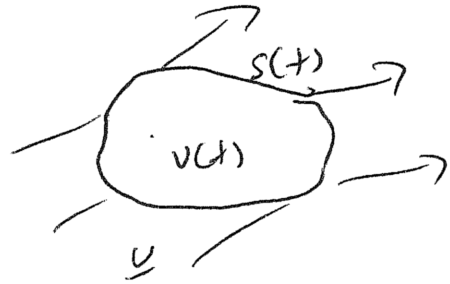
Reynolds transport theorem

$$\frac{D}{Dt}\left[\int_{V(t)} \phi \, dv\right] = \int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\underline{u}\phi)\right] dV$$

revisit mass conservation

$$\frac{D}{Dt}\left[\int_{V(t)} \rho \, dv\right] = 0$$

$$= \int_{V(t)} \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u})\right]}_{=0} dv$$



Conservation of momentum

$m\underline{a}$	$=$	$\underline{f}$	Newton's 2nd law $\underline{F} = m\underline{a}$
time-rate-of-change of momentum	$=$	sum of forces acting on body	

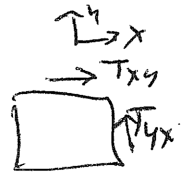
$$\frac{D}{Dt}\left\{\int_{V(t)} \rho \underline{u} \, dV\right\} = \underbrace{\int_{V(t)} \rho \underline{g} \, dV}_{\text{body forces (gravity)}} + \int_{S(t)} \underline{t} \, dS$$

surface forces

let $\underline{t} = \underline{n} \cdot \underline{\underline{T}}$

$\underline{\underline{T}} = T_{ij}$ ← direction of surface

↑ direction of force



LHS use RTT

$$\int_{V(t)} \left\{ \frac{\partial}{\partial t}(\rho \underline{v}) + \nabla \cdot (\rho \underline{v}\,\underline{v}) \right] dV$$

$$\int_{S(t)} (\underline{n} \cdot \underline{\underline{T}})\, dS = \int_{V(t)} \nabla \cdot \underline{\underline{T}}\, dV$$

since V is arbitrary

$$\underbrace{\frac{\partial(\rho \underline{v})}{\partial t} + \nabla \cdot (\rho \underline{v}\,\underline{v})}_{\substack{\text{acceleration} \\ \text{inertial forces}}} = \rho \underline{g} + \nabla \cdot \underline{\underline{T}}$$

$$\nabla \cdot (\underline{v}\,\underline{v}) = \underline{v}(\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \underline{v}$$

if $\rho =$ constant

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) &= \rho \underline{g} + \nabla \cdot \underline{\underline{T}} \\ \nabla \cdot \underline{v} &= 0 \end{aligned}\right\} \text{Cauchy equation of motion}$$

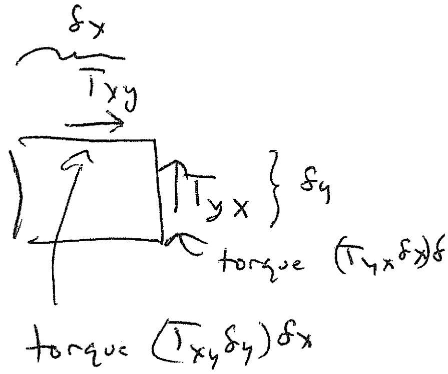
Are we ready to solve problems ?

\# unknowns : $\underline{\underline{T}}$ has 9 $\underline{v}$ has 3 } 12 total

\# equations : 4

$\boxed{\text{oops}}$

Conserve angular momentum



$T_{xy} = T_{yx}$ so there is no net torque

$\underline{\underline{T}}$ is symmetric

$(\underline{\underline{T}} = \underline{\underline{T}}^T$ or $T_{ij} = T_{ji})$

$\underline{\underline{T}}$ has 6 indep components

only hope ...

$$\underbrace{\text{express } \underline{\underline{T}} \text{ in terms of } \underline{u}}_{\text{constitutive relationship}}$$

