Lecture 3 : ultracold Fermi Gases

The ideal Fermi gas: a reminder

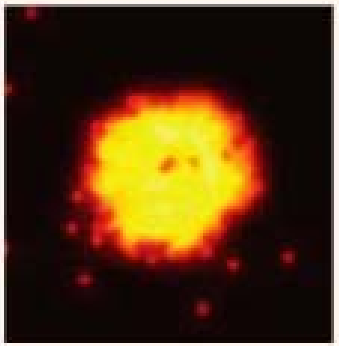
Interacting Fermions

BCS theory in a nutshell

The BCS-BEC crossover and quantum simulation

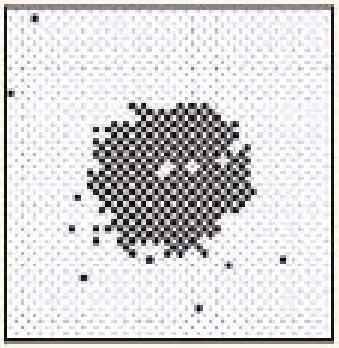
Many-Body Physics with Cold Gases

Diluteness: atom-atom interactions described by 2-body (and 3 body) physics. At low energy: a single parameter, the scattering length *a*

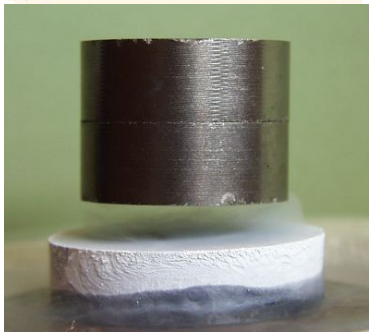


Sherson et al., MPQ 2010

Control of the sign and magnitude of interaction
Control of trapping parameters:
access to time dependent phenomena,
out of equilibrium situations, 1D, 2D, 3D
Simplicity of detection



Comparison with quantum Many-Body theories:
Gross-Pitaevskii, Bose and Fermi Hubbard models,
search for exotic phases, dipolar gases
disorder effects, Anderson localization, …



Link with condensed matter (high Tc superconductors),
astrophysics (neutron stars), Nuclear physics,
high energy physics (quark-gluon plasma),

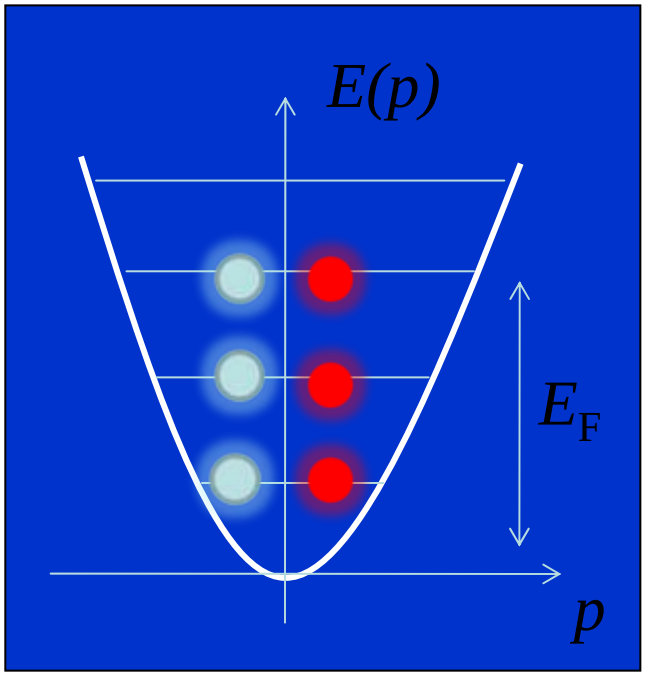
Quantum simulation with cold atoms « a la Feynman »

The ideal Fermi gas: a reminder

Zero temperature Fermi sea:

$$E_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2m} = k_{\mathrm{B}} T_{\mathrm{F}}$$

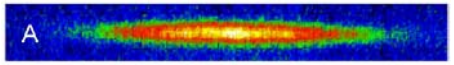
$$k_F = (6\pi^2 n)^{1/3} \sim (\text{particle distance})^{-1}$$



Fermi pressure $\frac{1}{15\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{5/2}$

Approximation valid as long as $T \ll T_{\mathrm{F}}$

Electrons vs cold atoms

	Electrons in metal	Ultra cold atoms
		
Density	10^{30} /m^3	10^{20} /m^3
Mass	10^{-30} kg	10^{-26} kg
Fermi temperature	10^4 K	1µK
T/T_F	$<10^{-5}$	~ 10^{-2}
Lifetime	Infinite	~10s
Size	1cm	10 µm
Particle number	10^{24}	10^5

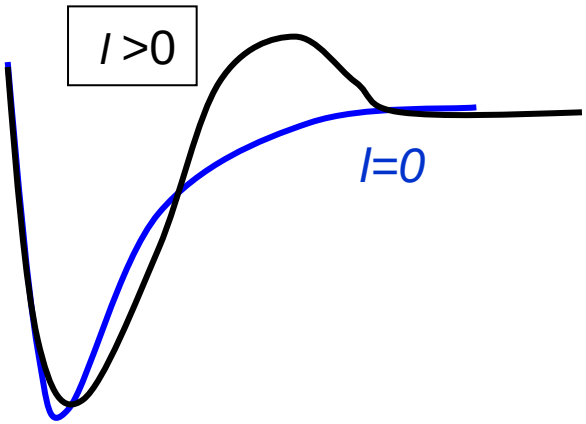
Pauli Exclusion Principle and evaporative cooling of ultra-cold Fermi gases

Collision between two atoms. Effective potential in the *l*-wave:

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}$$

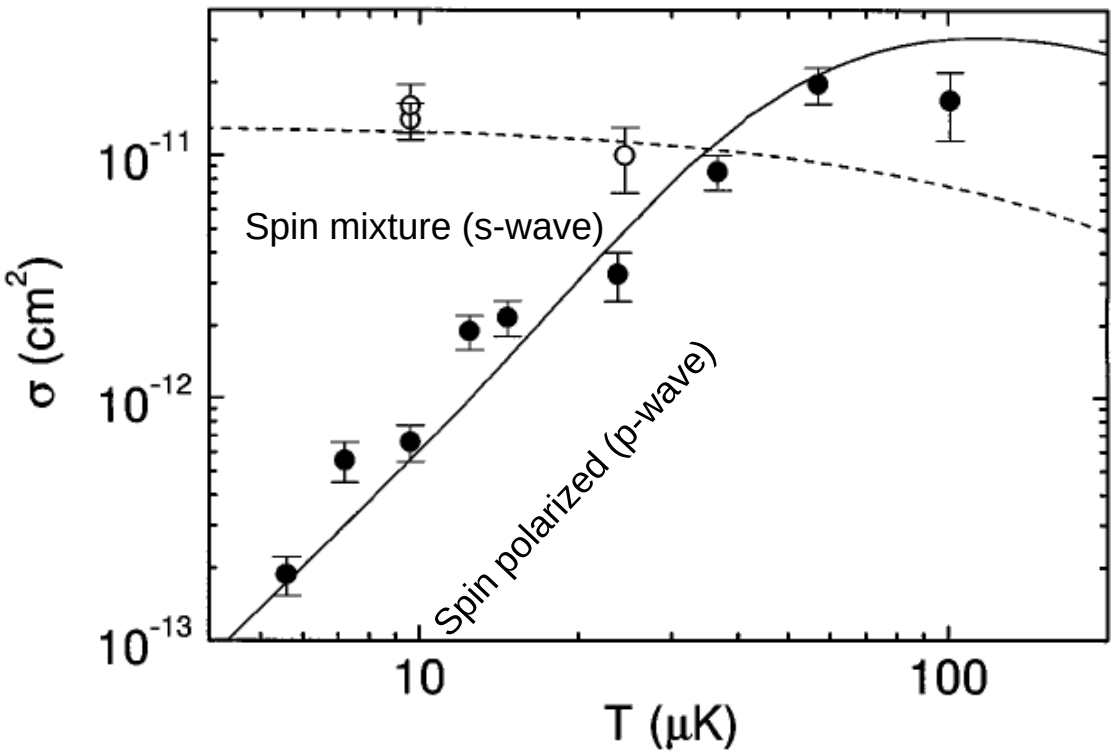
Interatomic potential
(long range~-1/r^6)

centrifugal potential



At low temperature, atoms cannot cross the centrifugal barrier: only s-wave collisions. Symmetrization for identical particles: even l-wave collisions forbidden for polarized fermions.

Suppression of elastic collisions in a spin polarized Fermi gas

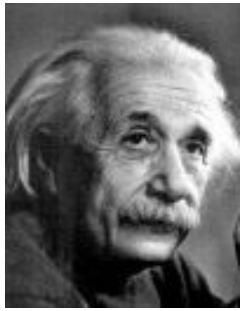


B. DeMarco, J. L. Bohn, J.P. Burke, Jr., M. Holland, and D.S. Jin, Phys. Rev. Lett. **82**, 4208 (1999).

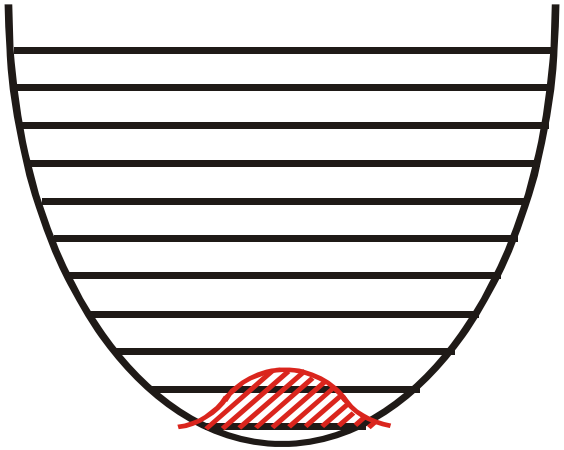
Use spin mixtures or several atomic species (eg ^{6}Li-^{7}Li, K-Rb, different spin states…)

Quantum gases in harmonic traps

- Bose-Einstein statistics (1924)

