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Holographic entanglement entropy

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Holographic Entanglement Entropy

(with H. Casini, M. Huerta, J. Hung, A. Sinha, M. Smolkin & A. Yale)

(arXiv:1101.5813, arXiv:1102.0440, arXiv:1109.0???)

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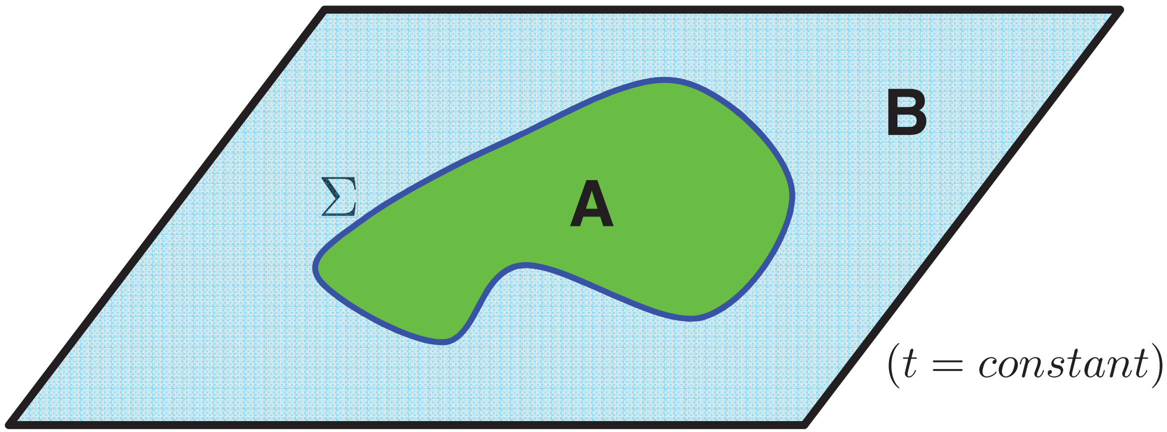
Entanglement Entropy

- what is entanglement entropy?
  very general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface Σ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix ρ_A

$\longrightarrow$ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$



Entanglement Entropy

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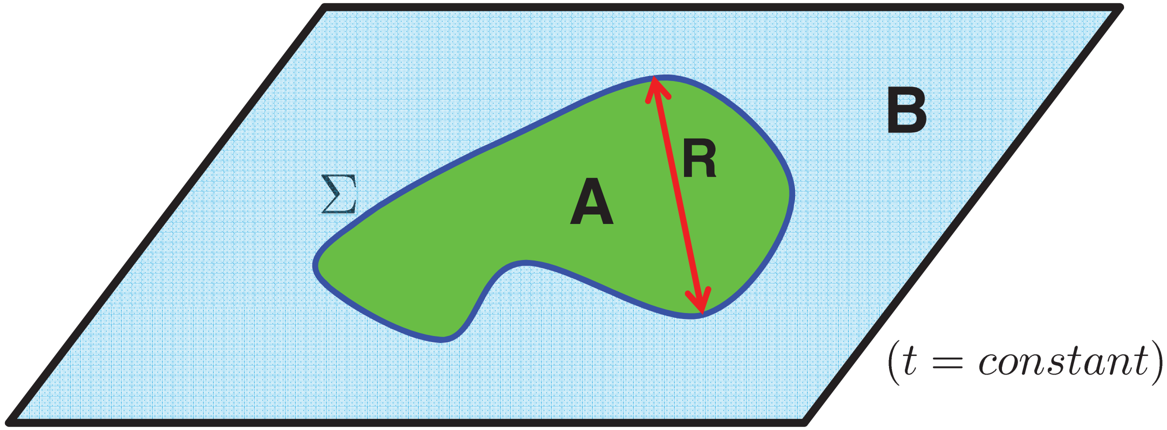


- result is UV divergent!

$$S = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_1 \frac{R^{d-3}}{\delta^{d-3}} + \cdots$$

d = spacetime dimension
δ = short-distance cut-off

- find universal information characterizing underlying QFT in subleading terms: $S = \cdots + c_d \log\left(R/\delta\right) + \cdots$ (for even d)

Entanglement Entropy

- remaining dof are described by a density matrix ρ_A

$\longrightarrow$ calculate von Neumann entropy: $S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$

- nonlocal quantity which is (at best) very difficult to measure
- in condensed matter theory: diagnostic to characterize quantum critical points or topological phases (eg, quantum hall fluids)
- in quantum information theory: useful measure of quantum entanglement (a computational resource)
- in black hole physics: leading term obeys "area law" $S \simeq c_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}}$

$\longrightarrow$ reminiscent of black hole entropy (eg, $\delta \simeq \ell_P$)

(Bombelli, Koul, Lee & Sorkin; Srednicki; Callan & Wilczek; Frolov; Susskind; . . . .)

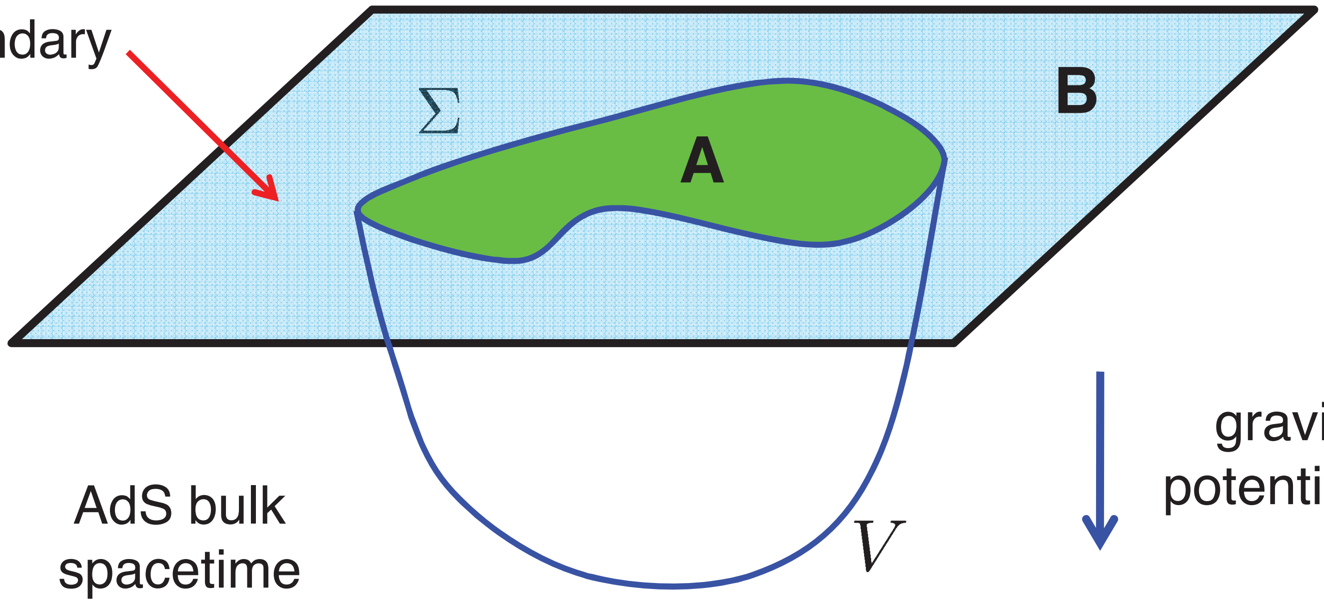
- recently considered in AdS/CFT correspondence