<u>Constitutive relationships</u>

$$\underline{\underline{T}} = \underbrace{\underbrace{-p\,\underline{\underline{I}}}_{\substack{\text{isotropic part}\\ \text{(pressure)}}} + \underbrace{\underline{\underline{\tau}}}_{\substack{\text{deviatoric}\\ \text{part}}}(\underline{u}, \nabla\underline{u}, \nabla^2\underline{u}, \int \underline{u}\,dt, \ldots)}_{\text{general}}$$

assume $\underline{\underline{\tau}} = \underline{\underline{\tau}}(\nabla\underline{u})$

but $\underline{\underline{\tau}}$ is symmetric so

$$\nabla\underline{u} = \underbrace{\overset{\text{symmetric}}{\tfrac{1}{2}(\nabla\underline{u} + \nabla\underline{u}^T)}}_{\substack{\underline{\underline{E}}\\ \text{rate-of-strain}\\ \text{tensor}}} + \underbrace{\overset{\text{antisymm}}{\tfrac{1}{2}(\nabla\underline{u} + \nabla \ldots}}_{\substack{\underline{\underline{\Omega}}\\ \text{vorticity}\\ \text{tensor}}}$$

Assume $\underline{\underline{\tau}} \propto \underline{\underline{E}}$ (Newtonian)

$$= \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{E}}$$

81 components, only 2 independent ones for an isotropic fluid

$$\underline{\underline{T}} = (-p + \lambda \nabla\cdot\underline{u})\, \underline{\underline{I}} + 2\mu \underline{\underline{E}}$$

λ: bulk viscosity; μ: shear viscosity

if $\nabla\cdot\underline{u} = 0$ $\quad \underline{\underline{T}} = -p\,\underline{\underline{I}} + 2\mu \underline{\underline{E}}$

$$\left.\begin{aligned} \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) &= \rho \underline{g} - \nabla p + \mu \nabla^2 \underline{u} \\ \nabla\cdot\underline{u} &= 0 \end{aligned}\right\} \text{Navier Stokes equations}$$

4 unknowns $(p, \underline{u})$
4 equations

assumptions
- $\rho =$ const (λ does not matter)
- isotropic
- $\mu =$ constant
- Newtonian ($\underline{\underline{\tau}} \propto \nabla\underline{u}$)

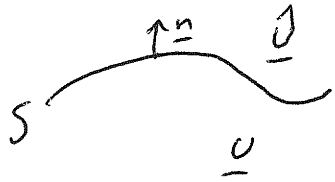
Boundary conditions

$y \uparrow \rightarrow x$

1) $\underline{u} = \underline{0}$ on a solid surface (noslip)

2) free-slip $u_y = 0$, $\frac{\partial u_x}{\partial y} = 0$ (no shear stress)

3)



$\hat{\underline{U}} \cdot \underline{n} = \underline{u} \cdot \underline{n}$

kinematic condition

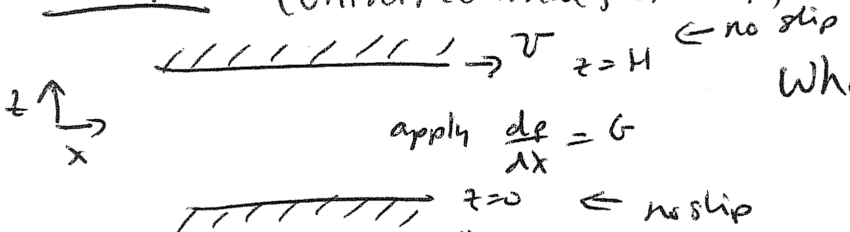
4) dynamic condition

tangential $\underline{u}$ is continuous

$3+4 \Rightarrow \quad \underline{u} = \hat{\underline{U}}$ on S

not true at contact lines

Example (unidirectional, steady)



What is $u(x,z)$?

$$\nabla p = \mu \nabla^2 \underline{v}$$

$$\frac{\partial p}{\partial z} = \mu \left(\cancel{\frac{\partial^2 v_z}{\partial x^2}}^{\,0} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}^{\,0} \right)$$

$$\Rightarrow \frac{dp}{dz} = 0 \quad \Rightarrow p \text{ constant with respect to } z$$

$$\frac{\partial p}{\partial x} = \mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}^{\,0} + \frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\frac{dp}{dx} = \mu \frac{d^2 v_x}{dz^2}$$

integrate wrt to z

$$\frac{dp}{dx} z = \mu \frac{dv_x}{dz} + C_1$$

integrate again

$$v_x = \frac{1}{2\mu}\frac{dp}{dx} z^2 + C_1 x + C_2$$

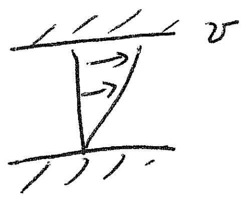
at $z = 0$, $v_x = 0 \Rightarrow C_2 = 0$