Bose-Einstein condensate

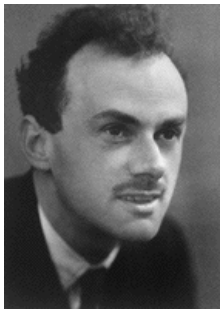


Bose enhancement

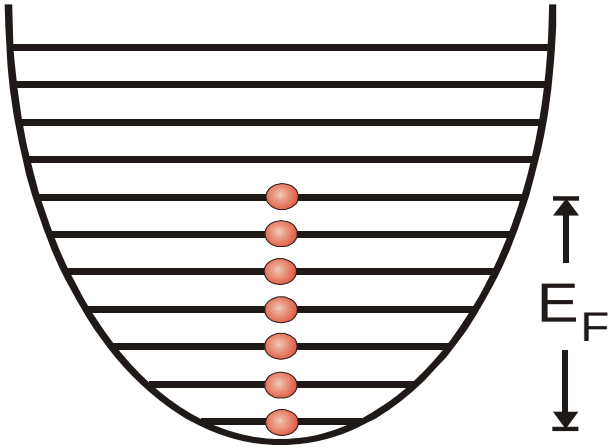
$$T_C = \frac{\hbar\omega}{k_B}(0.83\ N)^{1/3}$$

Dilute gases: 1995, JILA, MIT

- Fermi-Dirac statistics (1926)

