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Large-scale Structure - Lecture Note 3

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Linear PT solutions (Newtonian case)

(1)

The equations of motion in Newtonian case are:

$$\left\{ \begin{array}{l} \frac{\partial \delta}{\partial t} + \bar{\nabla}_r \cdot [(1+\delta)\vec{v}] = 0 \\ \frac{\partial \vec{v}}{\partial t} + H\vec{v} + (\vec{v} \cdot \bar{\nabla}_r)\vec{v} = -\bar{\nabla}_r \Phi - \frac{\bar{\nabla}_r \rho}{\bar{\rho}} \\ \nabla_r^2 \Phi = \frac{3}{2} H^2 \Omega_m \delta \end{array} \right. \quad \left\{ \begin{array}{l} \frac{\partial \delta}{\partial \tau} + \bar{\nabla} \cdot [(1+\delta)\vec{v}] = 0 \\ \frac{\partial \vec{v}}{\partial \tau} + H\vec{v} + (\vec{v} \cdot \bar{\nabla})\vec{v} = -\bar{\nabla} \Phi - \frac{\bar{\nabla} \rho}{\bar{\rho}} \\ \nabla^2 \Phi = \frac{3}{2} H^2 \Omega_m \delta \end{array} \right.$$

t : cosmic time

$$\bar{\nabla}_r = \frac{\partial}{\partial \vec{r}} = \frac{1}{a} \frac{\partial}{\partial \vec{x}}$$

r : physical x : comoving

τ : conformal time

$$a d\tau = dt$$

$$H^* = \frac{d \ln a}{d\tau} = aH$$

E.g., in $\Omega_m = 1$ case, $a(t) \sim t^{2/3}$ [Throughout this part we assume matter domination]

$$\Rightarrow d\tau \propto \frac{dt}{t^{1/3}} \Rightarrow \tau \sim t^{2/3} \Rightarrow a(\tau) \sim \tau^2$$

$$H = \frac{2}{3\tau}, \quad \mathcal{H} = \frac{2}{\tau}, \quad \frac{d\mathcal{H}}{d\tau} \equiv \dot{\mathcal{H}} = -\frac{2}{3\tau^2} = -\frac{3}{2}\mathcal{H}^2, \quad \frac{dH}{d\tau} = \mathcal{H}' = -\frac{2}{\tau^2} = -\frac{\mathcal{H}^2}{2}$$

The equations above are fully non-linear - let's linearize (for personal reasons I prefer to work in conformal time),

$$\frac{\partial \delta}{\partial \tau} + \bar{\nabla} \cdot \vec{v} \equiv \boxed{\frac{\partial \delta}{\partial \tau} + \theta = 0}$$

$\theta \equiv \bar{\nabla} \cdot \vec{v}$: velocity divergence

$$\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + \vec{v} \cdot \bar{\nabla} \vec{v} = -\bar{\nabla} \Phi - \frac{\bar{\nabla} \rho}{\bar{\rho}}$$

$$\frac{\bar{\nabla} \rho}{\bar{\rho}} = \frac{\bar{\nabla} \rho}{\bar{\rho}(1+\delta)} \approx \frac{\bar{\nabla} \rho}{\bar{\rho}} (1-\delta) \approx \frac{\bar{\nabla} \rho}{\bar{\rho}}$$

2nd order in PT

taking $\bar{\nabla} \cdot$ of this equation:

$\bar{\nabla} \rho$ is 1st order in PT because homogeneous part does not contribute

$$\boxed{\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta = -\nabla^2 \Phi - \frac{\bar{\nabla} \rho}{\bar{\rho}}}$$

$$\uparrow$$

$$\text{Poisson} \quad -\frac{3}{2} H^2 \Omega_m \delta - \frac{\nabla^2 \rho}{\bar{\rho}}$$

Let's Fourier transform,

$$A(\vec{k}) = \int \frac{d^3x}{(2\pi)^3} A(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} \quad (2)$$

$$\vec{\nabla} \rightarrow i\vec{k}$$

$$A(\vec{x}) = \int d^3k A(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

$$\Rightarrow \nabla^2 \rightarrow -k^2$$

Thus, we have:

$$\frac{\partial \delta_k}{\partial \tau} + \theta_k = 0$$

$$\frac{\partial \theta_k}{\partial \tau} + \mathcal{H} \theta_k = -\frac{3}{2} \Omega_m \mathcal{H}^2 \delta_k + \frac{k^2 p_k}{\bar{f}}$$

From this we find 2nd order differential equation for δ_k :

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \frac{\partial \theta_k}{\partial \tau} = 0 \Rightarrow$$

$$\boxed{\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta_k - \frac{k^2 p_k}{\bar{f}}}$$

friction due to expansion of the universe (suppresses growth of δ)

