



**The Abdus Salam  
International Centre for Theoretical Physics**



**2354-10**

## **Summer School on Cosmology**

*16 - 27 July 2012*

### **Large-scale Structure - Lecture Note 6**

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## Non-linear evolution of the power spectrum

### One-loop PT

First, let's consider the one-loop diagrams for the power spectrum, in terms of PT solutions the non-linear corrections are given by

$$\langle \delta^2 \rangle = \underbrace{\langle \delta_1^2 \rangle}_{\text{Linear}} + \underbrace{\langle \delta_2^2 + 2\delta_1\delta_3 \rangle}_{\text{Non-linear}} + \dots$$

In terms of Fourier components

$$\langle \delta_2(\vec{k}) \delta_2(\vec{k}') \rangle = \int F_2(k_1, k_2) \delta_D(\vec{k} - \vec{k}'_1) F_2(k_3, k_4) \delta_D(\vec{k}' - \vec{k}'_3) \langle \delta_1(k_1) \dots \delta_1(k_4) \rangle d^3k_1 \dots d^3k_4$$

Again, contracting  $\vec{k}_1$  with  $\vec{k}'_1$ , leads to no contributions, so there is only 2 terms that are equal by symmetry, then:

$$\langle \delta_2(\vec{k}) \delta_2(\vec{k}') \rangle = \delta_D(\vec{k} + \vec{k}') \underbrace{\int [F_2(\vec{k} - \vec{q}, \vec{q})]^2 P(\vec{k} - \vec{q}) P(\vec{q}) d^3q}_{= P_{22}(\vec{k})}$$

The other term is

$$\langle \delta_1(\vec{k}) \delta_3(\vec{k}') + \delta_1(\vec{k}') \delta_3(\vec{k}) \rangle = \delta_D(\vec{k} + \vec{k}') P_{13}(\vec{k})$$

Calculate one and symmetrize (factor of 2):

$$\begin{aligned} \langle \delta_1(\vec{k}) \delta_3(\vec{k}') \rangle &= \int F_3(k_1, k_2, k_3) \delta_D(\vec{k}' - \vec{k}'_3) \langle \delta_1(\vec{k}) \delta_1(\vec{k}_1) \delta_1(\vec{k}_2) \delta_1(\vec{k}_3) \rangle \\ &\stackrel{3 \text{ possible}}{=} \delta_D(\vec{k} + \vec{k}') 3P_{13} \int F_3(\vec{k}, \vec{q}, -\vec{q}) P(\vec{q}) d^3q \end{aligned}$$

Then, we have:

$$P(h) = P_{1m}(h) + [P_{22}(h) + P_{13}(h)]$$

with

$$\left\{ \begin{array}{l} P_{22}(h) = 2 \int [F_2(\vec{k} - \vec{q}, \vec{q})]^2 P(\vec{k} - \vec{q}) P(\vec{q}) d^3q \\ P_{13}(h) = P_{1m}(h) G \int F_3(\vec{k}, \vec{q}, -\vec{q}) P(\vec{q}) d^3q \end{array} \right.$$

What is the physical interpretation of these two terms? Notice their behavior is very different. For example, if  $P_{1m}(k) = 0$  for  $k > k_c$ , then  $P_{13} = 0$  for  $k > k_c$ , but  $P_{22}(k) = 0$  only for  $k > 2k_c$ . In other words,  $P_{22}$  reflects the power generated by mode-mode coupling, in particular it couples modes  $\vec{k}-\vec{q}$  and  $\vec{q}$  - ~~electrons~~. Note also that  $P_{22}$  is always positive. On the other hand,  $P_{13}$  can be negative and it does not describe mode-mode coupling, but rather the non-linear corrections to the growth factor, notice that it multiplies the linear power.

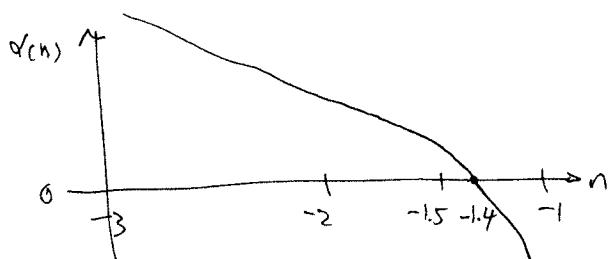
Although for CDM spectra the calculations have to be done numerically, in the case of power-law spectra it can be done analytically, and it is useful to illustrate the more general complicated CDM case. If  $P(k) \propto k^n$  then:

$$P(k) = P_{1m}(k) \left[ 1 + \alpha(n) \left( \frac{k}{k_{1e}} \right)^{n+3} \right]$$

where  $k_{1e}$  is defined from the linear power spectrum as

$$4\pi k_{1e}^3 P_{1m}(k_{1e}) = 1$$

And  $\alpha(n)$  is a coefficient that depends only on spectral index, it has the form