Fermi sea

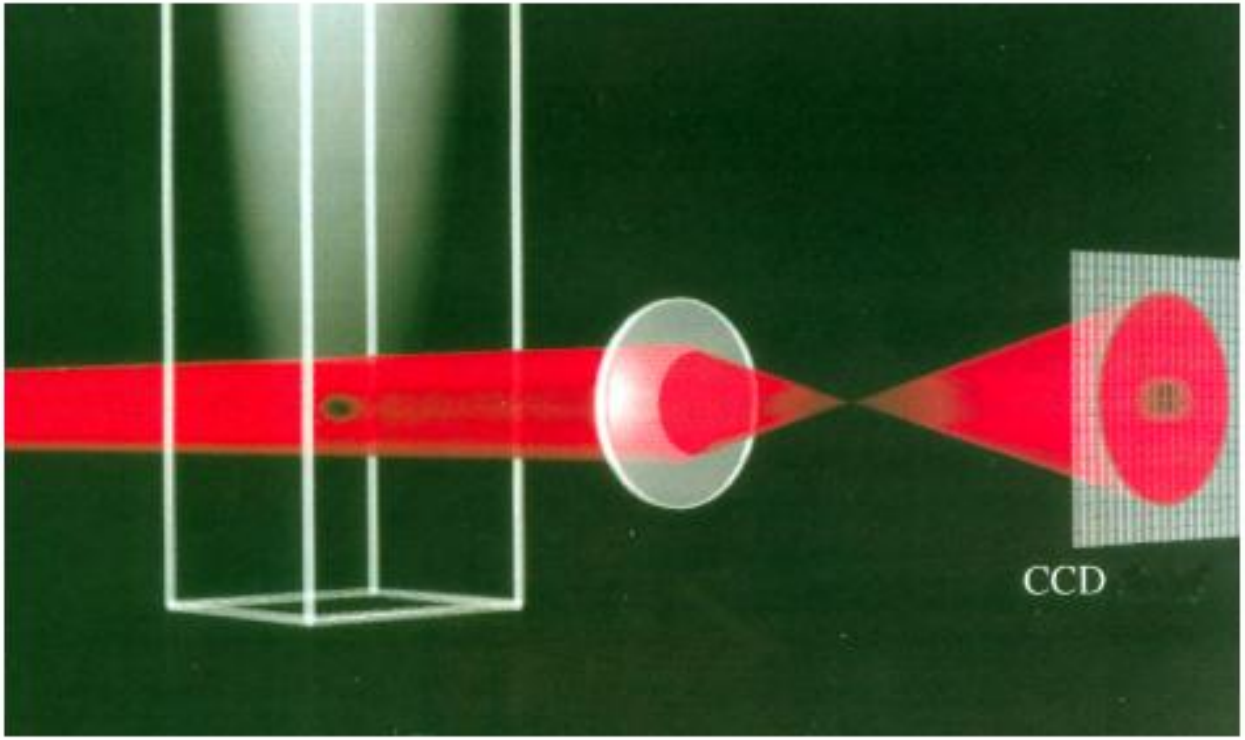


Pauli Exclusion

$$T << T_F = \frac{\hbar\omega}{k_B}(6\ N)^{1/3}$$

Dilute gases: 1999, JILA

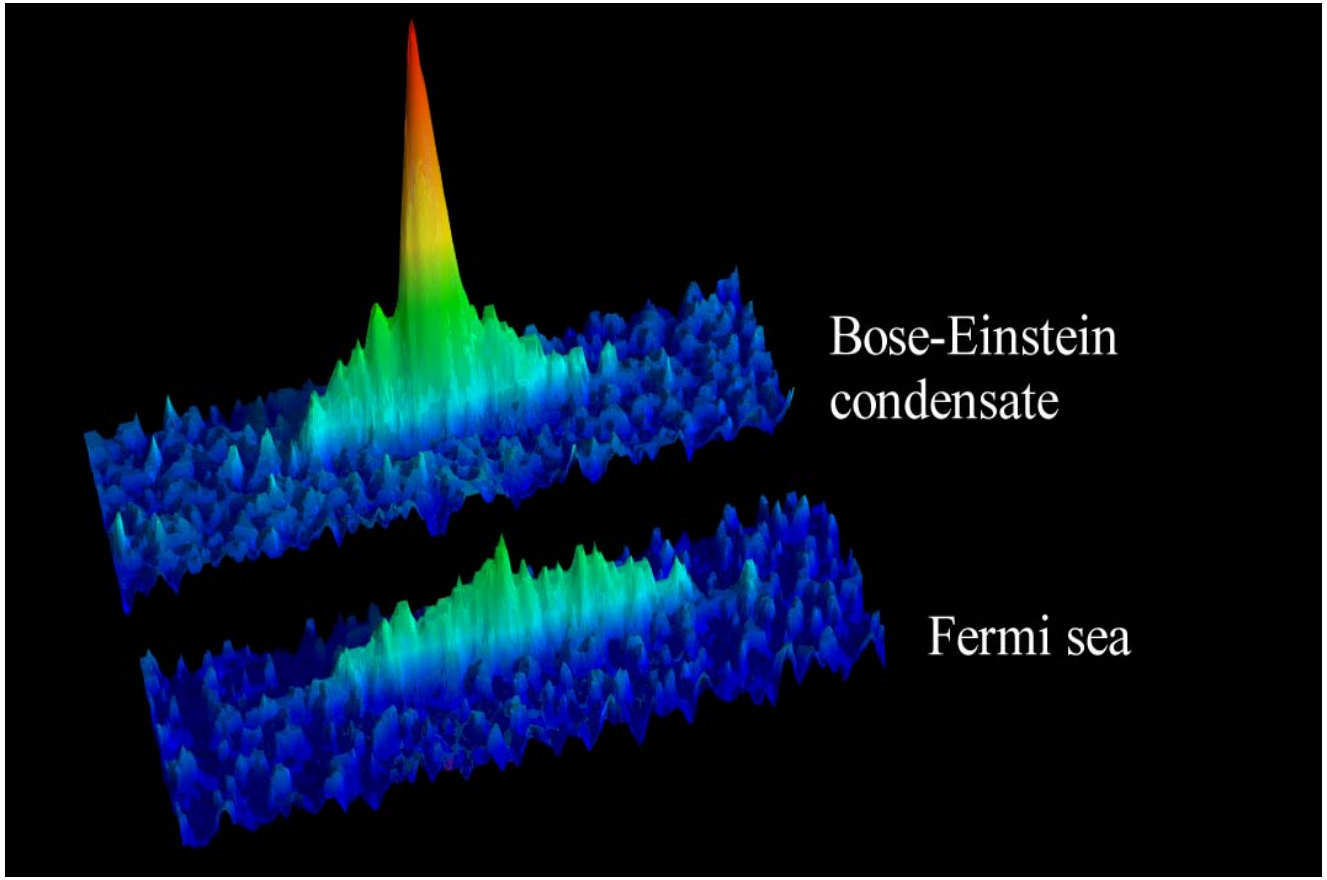
Absorption imaging



in situ: cloud size

Bose-Einstein condensate and Fermi sea

2001
ENS



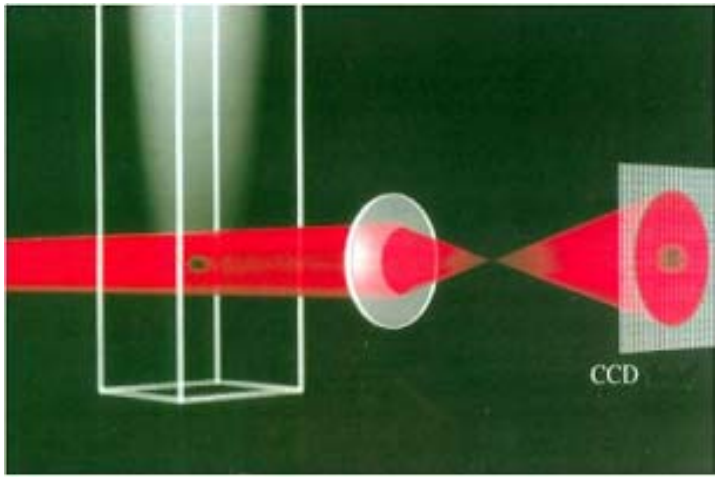
Lithium 7

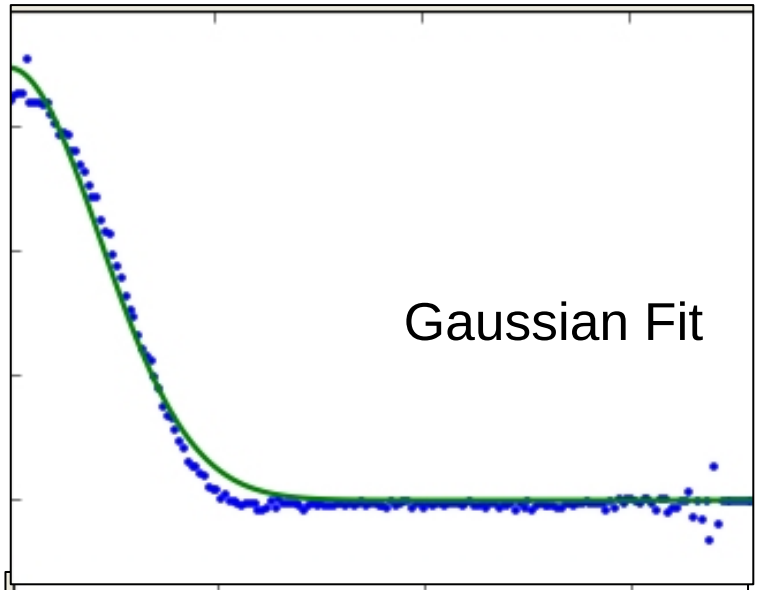
Lithium 6

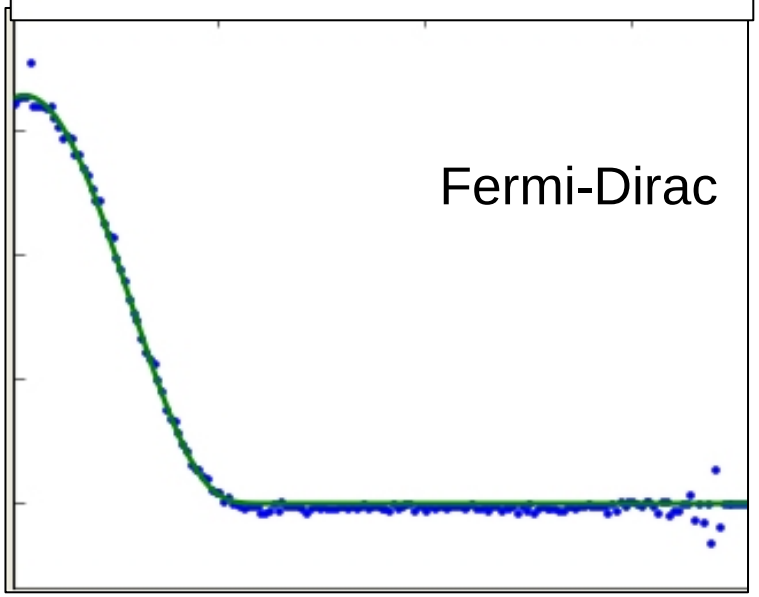
10^4 Li 7 atoms, in thermal equilibrium with
10^4 Li 6 atoms in a Fermi sea.

Quantum degeneracy: $T = 0.28\ \mu K = 0.2(1)\ T_C = 0.2\ T_F$
Now: $T = 0.03\ T_F$

The non-interacting Fermi gas







$T/T_F<0.05$
Atom number~10^5

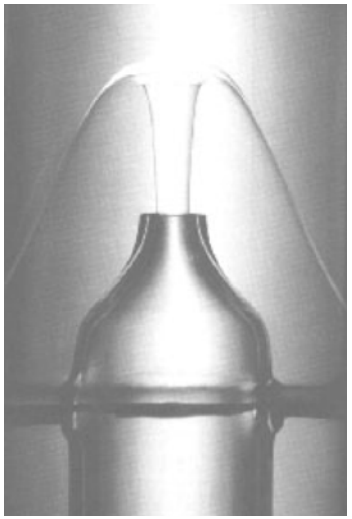
Quantum simulation of strongly interacting Fermions

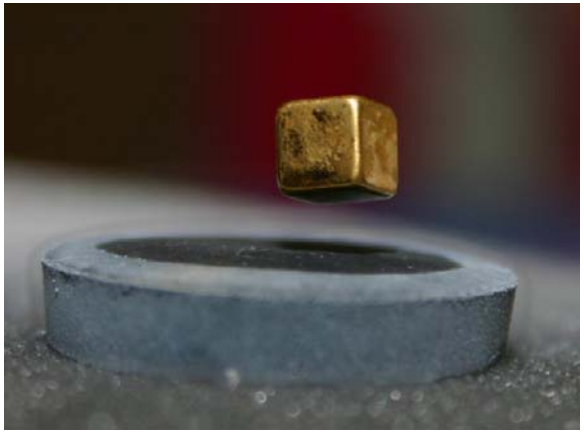
Quantum fluids

Bose Einstein condensates

Superconductivity and helium 3

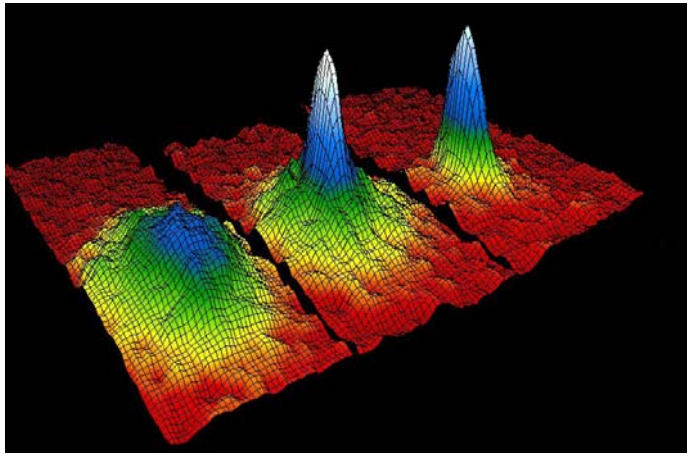
^{4}He



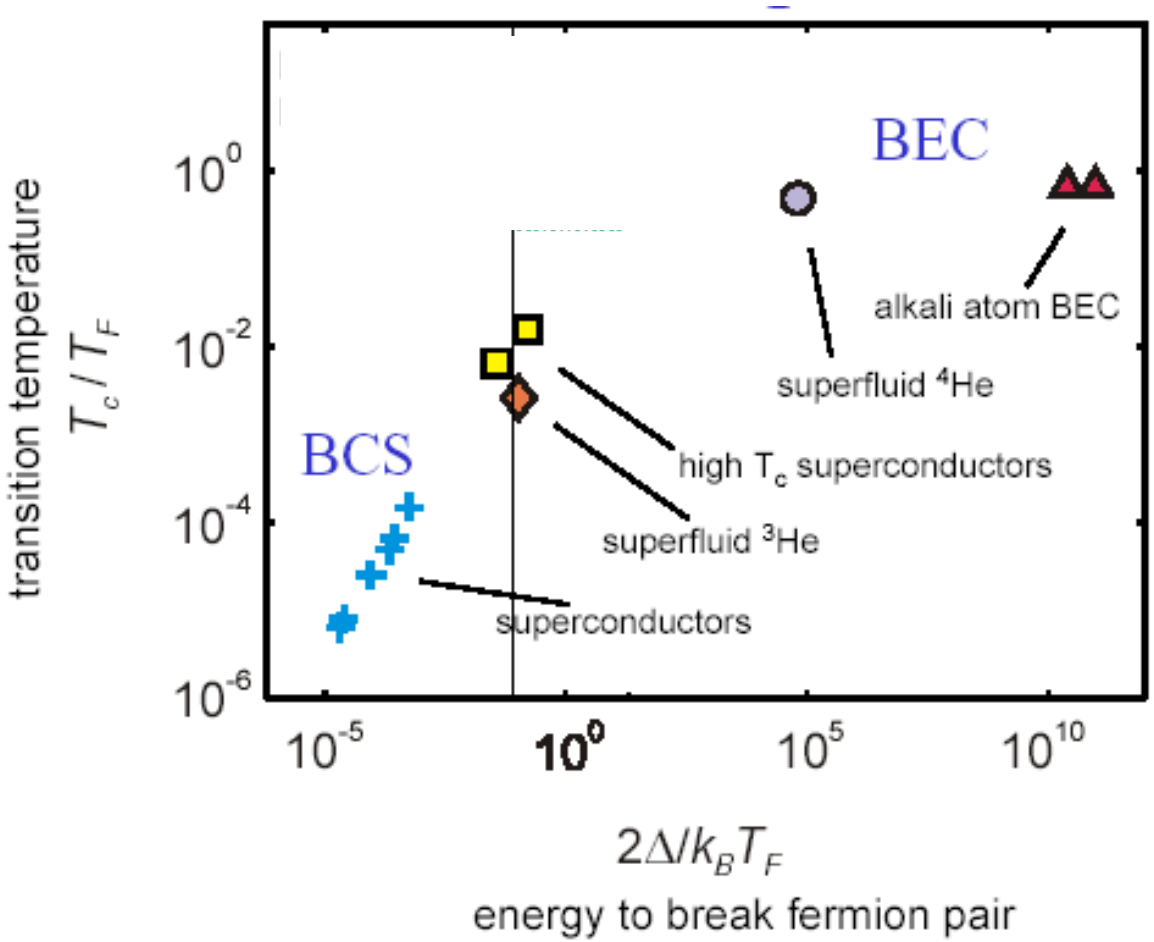


High Tc and 3He

Dilute gas BEC



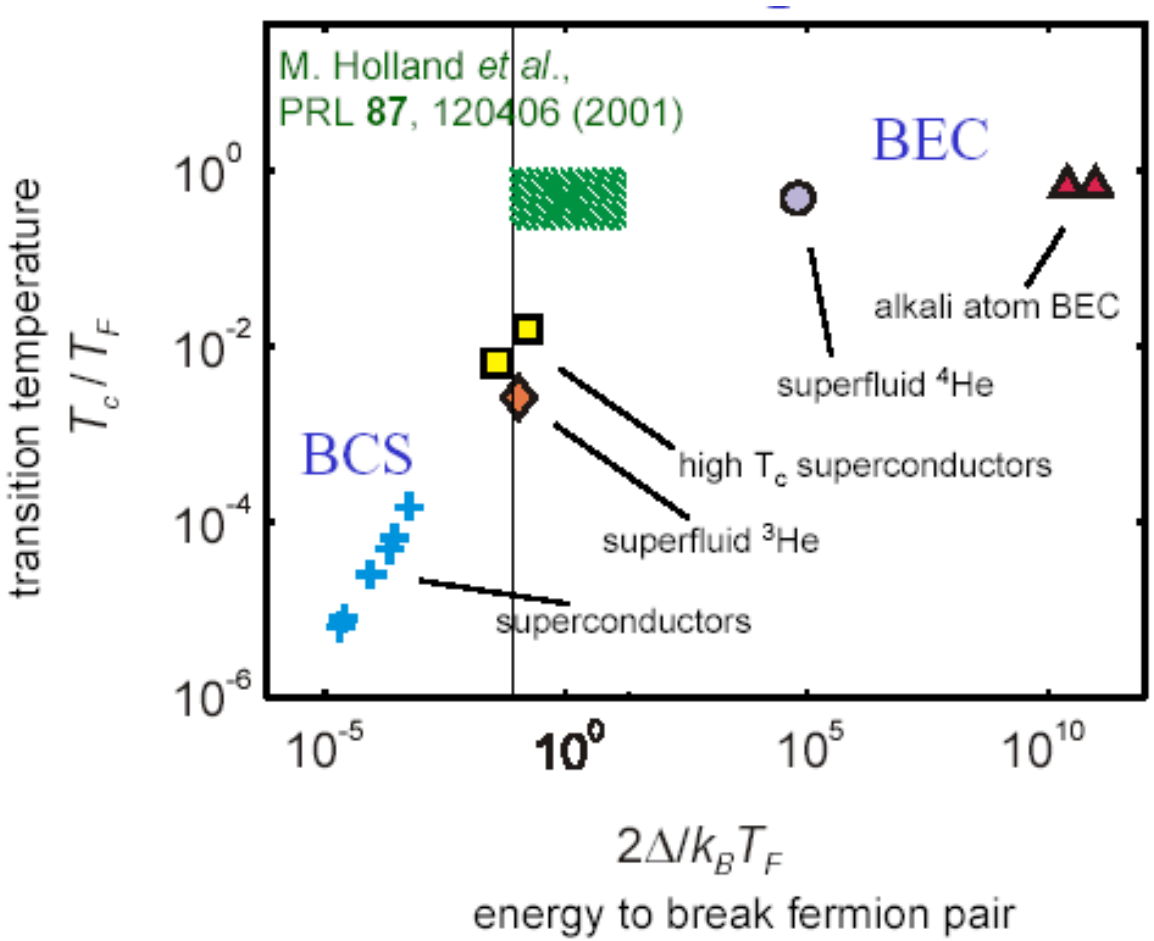
BEC of strongly bound fermions



Connecting the two regimes ?