Growth (enhances growth)

pressure suppresses growth at high k , small scales

~~Notes~~

a) Pressureless Case : $p_k = 0$

This is a very important case, as it describes very well the behavior of CDM (cold dark matter) -

$$\frac{\partial^2 \delta_k}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta_k}{\partial \tau} = \frac{3}{2} \Omega_m \mathcal{H}^2 \delta_k$$

[recall, $\Omega_m(\tau), \mathcal{H}(\tau)$]

Since nothing depends on k except δ_k , we see immediately that this admits factorizable solutions:

$$\delta_k(\tau) = D(\tau) A_k$$

$$\Rightarrow \boxed{\frac{d^2 D}{d\tau^2} + \mathcal{H} \frac{dD}{d\tau} = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2 D}$$

This is the equation that describes the growth factor $D(\tau)$ -

In order to solve it, we need $\mathcal{H}(\tau)$ and $\Omega_m(\tau)$, which are coupled through Friedmann equations:

$$\mathcal{H}^2 (1 - \Omega_m) = k$$

$$\frac{\ddot{a}}{a} = \frac{1}{a} \frac{d\dot{a}}{d\tau} = \frac{1}{a} \frac{d\mathcal{H}}{d\tau} = \frac{1}{a^2} \frac{d\mathcal{H}}{d\tau} = \frac{\mathcal{H}'}{a^2} = -\frac{4\pi G}{3} \rho_m = -\frac{1}{2} \frac{\mathcal{H}}{a^2} \Omega_m \Rightarrow \mathcal{H}' = -\frac{\mathcal{H}^2}{2} \Omega_m$$

In std. variables these are:

$$\left\{ \begin{aligned} H^2 (1 - \Omega_m(H)) &= \frac{k}{a^2} \\ \dot{H} + H^2 &= -\frac{1}{2} H^2 \Omega_m \end{aligned} \right.$$

i) let's consider simplest case, $\Omega_m = 1$ ($k=0$) $\Rightarrow \mathcal{H}' = -\frac{\mathcal{H}^2}{2}$, $a \sim \tau^2$

Then, it's easy to check that a^m is a solution,

$$\frac{d^2 a^m}{d\tau^2} = m(m-1) a^{m-2} (a')^2 + m a^{m-1} a'' \quad \mathcal{H} = \frac{da^m}{d\tau} = \frac{1}{a} a' = \frac{2}{\tau}$$

$$\frac{d^2}{d\tau^2} a^m = m(m-1) a^{m-2} (a')^2 + m a^{m-1} a''$$

$$\frac{a'}{a} = \frac{2}{\tau} \quad \frac{a''}{a} = \frac{2}{\tau^2}$$

$$\Rightarrow m(m-1) \left(\frac{2}{\tau}\right)^2 + m \left(\frac{2}{\tau^2}\right) a^m = \frac{3}{2} \left(\frac{2}{\tau}\right)^2 a^m$$

$$\Rightarrow m(m-1) + \frac{3m}{2} = \frac{3}{2} \Rightarrow m = 1, -3/2$$

$$\Rightarrow \begin{cases} D_+ \sim a & \text{"growing mode"} \\ D_- \sim a^{-3/2} & \text{"decaying mode"} \end{cases}$$

(So, unlike static case of no expansion (where perturbations grow exponentially) grow is only power-law)

$$\Rightarrow \boxed{\delta_k(\tau) = A_k a + B_k a^{-3/2}}$$

(initial conditions will tell whether a perturbation decays or grows, we will see soon)

What about velocity perturbations?

For velocity divergence we have in linear PT: $\Theta_k = -\frac{\partial \delta_k}{\partial \tau}$

$$\Rightarrow \boxed{\Theta_k(\tau) = -\mathcal{H} \left(A_k a - \frac{3}{2} B_k a^{-3/2} \right)}$$

This is not the whole story for velocity perturbations though, as usual, a vector field can be decomposed into a divergence ("scalar part") and a vorticity ("vector part"),

$$\vec{v} = \nabla\psi_v + \nabla \times \vec{A}_v$$

So that

$$\begin{cases} \nabla \cdot \vec{v} = \nabla^2 \psi_v \equiv \theta & (\nabla \cdot (\nabla \times \vec{A}) = 0) & \text{divergence} \\ \nabla \times \vec{v} = -\nabla^2 \vec{A}_v \equiv \vec{\omega} & (\nabla \times \nabla \psi = 0; \nabla \cdot \vec{A} = 0) & \text{vorticity} \end{cases}$$

