On rigorous integration of piece-wise linear systems

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Rigorous integration of nonlinear dynamical systems

- Considerable interest in using computers for obtaining rigorous results in the field of continuous dynamical systems,
  - computing rigorous enclosures of trajectories,
  - finding accurate positions of periodic solutions,
  - finding all short periodic orbits,
  - proving the existence of topological chaos,
  - proving the existence of chaotic attractors.
- Interval arithmetic: all calculations are performed on intervals in such a way that the true result is always enclosed within the interval found by a computer, notations:
  - boldface is used to denote intervals, $\mathbf{x} = [a, b]$
  - by $\underline{\mathbf{x}}$ and $\overline{\mathbf{x}}$ we denote left and right end points of $\mathbf{x}$,
  - the diameter of the interval $\mathbf{x}$: $\mathrm{diam}(\mathbf{x}) = \overline{\mathbf{x}} - \underline{\mathbf{x}}$.
- Rigorous integration — the basic tool needed to study continuous systems,
- Most of methods for rigorous integration work under the assumption that the vector field is smooth.

Rigorous integration of piecewise linear systems

- The methods developed for smooth systems are not directly applicable to piece-wise linear (PWL) (or piece-wise smooth) systems, which are an important class of nonlinear dynamical systems,
- When intersections of trajectories with hyperplanes separating linear regions (C^0 hyperplanes) are transversal it is possible to extend general methods to integration of PWL systems:
  - C^0 hyperplanes are used as transversal sections,
  - when a trajectory intersects a C^0 hyperplane, its intersection with the transversal plane is computed and the resulting set is used as a starting set for further computations.
- What to do when trajectories are tangent to C^0 hyperplanes?

Piece-wise linear systems

- The continuous piecewise linear system is defined by

$$\dot{x} = f(x),$$

where $f\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is a piece-wise linear continuous map.

- By $x(t) = \varphi(t, \hat{x})$ we denote the solution of $\dot{x} = f(x)$ satisfying the initial condition $x(0) = \hat{x}$.
- Let us assume that the state space $\mathbb{R}^n$ is composed of m linear regions $R_1, R_2, \ldots R_m$, separated by hyperplanes $\Sigma_1, \Sigma_2, \ldots, \Sigma_p$ (the C^0 hyperplanes).
- In the region R_k the state equation has the form $\dot{x} = A_k x + v_k$, where $A_k \in \mathbb{R}^{n\times n}$, $v_k \in \mathbb{R}^n$. If A_k is invertible then in the linear region R_k solutions can be computed as

$$x(t) = \varphi_k(t, \hat{x}) = \mathrm{e}^{A_k t}(\hat{x} - p_k) + p_k,$$

where $p_k = -A_k^{-1} v_k$.

Rigorous integration of PWL systems

- The problem is how to rigorously calculate an enclosure of the set $\varphi(\mathbf{t}, \mathbf{x})$ for a given interval $\mathbf{t}$ and an interval vector $\mathbf{x} \in \mathbb{R}^n$. Without loss of generality we can assume that $\mathbf{x} \subset R_k$.
- If all trajectories based at $\mathbf{x}$ remain in R_k for $s \in [0, \bar{\mathbf{t}}]$ the problem is simple. The enclosure can be found by evaluating the solution of a linear system in interval arithmetic:

$$\mathbf{y} = \varphi_k(\mathbf{t}, \mathbf{x}) = \mathrm{e}^{A_k \mathbf{t}}(\mathbf{x} - p_k) + p_k.$$

- For the evaluation of the above formula one can use the mean value form to obtain a narrower enclosure of the set of solutions.

Transversal intersection — rigorous integration procedure

- Another relatively easy case is when all trajectories based at $\mathbf{x}$ enter another linear region R_l through the plane Σ, and intersections of trajectories with Σ are transversal.
- In this case the first step is to find $s_1 > 0$ such that $\varphi_k([0, s_1], \mathbf{x}) \in R_k$, s_1 should be as large as possible.
- Then we find $s_2 > s_1$ such that $\varphi_k(s_2, \mathbf{x}) \subset R_l$, s_2 should be as small as possible.
- Next, one evaluates $\mathbf{y} = \varphi_k(\mathbf{s}, \mathbf{x})$, where $\mathbf{s} = [s_1, s_2]$.
- Finally, the intersection of $\mathbf{y}$ and Σ is computed. The intersection serves as a set of initial conditions for further computations. The problem of finding $\varphi(\mathbf{t}, \mathbf{x})$ has been reduced to the problem of finding $\varphi(\mathbf{t} - \mathbf{s}, \mathbf{y} \cap \Sigma)$.