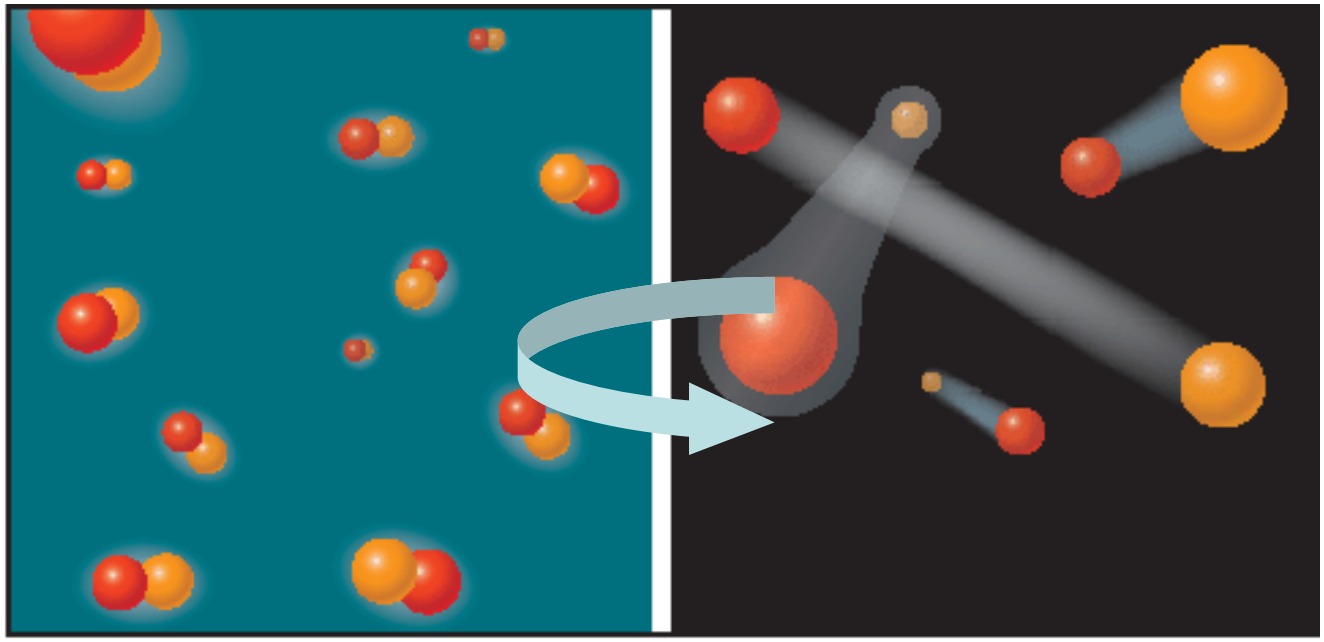
Theory since 80': Leggett, Randéria, Nozières,Schmidt-Rink, Holland, Kokkelmans, Levin, Ohashi, Griffin, Strinati, Falco, Stoof, Bruun, Pethick, Combescot, Giorgini….

intermediate regime



Near Feshbach resonance,
interactions in cold Fermi gas can be enhanced
$T_c \sim T_F/5$

Fermions with two spin states with attractive interaction



$$T_c \approx T_F \; e^{-\pi / 2k_F |a|}$$

BEC of molecules ⇐ BCS fermionic superfluid

Bound state Interaction strength No bound state

Dilute gases: Feshbach resonance

Cooper Pairing

Bardeen, Cooper, Schrieffer, 1957

100 years of supraconductivity.

Naive interpretation: Take a homogeneous T=0 Fermi gas, k_F, E_F

Add two fermions, 1 et 2, which display attractive interaction $a_{\uparrow\downarrow} < 0$

$$V(\vec{r}_1 - \vec{r}_2) = V\delta(\vec{r}_1 - \vec{r}_2)$$ avec $V < 0$

Then these particles will always form at state with an energy lower than E_F , a bound state.

Pairs $\vec{k}, -\vec{k}$ at Fermi surface: $|k| \geq k_F$

These pairs form a superfluid phase

$$T_{BCS} \sim 0.3\, T_F\, e^{-\frac{\pi}{2\, k_F\, |a|_{\uparrow\downarrow}}} \qquad k_F a << 1$$

Mean-field Theory (BCS) at T=0

Bardeen, Cooper, Schrieffer, 1957

Many-body hamiltonian:

$$\hat{H} = \int d^3\mathbf{r} \sum_\sigma \left(\hat{\Psi}_\sigma^+(\mathbf{r}) h_0 \hat{\Psi}_\sigma(\mathbf{r}) \right) + \int d^3\mathbf{r} d^3\mathbf{r}' \hat{\Psi}_\uparrow^+(\mathbf{r}) \hat{\Psi}_\downarrow^+(\mathbf{r}') V(\mathbf{r}-\mathbf{r}') \hat{\Psi}_\downarrow(\mathbf{r}') \hat{\Psi}_\uparrow(\mathbf{r})$$

$$= \int d^3\mathbf{r} \sum_\sigma \left(\hat{\Psi}_\sigma^+(\mathbf{r}) h_0 \hat{\Psi}_\sigma(\mathbf{r}) \right) + g_b \int d^3\mathbf{r} \hat{\Psi}_\uparrow^+(\mathbf{r}) \hat{\Psi}_\downarrow^+(\mathbf{r}) \hat{\Psi}_\downarrow(\mathbf{r}) \hat{\Psi}_\uparrow(\mathbf{r})$$

$$h_0 = -\frac{\hbar^2}{2m}\Delta - \mu$$

In momentum space: $\hat{\Psi}_\sigma(r) = \sum_{k\sigma} \frac{e^{i\mathbf{k}.\mathbf{r}}}{\sqrt{L^3}} \hat{a}_{\mathbf{k},\sigma}$

$$\hat{H} = \sum_{k\sigma} \xi_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma}^+ \hat{a}_{\mathbf{k}\sigma} + \frac{g_b}{L^3} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} \hat{a}_{\mathbf{k}+\mathbf{q}\uparrow}^+ \hat{a}_{\mathbf{k}'-\mathbf{q}\downarrow}^+ \hat{a}_{\mathbf{k}'\downarrow} \hat{a}_{\mathbf{k}\uparrow}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$$

Mean-field Theory (2)

Search for a solution in the form: $\psi_N = N'\left(\sum_k c_k a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow}\right)^{N/2} |vac>$

$c_k = c_{-k}$ Fourier components of spatial function $\varphi(r_i - r_j)$

Assumption: zero momentum for BCS pairs:

$\chi_{\mathbf{k}} = \langle \hat{a}_{-\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\uparrow} \rangle$ ~ order parameter for a gas of bosonic molecules with zero center of mass momentum.

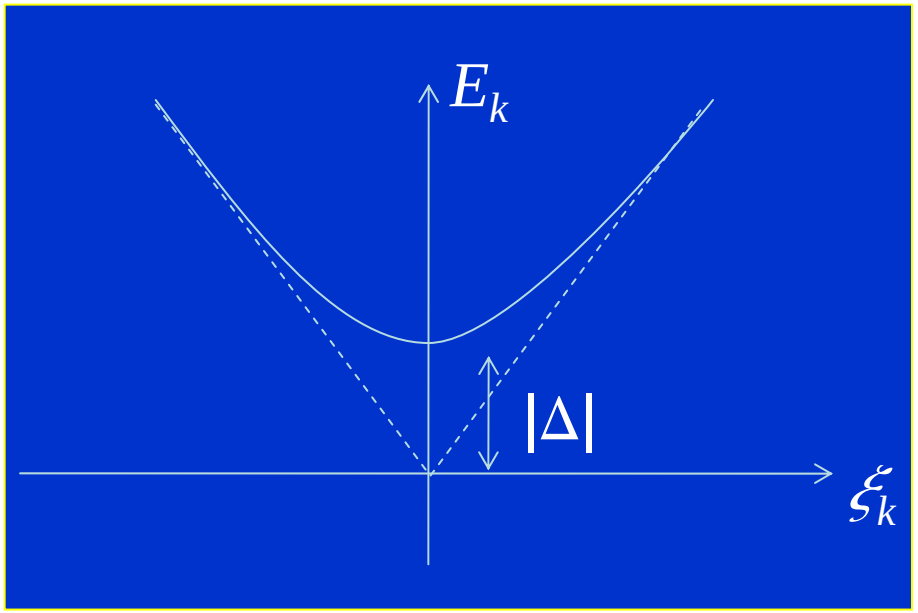
BCS order parameter

$$\hat{H} = \text{cte} + \sum_{k\sigma} \xi_{\mathbf{k}\sigma} \hat{a}^+_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma} + \Delta^* \sum_{\mathbf{k}} \hat{a}_{-\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\uparrow} + \Delta \sum_{\mathbf{k}} \hat{a}^+_{\mathbf{k}\uparrow} \hat{a}^+_{-\mathbf{k}\downarrow} + \ldots$$

$$\Delta = \frac{g_b}{L^3} \sum_{\mathbf{k}} \chi_{\mathbf{k}} = \frac{g_b}{L^3} \sum_{\mathbf{k}} \langle \hat{a}_{-\mathbf{k}\downarrow} \hat{a}_{\mathbf{k}\uparrow} \rangle$$