In Fourier space these relations take a very simple form:

$$\vec{v}_k = \underbrace{\left(\frac{-i\vec{k}}{k^2} \theta_k \right)}_{\text{longitudinal (|| to } \vec{k})} + i \underbrace{\left(\frac{\vec{k} \times \vec{\omega}_k}{k^2} \right)}_{\text{transverse (\perp to } \vec{k})}$$

Since then:

$$\begin{cases} i\vec{k} \cdot \vec{v}_k = \theta_k \\ i\vec{k} \times \vec{\omega}_k = -\frac{1}{k^2} \vec{k} \times (\vec{k} \times \vec{\omega}_k) = \frac{k^2}{k^2} \vec{\omega}_k = \vec{\omega}_k \end{cases}$$

(recall $i\vec{k}$ is ∇ in Fourier)

A velocity field with $\theta \neq 0$ and $\vec{\omega} = 0$ is said to be potential (irrotational)
 " " " " $\theta = 0$ " $\vec{\omega} \neq 0$ " " " " solenoidal

How does vorticity perturbations evolve?

Since $\frac{\partial \vec{v}}{\partial t} + \mathcal{H}\vec{v} = -\nabla\Phi - \frac{\nabla\Phi}{\rho}$

we see that $\frac{\partial \vec{\omega}}{\partial t} + \mathcal{H}\vec{\omega} = 0$

Vorticity is not sourced by density perturbations, to the expansion of the universe (in linear PT).

This evolves so it conserves angular momentum $\vec{L} = \vec{r} \times \vec{p}$
 $\Rightarrow \vec{L} = \rho d^3r \vec{v} \times \vec{r}$ (all physical)
 $\rho = \bar{\rho}(1+\delta) \approx \bar{\rho} \frac{1}{a^3} dr^3 \sim a^{-3}$
 $\vec{v} = \vec{v}_{||} + \vec{v}_{\perp}$ where $\vec{v}_{||}$ is from θ and \vec{v}_{\perp} is from $\vec{\omega}$
 $L_{\perp} \propto \vec{v}_{\perp} \times \vec{r} \sim \frac{1}{a} a = \text{const.}$
 then it decays due

$$\frac{1}{\mathcal{H}} \frac{\partial \vec{\omega}}{\partial t} = \frac{\partial \vec{\omega}}{\partial \ln a} = -\vec{\omega} \Rightarrow \boxed{\vec{\omega} \sim a^{-1}}$$

(Note this is independent of δ_m ,

Therefore, we will deal with a potential flow and only care about θ .

Therefore, we can understand perturbations in terms of just 2 scalar fields, δ and θ (θ is linearly related to δ through Poisson equation). The eqs. of motion are,

$$\begin{cases} \frac{\partial \delta}{\partial \tau} + \theta = 0 \\ \frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta = -\frac{3}{2}\mathcal{H}^2\delta \end{cases}$$

To better understand growing and decaying modes, and how are they excited by initial conditions let's do the following tricks:

- since everything depends on time through a , change time variable

from τ to $t \equiv \ln a$

$$\frac{1}{\mathcal{H}} \frac{d}{d\tau} = \frac{\partial}{\partial \ln a} = \partial_t$$

- since θ scales as \mathcal{H} , and in linear PT θ is opposite in sign to δ , lets define: $\theta = -\mathcal{H}\Phi$

Then, we have:

$$\frac{\partial \delta}{\partial \tau} - \mathcal{H}\Phi = 0 \Rightarrow \boxed{\partial_t \delta - \Phi = 0}$$

$$\frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta = -\frac{\partial}{\partial \tau}(\mathcal{H}\Phi) - \mathcal{H}^2\Phi = -\frac{\mathcal{H}^2}{2}\Phi - \mathcal{H}\frac{\partial \Phi}{\partial \tau} = -\frac{3}{2}\mathcal{H}^2\delta$$

dividing by \mathcal{H}^2 : $\boxed{\partial_t \Phi + \frac{1}{2}\Phi - \frac{3}{2}\delta = 0}$

We will solve these equations again by one more powerful method to simplify algebra* - let's consider a vector with two components,

$$\Psi_a = \begin{pmatrix} \delta \\ \Phi \end{pmatrix} \quad a = 1, 2$$

*: The reason we are doing this is that the same method will be easily extended to non-linear solutions

Then the 2 equations of motion can be written as:

$$\partial_t \psi_a + \Omega_{ab} \psi_b = 0, \quad \Omega_{ab} \equiv \begin{bmatrix} 0 & -1 \\ -3/2 & 1/2 \end{bmatrix}$$

