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DEEP CONVECTION
(Compressible Mantle Convection)

Shijie ZHONG

Department of Physics, University of Colorado
Boulder
Colorado
U.S.A.

Compressible mantle convection

1) Adams-Williamson relation and $\rho_r = \rho_r(r)$ as reference density

Consider density ρ_r as a function of Pressure P and entropy S, $\rho_r(S, P)$.

$$\nabla \rho_r(S, P) = \left(\frac{\partial \rho_r}{\partial S}\right)_P \nabla S + \left(\frac{\partial \rho_r}{\partial P}\right)_S \nabla P. \quad ①$$

Assume an adiabatic process, $\nabla S = 0$. $\nabla P = -\rho_r g \vec{e}_r$

$$\frac{1}{\rho_r}\frac{d\rho_r}{dr} = -g\left(\frac{\partial \rho_r}{\partial P}\right)_S \quad ②$$

$K_S = \rho_r \left(\frac{\partial P}{\partial \rho_r}\right)_S$ defines the isentropic bulk moduli.

$$\frac{1}{\rho_r}\frac{d\rho_r}{dr} = -\frac{\rho_r g}{K_S} \quad ③$$

Define Gruneisen's parameter $\Gamma = \frac{\alpha K_S}{\rho C_p}$, and

③ becomes

$$\frac{1}{\rho_r}\frac{d\rho_r}{dr} = -\frac{g\alpha}{\Gamma C_p} \quad ④$$

Assume that parameters on the RHS of ④ are constant, and integrate ④.

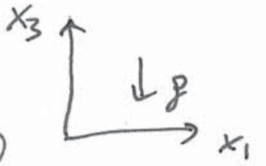
$$\rho_r(r) = \rho_0 e^{\frac{g\alpha}{\Gamma C_p}(d-r)}, \quad ⑤$$

where ρ_0 is the surface density at $r = d$.

Reference density ρ_r increases with pressure or depth, i.e., the fluid is compressible. This affects mantle convection.

2) Governing equation for compressible convection

$$\nabla\cdot(\rho_r \vec{V}) = 0 \qquad (6)$$



$$-\nabla p + \nabla\cdot\overleftrightarrow{\tau} - \delta\rho g \vec{e}_3 = 0 \qquad (7)$$

$$\rho_r C_p\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T\right) + \rho_r g \alpha T v_3 = \nabla\cdot(k\nabla T) + \tau_{ij} u_{ij} + \rho_r H. \qquad (8)$$

where $\delta\rho = \rho_r[-\alpha(T - T_r) + K_T^{-1} p]$

and p is the dynamic pressure and K_T is isothermal modulus, T_r is reference temperature.

Non-dimensionlize the equations, using d, $\frac{\kappa_0}{d}$, ΔT, ρ_0, η_0, α_0, C_{p_0}, T_0, and k_0 to normalize x, v, T, ρ, η, α, C_p, T and k. Time, pressure and internal heating scales are $\frac{d^2}{\kappa_0}$, $\frac{\eta_0 \kappa_0}{d^2}$, and $\frac{\kappa_0 \Delta T C_{p\text{-}0}}{d^2}$, respectively.

$$\nabla\cdot(\rho_r \vec{V}) = 0 \qquad (9)$$

$$-\nabla p + \nabla\cdot\overleftrightarrow{\tau} + \left[\rho_r \alpha g Ra (T - T_r) - \frac{\alpha g D_i}{C_p T_0} p\right]\vec{e}_3 = 0 \qquad (10)$$

$$\rho_r c_p\left(\frac{\partial T}{\partial t} + \vec{V}\cdot\nabla T\right) + \rho_r g \alpha D_i (T + T_s) u_3 = \nabla\cdot(k\nabla T) + \frac{D_i}{Ra}\overleftrightarrow{\tau}\cdot\vec{\varepsilon} + \rho_r H \qquad (11)$$

$$Di = \frac{\alpha_0 g_0 d}{c_{p,0}} \qquad \text{Dissipation number}$$

$$Ra = \frac{\rho_0 g_0 \alpha_0 \Delta T d^3}{\kappa_0 \eta_0}.$$

Notice that if we assume constant parameters such that gravitational acceleration ..... are constant, $\alpha = g = c_p = T = k = 1$ in the non dimensional equations 9, 10 and 11.

Remarks:

A $$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\eta\delta_{ij}\frac{\partial v_k}{\partial x_k} + \eta_2\delta_{ij}\frac{\partial v_k}{\partial x_k}$$

η_2 : bulk viscosity, often ignored.

$$\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \frac{2}{3}\eta\delta_{ij}\frac{\partial v_k}{\partial x_k}.$$

B $$\frac{dT_r}{dr} = -\frac{\alpha T_r g}{c_p},$$ adiabatic temperature gradient.