(Ryu & Takayanagi `06)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

AdS boundary



$r = \infty$

AdS bulk
spacetime

gravitational
potential/redshift

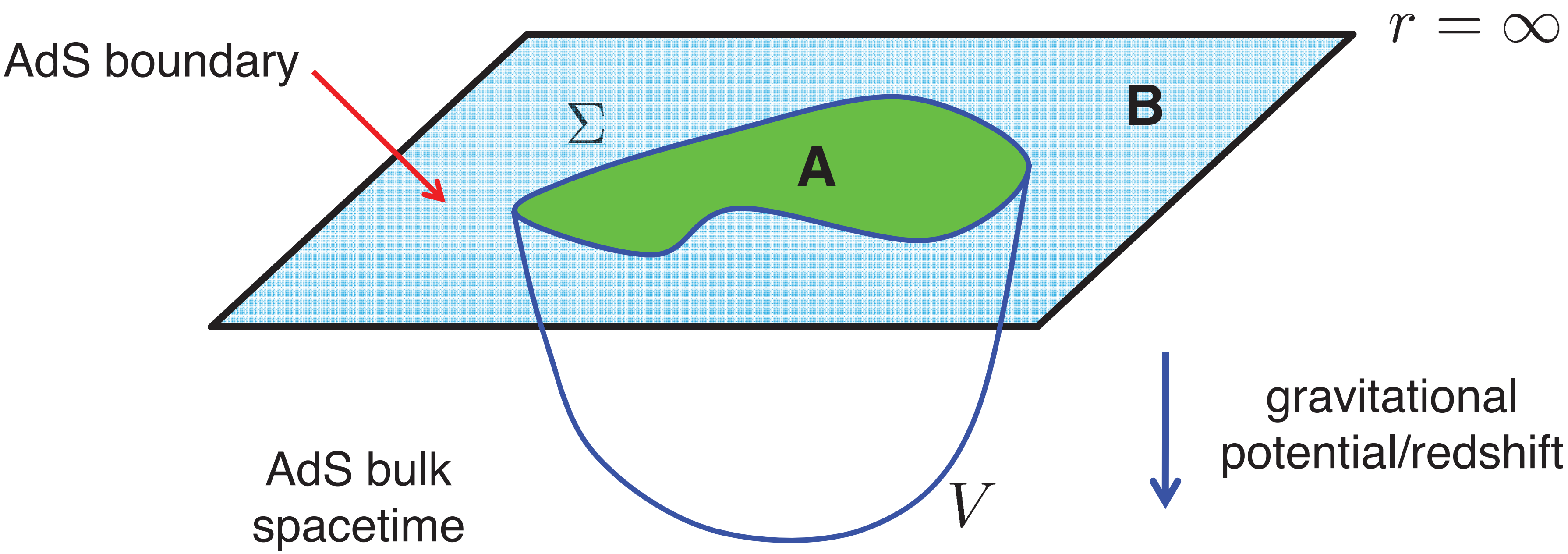
$(d-1)$ dimensional

$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

looks like
BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



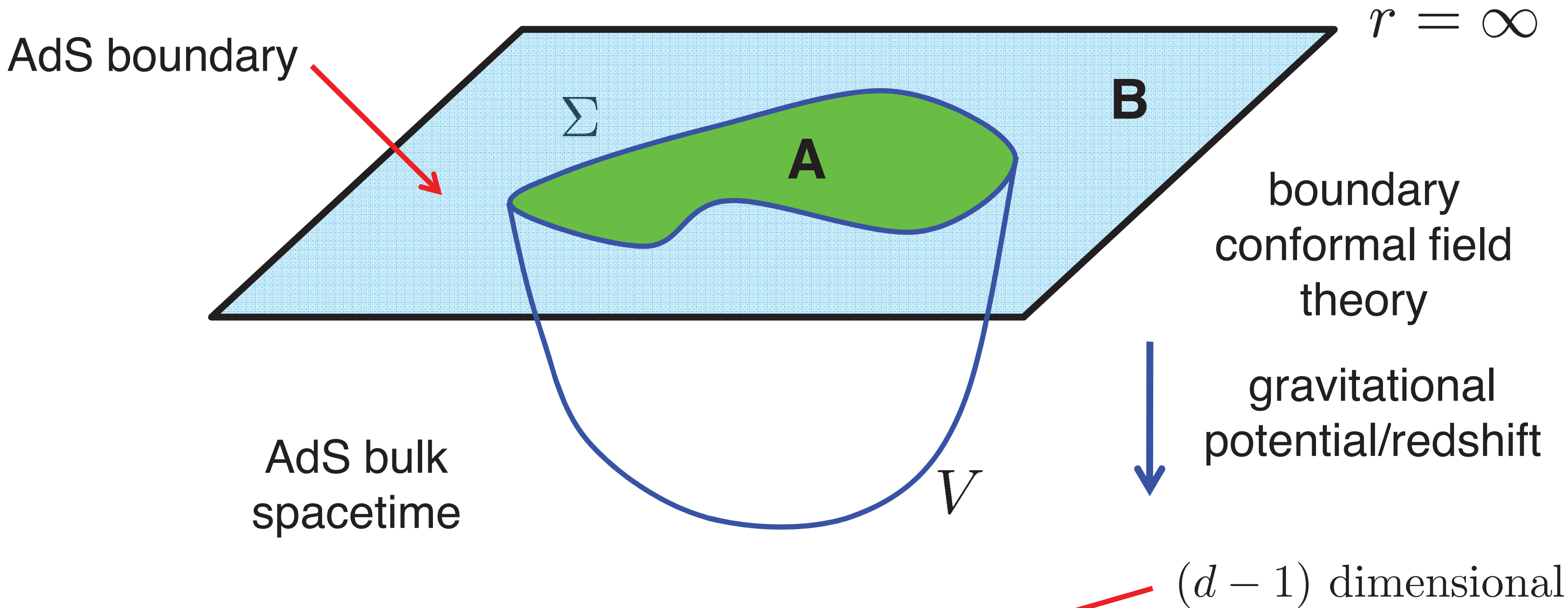
$(d-1)$ dimensional

$$S(A) \; = \; \operatorname*{ext}_{V \sim A} \; \frac{A_V}{4G_N}$$

looks like
BH entropy!