(summation convention is used) -

We solve this linear equation by Laplace's transforms (which properly take into account initial conditions) - Recall that

$$\begin{aligned} \text{Laplace transform: } \mathcal{L}(\psi_a)_{(\omega)} \equiv \tilde{\psi}_a(\omega) &\equiv \int_0^\infty e^{-\omega t} \psi_a(t) dt \\ \Rightarrow \psi_a(t) &= \oint_{C-i\infty}^{C+i\infty} \frac{d\omega}{2\pi i} e^{\omega t} \tilde{\psi}_a(\omega) \end{aligned}$$

The important property is that $\mathcal{L}(\partial_t \psi_a(t))_{(\omega)} = \omega \tilde{\psi}_a(\omega) - \phi_a$

where $\phi_a \equiv \psi_a(t=0)$ is the initial condition - (initial density and velocity)

Then, we have.

$$\omega \tilde{\psi}_a(\omega) - \phi_a + \Omega_{ab} \tilde{\psi}_b(\omega) = 0$$

$$\Rightarrow (\omega \delta_{ab} + \Omega_{ab}) \tilde{\psi}_b(\omega) = \phi_a$$

$$\text{let } \sigma_{ab}^{-1}(\omega) \equiv \omega \delta_{ab} + \Omega_{ab} = \begin{bmatrix} \omega & -1 \\ -3/2 & 1/2 + \omega \end{bmatrix}$$

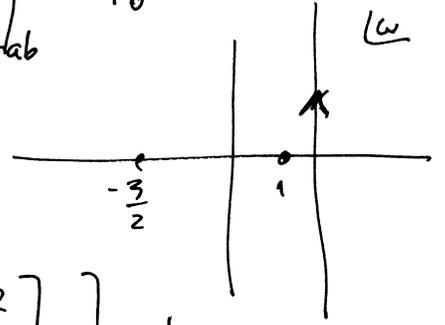
$$\Rightarrow \sigma_{ab} = \frac{1}{(2\omega+3)(\omega-1)} \begin{bmatrix} 2\omega+1 & 2 \\ 3 & 2\omega \end{bmatrix}$$

$\Rightarrow \tilde{\psi}_a(\omega) = \sigma_{ab}(\omega) \phi_b$ is the solution in terms of the initial condition.

We Laplace transform back to time-domain,

$$\psi_a(t) = \oint_{C-i\infty}^{C+i\infty} \frac{d\omega}{2\pi i} \frac{e^{\omega t}}{(2\omega+3)(\omega-1)} \begin{bmatrix} 2\omega+1 & 2 \\ 3 & 2\omega \end{bmatrix} \times \phi_b \quad (7)$$

This has two poles @ $\omega = +1, -3/2$



$$\Rightarrow \psi_a(t) = \left[e^{t} \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} - e^{-3t/2} \frac{1}{5} \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \right] \phi_b$$

the pole $\omega = +1$ corresponds to growing mode, $\omega = -3/2$ to decaying mode

Recall $t = \ln a$, then, we have:

$$\psi_a(a) = g_{ab}(a) \times \phi_b$$

Of course, you can derive this trivially from $\delta = Aa + Ba^{-3/2}$, $\theta = Aa - \frac{3}{2}Ba^{-3/2}$
 $\Rightarrow \delta_0 = A+B$ $\theta_0 = A - \frac{3}{2}B$ then get A, B
 $\Rightarrow \begin{pmatrix} \delta \\ \theta \end{pmatrix} = g_{ab} \begin{pmatrix} \delta_0 \\ \theta_0 \end{pmatrix}$ - However, the method using \mathcal{L} extends nicely to Non-linear solutions, we'll discuss this later

where the linear propagator $g_{ab}(a)$ is given by

$$g_{ab}(a) = \frac{a}{5} \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \frac{1}{5} a^{-3/2} \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

[Note that as $a \rightarrow 1$ $g_{ab} \rightarrow \delta_{ab}$, as it should]

Now, we are ready to discuss excitations of growing and decaying modes. ~~the~~ Mathematically, we see that if the initial condition has the form:

$$\phi_a^+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

only the growing mode will be excited,

$$g_{ab}(a) \phi_b^+ = a$$

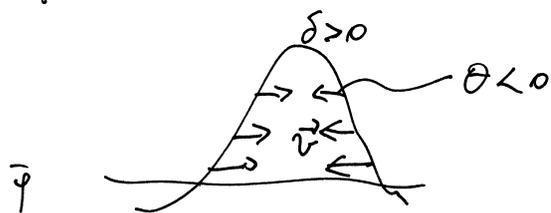
whereas if $\phi_a^- = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$, then only decaying mode is excited,

$$g_{ab}(a) \phi_b^- = a^{-3/2}$$

What does this mean physically?

(8)

(1) corresponds to $\delta = \theta = -\frac{\theta}{H}$, so velocity divergence is opposite to density:



So, it corresponds to an initial condition where initial velocities are going toward a high density region.