Excitation Gap

In the ground state, no excitation of the Bogoliubov modes



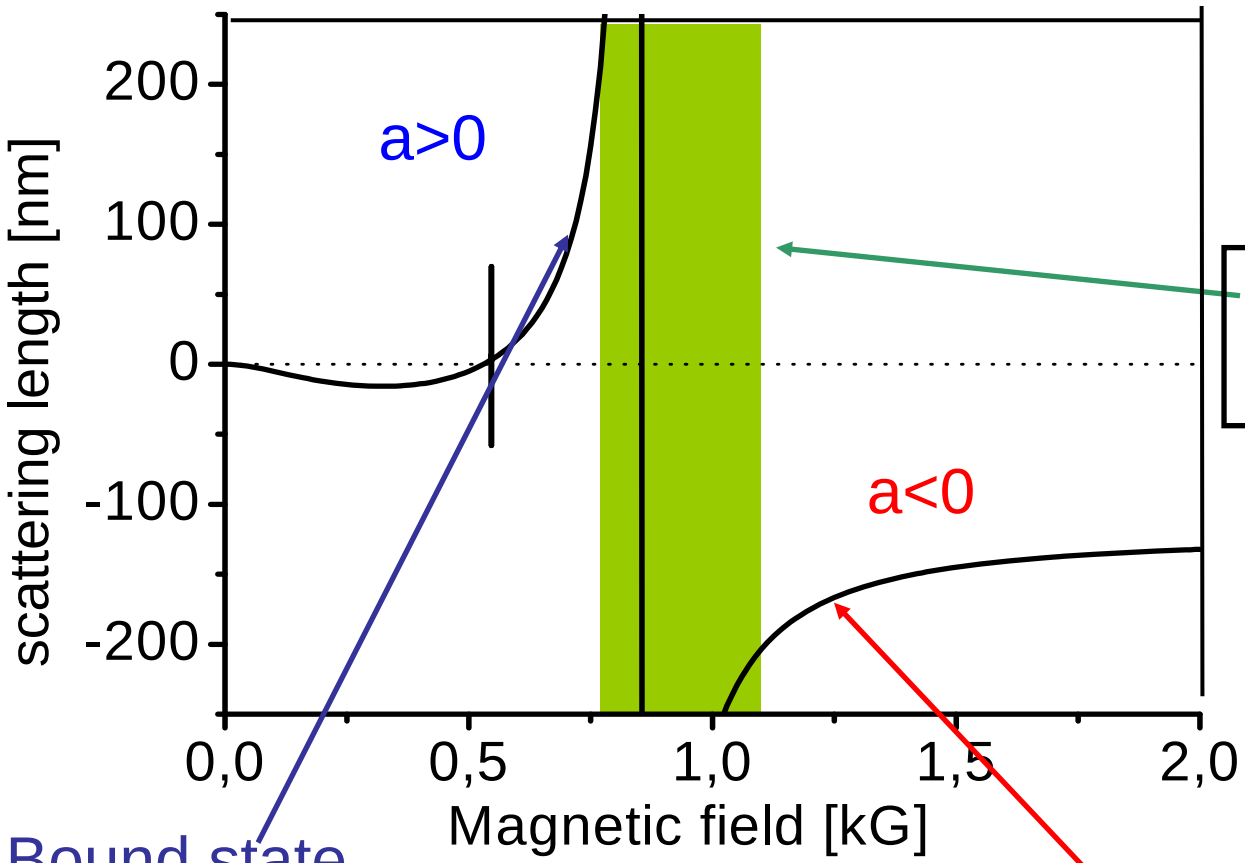
$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta|^2}$$

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu$$

BCS wavefunction: $|\psi_{BCS}> = \prod_k (|u_k| + |v_k| e^{i\varphi} a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow}) |vac>$