$z = H$, $v_x = \frac{H^2}{2\mu}\frac{dp}{dx} H^2 + C_1 H = 0$

$$\therefore\ v_x = \frac{1}{2\mu}(z^2 - Hz)\frac{dp}{dx} + v\, z/H$$

a) if $dp/dx = 0$

$v_x = v\, z/H$



linear

b) if $v = 0$

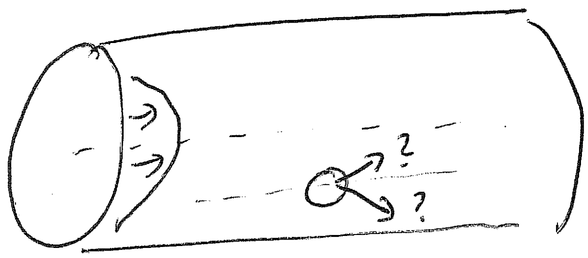
$$v_x = \frac{1}{2\mu}(z^2 - Hz)\frac{dp}{dx}$$

parabolic

superimpose owing to linearity

solve equivalent problem for a cylinder

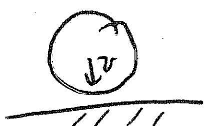
Linearity + reversibility of Stokes flows

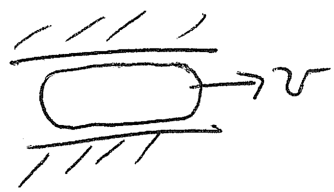


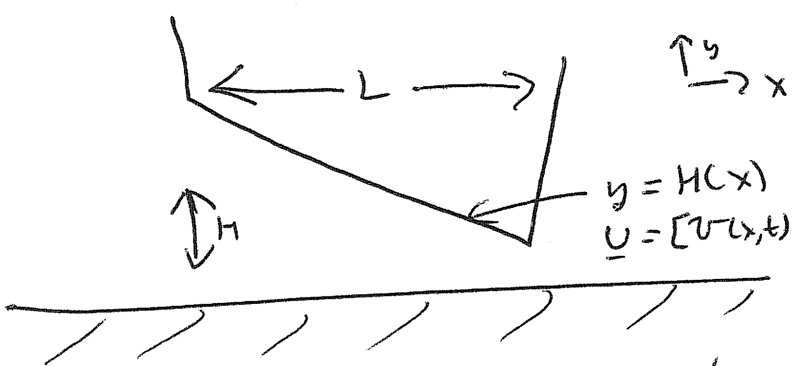
$Re \ll 1$; no buoyancy

Does sphere migrate towards or away from center line?

"Lubrication" model







$\epsilon = \frac{H}{L} \ll 1$

$y = H(x)$

$\underline{U} = [U(x,t), V(x,t), 0]$ steady flow

$v = 0$

let $u = U_c u'$; $v = V_c v'$

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \; ; \; \frac{\partial}{\partial y} = \frac{1}{H}\frac{\partial}{\partial y'}$$

continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_c}{L}\frac{\partial u'}{\partial x'} + \frac{V_c}{H}\frac{\partial v'}{\partial y'} = 0$$

$$V_c \sim \frac{H}{L} U_c = \epsilon U_c$$

X - component N.S.

$$\rho U \frac{\partial u}{\partial x} + \rho V \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho \frac{v_c^2}{L} u' \frac{\partial u'}{\partial x'} + \rho \frac{v_c}{L^2} v' \frac{\partial u'}{\partial y'} = -\frac{p_c}{L}\frac{\partial p'}{\partial x'} + \mu \frac{v_c}{L^2}\frac{\partial^2 u'}{\partial x'^2} + \mu \frac{v_c}{H^2}\frac{\partial^2 u'}{\partial y'^2}$$

multiply by $H^2/\mu v_c$

$$\underbrace{\rho \frac{H^2 v_c}{L\mu}}\left(u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'}\right) = -\frac{p_c H^2}{L \mu v_c}\frac{\partial p'}{\partial x'} + \cancel{\epsilon^2 \frac{\partial^2 u'}{\partial x'^2}}^{\,0} + \frac{\partial^2 u'}{\partial y'^2}$$