On the contrary, a $(\frac{1}{-3/2})$ mode corresponds to velocity flowing out of high density, so that perturbation can fully decay (if properly arranged), otherwise for a general case some amplitude will be lost to decaying part, but eventually grows.

ii) Now, let's go back to case where $\Omega_m \neq 1$ - ~~all anything was~~ Observationally, we are only interested in case where $\Omega_m < 1$.

In this case solution can be obtained by change of variables,

$$\chi = \frac{1}{\Omega_m} - 1 \geq 0$$

Then, it follows

$$\begin{cases} D_+ = 1 + \frac{3}{\chi} + 3\sqrt{\frac{1+\chi}{\chi^3}} \ln \left[\sqrt{1+\chi} - \sqrt{\chi} \right] \\ D_- = \sqrt{\frac{1+\chi}{\chi^3}} \end{cases}$$

For small χ , $\Omega_m \approx 1$, we recover $D_+ \sim \frac{2}{5}\chi$, $D_- \sim \chi^{-3/2}$

and $\chi \sim a$ - For large $\chi \gg 1$, we see that $D_+ \sim 1$, $D_- \sim \frac{1}{\chi}$

So, fluctuations stop growing when $\Omega_m \rightarrow 0$, as expected ("not enough gravity").

So, in general, if $\Omega_m > 1$, we have:

(9)

$$\delta_k = A_k D_+(a) + B_k D_-(a) \approx A_k D_+(a)$$

↑
neglect decaying modes

$$\Rightarrow \vec{v}_k = -\frac{i\vec{k}}{k^2} \theta_k$$

now: $\theta = -\frac{\partial \delta}{\partial \tau} = -H \frac{\partial \delta}{\partial \ln a}$

$$\Rightarrow \vec{v}_k = \left(\frac{i\vec{k}}{k^2} A_k\right) H(a) \frac{dD_+(a)}{d \ln a}$$

It is customary to define $f \equiv \frac{d \ln D_+}{d \ln a}$ $f \rightarrow 1$
 $\Omega_m \rightarrow 1$

$$\Rightarrow \boxed{\theta = -H f \delta \quad \left| \quad \vec{v}_k = \frac{i\vec{k}}{k^2} H f \delta_k \right.}$$

$f(z)$ can be actually written in terms of $\Omega_m(z)$. A useful analytic approximation is

$$f(z) \sim \Omega_m^{0.6}$$

$$\Rightarrow \theta \approx -H \Omega_m^{0.6} \delta$$

This can be used to estimate Ω_m from observations: θ can be obtained from radial motions of galaxies (since velocity field should be potential, one component is enough), δ can be traced by galaxies (we will discuss this in more detail later) ~~and that is what we want~~
~~really saying is~~

b) Non-zero pressure

We now go back to equations of motion for δ_k and put pressure in,

$$\frac{\partial^2 \delta_k}{\partial t^2} + \mathcal{H} \frac{\partial \delta_k}{\partial t} = \frac{3}{2} \Omega_B \mathcal{H}^2 \delta_k - k^2 \frac{\rho k}{\bar{\rho}}$$

(we are now assuming there are only baryons)

In order to work with the pressure term, we need to know the equation of state for perturbations.

Remember, going back from Fourier space to real space that

$$-k^2 \frac{\rho k}{\bar{\rho}} \rightarrow \nabla^2 \frac{\delta p}{\bar{\rho}}$$

We define the sound speed c_s ,

$$\left(\frac{\delta p}{\delta \rho} \right)_s = c_s^2 \Rightarrow \nabla^2 \frac{\delta p}{\bar{\rho}} = c_s^2 \frac{\nabla^2 \delta \rho}{\bar{\rho}} \rightarrow -k^2 c_s^2 \delta_k$$

↑
at constant entropy

↑
we are assuming that pressure is a function of density alone

For radiation, $p = \frac{1}{3} \rho \Rightarrow c_s^2 = 1/3$

For matter (baryons), $\rho_B \approx n m_B + \frac{3}{2} n k T$ $p = n k T$ ($s = \frac{p}{\rho}$)

$\Rightarrow c_s^2 = \frac{5}{3} \frac{kT}{m_B}$ (i.e. a monoatomic ideal gas with $\gamma = 5/3$)
(we are dominated by hydrogen after rec., which is monoatomic)

Going back to eq. of motion,

$$\frac{\partial^2 \delta_k}{\partial t^2} + \mathcal{H} \frac{\partial \delta_k}{\partial t} = \left(\frac{3}{2} \Omega_B \mathcal{H}^2 - k^2 c_s^2 \right) \delta_k$$

(c_s^2 can be a function of time, of course)