3) Dissipation ϕ and adiabatic heating Q_a

Let's use the dimensional forms of equations. (i.e., 4, 6–8)

Multiplying 7 with $\vec{v}$ and integrating,

$$\int -\vec{v}\cdot\nabla p\,dV + \int \vec{v}\cdot\nabla\cdot\overleftrightarrow{\tau}\,dV + \int \rho_r g\left[\alpha(T-T_r) - \frac{p}{K_T}\right]v_3\,dV = 0 \quad (12)$$

The 2nd term:

$$\int \vec{v}\cdot\nabla\cdot\overleftrightarrow{\tau}\,dV = \oint \overleftrightarrow{\tau}\cdot\vec{v}\cdot d\vec{S} - \int \overleftrightarrow{\tau}\,\nabla\vec{v}\,dV \quad (13)$$

For a closed box with either free-slip or no-slip B.C.;

$$\oint \overleftrightarrow{\tau}\cdot\vec{v}\cdot d\vec{S} = 0.$$

Then (13) $\rightarrow$

$$\int \vec{v}\cdot\nabla\cdot\overleftrightarrow{\tau}\,dV = -\int \overleftrightarrow{\tau}\,\nabla\vec{v}\,dV = -\phi \quad (14)$$

ϕ: total viscous dissipation.

Likewise, the 1st term of (12) $\rightarrow$

$$-\int \vec{v}\cdot\nabla p\,dV = \int p\,\nabla\cdot\vec{v}\,dV$$

(12) $\rightarrow$

$$\int p\,\nabla\cdot\vec{v}\,dV - \int \overleftrightarrow{\tau}\cdot\nabla\vec{v}\,dV + \int \rho_r g\left[\alpha(T-T_r) - \frac{p}{K_T}\right]v_3\,dV = 0 \quad (15)$$

Now consider Adams-Williamson relation

$$\frac{1}{\rho_r}\frac{d\rho_r}{dx_3} = \quad - \frac{\rho_r g}{K_s} \quad (16)$$

And (6) $\rightarrow$

$$\nabla\cdot\vec{v} = -\frac{\nabla\rho_r\cdot\vec{v}}{\rho_r} = -\frac{1}{\rho_r}\frac{d\rho_r}{dx_3}v_3 \quad (17)$$

plug 16 to 17,

$$\nabla \cdot \vec{v} = \frac{\rho_r g}{K_S} v_3 \qquad (18)$$

plug 18 to 15, and assume $K_T = K_S$,

$$\underline{\int \frac{\rho_r g}{K_T} p\, v_3\, dV} - \int \overset{\Rightarrow}{\tau} \cdot \nabla \vec{v}\, dV + \int \rho_r g \Big[\alpha (T - T_r) - \underline{\frac{p}{K_T}}\Big] v_3\, dV = 0$$

$$\int \overset{\Rightarrow}{\tau} \cdot \nabla \vec{v}\, dV = \int \rho_r g \alpha (T - T_r) v_3\, dV. \qquad (19)$$

From mass conservation equation,

$\int_A \rho_r v_3\, dA = 0$ for any horizontal plane A through the box.

This implies $\int_0^d \int_A \rho_r v_3 T_r\, dA\, dx_3 = 0$

$$\text{or } \int \rho_r v_3 T_r\, dV = 0 \qquad (20)$$

Put 20 to 19

$$\underbrace{\int \overset{\Rightarrow}{\tau} \cdot \nabla \vec{v}\, dV}_{\Phi} = \underbrace{\int \rho_r g \alpha T v_3\, dV}_{\dot{Q}_a} \qquad (21)$$

total viscous dissipation = total adiabatic heating

Remark: The effect of p on $\delta\rho$ is the key to assuring the balance [Leng and Zhong, 2008]

From (21), $\phi = \int \rho_r g \alpha T v_3 \, dV$

$$\phi = \int \frac{g\alpha d}{d\, C_p} \cdot \rho_r C_p T v_3 \, dV$$

$$\phi = D_i \frac{1}{d} \int \rho_r C_p T v_3 \, dV$$

$$\phi = D_i \frac{1}{d} \int_0^d \int_A \rho_r C_p T v_3 \, dA \, dx_3 \quad (22)$$

$\int_A \rho_r C_p T v_3 \, dA$ is convective heat flux and in the convection-dominated regions (i.e., outside the top and bottom thermal boundaries) it is equal to F_s that is uniform at all depths in the convection-dominated regions and is equal to the surface heat flux ($H=0$),

$$(22) \rightarrow \phi = D_i \frac{1}{d} F_s (d - 2\delta)$$

$$\frac{\phi}{F_s} = D_i \left(1 - 2\frac{\delta}{d}\right) \approx D_i \quad \text{for } \delta \ll d \text{, or large } Ra.$$

$$\frac{\phi}{F_s} \approx D_i \quad (23)$$

Hewitt et al. [1975] proposed that $\frac{\phi}{F_s} = D_i \left(1 - \frac{1}{2}\zeta\right)$

ζ : internal heating ratio.