$$\frac{H^2}{L^2} Re = \epsilon^2 Re$$

$$p_c \sim \mu v_c \frac{L}{H^2} = \frac{1}{\epsilon^2}\left(\frac{\mu v_c}{L}\right)$$ normal characteristic pressure

y - component

$$\rho\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho v_c^2 \frac{H}{L^2}\left(u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'}\right) = -\mu \frac{v_c L}{H^3}\frac{\partial p'}{\partial y'} + \mu \frac{v_c H}{L^3}\left(\frac{\partial^2 v'}{\partial x'^2}\right) + \frac{\mu v_c}{L H^2}\frac{\partial^2 v'}{\partial y'}$$

multiply by $H^3/\mu v_c L$

$$\underbrace{\rho \frac{v_c H^4}{\mu L^3}}_{\epsilon^4 Re}\left(\qquad \right) = -\frac{\partial p'}{\partial y'} + \overbrace{\left(\frac{H^4}{L^4}\right)}^{\epsilon^4}\frac{\partial^2 v'}{\partial x'^2} + \overbrace{\left(\frac{H^2}{L^2}\right)}^{\epsilon^2}\frac{\partial^2 v'}{\partial y'^2}$$

$$\Rightarrow \frac{\partial p'}{\partial y'} = 0$$

Governing equations

$$
\left.\begin{array}{ll}
(1) & \dfrac{dp}{dy} = 0 \\[2ex]
(2) & \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\[2ex]
(3) & \dfrac{\partial p}{\partial x} = \mu \dfrac{\partial^2 u}{\partial y^2}
\end{array}\right\}
$$

$Re\,\epsilon^2$ is small

ϵ is small

Q: Given $\mathcal{V}$, what is Force?
-or- given Force, what is $\mathcal{V}$?

integrate (3) — can do because $\frac{\partial p}{\partial x}$ is indep of y (equation 1)

$$U(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + c_1(x)\,y + c_2(x)$$

as before

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - H(x,t)\,y\right) + \frac{\mathcal{V}(x,t)}{H}\,y$$

but what is dp/dx?

Have not used boundary conditions on v (just on u)

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{1}{2\mu} y(y-H)\frac{d^2 p}{dx^2} + \frac{1}{2\mu}\frac{dp}{dx} y \frac{dH}{dx} - \frac{dU}{dx}\frac{y}{H} + \frac{Uy}{H^2}\frac{dH}{dx}$$

integrate wrt to y to get v

$$v = \underbrace{-\frac{1}{2\mu}\frac{d^2 p}{dx^2}\left(\frac{y^3}{3} - \frac{y^2 H}{2}\right) + \frac{1}{2\mu}\frac{dp}{dx}\frac{dH}{dx}\frac{y^2}{2}} - \left(\frac{dU}{dx}\frac{1}{H} - \frac{U}{H^2}\frac{dH}{dx}\right)\frac{y^2}{2} + C_3$$

$v = 0$ at $y = 0 \Rightarrow C_3 = 0$

at $y = H$, $v = V$

$$V = \underbrace{\frac{1}{12\mu}\frac{d^2 p}{dx^2}H^3 + \frac{1}{4\mu}\frac{dp}{dx}H^2\frac{dH}{dx}}_{\frac{1}{12\mu}\frac{d}{dx}\left(H^3\frac{dp}{dx}\right)} - \frac{1}{2}\frac{dU}{dx}H - \frac{1}{2}U\frac{dH}{dx}$$

$$\boxed{\frac{d}{dx}\left(H(x)^3\frac{dp}{dx}\right) = 6\mu\left[H(x)\frac{dU}{dx} - U\frac{dH(x)}{dx} + 2V\right]}$$

2^{nd} order equation for $p(x)$

Reynolds equation

solve for p in terms of $H(x)$, U, V

need B.C. on P at edges of gap

let v = constand

$V = 0$

$$\frac{d}{dx}\left(H^3 \frac{dp}{dx}\right) = -6\mu v \frac{dH}{dx}$$

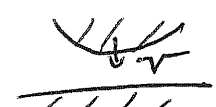
integrate once

$$\frac{H^3}{6\mu}\frac{dp}{dx} = -vH + c_1$$

integrate again

$$p(x) = -6\mu v \int^x \frac{H(\xi)+a_1}{H^3(\xi)} + a_2$$

let $v = 0$

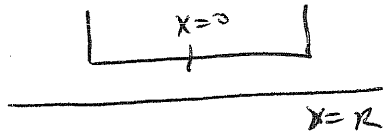


$$\frac{1}{12\mu}\frac{d}{dx}\left(H^3 \frac{dp}{dx}\right) = V(t)$$

$$\frac{dp}{dx} = \frac{12\mu V(t)}{H^3}\, x + c \leftarrow 0$$ from symmetry

∇p scales with $\frac{1}{H^3}$!!