The RHS of this equation defines the Jeans wavenumber k_J :

$$k_J^2 c_s^2 = \frac{3}{2} \Omega_B \mathcal{H}^2 = 4\pi G \bar{\rho}_B a^2$$

remember k 's are comoving, $k^{phys} = a^{-1} k$

So,

$$k_{JH}^{phys} = \frac{\sqrt{4\pi G \bar{\rho}}}{c_s}$$

$$\lambda_J = \frac{2\pi}{k_J^{phys}} = \frac{2\pi c_s}{\sqrt{4\pi G \bar{\rho}}} \quad (\text{Jeans scale})$$

So, ~~for~~ we can write,

$$\frac{\partial^2 \delta h}{\partial t^2} + H \frac{\partial \delta h}{\partial t} = c_s^2 (k_J^2 - k^2) \delta h$$

Thus, for $k \ll k_J$ ($\lambda \gg \lambda_J$), perturbations evolve like in the pressureless case; however at small scales (where pressure takes over) perturbations will be suppressed - To understand physically what's going on, recall that

$$H^2 \propto G \bar{\rho} \Rightarrow (G \bar{\rho})^{-1/2} \sim H^{-1} = \text{Hubble time}$$

$$\lambda \gg \lambda_J \text{ means } \lambda \gg \frac{c_s}{\sqrt{G \bar{\rho}}} \Rightarrow t_s = \frac{\lambda}{c_s} \gg H^{-1}$$

t_s measures time a ^{sound} (pressure) wave can travel distance λ ; since that is larger than Hubble time, sound waves cannot do anything against gravity, so perturbations grow -

In the opposite limit, when $\lambda \ll \lambda_J$ $t_s = \frac{\lambda}{c_s} \ll H^{-1}$ so, in a Hubble time sound waves can travel distance λ , so pressure can react and support perturbations against gravity -

Notice that in the absence of expansion ($\dot{a}=0$), $\delta''_k = -c_s^2 k^2 \delta_k$ at $\lambda \ll \lambda_J$,

so $\delta_k \propto e^{\pm i k c_s t}$, simple sound waves. (assuming c_s is indep of time, for simplicity)

Obviously, when we add expansion there will be a suppression (12)
 on top of this oscillation (details will also depend on $c_s(t)$, before
 decoupling $c_s^2 = \frac{5}{3} \frac{kT}{m_B} \sim \frac{1}{2}$ since $T_B = T_r$)

It is customary to define the Jean's mass, the mass in a $\lambda_{J/2}$ radius:

$$M_J = \frac{4\pi}{3} \bar{\rho} \left(\frac{\lambda_J}{2}\right)^3 = \frac{\pi^{5/2} c_s^3}{6 G^{3/2} \bar{\rho}^{1/2}}$$

The evolution of λ_J and M_J depends on $c_s(a)$, we have

$$\begin{cases} \lambda_J \sim c_s a^{3/2} \\ M_J \sim c_s^3 a^{3/2} \end{cases}$$

~~More on the evolution of the Jeans mass depends on the baryons
 and photons, the baryons.~~

During the RAD era, $c_s^2 = 1/3$, then the Jeans mass in
 baryons, $\left\{ \begin{array}{l} \text{since baryons are tightly coupled to photons, baryon} \\ \text{pressure is provided by photons} \end{array} \right.$

$$M_J^B = \frac{4\pi}{3} \bar{\rho}_B \left(\frac{\lambda}{2}\right)^3 \approx 5.4 \times 10^{18} \Omega_B h^2 \left(\frac{T}{1\text{eV}}\right)^{-3} M_\odot$$

where $M_\odot = 1.989 \times 10^{33} \text{g} \approx 2 \times 10^{30} \text{kg}$ or $1.1 \times 10^{57} \text{eV}$ is sun's mass
 a useful thing is also $\rho_{crit} = 2.775 \times 10^{11} M_\odot/h^3 / (\text{Mpc}/h)^3$
 then density of some species i is $\rho_i = 27.75 \Omega_i \times 10^{10} \frac{M_\odot/h^3}{(\text{Mpc}/h)^3}$

Let's compare this with baryon mass within H^{-1} ,

$$M_H^B = \frac{4\pi}{3} \bar{\rho}_B \left(\frac{1}{H}\right)^3 = \left(\frac{2H^{-1}}{\lambda_J}\right)^3 M_J^B$$

$$\Rightarrow \frac{M_J^B}{M_H^B} = \left(\frac{\lambda_J}{2H^{-1}}\right)^3 = \left[\frac{2\pi c_s}{\sqrt{4\pi G \bar{\rho}_B}} \sqrt{\frac{3\pi G \bar{\rho}_B}{3}} \right]^3 \quad \rho = \rho_r + \rho_B \stackrel{\text{RAD}}{\approx} \rho_r$$

$$\Rightarrow \frac{M_J^B}{M_H^B} = (\pi c_s)^3 \left(\frac{2\rho_0}{3\rho_B}\right)^{3/2} \gg 1$$

Thus, during radiation-dominated era, baryons cannot grow for scales less than H^{-1} (for scales larger than that we live to see here); pressure due to photon is enough to counteract gravity.