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

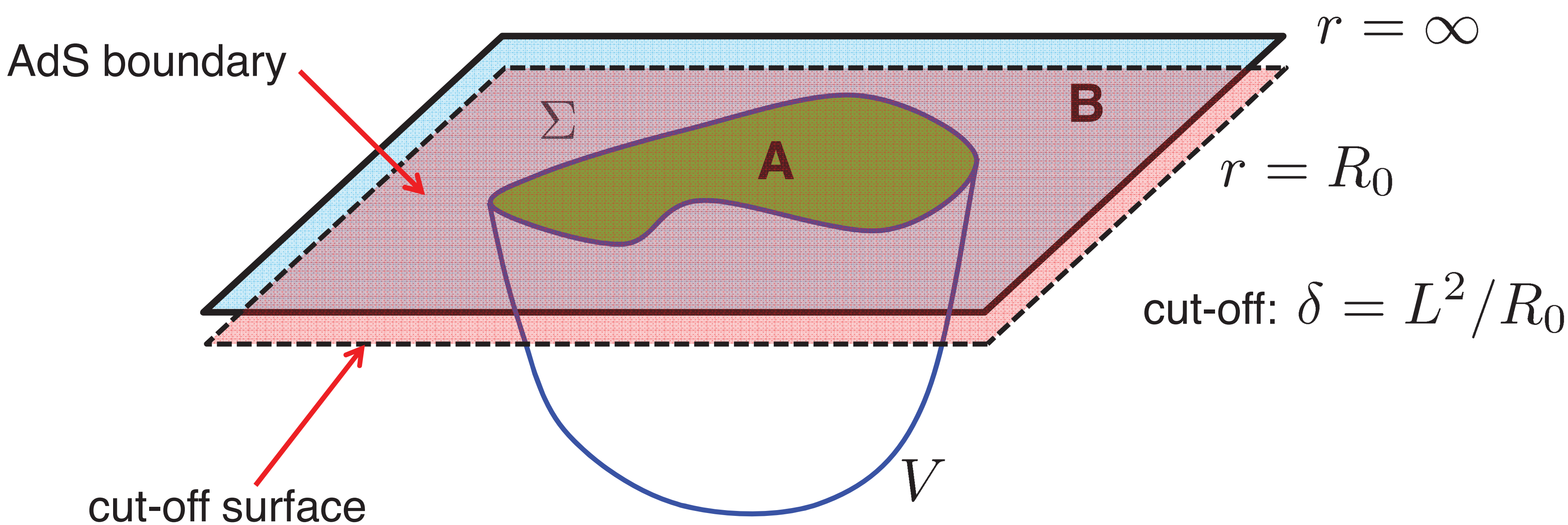


$$S(A) = \operatorname*{ext}_{V \sim A} \frac{A_V}{4G_N} = \infty!!$$

- "UV divergence" because area integral extends to $r = \infty$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



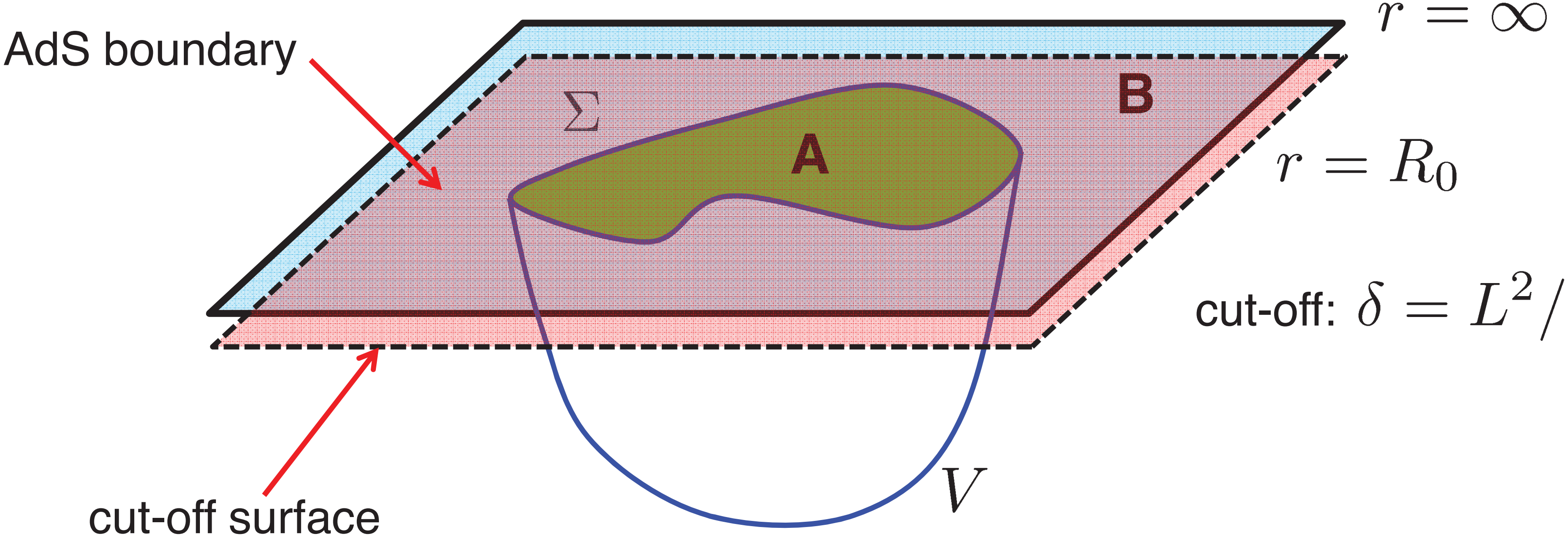
$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N} \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

- "UV divergence" because area integral extends to $r = \infty$
- finite result by stopping radial integral at large radius: $r = R_0$

$\longrightarrow$ short-distance cut-off in boundary theory: $\delta = L^2/R_0$

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



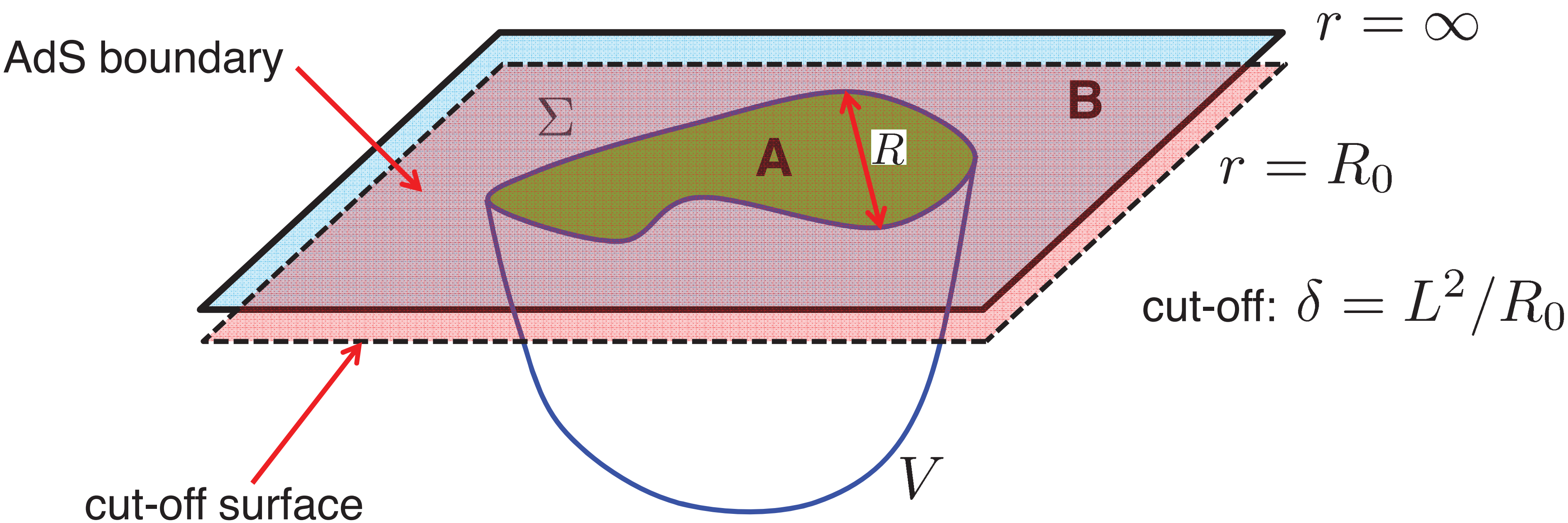
$$S(A) \; = \; \mathop{\mathrm{ext}}_{V \sim A} \; \frac{A_V}{4G_N} \simeq \frac{L^{d-1}}{G_N} \, \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$$

central charge
(counts dof) $\quad (L/\ell_{Planck})^{d-1}$

"Area Law"