Transversal intersection — algorithm

Algorithm 1. Computation of $\varphi(\mathbf{t}, \mathbf{x})$, transversal case:

1. find s_1 such that $\varphi_k(s_1, \mathbf{x}) \subset R_k$,
2. if $s_1 > \bar{\mathbf{t}}$ return $\mathbf{y} = \varphi_k(\mathbf{t}, \mathbf{x})$,
3. find $s_2 > s_1$ such that $\varphi_k(\mathbf{t}, \mathbf{x}) \subset R_l$,
4. define $\mathbf{s} = [s_1, s_2]$ and compute $\mathbf{y} = \varphi_k(\mathbf{s}, \mathbf{x})$,
5. go to step 1 with $\mathbf{x} = \mathbf{y} \cap \Sigma$, $\mathbf{t} = \mathbf{t} - \mathbf{s}$.

- The algorithms works when trajectories transversally intersect the C^0 hyperplanes.
- It has been successfully applied to the analysis of the Chua's circuit for parameter values, for which the attractor does not contain trajectories tangent to the C^0 hyperplanes.

- Consider an ordinary differential equation $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$.
- Assume that we know how to rigorously integrate $\dot{x} = g(x)$.

Theorem

Let $x(t)$ and $y(t)$ be solutions of $\dot{x} = f(x)$ and $\dot{x} = g(x)$, respectively. Let us assume that $x(0) = y(0)$, and $x(t), y(t) \in D \subset \mathbb{R}^n$ for $t \in [0, h]$, where D is a bounded, closed, convex set, and the map g is C^1. Then for $t \in [0, h]$

$$|y_i(t) - x_i(t)| \leq \Delta_i,$$

where $\Delta = \int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}s$, $b_{ij} \geq \sup_{x \in D} \left| \frac{\partial g_i}{\partial x_j}(x) \right|$ for $i \neq j$, $b_{ii} \geq \sup_{x \in D} \frac{\partial g_i}{\partial x_i}(x)$, and $c_i \geq |g_i(x(t)) - f_i(x(t))|$, for $t \in [0, h]$.

Rigorous integration — tangent intersection case

- Let us assume that $\mathbf{x} \subset R_k$, and that some trajectories based at $\mathbf{x}$ are tangent to the C^0 hyperplane Σ separating the linear regions R_k and R_l.
- The goal is to compute an enclosure of the set $\varphi(\mathbf{t}, \mathbf{x}) = \{\varphi(t, x) \colon x \in \mathbf{x}, t \in \mathbf{t}\}$.
- The PWL system is considered as a perturbed linear system:

$$\dot{x} = g(x) = A_k x + v_k.$$

- We use the main theorem with $b_{ij} = |a_{ij}|$ for $i \neq j$ and $b_{ii} = a_{ii}$.
- $g(x) - f(x) = 0$ over the region R_k, and $g(x) - f(x) = (A_k - A_l)x + v_k - v_l$ for $x \in R_l$. Close Σ this difference is small (f is continuous).
- When B is invertible

$$\Delta = \int_0^t \mathrm{e}^{B(t-s)} c \mathrm{d}s = B^{-1} \left(\mathrm{e}^{Bt} - I \right) c.$$

Tangent intersection — rigorous integration procedure

- Find $s_1 > 0$ such that $\varphi_k([0, s_1], \mathbf{x}) \subset R_k$. The set $\mathbf{u} = \varphi_k(s_1, \mathbf{x})$ serves as an initial condition for integration along the tangency. To reduce overestimation s_1 should be as large as possible.
- Select s_2, compute enclosure $\mathbf{v}$ of the solution $\varphi_k([0, s_2], \mathbf{u})$ of the linear system.
- Choose $\mathbf{w} \supset \mathbf{v}$, $\mathbf{w}$ serves as a guess of the set containing the solution $\varphi([0, s_2], \mathbf{u})$ of the PWL system.
- Compute $c = \sup_{x \in \mathbf{w}} |g(x) - f(x)|$ and the vector Δ.
- If $\mathbf{v} + [-1, 1]\Delta \subset \mathbf{w}$ then the solution $\varphi([0, s_2], \mathbf{u})$ of the PWL system is enclosed in $\mathbf{v} + [-1, 1]\Delta$. It follows that $\varphi(s_2, \mathbf{u}) \subset \mathbf{z} = \varphi_k(s_2, \mathbf{u}) + [-1, 1]\Delta$.
- If $\mathbf{z} \subset R_k$ and the vector field f over the set $\mathbf{z}$ points away from the plane Σ, then we continue integration using the Algorithm 1.