ii) $H =$ constant, $V = 0$



$$p = \frac{6\mu V}{H^3}x^2 + C_2$$

$$p_0 = \frac{6\mu V R^2}{H^3} + C_2 = 0$$

so $$p(x) = \frac{6\mu V}{H^3}(x^2 - R^2)$$

$$F = 2\int_0^R p(x)\,dx$$

$$= \frac{12\mu V}{H^3}\left(\frac{R^3}{3} - \frac{R^3}{2}\right)$$

$$= 2\mu V \frac{R^3}{H^3}$$

Beyond Newtonian Fluids

Newtonian fluid

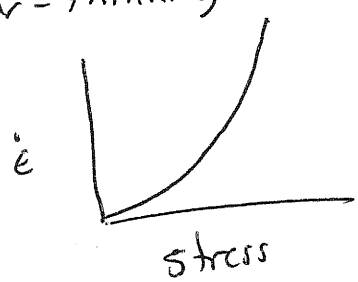
$\sigma = 2\mu\dot{\epsilon}$

Linear elastic material (Hooke's law)

stess $\propto$ strain

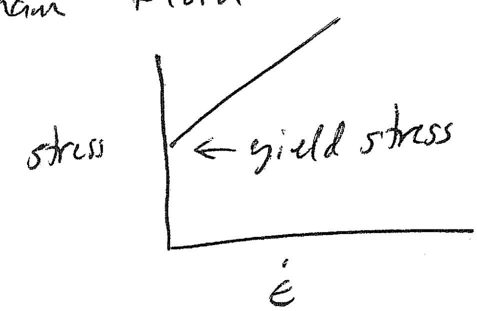
$\sigma = E\epsilon$

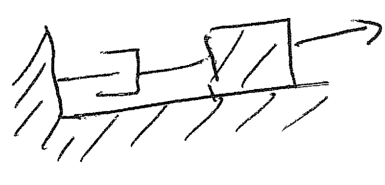
Shear-thinning fluid



$\Rightarrow \quad \mu = \dfrac{\text{stress}}{\dot{\epsilon}}$

Bingham fluid





Maxwell viscoelastic

short time : elastic

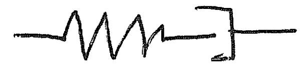
long time : fluid

superimpose strain rates

$$\epsilon_e = \sigma / E$$

$$\dot{\epsilon}_f = \frac{d\epsilon_f}{dt} = \frac{\sigma}{2\mu}$$

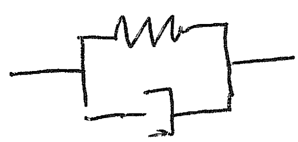
$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_f$$



$$\underbrace{\frac{d\epsilon}{dt} = \frac{\sigma}{2\mu} + \frac{1}{E}\frac{d\sigma}{dt}}_{\text{constitutive law}}$$

$$t_{char} = t_{Maxwell} = 2\mu / E$$

Kelvin model



$$\sigma = \sigma_f + \sigma_e$$

$$= 2\mu \frac{d\epsilon}{dt} + \epsilon E$$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k};\ d\tau = dx\,dy\,dz$

Gradient. $\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl. $\nabla\times\mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$

Laplacian. $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi};\ d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl. $\nabla\times\mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r}$

$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$

Laplacian. $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z};\ d\tau = r\,dr\,d\phi\,dz$

Gradient. $\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial\phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$

Divergence. $\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl. $\nabla\times\mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$

$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial\phi}\right]\hat{z}$

Laplacian. $\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_a^b (\nabla f)\cdot d\mathbf{l} = f(b) - f(a)$

Divergence Theorem: $\int_{\text{volume}} (\nabla\cdot\mathbf{A})\,d\tau = \oint_{\text{surface}} \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem: $\int_{\text{surface}} (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint_{\text{line}} \mathbf{A}\cdot d\mathbf{l}$

Scaling in Geophysics
Michael Manga
May 2011

Some notes for the derivations of equations from conservation of mass, energy and momentum

The governing equations of fluid mechanics follow from conservation principles of classical physics (conservation of mass, linear and angular momentum, energy). These equations are in general not sufficient to describe fluid motions, and we will additionally need to use equations of state and constitutive relations.