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



general expression (as desired):

$$S(A) \simeq c_0 (R/\delta)^{d-2} + c_1 (R/\delta)^{d-4} + \cdots$$

$$\left\{ \begin{array}{ll} +c_{d-2} \log(R/\delta) + \cdots & \text{(d even)} \\ + \underbrace{c_{d-2} \;\; + \cdots}_{} & \text{(d odd)} \end{array} \right.$$

universal contributions

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \operatorname*{ext}_{V \sim A} \frac{A_V}{4G_N}$$

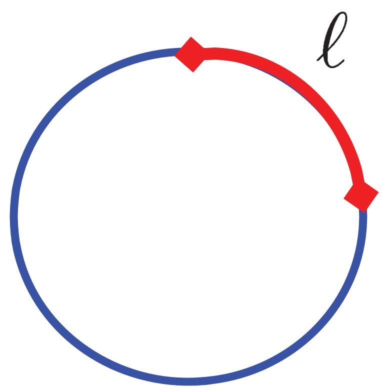
conjecture

Extensive consistency tests:

1) leading contribution yields "area law"

2) recover known results of Calabrese & Cardy for d=2 CFT

$$S = \frac{c}{3} \log\left(\frac{C}{\pi\,\delta} \sin\frac{\pi\ell}{C}\right)$$



$C = \text{circumference}$

(also result for thermal ensemble)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) \;=\; \underset{V\sim A}{\mathrm{ext}}\;\frac{A_V}{4G_N}$$

<u>conjecture</u>

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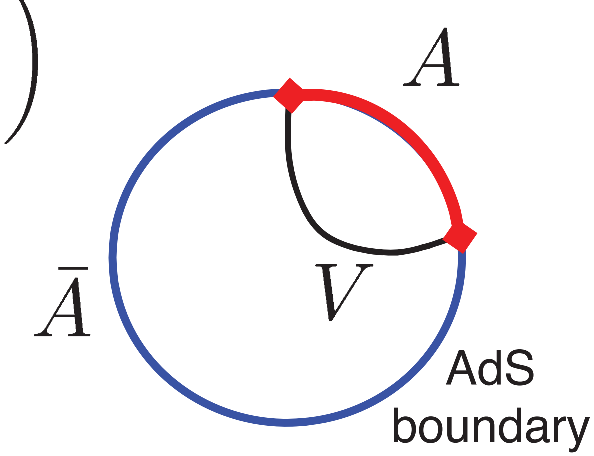


3) $S(A) = S(\bar{A})$ in a pure state

$\longrightarrow$ A and $\bar{A}$ both yield same bulk surface V

(not pure state $\longrightarrow$ horizon in bulk; $S(A) \neq S(\bar{A})$ for thermal state)

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

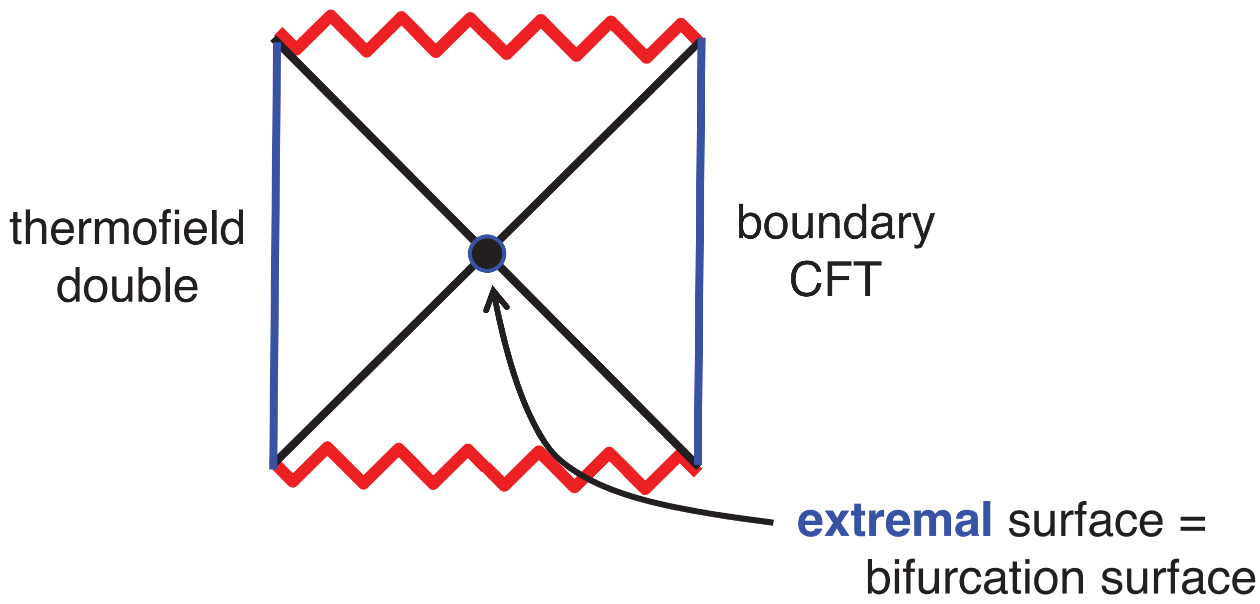
$$S(A) = \underset{V \sim A}{\mathrm{ext}} \frac{A_V}{4G_N}$$

<u>conjecture</u>

Extensive consistency tests:

4) Entropy of eternal black hole =
entanglement entropy of boundary CFT & thermofield double

(Headrick)



(Ryu & Takayanagi `06)