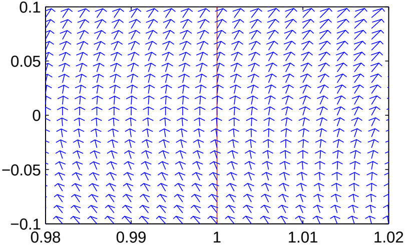
Algorithm 2. Computation of $\varphi(\mathbf{t}, \mathbf{x})$, tangent case:

1. Find maximum s_1 such that $\varphi_k(s_1, \mathbf{x}) \subset R_k$,
2. Compute $\mathbf{u} = \varphi_k(s_1, \mathbf{x})$,
3. Select $s_2 > 0$ and compute $\mathbf{v} = \varphi_k([0, s_2], \mathbf{u})$,
4. Select $\mathbf{w} \supset \mathbf{v}$,
5. Compute $c = \sup_{x \in \mathbf{w}} |g(x) - f(x)|$,
6. Compute $\Delta = B^{-1}\left(\mathrm{e}^{Bt} - I\right) c$,
7. Compute $\mathbf{z} = \varphi_k(s_2, \mathbf{u}) + [-1, 1]\Delta$,
8. If $\mathbf{v} + [-1, 1]\Delta \subset \mathbf{w}$, $\mathbf{z} \subset R_k$ and the vector field f over the set $\mathbf{z}$ points away from the plane Σ call the Algorithm 1 with $\mathbf{x} = \mathbf{z}$ and $\mathbf{t} = \mathbf{t} - s_1 - s_2$,
9. Go back to step 4 and select larger $\mathbf{w}$ or go back to step 3 and select larger s_2.

- A simple piecewise-linear planar system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = g\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + (|x_1 - 1| - 1)e \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

- Parameter values: $a_{11} = 2,\ a_{12} = 1,\ a_{21} = 1,\ a_{22} = 1,\ e = 2.$
- The line $\Sigma_1 = \{x\colon x_1 = 1\}$ separates the two linear regions $U_1 = \{x\colon x_1 < 1\}$ and $U_2 = \{x\colon x_1 > 1\}$,
- Trajectories are tangent to Σ_1 at $(1, e - a_{11}/a_{12}) = (1, 0)$.



The PWL system as a perturbed linear system

- We treat the planar PWL system as a perturbed linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + (x_1 - 2)e \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix},$$

  for which the vector field is equal to the vector field of the nonlinear system when $x_1 > 1$.

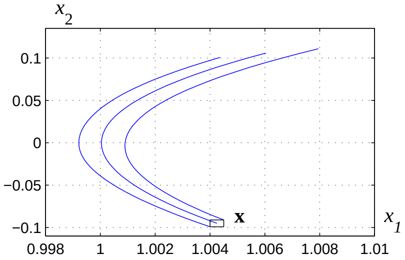
- Hence, we can get bounds for the solution $y(t)$ of the PWL system from the solution $x(t)$ of the linear system using bounds with the following constants:

$$B = \begin{pmatrix} a_{11}+e & |a_{12}| \\ |a_{21}| & a_{22} \end{pmatrix}, c = \begin{pmatrix} \sup_{x \in \mathbf{w}} |(|x_1 - 1| - x_1 + 1)e| \\ 0 \end{pmatrix}.$$

Rigorous integration of the planar PWL system

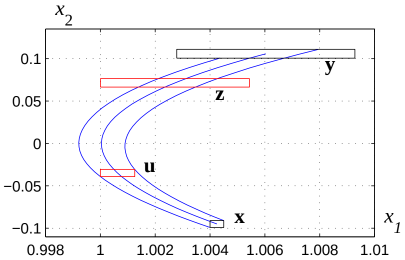
- Example: find of enclosure of $\varphi(t, \mathbf{x})$ for $\mathbf{x} = ([1.004, 1.0045], [-0.099, -0.091]) \subset U_2$ and $t = 0.2$.

- three types of trajectories: tangent to Σ_1, with no intersections, and with two intersections.



