On rigorous integration of piece-wise linear systems

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06.2011, Trieste, Italy
Considerable interest in using computers for obtaining rigorous results in the field of continuous dynamical systems,

- computing rigorous enclosures of trajectories,
- finding accurate positions of periodic solutions,
- finding all short periodic orbits,
- proving the existence of topological chaos,
- proving the existence of chaotic attractors.

Interval arithmetic: all calculations are performed on intervals in such a way that the true result is always enclosed within the interval found by a computer, notations:

- boldface is used to denote intervals, \( \mathbf{x} = [a, b] \)
- by \( \underline{x} \) and \( \overline{x} \) we denote left and right end points of \( x \),
- the diameter of the interval \( x \): \( \text{diam}(x) = \overline{x} - \underline{x} \).

Rigorous integration — the basic tool needed to study continuous systems,

Most of methods for rigorous integration work under the assumption that the vector field is smooth.
The methods developed for smooth systems are not directly applicable to piece-wise linear (PWL) (or piece-wise smooth) systems, which are an important class of nonlinear dynamical systems.

When intersections of trajectories with hyperplanes separating linear regions ($C^0$ hyperplanes) are transversal it is possible to extend general methods to integration of PWL systems:

- $C^0$ hyperplanes are used as transversal sections,
- when a trajectory intersects a $C^0$ hyperplane, its intersection with the transversal plane is computed and the resulting set is used as a starting set for further computations.

What to do when trajectories are tangent to $C^0$ hyperplanes?
The continuous piecewise linear system is defined by

\[ \dot{x} = f(x), \]

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a piece-wise linear continuous map.

By \( x(t) = \varphi(t, \hat{x}) \) we denote the solution of \( \dot{x} = f(x) \) satisfying the initial condition \( x(0) = \hat{x} \).

Let us assume that the state space \( \mathbb{R}^n \) is composed of \( m \) linear regions \( R_1, R_2, \ldots, R_m \), separated by hyperplanes \( \Sigma_1, \Sigma_2, \ldots, \Sigma_p \) (the \( C^0 \) hyperplanes).

In the region \( R_k \) the state equation has the form

\[ \dot{x} = A_k x + \nu_k, \]

where \( A_k \in \mathbb{R}^{n \times n} \), \( \nu_k \in \mathbb{R}^n \). If \( A_k \) is invertible then in the linear region \( R_k \) solutions can be computed as

\[ x(t) = \varphi_k(t, \hat{x}) = e^{A_k t} (\hat{x} - p_k) + p_k, \]

where \( p_k = -A_k^{-1} \nu_k \).
The problem is how to rigorously calculate an enclosure of the set \( \varphi(t, x) \) for a given interval \( t \) and an interval vector \( x \in \mathbb{R}^n \). Without loss of generality we can assume that \( x \subset R_k \).

If all trajectories based at \( x \) remain in \( R_k \) for \( s \in [0, t] \) the problem is simple. The enclosure can be found by evaluating the solution of a linear system in interval arithmetic:

\[
y = \varphi_k(t, x) = e^{A_k t}(x - p_k) + p_k.
\]

For the evaluation of the above formula one can use the mean value form to obtain a narrower enclosure of the set of solutions.
Another relatively easy case is when all trajectories based at \( \mathbf{x} \) enter another linear region \( R_l \) through the plane \( \Sigma \), and intersections of trajectories with \( \Sigma \) are transversal.

In this case the first step is to find \( s_1 > 0 \) such that \( \varphi_k([0, s_1], \mathbf{x}) \in R_k \), \( s_1 \) should be as large as possible.

Then we find \( s_2 > s_1 \) such that \( \varphi_k(s_2, \mathbf{x}) \subset R_l \), \( s_2 \) should be as small as possible.

Next, one evaluates \( \mathbf{y} = \varphi_k(\mathbf{s}, \mathbf{x}) \), where \( \mathbf{s} = [s_1, s_2] \).

Finally, the intersection of \( \mathbf{y} \) and \( \Sigma \) is computed. The intersection serves as a set of initial conditions for further computations. The problem of finding \( \varphi(t, \mathbf{x}) \) has been reduced to the problem of finding \( \varphi(t - \mathbf{s}, \mathbf{y} \cap \Sigma) \).
Algorithm 1. Computation of $\varphi(t, x)$, transversal case:

1. find $s_1$ such that $\varphi_k(s_1, x) \subset R_k$,
2. if $s_1 > \bar{t}$ return $y = \varphi_k(t, x)$,
3. find $s_2 > s_1$ such that $\varphi_k(t, x) \subset R_l$,
4. define $s = [s_1, s_2]$ and compute $y = \varphi_k(s, x)$,
5. go to step 1 with $x = y \cap \Sigma$, $t = t - s$.