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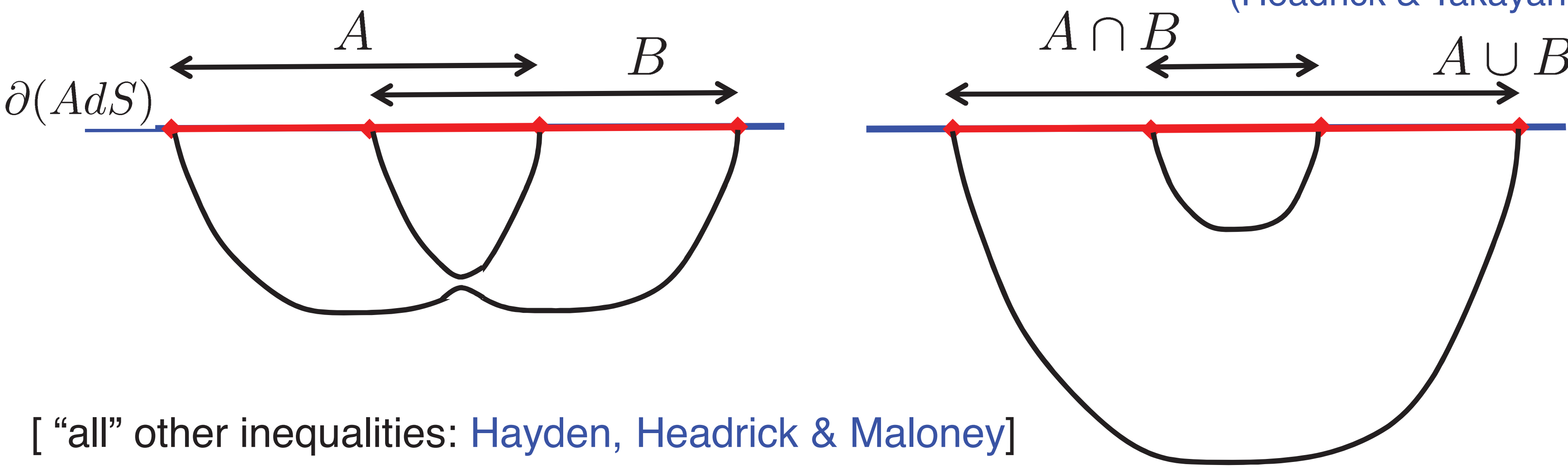
5) sub-additivity: $S(A \cup B) + S(A \cap B) \leq S(A) + S(B)$

(Headrick & Takayanagi)



[“all” other inequalities: Hayden, Headrick & Maloney]

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

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(Headrick & Takayanagi)

6) for general even d, connection to central charges of CFT

(Hung, RCM & Smolkin, arXiv:1101.5813)

7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

Central charges and trace anomaly:

d=2: $$\langle\, T_\mu{}^\mu \,\rangle = -\frac{\mathbf{c}}{12}\, R$$

d=4: $$\langle\, T_\mu{}^\mu \,\rangle = \frac{\mathbf{c}}{16\pi^2}\, I_4 \;-\; \frac{\mathbf{a}}{16\pi^2} E_4$$

$$I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \quad \text{and} \quad E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

- in higher (even) dimensions, number of central charges grows

$$\langle\, T_\mu{}^\mu \,\rangle = \sum \mathbf{B}_i\,(\text{Weyl invariants})_i \;-\; 2(-)^{d/2}\,\mathbf{A}\,(\text{Euler density})_d$$

(Deser & Schwimmer)

- universal contribution to entanglement entropy determined using trace anomaly (for even d)

(Holzhey, Larsen & Wilczek;Calabrese & Cardy;Takayanagi & Ryu;Schwimmer & Theisen)

$$S_{univ} = \log\left(R/\delta\right)\, 2\pi \int_\Sigma d^{d-2}x\,\sqrt{h}\, \frac{\partial \langle\, T_\lambda{}^\lambda \,\rangle}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$

(RCM & Sinha)

- partial result! needs rotational symmetry on entangling surface Σ

Central charges and trace anomaly:

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{\mathbf{c}}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{\mathbf{c}}{16\pi^2} I_4 - \frac{\mathbf{a}}{16\pi^2} E_4$$

$I_4 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ and $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

- in higher (even) dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even d)

d=2: $$S = \frac{\mathbf{c}}{3} \log\left(\frac{C}{\pi\,\delta} \sin\frac{\pi\ell}{C}\right)$$ (Holzhey, Larsen & Wilczek; Calabrese & Cardy)

d=4:
$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^\perp_{ik}\,\tilde{g}^\perp_{jl} \underline{- K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}}\right) - \mathbf{a}\,\mathcal{R}\right]$$

corrections for general (smooth) Σ (Solodukhin)

Central charges and trace anomaly:

d=2: $$\langle T_\mu{}^\mu \rangle = -\frac{c}{12} R$$

d=4: $$\langle T_\mu{}^\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4$$

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- in higher dimensions, numbers of central charges grows
- universal contribution to entanglement entropy determined using trace anomaly (for even d)
- central charges identified in AdS/CFT using holographic trace anomaly: (Henningson &Skenderis)

  e.g., for (boundary) d=4: $\boxed{a = c} = \pi^2 L^3/\ell_P^3$
- for general d, central charges $\propto (L/\ell_P)^{d-1}$
- for Einstein gravity, all central charges equal for any d
- distinguishing central charges requires higher curvature gravity

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

- consider more general gravity theory in AdS:

$$I = \int d^{d+1}x \sqrt{-g}\, \mathcal{L}(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \ldots, matter)$$

- how do we evaluate holographic entanglement entropy?

$\longrightarrow$ take direction from tests of R&T prescription

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

$$S(A) = \underset{V \sim A}{\text{ext}} \frac{A_V}{4G_N}$$

<u>conjecture</u>

Extensive consistency tests:

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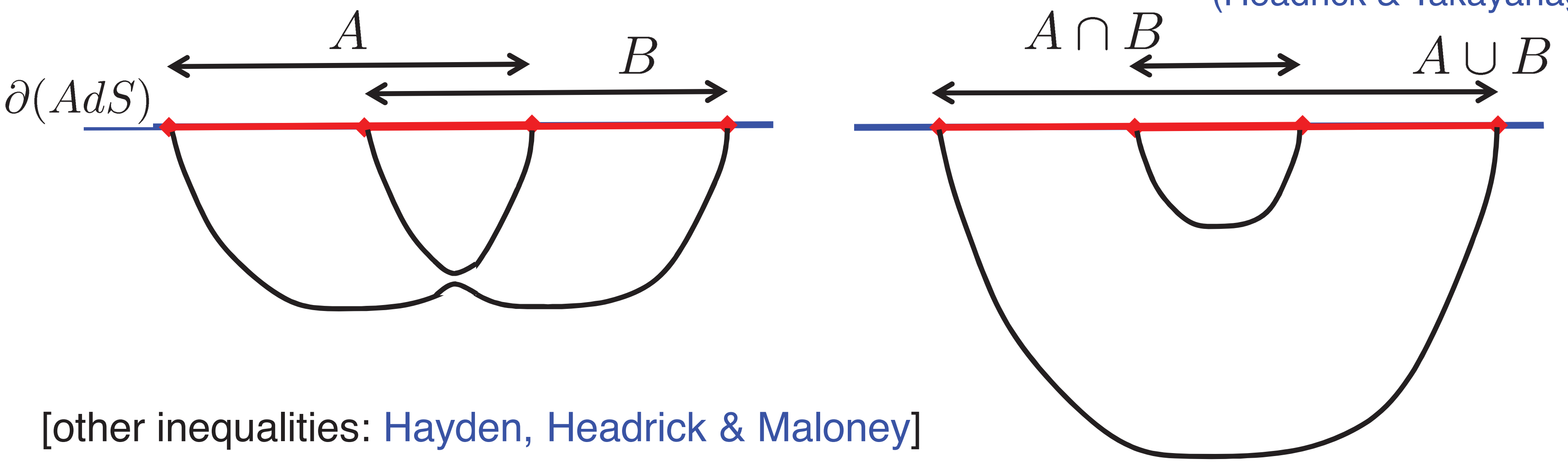

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$$I = \int d^{d+1}x\sqrt{-g}\,\mathcal{L}(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \ldots, matter)$$

- natural conjecture: extremize Wald's entropy formula

$$S = -2\pi \int d^{d-1}x\sqrt{h}\,\frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\,\hat{\varepsilon}^{\,\mu\nu}\,\hat{\varepsilon}_{\rho\sigma}$$

- focus on universal term for d=4: (Solodukhin)

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

- holographic calculation following above conjecture yields

$$S_{uni} = \log(R/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\left[\mathbf{a}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

$\longrightarrow$ conjecture wrong 

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

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$$I = \int d^{d+1}x \sqrt{-g}\, \mathcal{L}(g^{ab}, R^{ab}{}_{cd}, \nabla_e R^{ab}{}_{cd}, \dots, matter)$$

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- triumph of R&T prescription in Einstein gravity!! (**c** = **a**)
- for general gravity action, conjecture is wrong
- there is nothing wrong with Wald's formula!!