Notice that for  $n < -1.4$ ,  $\alpha(n) > 0$ , thus  $P_{22}$  "wins over"  $P_{13}$  and the non-linear power spectrum becomes larger than the linear one; however for  $n > -1.4$   $\alpha < 0$  and the correction is negative, the non-linear power spectrum is suppressed compared to the linear one. For CDM where  $n$  is changing one sees both behaviors, close to the non-linear scale  $n > -1.4$  and there is a slight suppression but at higher

$k^3 \sim -1.4$  and the correction quickly becomes positive. (3)

### Stable clustering and small scale behavior

Stable clustering says that at small scales, high-density regions decouple from the Hubble flow, in other words, the relative motion between gravitationally bound particles exactly cancels the Hubble flow. Let's explore what this means.

Start from the continuity equation

$$\frac{\partial \delta}{\partial \tau} + \bar{V} [(1+\delta) \vec{v}] = 0$$

From here one can find an evolution equation for the 2pt function:

$$\begin{aligned} \frac{\partial \xi(r)}{\partial \tau} &= \frac{\partial}{\partial \tau} \langle (1+\delta_1)(1+\delta_2^*) \rangle = - \langle (1+\delta_1) \bar{\nabla}_2 [ (1+\delta_2) \vec{v}_2 ] \rangle \\ &\quad + (1+\delta_2) \bar{\nabla}_1 [ (1+\delta_1) \vec{v}_1 ] \rangle \\ &= - \bar{\nabla}_r \langle (1+\delta_1)(1+\delta_2) \vec{v}_1 \rangle + \bar{\nabla}_r^* \langle (1+\delta_1)(1+\delta_2) \vec{v}_2 \rangle \\ &\quad \stackrel{\text{statistical homogeneity}}{=} - \bar{\nabla} \underbrace{\langle (1+\delta_1)(1+\delta_2)(\vec{v}_1 - \vec{v}_2) \rangle}_{\cancel{\text{ dabei}}} = - \bar{\nabla} \cdot [\vec{v}_{12} (1+\xi)] \\ &\quad \uparrow \text{pairwise streaming velocity} \\ \Rightarrow \frac{\partial \xi}{\partial \tau} + \bar{\nabla} \cdot [ (1+\xi) \vec{v}_{12} ] &= 0 \end{aligned}$$

This is known as the pair-conservation equation. Remember that  $(1+\xi) \bar{n} dr^3$  tells us how many particles are separated by distance  $r$  from a given particle, that is, the number of conditional pairs.

Now, the stable clustering ansatz is that the pairwise velocity is exactly cancelled by the Hubble flow,

$$\vec{v}_{12}^2 + \nabla H \vec{r} = 0$$

( $\vec{r}$  is comoving here)

$$\Rightarrow \frac{\partial \xi}{\partial z} - H \nabla \cdot [\vec{r}(1+\xi)] = 0$$

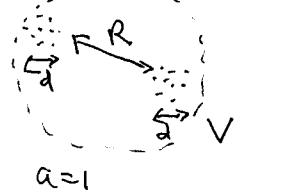
$$\xi = \xi(r)$$

$$\Rightarrow \frac{\partial \xi}{\partial \ln a} = \nabla \cdot [\vec{r}(1+\xi)] = (3 + r \partial_r) (1+\xi)$$