- The algorithms works when trajectories transversally intersect the $C^0$ hyperplanes.
- It has been successfully applied to the analysis of the Chua’s circuit for parameter values, for which the attractor does not contain trajectories tangent to the $C^0$ hyperplanes.
Consider an ordinary differential equation \( \dot{x} = f(x) \), where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R}^n \mapsto \mathbb{R}^n \).

Assume that we know how to rigorously integrate \( \dot{x} = g(x) \).

**Theorem**

Let \( x(t) \) and \( y(t) \) be solutions of \( \dot{x} = f(x) \) and \( \dot{x} = g(x) \), respectively. Let us assume that \( x(0) = y(0) \), and \( x(t), y(t) \in D \subset \mathbb{R}^n \) for \( t \in [0, h] \), where \( D \) is a bounded, closed, convex set, and the map \( g \) is \( C^1 \). Then for \( t \in [0, h] \)

\[
|y_i(t) - x_i(t)| \leq \Delta_i,
\]

where \( \Delta = \int_0^t e^{B(t-s)} c \, ds \), \( b_{ij} \geq \sup_{x \in D} \left| \frac{\partial g_i}{\partial x_j}(x) \right| \) for \( i \neq j \), \( b_{ii} \geq \sup_{x \in D} \frac{\partial g_i}{\partial x_i}(x) \), and \( c_i \geq |g_i(x(t)) - f_i(x(t))| \), for \( t \in [0, h] \).
Rigorous integration — tangent intersection case

- Let us assume that \( x \subset R_k \), and that some trajectories based at \( x \) are tangent to the \( C^0 \) hyperplane \( \Sigma \) separating the linear regions \( R_k \) and \( R_l \).
- The goal is to compute an enclosure of the set \( \varphi(t, x) = \{ \varphi(t, x) : x \in x, t \in t \} \).
- The PWL system is considered as a perturbed linear system:
  \[
  \dot{x} = g(x) = A_k x + v_k.
  \]
- We use the main theorem with \( b_{ij} = |a_{ij}| \) for \( i \neq j \) and \( b_{ii} = a_{ii} \).
- \( g(x) - f(x) = 0 \) over the region \( R_k \), and \( g(x) - f(x) = (A_k - A_l)x + v_k - v_l \) for \( x \in R_l \). Close \( \Sigma \) this difference is small (\( f \) is continuous).
- When \( B \) is invertible
  \[
  \Delta = \int_0^t e^{B(t-s)} c ds = B^{-1} \left( e^{Bt} - I \right) c.
  \]
Find $s_1 > 0$ such that $\varphi_k([0, s_1], x) \subset R_k$. The set $u = \varphi_k(s_1, x)$ serves as an initial condition for integration along the tangency. To reduce overestimation $s_1$ should be as large as possible.

Select $s_2$, compute enclosure $v$ of the solution $\varphi_k([0, s_2], u)$ of the linear system.

Choose $w \supset v$, $w$ serves as a guess of the set containing the solution $\varphi([0, s_2], u)$ of the PWL system.

Compute $c = \sup_{x \in w} |g(x) - f(x)|$ and the vector $\Delta$.

If $v + [-1, 1]\Delta \subset w$ then the solution $\varphi([0, s_2], u)$ of the PWL system is enclosed in $v + [-1, 1]\Delta$. It follows that $\varphi(s_2, u) \subset z = \varphi_k(s_2, u) + [-1, 1]\Delta$.

If $z \subset R_k$ and the vector field $f$ over the set $z$ points away from the plane $\Sigma$, then we continue integration using the Algorithm 1.
Algorithm 2. Computation of $\varphi(t, x)$, tangent case:

1. Find maximum $s_1$ such that $\varphi_k(s_1, x) \subset R_k$,
2. Compute $u = \varphi_k(s_1, x)$,
3. Select $s_2 > 0$ and compute $v = \varphi_k([0, s_2], u)$,
4. Select $w \supset v$,
5. Compute $c = \sup_{x \in w} |g(x) - f(x)|$,
6. Compute $\Delta = B^{-1} (e^{Bt} - I) c$,
7. Compute $z = \varphi_k(s_2, u) + [-1, 1] \Delta$,
8. If $v + [-1, 1] \Delta \subset w$, $z \subset R_k$ and the vector field $f$ over the set $z$ points away from the plane $\Sigma$ call the Algorithm 1 with $x = z$ and $t = t - s_1 - s_2$,
9. Go back to step 4 and select larger $w$ or go back to step 3 and select larger $s_2$. 
Example 1: A planar PWL system

- A simple piecewise-linear planar system:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = \begin{pmatrix} a_{11} x_1 + a_{12} x_2 + (|x_1 - 1| - 1)e \\ a_{21} x_1 + a_{22} x_2 \end{pmatrix}.
\]

- Parameter values: \(a_{11} = 2, a_{12} = 1, a_{21} = 1, a_{22} = 1, e = 2\).