$\longrightarrow$ to proceed further, focus on special gravity actions

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy:

- consider special case of Gauss-Bonnet gravity:

$$I=\frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2}+R+L^2\,\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd}-4R_{ab}R^{ab}+R^2\right)}_{\text{4d Euler density}}\right]$$

- higher curvature but eom are still **second order!!** (Lovelock)
- studied in detail for stringy gravity in 1980's

(Zwiebach; Boulware & Deser; Wheeler; Myers & Simon; . . . .)

- interest recently in AdS/CFT studies – a toy model with $c\neq a$

(eg, Brigante, Liu, Myers, Shenker, Yaida, de Boer, Kulaxizi, Parnachev, Camanho, Edelstein, Buchel, Sinha, Paulos, Escobedo, Smolkin, Cremonini, Hofman, ...)

- black hole entropy: (Jacobson & Myers)

$$S_{JM}=\frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1+\lambda\,L^2\,\mathcal{R}\right]$$

- not precisely same as Wald entropy – agree when K^i_{ab} vanish

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy: (deBoer, Kulaxizi & Parnachev)

- consider special case of Gauss-Bonnet gravity:

$$I = \frac{1}{2\ell_p^3}\int d^5x\,\sqrt{-g}\left[\frac{12}{L^2} + R + L^2\frac{\lambda}{2}\,\underbrace{\left(R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2\right)}_{\text{4d Euler density}}\right]$$

- second conjecture: extremize JM entropy formula

$$S_{JM} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1 + \lambda\,L^2\,\mathcal{R}\right]$$

- again consider universal term for d=4: (Solodukhin)

$$S_{univ} = \log(\ell/\delta)\,\frac{1}{2\pi}\int_\Sigma d^2x\sqrt{h}\,\left[\mathbf{c}\left(C^{ijkl}\,\tilde{g}^{\perp}_{ik}\,\tilde{g}^{\perp}_{jl} - K^{i\,b}_{a}K^{ia}_{b} + \frac{1}{2}K^{i\,a}_{a}K^{i\,b}_{b}\right) - \mathbf{a}\,\mathcal{R}\right]$$

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$\longrightarrow$ passes nontrivial test

(Hung, Myers & Smolkin)

Holographic Entanglement Entropy: (deBoer, Kulaxizi & Parnachev)

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- second conjecture: extremize JM entropy formula

$$S_{JM} = \frac{2\pi}{\ell_p^3}\int d^3x\sqrt{h}\,\left[1 + \lambda\,L^2\,\mathcal{R}\right]$$

✓ reproduces universal term for any smooth surface in d=4

✓ partial results for d=6 (geometries with rotational symmetry; found new curvature corrections when $K^i_{ab} = 0$)

✓ extends to general Lovelock theories for d≥6

? still no general result for completely general gravity action ?

⟶ with sufficient symmetry, Wald entropy seems correct

? curious instability to adding handles for $\lambda > 0$? (Ogawa & Takayanagi)

(Ryu & Takayanagi `06)

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<u>conjecture</u>

Extensive consistency tests:

4) Entropy of eternal black hole = entanglement entropy of boundary CFT & thermofield double

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7) derivation of holographic EE for spherical entangling surfaces

(Casini, Huerta & RCM, arXiv:1102.044)

(see also: RCM & Sinha, arXiv:1011.5819)

Calculating Entanglement Entropy:

$$S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$$

- "standard" approach relies on replica trick and calculating Renyi entropy first and taking n $\longrightarrow$ 1 limit

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n \to 1} S_n$$

- replica trick involves path integral of QFT in singular n-fold cover of background spacetime

- problematic in holographic framework
  $\longrightarrow$ produce singularity in dual gravity description
  (resolved by quantum gravity/string theory?)

(Fursaev; Headrick)

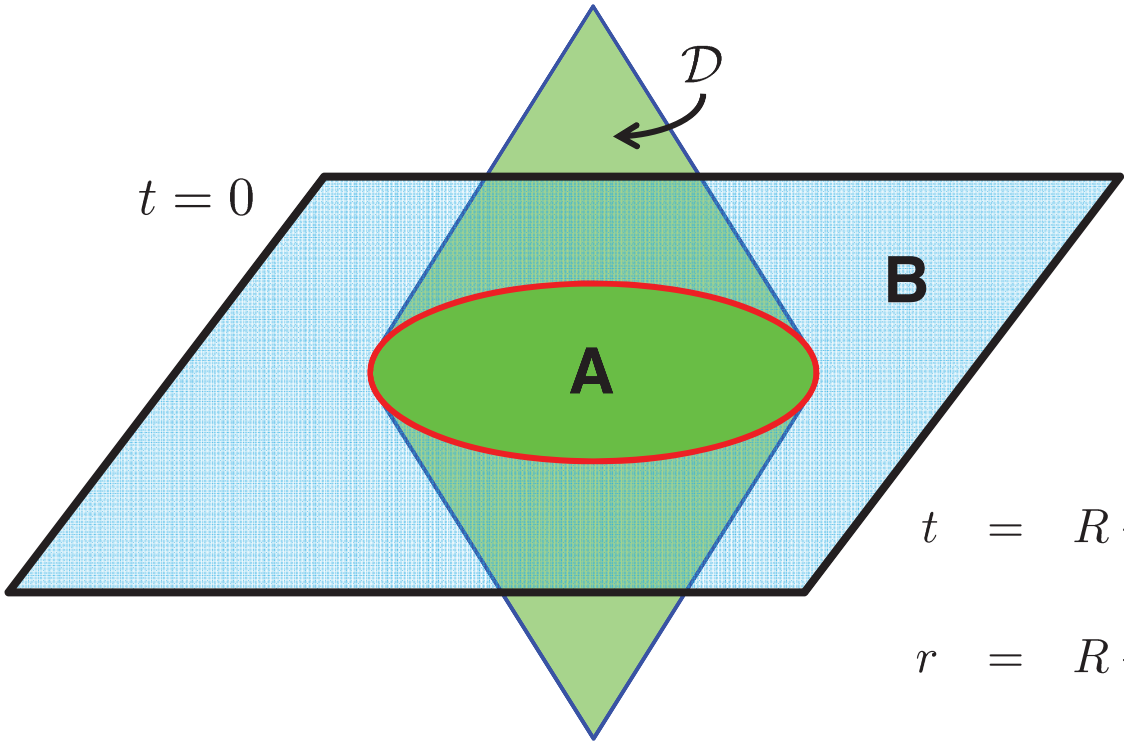
- need another calculation with simpler holographic translation