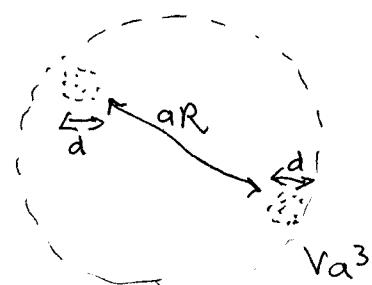
which can be solved by  $\xi(r, z) \underset{\substack{\text{at small} \\ \text{scale } \xi \gg 1}}{\approx} 1 + \xi(r, z) = a^3 f(ar) = a^3 f(r_p)$

$$r_p = ar \text{ is physical separation}$$

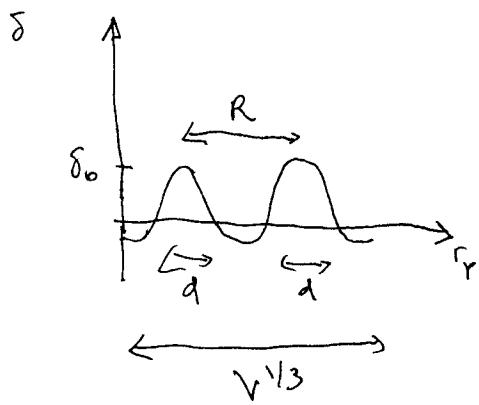
we see that at small scales, therefore, stable clustering predicts that  $\xi \propto a^3 f(r_p)$ , i.e. at fixed physical separation  $\xi$  increases as  $a^3$ . Does this make sense? Our intuition of stable clustering, pictorially is that inside virialized clusters things stay constant @ fixed  $r_p$ , but cluster themselves separate with the hubble flow, i.e. in physical coordinates:



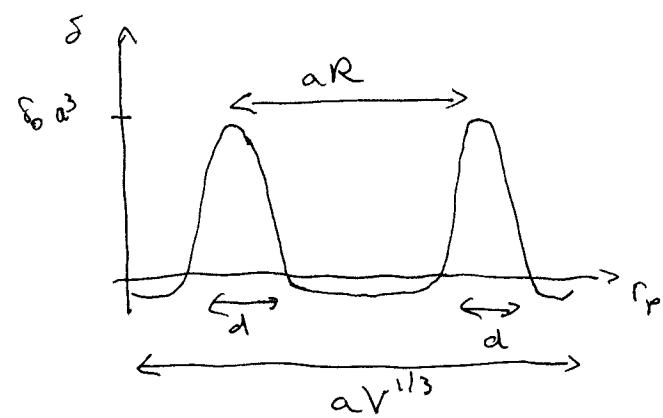
evolve by  $a$



all distances being physical here. In terms of density contrasts this means, doing a 1D cut:



evolve by  $a$



Notice that  $\delta$  in each peak increases by  $a^3$ , because the same mass is enclosed in the same volume and thus same  $\delta$  but  $\bar{\rho}$  has decreased by  $1/a^3$ , so fluctuation is larger by  $a^3$  - What does this imply for the correlation function? Let's compute it at <sup>fixed</sup> physical separation and see that

it gets amplified by  $a^3$ :

this is a "dummy variable"

$$\xi_0(r_p) = \frac{1}{V} \int \delta(x_p) \delta(x_p + \vec{r}_p) d^3x_p$$

at later time, have  $\xi(r_p) = \frac{1}{Va^3} \int \delta(x_p) \delta(x_p + \vec{r}_p) d^3x_p$

Now, at small scales, all that matters is small  $r_p$  enough so that we are inside one of the "halos" where  $\delta(\vec{x}) = a^3 \delta_0(\vec{x})$ , so all contribution is coming from ~~outside~~ both  $\delta$ 's inside each patch and therefore

$$\xi(r_p) \approx \frac{a^6}{Va^3} \int \delta_0(x_p) \delta_0(x_p + \vec{r}_p) d^3x_p = a^3 \xi_0(r_p)$$

which gives the stable clustering result from the continuity equation-

How good is stable clustering in simulations at small scales? In order to discuss this is useful to rewrite the pair conservation using the averaged two-point function (over a ball of radius  $r$ ):

$$\bar{\xi}(r) = \frac{3}{r^3} \int_0^r r^2 dr \xi(r)$$

Using that  $\vec{v}_{12}$  is directed along  $\vec{r}$ , one can rewrite pair conservation as:

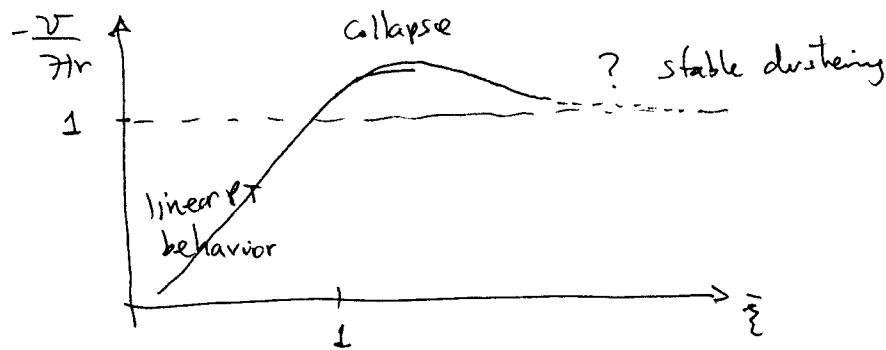
$$-\frac{\partial v_{12}}{\partial r} = \frac{1}{3(1+\xi)} \frac{\partial \bar{\xi}}{\partial \ln a}$$