- The line \(\Sigma_1 = \{x : x_1 = 1\}\) separates the two linear regions \(U_1 = \{x : x_1 < 1\}\) and \(U_2 = \{x : x_1 > 1\}\).

- Trajectories are tangent to \(\Sigma_1\) at \((1, e - a_{11}/a_{12}) = (1, 0)\).
The PWL system as a perturbed linear system

- We treat the planar PWL system as a perturbed linear system:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} = f
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
a_{11}x_1 + a_{12}x_2 + (x_1 - 2)e \\
a_{21}x_1 + a_{22}x_2
\end{pmatrix},
\]

for which the vector field is equal to the vector field of the nonlinear system when \( x_1 > 1 \).

- Hence, we can get bounds for the solution \( y(t) \) of the PWL system from the solution \( x(t) \) of the linear system using bounds with the following constants:

\[
B = \begin{pmatrix}
|a_{11} + e| & |a_{12}| \\
|a_{21}| & |a_{22}|
\end{pmatrix},
\]

\[
c = \begin{pmatrix}
\sup_{x \in w} |(|x_1 - 1| - x_1 + 1)e| \\
0
\end{pmatrix}.
\]
Example: find of enclosure of $\varphi(t, x)$ for $x = ([1.004, 1.0045], [-0.099, -0.091]) \subset U_2$ and $t = 0.2$.

three types of trajectories: tangent to $\Sigma_1$, with no intersections, and with two intersections.
Rigorous integration of the planar PWL system

- \( u = \varphi_2(s_1, x) \), \( s_1 \approx 0.0642 \), all trajectories are just before intersection with \( \Sigma_1 \), \( u \) is a very narrow enclosure of the set of true trajectories.

- \( z = \varphi_2(s_2, u) + \Delta \), \( s_2 = 0.1044 \), all trajectories has already passed the tangency area, \( u \) is relatively large and in consequence \( s_2 \) is also large. This results in a considerable overestimation.

- The final result: \( y = \varphi_2(0.2 - s_1 - s_2, z) \) is computed using formulas for solutions of linear systems.

- \( \text{diam}(x) = (0.0005, 0.008) \), \( \text{diam}(y) = (0.0065, 0.0104) \).

- when \( \text{diam}(x) = (10^{-5}, 10^{-5}) \) then \( s_2 \approx 0.0215 \), \( \text{diam}(y) = (6.63 \cdot 10^{-5}, 1.92 \cdot 10^{-5}) \) (reduced overestimation).
Example 2: The Chua’s circuit

- The state equation:

\[
\begin{align*}
C_1 \dot{x}_1 &= (x_2 - x_1)/R - g(x_1), \\
C_2 \dot{x}_2 &= (x_1 - x_2)/R + x_3, \\
L \dot{x}_3 &= -x_2 - R_0 x_3,
\end{align*}
\]

where

\[
g(z) = G_b z + 0.5(G_a - G_b)(|z+1| - |z-1|)
\]

is a three segment piecewise linear characteristics.

- Parameter values: \(C_1 = 1\), \(C_2 = 8.3\), \(G_a = -3.4429\), \(G_b = -2.1849\), \(L = 0.06913\), \(R = 0.33065\), \(R_0 = 0.00036\).
Roessler-type attractor

- Linear regions: $R_1 = \{ x \in \mathbb{R}^3 : x_1 < -1 \}$, $R_2 = \{ x : |x_1| < 1 \}$ and $R_3 = \{ x : x_1 > 1 \}$,
- $C^0$ hyperplanes: $\Sigma_1 = \{ x : x_1 = -1 \}$ and $\Sigma_2 = \{ x : x_1 = 1 \}$,

- Roessler-type attractor,
- intersections with $\Sigma_1$ are not always transversal.
Example: find an enclosure of $\varphi(t, \mathbf{x})$ for $t = 2$ and $\mathbf{x} = ([1.2412, 1.2432], [-0.2141, -0.2121], [-4.7623, -4.7603])$, $\text{diam}(\mathbf{x}) = (0.002, 0.002, 0.002)$.

- $\mathbf{x}$ has non-empty intersection with the numerically observed attractor and some trajectories based in $\mathbf{x}$ are tangent to $\Sigma_1$.

- $\text{diam}(\mathbf{y}) = (0.0098, 0.0042, 0.041)$.

- Integration time as a perturbed linear system: $s_2 = 0.1936$.

- The size of initial set is relatively large and the integration time is relatively long thus showing usefulness of the proposed method.
Conclusions

- We have studied rigorous integration methods for piece-wise linear systems.
- An algorithm handling the case of trajectories tangent to hyperplanes separating linear regions has been described.
- Several examples have been considered to show the effectiveness of this technique.
- The methods can be used without major modifications for rigorous integration of piece-wise smooth systems — one has to use standard techniques for rigorous integration of nonlinear systems in smooth regions.