Calculating Entanglement Entropy:

(Casini, Huerta & RCM)

- take $\boxed{\text{CFT}}$ in d-dim. flat space and choose $\boxed{\Sigma = S^{d-2}}$ with radius R

$\longrightarrow$ entanglement entropy: $S_{EE} = -Tr\,[\rho_A \log \rho_A]$



$$
\begin{aligned}
t &= R\,\frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)} \\
r &= R\,\frac{\sinh u}{\cosh u + \cosh(\tau/R)}
\end{aligned}
$$

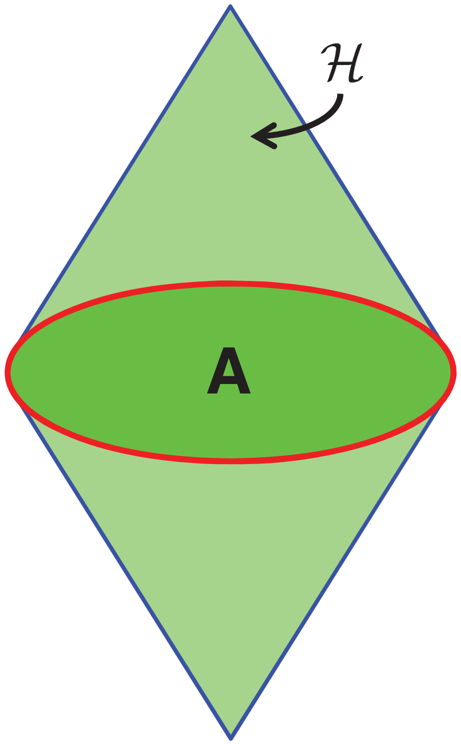
- density matrix ρ_A describes physics in entire causal domain $\mathcal{D}$
- conformal mapping: $\mathcal{D} \rightarrow \mathcal{H} = R \times H^{d-1}$

General result for any CFT

(Casini, Huerta & RCM)

- take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\,[\rho_A \log \rho_A]$



$$t = R\,\frac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)}$$
$$r = R\,\frac{\sinh u}{\cosh u + \cosh(\tau/R)}$$

- conformal mapping: $\mathcal{D} \to \mathcal{H} = R \times H^{d-1}$

curvature scale: 1/R   temperature: T=1/2πR !!

- for CFT: $\rho_{thermal} = U\,\rho_A\,U^{-1}$ → $\boxed{S_{EE} = S_{thermal}}$

General result for any CFT

(Casini, Huerta & RCM)

• take CFT in d-dim. flat space and choose S^{d-2} with radius R

→ entanglement entropy: $S_{EE} = -Tr\,[\rho_A \log \rho_A]$

→ by conformal mapping relate to thermal entropy on $\mathcal{H} = R \times H^{d-1}$ with $\mathcal{R} \sim 1/R^2$ and $T=1/2\pi R$

$$\boxed{S_{EE} = S_{thermal}}$$

AdS/CFT correspondence:

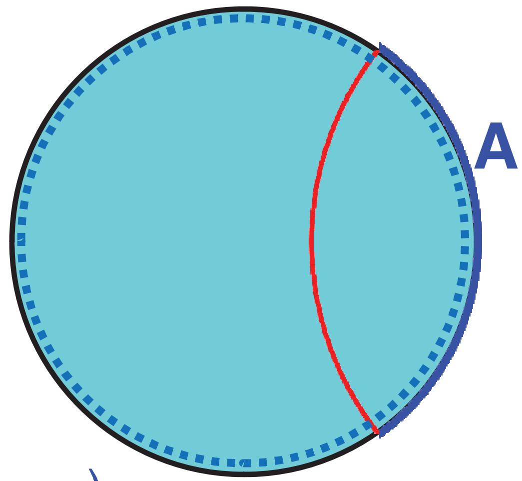
• thermal bath in CFT = black hole in AdS

$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

• only need to find appropriate black hole

→ topological BH with hyperbolic horizon which intersects A on AdS boundary

(Aminneborg et al; Emparan; Mann; . . .)

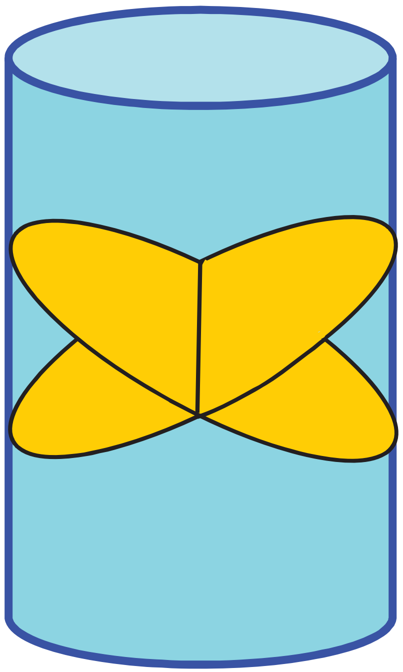


$$\boxed{S_{EE} = S_{thermal} = S_{horizon}}$$

- desired “black hole” is a hyperbolic foliation of empty AdS space

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right) d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- “Rindler coordinates” of AdS space



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$$ds^2 = \frac{L^2\, d\rho^2}{(\rho^2 - L^2)} - \frac{\rho^2 - L^2}{R^2}\, d\tau^2 + \rho^2\, d\Sigma_2^{d-1} \longrightarrow T = \frac{1}{2\pi R}$$

- apply Wald’s formula (for any gravity theory) for horizon entropy:

$$S = -2\pi \int d^{d-1}x\sqrt{h}\, \frac{\partial \mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\, \hat{\varepsilon}^{\,\mu\nu}\, \hat{\varepsilon}_{\rho\sigma}$$
$$= \frac{2\pi}{\pi^{d/2}} \Gamma\left(d/2\right)\, \frac{a_d^*}{R^{d-1}}\, V\left(H^{d-1}\right)$$

(RCM & Sinha)

where a_d^* = central charge for “A-type trace anomaly”

for even d

= entanglement entropy defines effective central charge

for odd d

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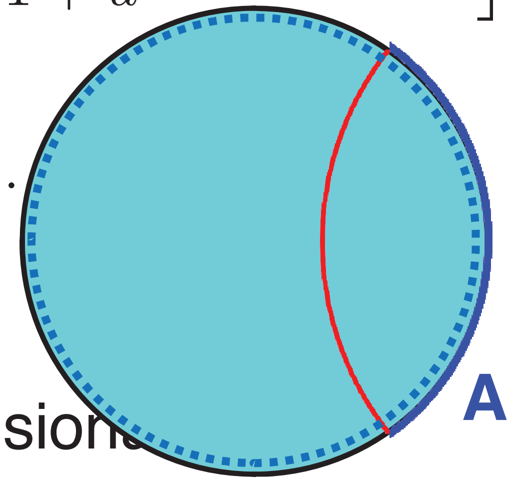
$$ds^2 = R^2\left[\frac{du^2}{1+u^2} + u^2\, d\Omega_2^{d-2}\right]$$

intersection with standard regulator surface: $z_{min} = \delta$

$$S = a_d^*\,\frac{4\pi^{\frac{d-3}{2}}}{(d-2)\Gamma\left(\frac{d-1}{2}\right)}\underbrace{\left(\frac{R}{\delta}\right)^{d-2}} + \cdots$$