At large scales, where linear theory can be used,  $\xi \ll 1$  and  $\frac{\partial \bar{\xi}}{\partial \ln a} = 2f \bar{\xi}$

$$\Rightarrow -\frac{\partial v_{12}}{\partial r} = \frac{2f}{3} \bar{\xi} \ll 1$$

recall in one-loop PT this is  $\sim D^4$

As  $\xi \approx 1$  the non-linear correlation grows faster than  $a^3$  and  $\frac{\partial v_{12}}{\partial r}$  becomes larger than 1, "overcompensating" for the Hubble flow, that's how things are able to collapse - Stable clustering that as  $r \rightarrow 0$   $\frac{\partial v_{12}}{\partial r}$  goes to 1 - In pictorial way,



It is difficult to see from simulations whether the stable clustering regime is obeyed or not [slow results], there is an alternative (due to constant merging)

way of looking at this problem through the scale dependence of  $P(k)$  or  $\xi(r)$  for power-law initial conditions using self-similarity, which we now discuss.

### Self-similarity

If the initial perturbations are scale-free  $P(k) \propto k^n$ , and  $\Omega_m = 1$  so there is no characteristic time scale either, since gravity is scale free and collisionless dark matter has no characteristic scale, then it is possible to show that the equations of motion admit self-similar solutions where statistical properties depend only on time and space  $\propto (or k)$  through a self-similarity variable, e.g.

$$\xi(r, t) = \hat{\xi}\left(\frac{r}{t^\alpha}\right)$$

at all scales and times. In fact  $\alpha$  follows just from using the evolution in linear PT (which is self-similar),

$$\xi_{lin} \propto \frac{\alpha^2}{r^{n+3}} \propto \left(\frac{r}{t^\alpha}\right)^{-(n+3)} \propto \left(\frac{\alpha^2}{r}\right)^{n+3}$$

$\alpha < t^{2/3}$

$$\Rightarrow \frac{3\alpha}{2}(n+3) = 2 \Rightarrow \boxed{\alpha = \frac{4}{3(n+3)}}$$

Now, let's see what happens when we combine self-similarity with

stable clustering at small scales - we have

$$\xi(r, t) = \hat{\xi}\left(\frac{r}{t^a}\right) = a^3 f(ar)$$

the only way this can be satisfied is if  $\hat{\xi}$  and  $f$  are power laws with their arguments, to match  $r$ -dependence both must scale as argument then:

$$\left(\frac{r}{a^{3/2}}\right)^{-\gamma} \propto a^3 (ar)^{-\gamma}$$

$$\alpha = \frac{4}{3(n+3)}$$

matching time dependence gives

$$3 - \gamma = \frac{3\alpha}{2} \Rightarrow \boxed{\gamma = \frac{3}{1+3\frac{\alpha}{2}} = \frac{3(n+3)}{n+5}}$$

Therefore, this says that at small scales things evolve like

$$\xi(r, a) \propto \left(\frac{r}{a^{\frac{2}{n+3}}}\right)^{-\frac{3(n+3)}{n+5}}$$

$$\text{or for } 4\pi h^3 P(h) \equiv \Delta(h) \propto (k a^{\frac{2}{n+3}})^{\frac{3(n+3)}{n+5}} \propto (\Delta_{lin}(k))^{\frac{3}{n+5}}$$

[show results from self-similarity]

### Fitting formulae

Self-similarity is useful because it reduces the problem from 2 variables to a single one - Thus if one wants to provide a fit to N-body results for the nonlinear evolution of say,  $\xi(r, a)$ , ~~or~~ or  $\Delta(h)$ , it makes sense (for  $\zeta_{lin}=1$  and scale-free initial conditions) to phrase it in terms of the search for a mapping  $\mathcal{F}$  so that

$$\Delta(k) = \mathcal{F}[\Delta_{lin}(k)]$$

where all time dependencies are inside  $\Delta_{lin}(h) = 4\pi k^3 a^2 P_{lin}(h) \sim a^2 k^{n+3}$  - The function  $\mathcal{F}$  satisfies the following asymptotics,

$$\mathcal{F}(x) \sim \begin{cases} x & x \ll 1 \\ x^{\frac{3}{n+5}} & x \gg 1 \end{cases} \quad \begin{array}{l} \text{(linear theory)} \\ \text{(assuming stable clustering holds)} \end{array}$$

Clearly the map  $F$  will depend on  $n$ .