These notes are only a short summary of results.

1. Useful integral relations

Gauss's divergence theorem

$$\int_S \mathbf{u}\cdot\mathbf{n}\,dS = \int_V \nabla\cdot\mathbf{u}\,dV$$

where S includes *all* surfaces bounding volume V and $\mathbf{n}$ is a unit normal vector to the surface S pointing outwards from the volume V. This expression is also valid if $\mathbf{u}$ is a second-rank (or higher-order) tensor.

Stokes theorem

$$\oint_C \mathbf{u}\cdot d\mathbf{l} = \int_S \nabla\wedge\mathbf{u}\cdot\mathbf{n}\,dS$$

where $d\mathbf{l}$ is tangent to the curve C and points in a direction consistent with the right-hand rule, and $\mathbf{n}$ is a unit normal to the surface S: for example, if $\mathbf{n}$ points out of the page, $d\mathbf{l}$ is in the anticlockwise direction. In words, Stokes theorem states that the normal component of the curl of a vector field $\mathbf{u}$ over a surface S is equal to the integral of the tangential component of $\mathbf{u}$ around the boundary C.

2. Useful vector identities involving gradients (∇)

$$\nabla\cdot(\nabla a) = \nabla^2 a$$

$$\nabla(ab) = a\nabla b + b\nabla a$$

$$\nabla^2(ab) = a\nabla^2 b + 2(\nabla a)\cdot(\nabla b) + b\nabla^2 a$$

$$\nabla\cdot(a\mathbf{b}) = (\nabla a)\cdot\mathbf{b} + a\nabla\cdot\mathbf{b}$$

$$\nabla\cdot(a\nabla b) = a\nabla^2 b + \nabla a\cdot\nabla b$$

$$\nabla\wedge(\nabla a) = 0$$

$$\nabla\cdot(\nabla\wedge\mathbf{a}) = 0$$

$$\nabla\cdot(\mathbf{a}\wedge\mathbf{b}) = (\nabla\wedge\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\nabla\wedge\mathbf{b})$$

$$\nabla\wedge(a\mathbf{b}) = \nabla a\wedge\mathbf{b} + a\nabla\wedge\mathbf{b}$$

$$\nabla\wedge(\nabla\wedge\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

3. Reynolds transport theorem

In order to derive an equation for conservation of momentum, the Reynolds transport theorem will be useful:

$$\frac{D}{Dt}\left[\int_{V(t)} X dv\right] = \int_{V(t)}\left[\frac{\partial X}{\partial t} + \nabla\cdot(\mathbf{u}X)\right]dv$$

4. Conservation of mass

Conservation of mass leads to the *continuity equation*

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

where $\mathbf{u}$ is the fluid velocity, ρ is density, and t is time. If the fluid is incompressible (i.e. the density does not change)

$$\nabla\cdot\mathbf{u} = 0.$$

Fields (e.g. in this case $\mathbf{u}$) that are divergence-free are often called *solenoidal*. The magnetic field is another example of a solenoidal field.

Often it is useful to consider derivatives in a frame of reference moving with the fluid (these are called material or convective derivatives). Denoting the material derivative with a capital D we have

$$\frac{DX}{Dt} = \frac{\partial X}{\partial t} + \mathbf{u}\cdot\nabla X.$$

The material derivative of the variable X is the rate of change of X following the fluid.

5. Conservation of energy

The internal energy density at a point within a fluid is

$$\rho\frac{\mathbf{u}\cdot\mathbf{u}}{2}+\rho e.$$

Here the first term represents the kinetic energy of the fluid and the second describes molecular-level energy.

In a frame of reference moving with the fluid, conservation of energy requires that

$$\begin{aligned}\frac{D}{Dt}\left[\int_{V(t)}\left(\rho\frac{\mathbf{u}\cdot\mathbf{u}}{2}+\rho e\right)dV\right] &= \text{work done by external forces}\\ &+ \text{energy flux across boundaries}\end{aligned}$$

6. Conservation of linear momentum

Conservation of momentum leads to

time-rate-of-change of momentum of some body = sum of forces acting on body

(this is Newton's second law). The term of the right-hand side includes both body forces (e.g. gravity) and surface forces.