After recombination, matter decouples from radiation, so

$$c_s^2 = \frac{5}{3} \frac{T_B}{m} = \frac{5}{3} \frac{T}{m T_{REC}}$$

$$\left\{ \begin{array}{l} T_B \sim a^{-2} \text{ (NR species, decoupled)} \\ T \sim a^{-1} \text{ (for photons)} \end{array} \right\} T_B = \frac{T_{REC}}{a^2} = T_{REC} \left(\frac{T}{T_{REC}}\right)^2 = \frac{T^2}{T_{REC}}$$

$$\Rightarrow M_J^B \underset{\rho \sim \rho_B}{=} \frac{\pi^{5/2}}{6} \frac{c_s^3}{G^{3/2} \rho_B^{1/2}} \sim T^{3/2} \sim (1+z)^{3/2}$$

\uparrow
 $c_s^2 \sim T^2$
 $\rho_B \sim a^{-3} \sim T^3$

putting
 \Rightarrow
 numbers

$$M_J^B \approx 1.3 \times 10^5 (\Omega_B h^2)^{-1/2} \left(\frac{z}{1100}\right)^{3/2} M_\odot$$

Note the substantial drop in the Jeans mass (due to drop in c_s^2); now perturbations in baryons with $M > M_J^B$ can grow and collapse to form objects - The point here is that only after REC can sub-Hubble baryon perturbations grow (Before REC, baryons cannot move freely through plasma to collapse, due to photon pressure)

- So far we have assumed ideal fluid; we now describe 2 additional effects that can play a role in the evolution of fluctuations at small scales: free streaming and Silk damping.

Free Streaming (Collisionless "Landau" damping)

This operates in collisionless dark matter. The idea is that collisionless particles can stream out of overdense regions into underdense regions and thus smooth out perturbations. Let's estimate the physical scale of this process l_{FS} ,

Note: This is just due to their free motion, not the perturbed velocity field that responds to density perturbations.

$$l_{FS}(t) = a(t) \underbrace{\int_0^t \frac{v(t)}{a(t)} dt}_{\text{comoving length}} = a(t)$$

Let's assume that for $t < t_{NR}$ (when DM is RZ) universe is dominated by rad (this is usually good approx). Then,

$$l_{FS}(t) = a(t) \int_0^t \frac{dt}{a(t)} = 2t \propto a^2 \quad t < t_{NR}$$

If $t_{NR} < t < t_{EQ}$ (once it becomes NR, $v \sim 1/a$), $\Rightarrow v = a_{NR}/a$

$$l_{FS}(t) = \frac{a}{a_{NR}} 2t_{NR} + a(t) \int_{t_{NR}}^t \frac{dt}{a} \frac{a_{NR}}{a} \quad a = a_{NR} (t/t_{NR})^{1/2}$$

$$= \frac{a}{a_{NR}} 2t_{NR} + a(t) \frac{a_{NR}}{a_{NR}} \frac{t_{NR}}{a_{NR}} \int_{t_{NR}}^t \frac{dt}{t}$$

$$= \frac{a}{a_{NR}} 2t_{NR} + \left(\frac{a}{a_{NR}} \right) \frac{a_{NR}}{2} \ln(t/t_{NR})$$

$$\Rightarrow l_{FS}(t) = \frac{a}{a_{NR}} 2t_{NR} \left[1 + \frac{1}{2} \ln(a/a_{NR}) \right] \quad t_{NR} < t < t_{EQ}$$

For $t > t_{EQ}$ ($a \sim t^{2/3}$)

$$\Rightarrow l_{FS}(t) = \frac{a(t)}{a_{EQ}} l_{FS}(t_{EQ}) + a(t) \int_{t_{EQ}}^t \frac{a_{NR}}{a_{EQ}^2} \left(\frac{t_{EQ}}{t} \right)^{1/3} dt$$

Then:
$$l_{FS}(t) = \frac{2t_{NR} a}{a_{NR}} \left(1 + \ln \frac{a_{00}}{a_{NR}}\right) + \frac{2t_{NR} a}{a_{NR}} \left(1 - \sqrt{\frac{a_{00}}{a}}\right) \quad t < t_{NR}$$
 (15)