$$|u_k|^2 + |v_k|^2 = 1$$

Tuning interactions in Fermi gases
Lithium 6



BEC-BCS Crossover

Bound state

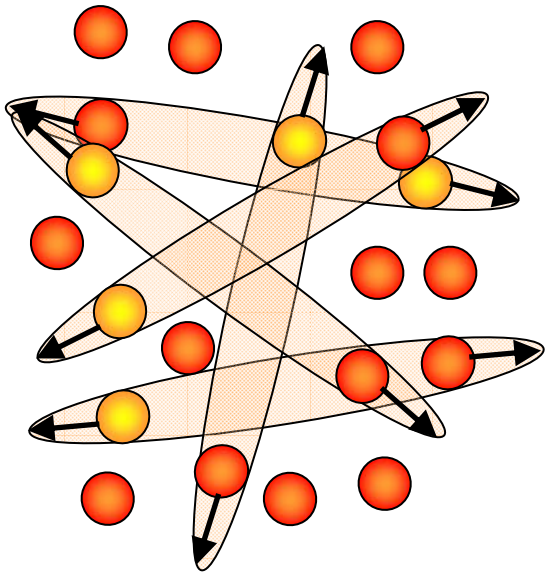
No bound state

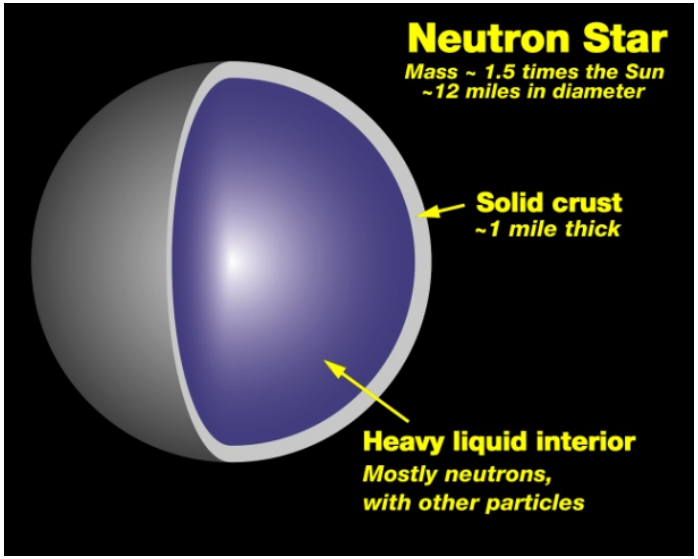
$$E_B = -\frac{\hbar^2}{ma^2}$$

condensate of molecules

BCS phase

The Equation of State of a Fermi Gas with Tunable Interactions





Cold atoms, Spin ½

Dilute gas : 10^{14} at/cm^{3}, T=100nK

BEC-BCS crossover

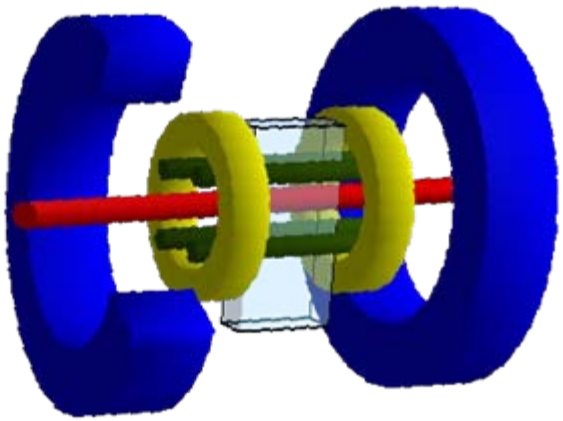
Spin imbalance, exotic phases

Neutron star, Spin ½

a = -18.6 fm, n~2 10^{36}cm^{-3}

- $T_c=10^{10}$ K , $T=T_F/100$
- $k_F a$ ~ -4,-10,…
- $k_F r_e \ll 1$

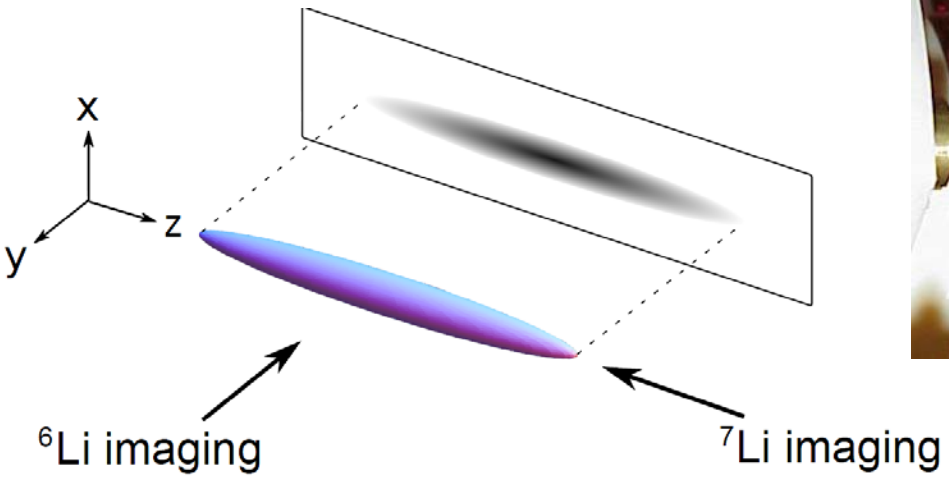
Experimental sequence



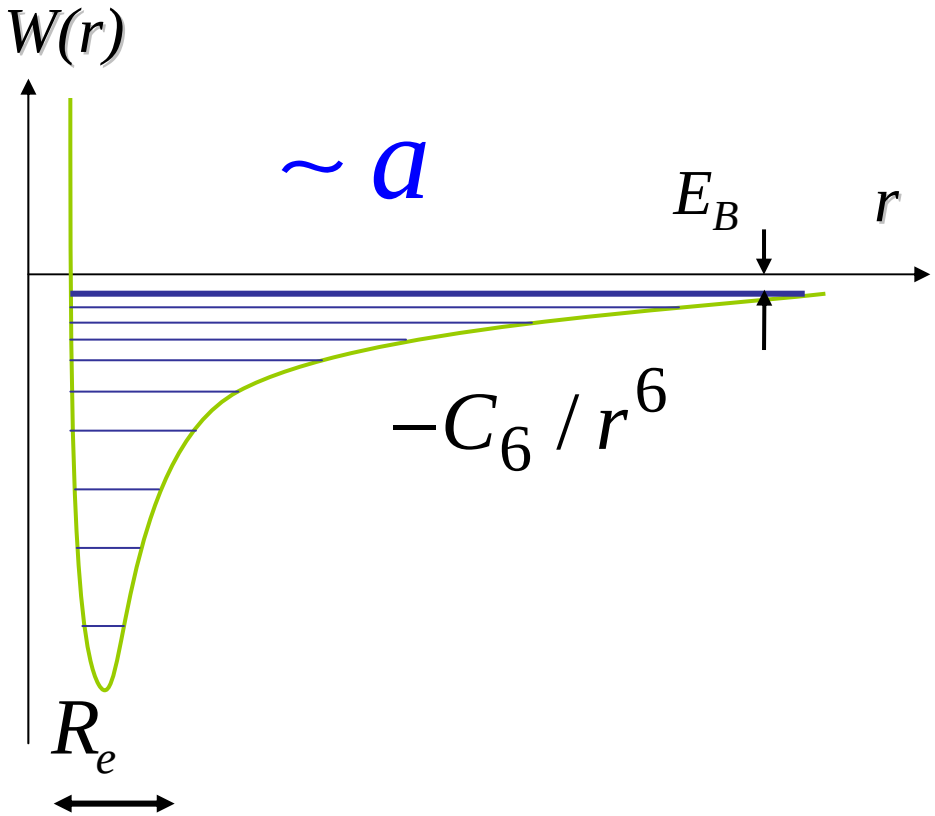
- Loading of ^{6}Li in the optical trap

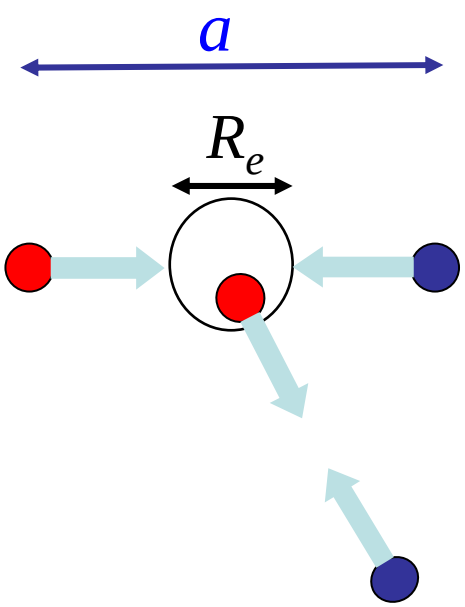
- Tune magnetic field to Feshbach resonance
- Evaporation of ^{6}Li

- Image of ^{6}Li *in-situ*



Suppression of vibrational relaxation for fermions





Pauli exclusion principle

Inhibition by factor $(a/R_e)^2$ >>1

Binding energy: $E_B = h^2/ma^2$

Momentum of each atom: $\hbar/a$

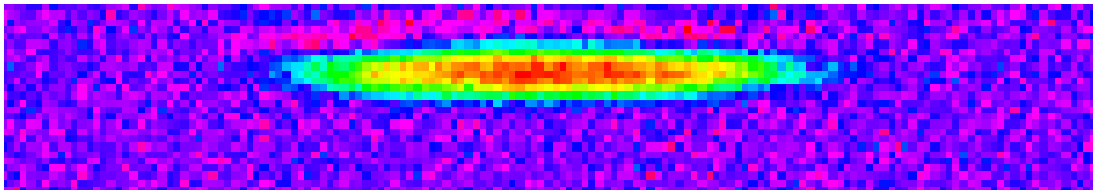
G~ $1/a^s$

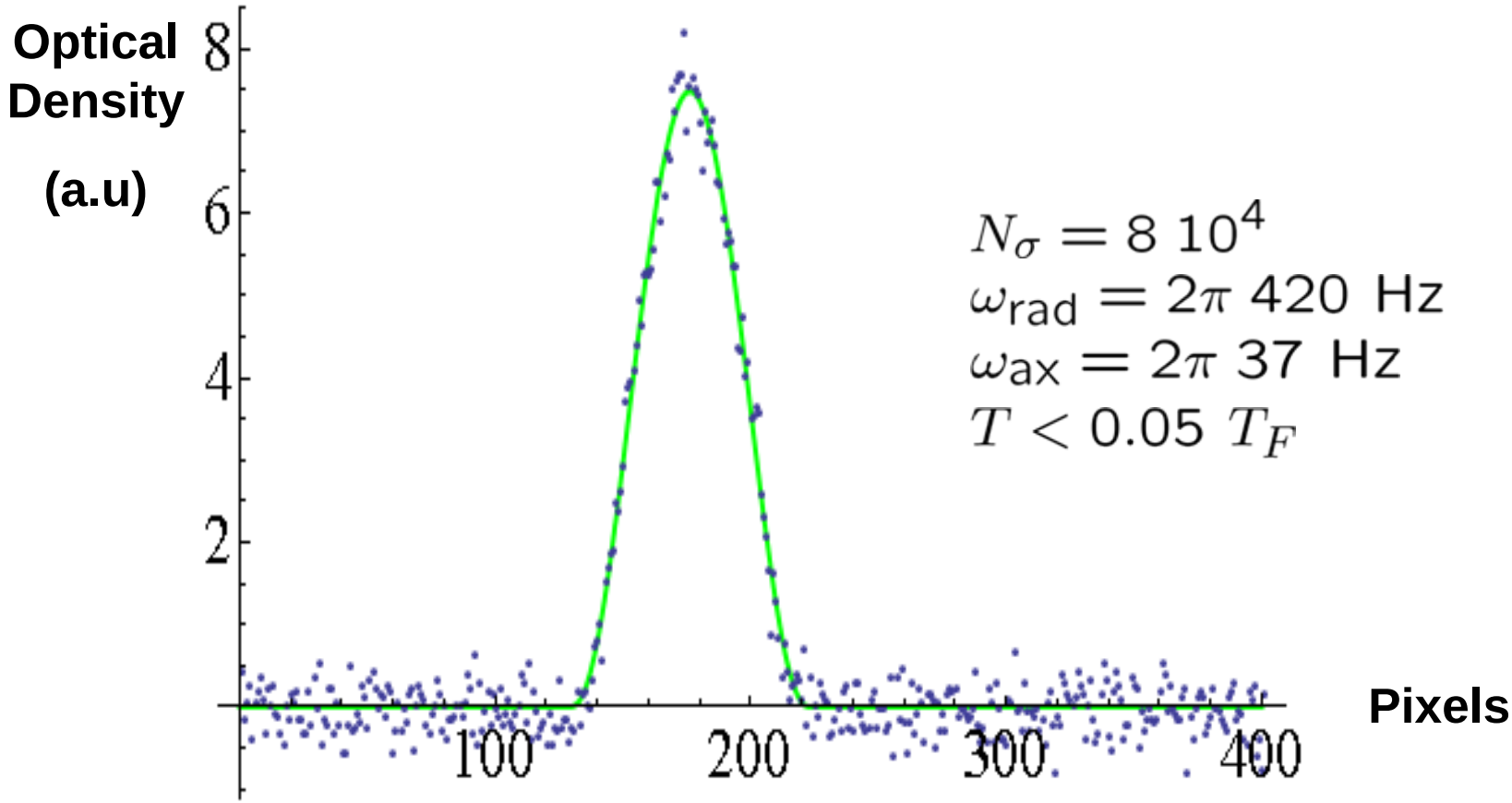
with s = 2.55 for dimer-dimer coll.

3.33 for dimer-atom coll.

D. Petrov, C. Salomon, G. Shlyapnikov, '04

Unitary Fermi Gas : $a = \infty$





Direct proof of superfluidity: classical vs. quantum rotation

Rotating classical gas

velocity field of a rigid body $\vec{v} = \vec{\Omega} \times \vec{r}$ → $\vec{\nabla} \times \vec{v} = 2\vec{\Omega}$

Rotating a quantum macroscopic object

macroscopic wave function: $\psi(\vec{r}) = \sqrt{\rho(\vec{r})}\, e^{i\phi(\vec{r})}$

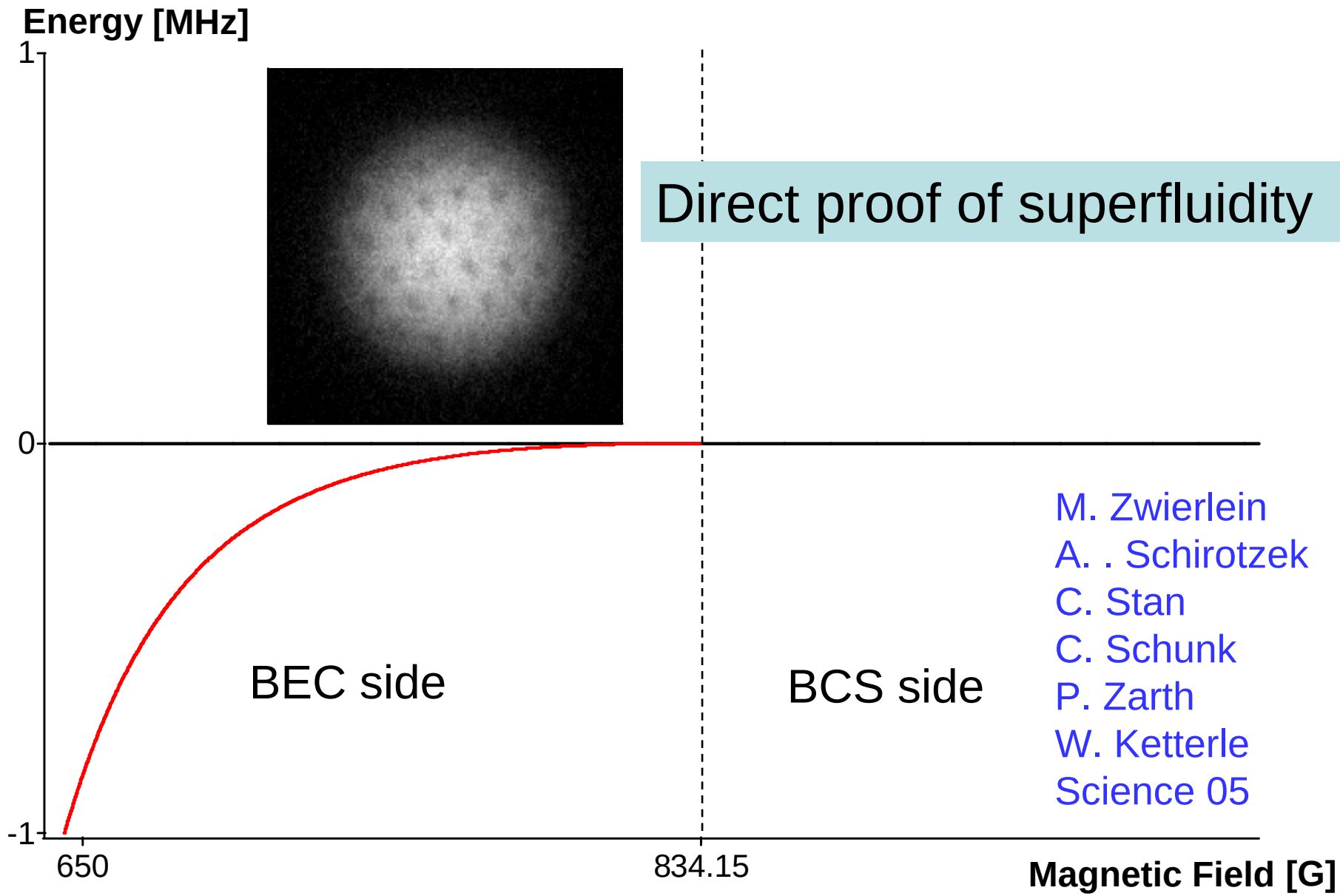
In a place where $\rho(\vec{r}) \neq 0$, irrotational velocity field: $\vec{v} = \frac{\hbar}{m}\vec{\nabla}\phi$