Again, we consider a volume element moving with the fluid:

$$\frac{D}{Dt}\left[\int_{V(t)}\rho\mathbf{u}dV\right]=\int_V\rho\mathbf{g}dV+\int_S\mathbf{t}dS.$$

Here S is the surface bounding V and $\mathbf{t}$ characterizes surface forces/area.

Let

$$\mathbf{t}=\mathbf{n}\cdot\mathbf{T}$$

where $\mathbf{n}$ is the unit normal vector and $\mathbf{T}$ is the stress tensor. This allows us to apply the divergence theorem to the surface integral. We use the Reynolds Transport theorem to bring the D/Dt under the integration sign on the LHS so that everything involves only volume integrals; since $V(t)$ is arbitrary we get

$$\frac{\partial(\rho\mathbf{u})}{\partial t}+\nabla\cdot(\rho\mathbf{u}\mathbf{u})=\rho\mathbf{g}+\nabla\cdot\mathbf{T}$$

If ρ is constant (i.e. the fluid is incompressible, $\nabla\cdot\mathbf{u}=0$), then

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right]=\rho\mathbf{g}+\nabla\cdot\mathbf{T}$$

This equation is called *Cauchy's equation of motion.*

This equations has 9 unknowns ($\mathbf{T}$ has 9 unknowns, and $\mathbf{u}$ has 3 unknowns), but there are only 4 equations (3 from the Cauchy equation, 1 from conservation of mass). This is clearly a problem. The solution is to use constitutive relations to relate $\mathbf{T}$ and $\mathbf{u}$.

7. Conservation of angular momentum

We can go through the math for conservation of angular momentum. We find that we get an additional constraint on $\mathbf{T}$, namely that $\mathbf{T}$ must be symmetric (and thus has "only" 6 components).

8. Constitutive relationship for a newtonian fluid

We just cite the result here.
First we write the stress tensor as the sum of an isotropic part and a deviatoric stress

$$\mathbf{T}=-p\mathbf{I}+\boldsymbol{\tau}(\mathbf{u},\nabla\mathbf{u},....)$$

where p is pressure, $\mathbf{I}$ is the identity matrix (δ_{ij} in index notation), and $\boldsymbol{\tau}$ is the deviatoric stress tensor. We can write $\nabla\mathbf{u}$ as the sum of symmetric and antisymmetric parts

$$\begin{aligned}\nabla\mathbf{u} &= \frac{1}{2}\left[\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right]+\frac{1}{2}\left[\nabla\mathbf{u}-(\nabla\mathbf{u})^T\right]\\ &= \mathbf{E}+\boldsymbol{\Omega}\end{aligned}$$

where $\mathbf{E}$ is the rate-of-strain tensor and $\boldsymbol{\Omega}$ is the vorticity tensor.

For a newtonian fluid we **assume** that (recall that we said $\mathbf{T}$ must be symmetric)

$$\begin{aligned}\boldsymbol{\tau} &\propto \mathbf{E}\\ &= \mathbf{c}:\mathbf{E}\end{aligned}$$

where $\mathbf{c}$ is a 4th rank tensor with 81 components. It turns we can simplify $\mathbf{c}$ to only 2 components so that in the end we get

$$\mathbf{T}=(-p+\lambda tr\mathbf{E})\mathbf{I}+2\mu\mathbf{E}$$

where λ and μ are coefficients of viscosity and it can be shown that they must both be positive.

Note that $tr\mathbf{E}=\nabla\cdot\mathbf{u}$ so that if the fluid is incompressible we get the famous *Navier-Stokes* equations

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right]=\rho\mathbf{g}-\nabla p+\mu\nabla^2\mathbf{u}$$

and

$$\nabla\cdot\mathbf{u}=0$$

We now have only 4 equations and 4 unknowns! Of course, solving these equations is still not easy, in part because the equations are *non-linear*.

9. Boundary conditions

1. On a solid surface we have a *no-slip* condition, $\mathbf{u}=\mathbf{0}$
2. Across a fluid-fluid interface, the normal component of velocity, $\mathbf{u}\cdot\mathbf{n}$, is continuous (this is called the kinematic condition)
3. Across an interface, the tangential component of velocity is usually continuous (dynamic condition)

These conditions are generally valid except near contact lines.