Another way of doing this map from  $\Delta \ln$  to  $\Delta$  suggested some time ago that may be one could find a way to make this map "Universal", i.e.  $n$ -independent. Although since then deviations from universality have been found, there is interesting physics involved, so let's discuss that.

We'll write the pair conservation equation in the following form:

$$\frac{\partial}{\partial t} (1+\xi) + \frac{3}{r} (1+\xi) v_{12} = 0$$

Note that  $\frac{\partial}{\partial r} [r^3 (1+\xi)] = 3r^2 (1+\xi)$

$$\Rightarrow \frac{\partial}{\partial t} [r^3 (1+\xi)] + v_{12} \frac{\partial}{\partial r} [r^3 (1+\xi)] = 0$$

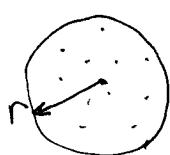
Note that  $r^3 (1+\xi) = r^3 + 3 \int_0^r r^2 dr \xi(r) = 3 \int_0^r r^2 dr (1+\xi)$

which is proportional to the number of neighbors ~~at~~ a particle at  $r=0$  has inside a sphere of radius  $r$ .

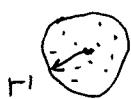
The pair conservation equation above says that if I pick a sphere of radius  $r$  such that

$$r_L^3 = r^3 (1+\xi)$$

is independent of time, this sphere will contain the same number of neighbors ~~at~~ throughout non-linear evolution, pictorially:



$\rightarrow$   
evolve



$a > 1$

$\xi(t) \gg 1 \Rightarrow r$  shrinks as things collapse  
to keep same # of neighbors

At early times, when  $\xi \ll 1$ ,  $r_L \sim r$ , once clustering develops  $r \ll r_L$  as things collapse - This motivates the ansatz that

$\tilde{\xi}(r, a)$  should be a function of  $\tilde{\xi}_{lin}(r_L, a)$ , then the

(9)

mapping should be done as follows

$$\tilde{\xi}(r, a) = \mathcal{F} [\tilde{\xi}_{lin}(r_L, a)] \quad r_L^3 = r^3(1 + \tilde{\xi})$$

or written in Fourier space (using just dimensional like analysis)

$$\Delta(k, a) = \mathcal{F} [\Delta_{lin}(k_L, a)] \quad k^3 = [1 + \Delta(k)] k_L^3$$

Again  $\mathcal{F}(x) \sim x^\gamma$  when  $x \ll 1$ , and  $k \approx k_L$  - however, in the highly non linear regime ( $\Delta \gg 1$ ),  $k_L^{n+3} \sim k^{n+3} \Delta^{-(n+3)/3}$  and say  $\mathcal{F}(x) \sim x^\gamma$  then

$$\Delta \sim [\alpha^2 k_L^{n+3}]^\gamma \sim \underbrace{(\alpha^2 k_L^{n+3})^\gamma}_{\Delta_{lin}^\gamma} \Delta^{-\frac{n+3}{3}\gamma}$$

$$\Rightarrow \Delta^{1 + \frac{\gamma(n+3)}{3}} \sim \Delta_{lin}^\gamma \Rightarrow \frac{\gamma}{1 + \frac{\gamma(n+3)}{3}} = \frac{3}{n+3}$$

stable clustering

$$\Rightarrow \gamma = 3/2$$

Thus, doing the mapping in this way reduces the dependence on  $n$ :

$$\mathcal{F}(x) = \begin{cases} x & x \ll 1 \\ x^{3/2} & x \gg 1 \end{cases}$$

However, it was found that there are indeed  $n$ -dependences at intermediate values of  $x$ . Also, simulations now show that at high  $k$  the slope is less than that predicted by stable clustering, implying  $\frac{\mathcal{F}(x)}{x}$  decreases below unity at small  $r$ .  
 [show results]