The only possibility to generate a non-trivial rotating motion is to nucleate quantized vortices (points in 2D or lines in 3D) with quantized circulation around vortex core. Feynman, Onsager

$\oint \vec{v}(\vec{r}) \cdot d\vec{r} = n\frac{h}{m}$ to keep $\psi(\vec{r}) = \sqrt{\rho(\vec{r})}\, e^{i\phi(\vec{r})}$ single-valued

Vortices now all have the same sign, imposed by the external rotation

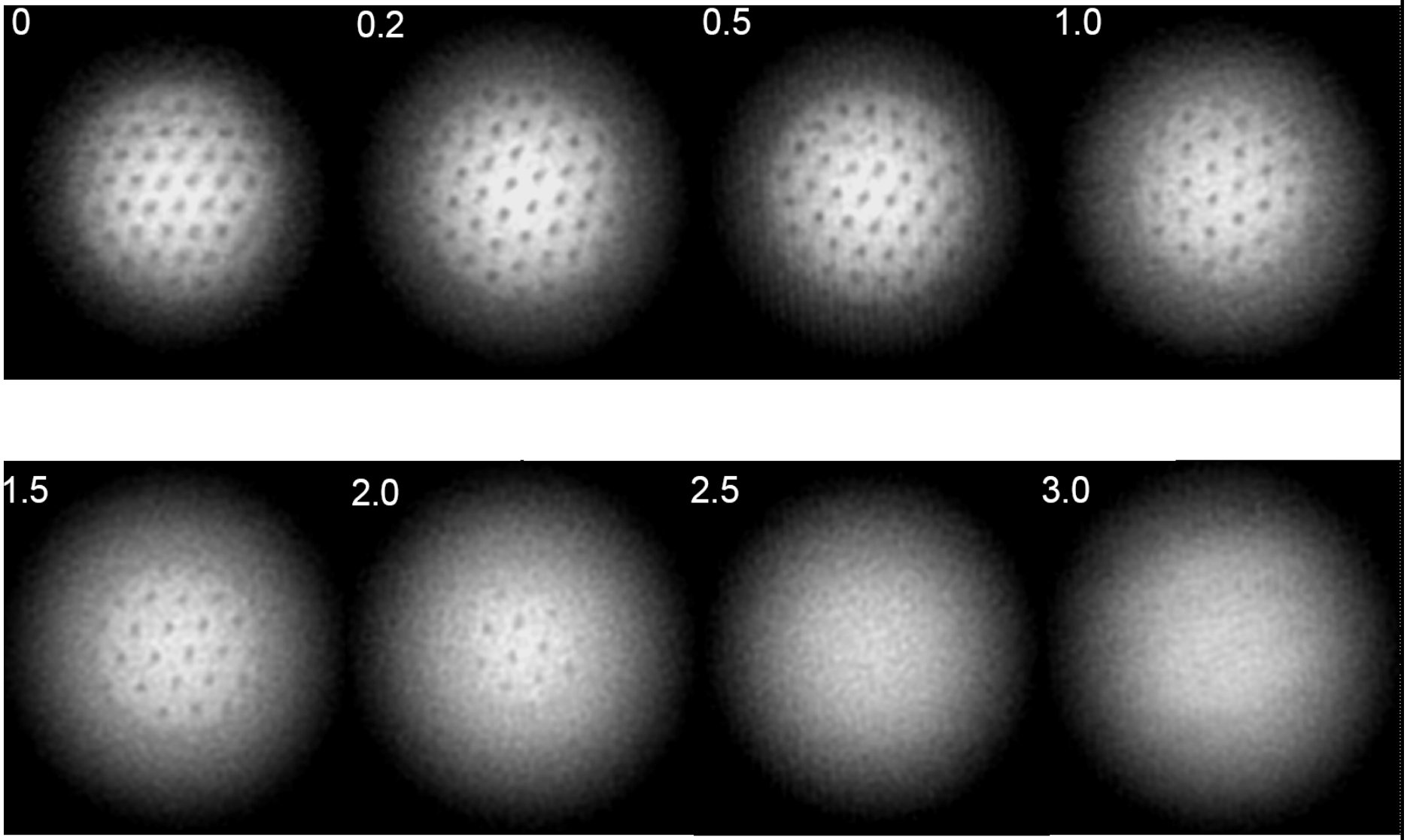
MIT 2005: Vortex lattices in the BEC-BCS Crossover



Direct proof of superfluidity

M. Zwierlein
A. . Schirotzek
C. Stan
C. Schunk
P. Zarth
W. Ketterle
Science 05

Direct proof of superfluidity

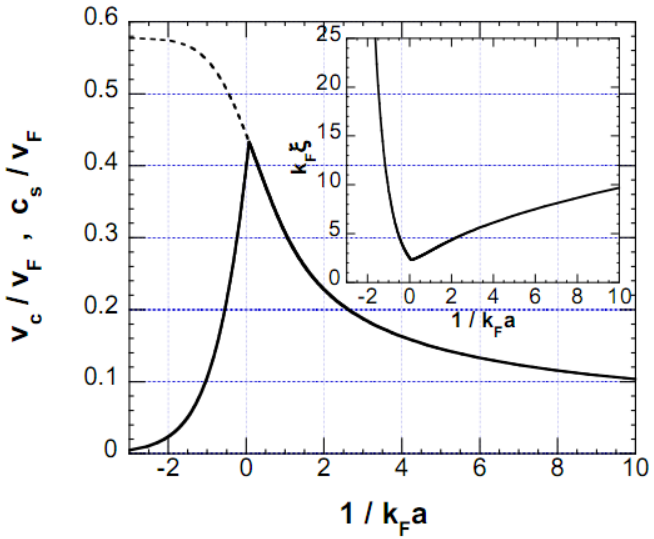


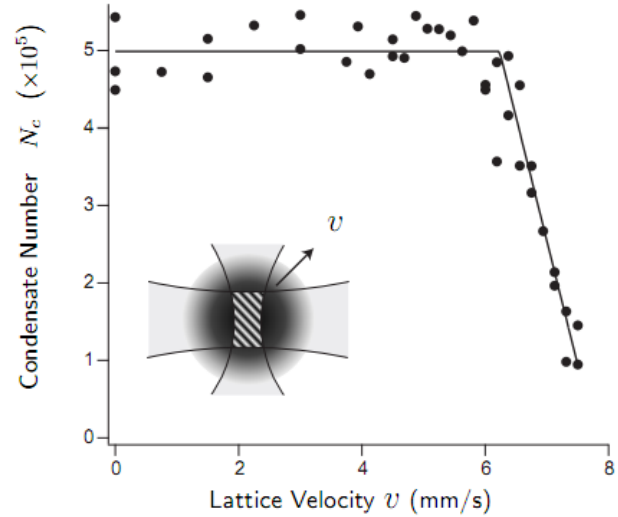
Critical SF temperature = 0.19 T_F

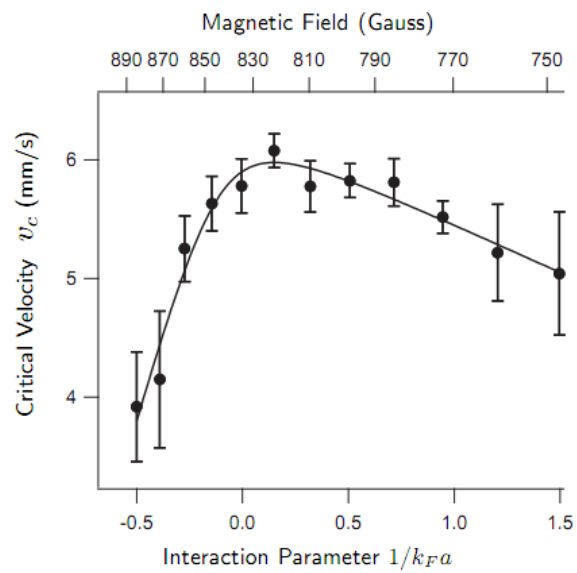
MIT 2006

Superflow in fermionic superfluids

Landau criterion: dissipation for $V>\omega_k/k$







Theory: Combescot, Kagan and Stringari.
Experiment: MIT, 2008.

strongly interacting Fermions
thermodynamic properties

Thermodynamics

$$PV = Nk_B T$$

Is a useful but incomplete equation of state !

Complete information is given by thermodynamic potentials:

Grand potential $$\Omega = -PV = E - TS - \mu N$$

Pressure
Volume

Temperature
Entropy

Chemical potential
Atom number

Internal energy

We have measured the grand potential of a tunable Fermi gas

S. Nascimbène et al., Nature, **463**, 1057, (2010), arxiv 0911.0747
N. Navon et al., Science **328**, 729 (2010)
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S. Nascimbène et al., arXiv 1012.4664: normal phase, to appear PRL 2011
M. Horikoshi et al., Science, 327, 442 (2010)