Rigorous integration of the planar PWL system

- $\mathbf{u} = \varphi_2(s_1, \mathbf{x})$, $s_1 \approx 0.0642$, all trajectories are just before intersection with Σ_1, $\mathbf{u}$ is a very narrow enclosure of the set of true trajectories.



- $\mathbf{z} = \varphi_2(s_2, \mathbf{u}) + \Delta$, $s_2 = 0.1044$, all trajectories has already passed the tangency area, $\mathbf{u}$ is relatively large and in consequence s_2 is also large. This results in a considerable overestimation.
- The final result: $\mathbf{y} = \varphi_2(0.2 - s_1 - s_2, \mathbf{z})$ is computed using formulas for solutions of linear systems.
- $\text{diam}(\mathbf{x}) = (0.0005, 0.008)$, $\text{diam}(\mathbf{y}) = (0.0065, 0.0104)$.
- when $\text{diam}(\mathbf{x}) = (10^{-5}, 10^{-5})$ then $s_2 \approx 0.0215$, $\text{diam}(\mathbf{y}) = (6.63 \cdot 10^{-5}, 1.92 \cdot 10^{-5})$ (reduced overestimation).

Example 2: The Chua's circuit

- The state equation:

$$C_1\dot{x}_1 = (x_2 - x_1)/R - g(x_1),$$
$$C_2\dot{x}_2 = (x_1 - x_2)/R + x_3,$$
$$L\dot{x}_3 = -x_2 - R_0 x_3,$$

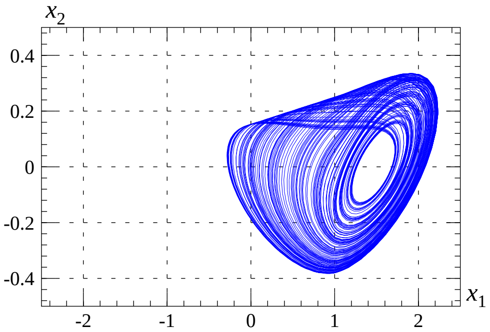
  where

$$g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)$$

  is a three segment piecewise linear characteristics.
- Parameter values: $C_1 = 1$, $C_2 = 8.3$, $G_a = -3.4429$, $G_b = -2.1849$, $L = 0.06913$, $R = 0.33065$, $R_0 = 0.00036$.

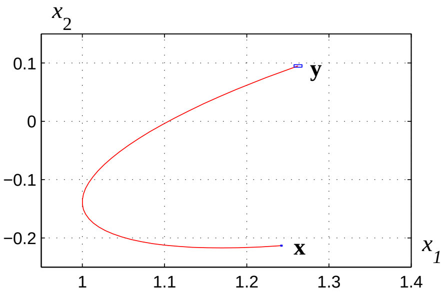
- Linear regions: $R_1 = \{x \in \mathbb{R}^3 \colon x_1 < -1\}$, $R_2 = \{x \colon |x_1| < 1\}$ and $R_3 = \{x \colon x_1 > 1\}$,
- C^0 hyperplanes: $\Sigma_1 = \{x \colon x_1 = -1\}$ and $\Sigma_2 = \{x \colon x_1 = 1\}$,

- Roessler-type attractor,
- intersections with Σ_1 are not always transversal.



Chua's circuit, rigorous integration along the tangency

- Example: find an enclosure of $\varphi(t, \mathbf{x})$ for $t = 2$ and $\mathbf{x} = ([1.2412, 1.2432], [-0.2141, -0.2121], [-4.7623, -4.7603])$, $\text{diam}(\mathbf{x}) = (0.002, 0.002, 0.002)$,
- $\mathbf{x}$ has non-empty intersection with the numerically observed attractor and some trajectories based in $\mathbf{x}$ are tangent to Σ_1.
- $\text{diam}(\mathbf{y}) = (0.0098,0.0042,0.041)$,
- integration time as a perturbed linear system: $s_2 = 0.1936$.



- the size of initial set is relatively large and the integration time is relatively long thus showing usefulness of the proposed method.

Conclusions

- We have studied rigorous integration methods for piece-wise linear systems.
- An algorithm handling the case of trajectories tangent to hyperplanes separating linear regions has been described.
- Several examples have been considered to show the effectiveness of this technique.
- The methods can be used without major modifications for rigorous integration of piece-wise smooth systems — one has to use standard techniques for rigorous integration of nonlinear systems in smooth regions.

References

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