“area law” for d-dimension



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universal contributions:

$$S = \cdots + (-)^{\frac{d}{2}-1} 4\, a_d^* \, \log\left(2R/\delta\right) + \cdots \quad \text{for even d}$$

$$\cdots + (-)^{\frac{d-1}{2}} 2\pi\, a_d^* + \cdots \quad \text{for odd d}$$

- discussion extends to case with background: $R^{1,d-1} \to R \times S^{d-1}$

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- turn to Renyi entropy (close cousin of entanglement entropy)

$$S_n = \frac{1}{1-n} \log Tr\left[\rho_A^n\right] \qquad\qquad S_{EE} = \lim_{n\to 1} S_n$$

- universal contribution (for even d)

$$S_n \;=\; \cdots \;+\; constant \times \log\left(R/\delta\right) \;+\; \cdots$$

(Hung, RCM, Smolkin & Yale)

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- universal contribution (for even d)

d=2: $$S_n = \cdots + \frac{c}{6}\left(1+\frac{1}{n}\right) \log\left(R/\delta\right) + \cdots$$

(Calabrese & Cardy)

- (almost) no calculations for d > 2

(Hung, RCM, Smolkin & Yale)

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- "standard" calculation involves singular n-fold cover of spacetime

$\longrightarrow$ problematic for translation to dual AdS gravity

- our previous derivation lead to thermal density matrix

$$\rho_A = U^{-1} \frac{e^{-H/T_0}}{Tr\left[e^{-H/T_0}\right]} U \qquad \text{with} \qquad T_0 = \frac{1}{2\pi R}$$

$$\Longrightarrow \quad Tr\left[\rho_A^n\right] = \frac{Tr\left[e^{-nH/T_0}\right]}{Tr\left[e^{-H/T_0}\right]^n} \quad \longleftarrow$$

partition function at new temperature, $T = T_0/n$

(Hung, RCM, Smolkin & Yale)

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  $\longrightarrow$ problematic for translation to dual AdS gravity

- with bit more work, find convenient formula:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)\,dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}$$

Renyi entropy for spherical Σ

thermal entropy on hyperbolic space H^{d-1}

- in holographic framework, need to know topological black hole solutions for arbitrary temperature

(Hung, RCM, Smolkin & Yale)

Holographic Renyi entropy:

- Renyi entropy of CFT for spherical entangling surface:

$$S_n = \frac{n}{n-1}\frac{1}{T_0}\int_{T_0/n}^{T_0} S(T)dT \quad \text{where} \quad T_0 = \frac{1}{2\pi R}$$

- need to know topological black holes for arbitrary temperature
- focus on gravity theories where we can calculate: Einstein, Gauss-Bonnet, Lovelock, quasi-topological, …..
- for example, with GB gravity and (boundary) d=4:

$$S_n = \frac{n}{n-1}\frac{V(H^3)}{4\pi}\frac{3c-a}{3a-c}(1-x^2)\left[(5a-c)x^2-(13a-5c)+4a\frac{2ax^2-(a-c)}{(3a-c)x^2-(a-c)}\right]$$

$$\text{where} \qquad 0 = x^3 - \frac{3a-c}{5a-c}\left(\frac{x^2}{n}+x\right)+\frac{1}{n}\frac{a-c}{5a-c}$$

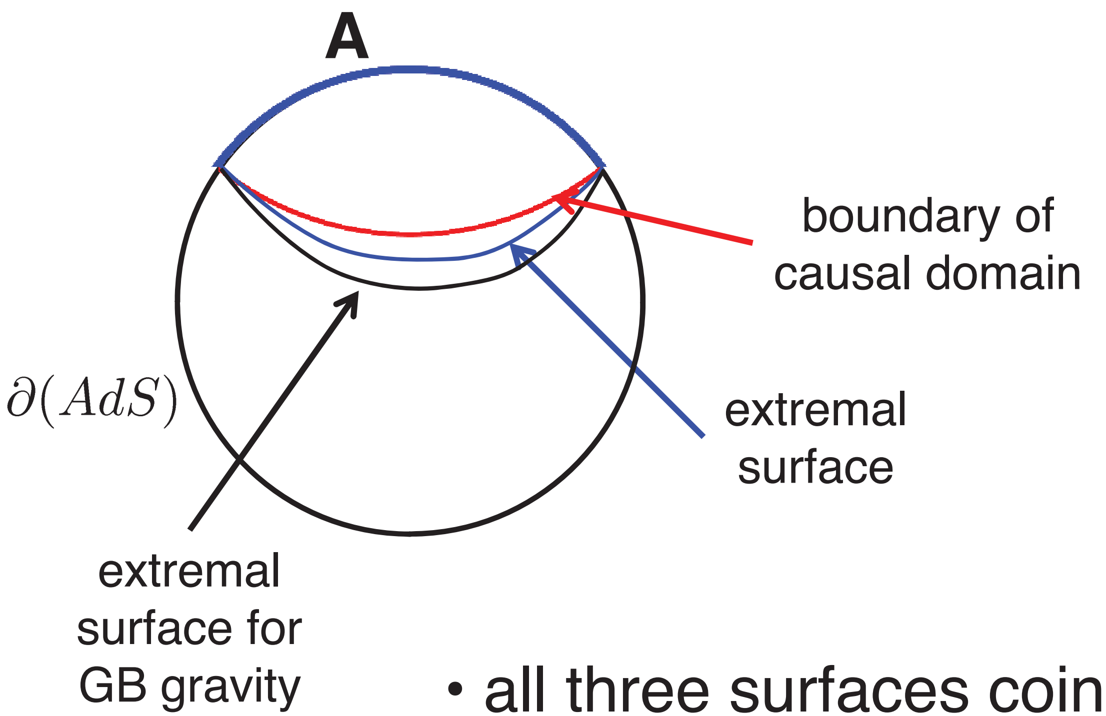
- further work shows the universal constant depends on more boundary data than central charges

General Lessons/Challenges:

- is there a "covariant" definition of holographic EE??

  ("covariant" = in terms of light sheets & causal structure)

(see, eg, Hubney, Rangamani & Takayanagi)



- differences in S_{EE} are infinite as $\delta \to 0$
- seems answer is: **NO**

- all three surfaces coincide for spherical Σ

→ Problem?: in general, entanglement H is nonlocal

What are the general rules?

Conclusions:

- AdS/CFT correspondence (gauge/gravity duality) has proven an excellent tool to study strongly coupled gauge theories

- holographic entanglement entropy is part of an interesting dialogue has opened between string theorists and physicists in a variety of fields (eg, condensed matter, nuclear physics, . . .)

- potential to learn lessons about issues in boundary theory
  eg, readily calculate Renyi entropies for wide class
  of theories in higher dimensions

- potential to learn lessons about issues in bulk gravity theory
  eg, holographic entanglement entropy may give new
  insight into quantum gravity or emergent spacetime

(eg, van Raamsdonk)

Lots to explore!