The ENS Fermi Gas Team

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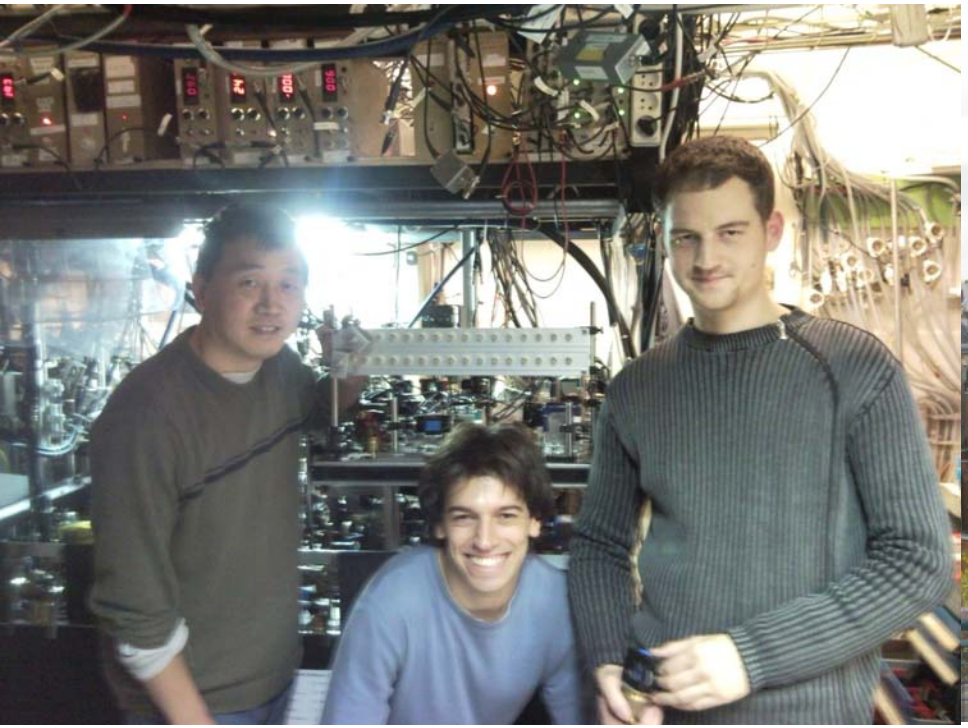

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Useful thermodynamic quantities

- Gibbs –Duhem relation: $VdP = SdT + Nd\mu$
- Density: $n = \dfrac{\partial P}{\partial \mu}$
- Entropy density: $S/V = \dfrac{\partial P}{\partial T}$
- Free energy in canonical ensemble: $P = -\dfrac{\partial F(T,\mu,V)}{\partial V}$
- Isothermal Compressibility: $1/\kappa_T = V\left(\dfrac{\partial^2 F}{\partial V^2}\right)_{T,N}$
- Specific heat: $C_V = -T\left(\dfrac{\partial^2 F}{\partial T^2}\right)_{N,V}$
- Legendre transform between grand canonical and canonical variables

Thermodynamics of a Fermi gas

The Equation of State of a uniform Fermi gas can be written as:

$$\Omega(\mu, T; a) = E - TS - \mu N = -P(\mu, T; a)\, V$$

Pressure contains all the thermodynamic information

Variables :

scattering length	$\boldsymbol{a}$
temperature	T
chemical potential	μ

We build the dimensionless parameters :

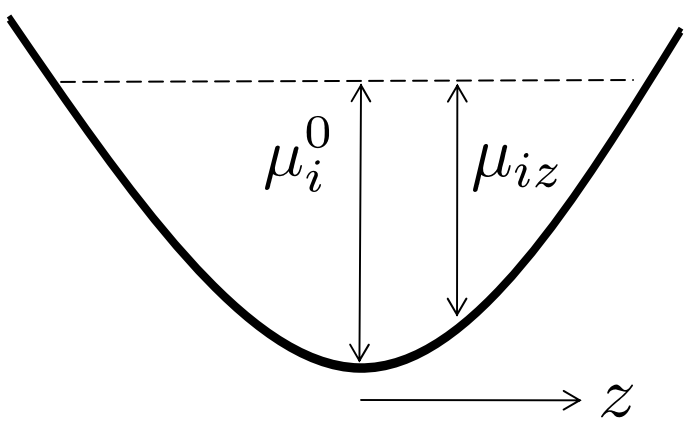
		Canonical analogs
Interaction parameter	$\delta = \frac{\hbar}{\sqrt{2m\mu}a}$	$(k_F a)^{-1}$
Fugacity (inverse)	$\zeta = \exp\left(-\frac{\mu}{k_B T}\right)$	T/T_F

Measuring the EoS of the Homogeneous Gas

Local density approximation:

gas locally homogeneous at

$$\mu_{iz} = \mu_i^0 - \frac{1}{2} m \omega_z^2 z^2$$



$$V(z) = \frac{1}{2} m \omega_z^2 z^2$$

Measuring the EoS of the Homogeneous Gas

Extraction of the pressure from *in situ* images

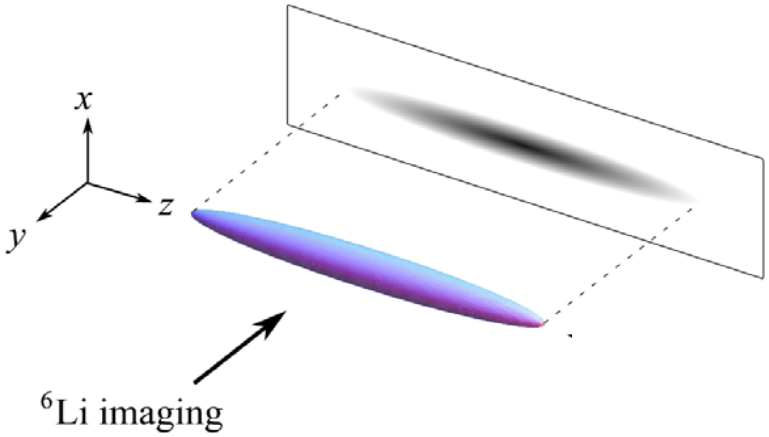
$$P(\mu_{1z}, \mu_{2z}, T) = \frac{m\omega_r^2}{2\pi}(\overline{n}_1(z) + \overline{n}_2(z))$$

Ho, T.L. & Zhou, Q., Nature Physics, 09
S. K Yip, 07

- $\overline{n}_i(z) = \int \mathrm{d}x\mathrm{d}y\, n_i(x, y, z)$

doubly-integrated density profiles

equation of state measured for all values of (μ_{1z}, μ_{2z}, T)



Derivation

LDA: $\mu(x,y,z) = \mu^0 - \frac{1}{2} m\omega_r{}^2 (x^2 + y^2) - \frac{1}{2} m\omega_z{}^2 z^2$

Cylindrical symmetry

Gibbs-Duhem: $dP = -SdT + nd\mu = nd\mu$ at constant temperature

Integrating over x and y between 0 and + ∞

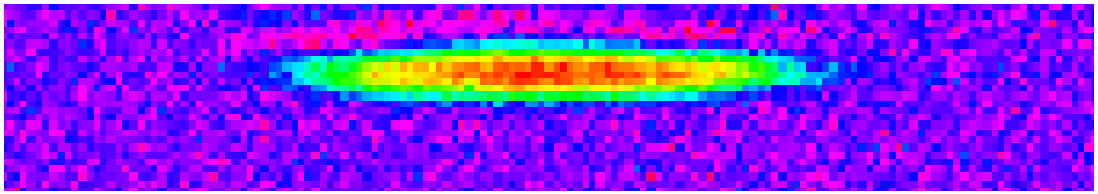
$$P(\mu_z) = \frac{1}{2\pi} m\omega_r^2 \int_0^\infty n(x,y,z) 2\pi r dr$$

$$P(\mu_z, T) = \frac{m\omega_r^2}{2\pi} \int_0^\infty n(x,y,z) dx dy = \frac{m\omega_r^2}{2\pi} \bar{n}(z)$$

Directly relates the pressure at abscissa z
to the doubly integrated absorption signal at z

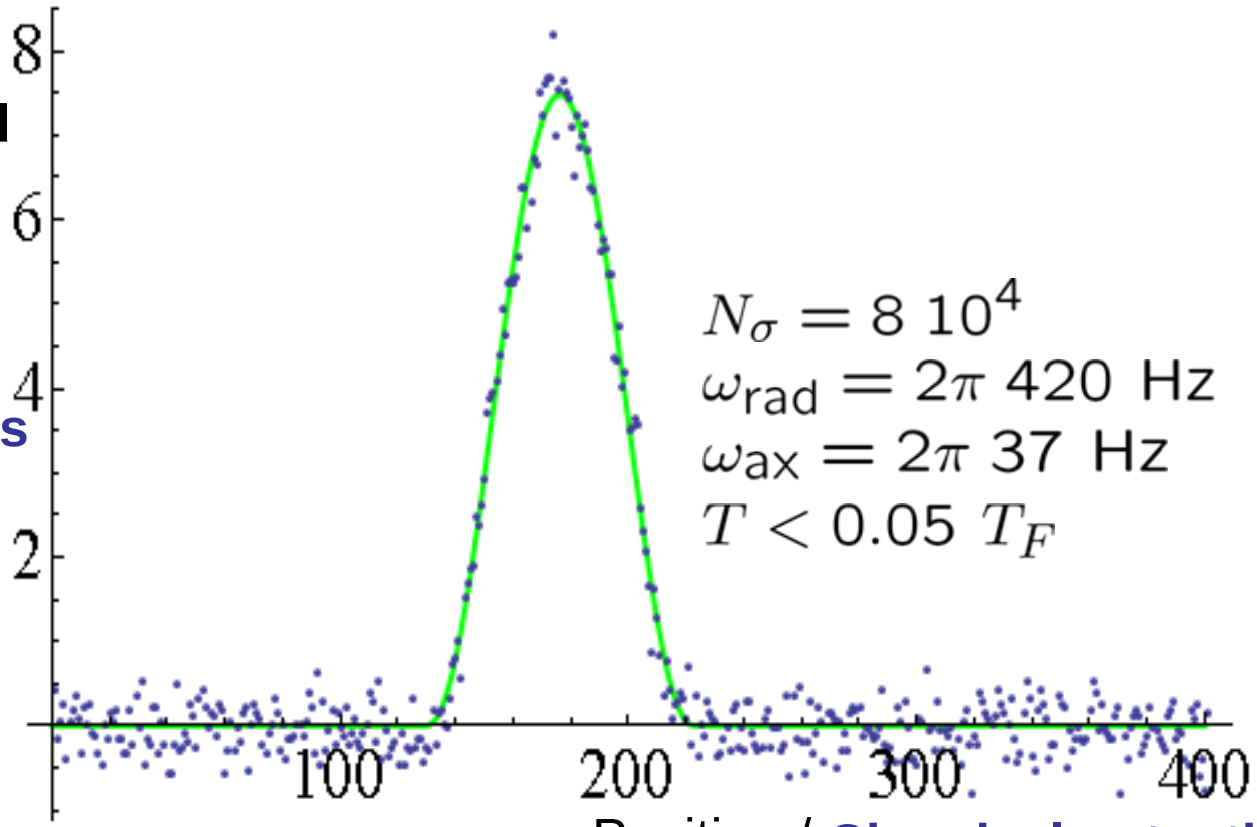
Unitary Fermi Gas

$a = \infty$



Doubly integrated Density

Pressure of the locally homogeneous gas



Position / **Chemical potential** μ_z

S. Nascimbène et al., Nature, **463**, 1057, (2010)

Next lecture:
measurement of the
Equation of State $P(\mu, T)$