Note that for $a \gg a_{00}$
$$l_{FS} \approx \frac{2t_{NR}}{a_{NR}} a \left[\frac{5}{2} + \ln \frac{a_{00}}{a_{NR}}\right]$$

To compare λ with l_{FS} it is easier to do it in comoving, because $\frac{\lambda}{a}$ is a constant. Then the comoving free streaming length is,

$$\frac{l_{FS}(t)}{a(t)} \approx \begin{cases} \frac{2t}{a} = \frac{2t_{NR}}{a} \left(\frac{t}{t_{NR}}\right)^2 = \frac{2t_{NR} a}{a_{NR}^2} & t < t_{NR} \\ \frac{2t_{NR}}{a_{NR}} [1 + \ln(a/a_{NR})] & t_{NR} < t < t_{00} \\ \frac{2t_{NR}}{a_{NR}} \left[\frac{5}{2} + \ln \frac{a_{00}}{a_{NR}}\right] & t > t_{00} \end{cases}$$

So, in $t < t_{NR}$, increases like a , then it only grows logarithmically and in MAT we saturates - This is the largest value it can take, so today

$$l_{FS}(t_0) = \frac{a_0}{a_{NR}} 2t_{NR} \left[\frac{5}{2} + \ln \frac{a_{00}}{a_{NR}}\right]$$

In order to evaluate this, we need to get NR time as a function of mass.

$$\left(\frac{T_{NR}}{T}\right)^3 = \frac{n}{n_{\gamma}}$$

$$\Omega_{DM} = \left(\frac{m n}{\rho_{crit}}\right)_{today} = \frac{m n_{\gamma}}{\rho_{crit}} \left(\frac{m}{n_{\gamma}}\right) \approx 30 \left(\frac{m}{1 \text{ keV}}\right) \left(\frac{n}{n_{\gamma}}\right) h^{-2}$$

Then,

$$\left(\frac{T_{NR}}{T}\right)^3 = \frac{n}{n_0} \approx \frac{\Omega_{DM} h^2}{30} \left(\frac{m}{1\text{KeV}}\right)^{-1}$$

numbers $\Rightarrow \frac{q_{NR}}{a_0} = 7 \times 10^{-7} \left(\frac{m}{1\text{KeV}}\right)^{-1} \left(\frac{T_{NR}}{T}\right) \quad t_{NR} = 1.2 \times 10^7 \left(\frac{m}{1\text{KeV}}\right)^{-2} \left(\frac{T_{NR}}{T}\right)^2$

$$\frac{q_{NR}}{a_{NR}} \approx \frac{m}{17\text{KeV}} (\Omega h^2)^{-1} \left(T/T_{NR}\right)$$

$$\Rightarrow \left[\lambda_{FS}(t_0) \approx 40 \text{ Mpc} (\Omega_{DM} h^2)^{-1} \left(\frac{T_{NR}}{T}\right)^4 = 0.5 \text{ Mpc} (\Omega_{DM} h^2)^{1/3} \left(\frac{m}{1\text{KeV}}\right)^{-4/3} \right]$$

massive neutrinos: $m \approx 30 \text{ eV} \quad \frac{T_{NR}}{T} \sim 0.7 \quad \Omega_{\nu} h^2 \approx m_{\nu} / 100 \text{ eV}$

$$\Rightarrow \lambda_{FS}(t_0) \sim 30 \text{ Mpc} \left(\frac{m_{\nu}}{30 \text{ eV}}\right)^{-1}$$

"heavy" cold relic: $m \sim 1 \text{ keV} \quad \Omega_{DM} h^2 \sim 1 \quad \Rightarrow \lambda_{FS}(t_0) = 0.5 \text{ Mpc} \left(\frac{m}{1\text{keV}}\right)^{-4/3}$

Since for dark matter perturbations cannot grow until they are matter dominated, and free streaming operates mostly before that (when DM is RER), this will modify the fluctuation spectrum at small scales (on top of other effects that enter) "prioritized"

Silk (Collisional) Damping

- Since baryons are not collisionless, they are not affected by previous effect (actually they are, but on scales compared to mean free path, small compared to the effect we are about to discuss) - As decoupling is approached λ_{γ} (mean free path in photons) becomes large enough that photons can diffuse out of overdense

into underdense regions, thus damping inhomogeneities in the photon-baryon plasma (recall γ 's and b 's are tightly coupled). ~~The~~ In between collisions, they ~~are~~ free-stream ~~where~~ where ~~the~~ mean free path is

$$\lambda_{\gamma} = \frac{1}{X_e n_e \sigma} \approx 1.5 \times 10^{29} X_e^{-1}(z) (\Omega_B h^2)^{-1} (1+z)^{-3} \text{ cm}$$

↑
(problem 5 a)
HMW #2)

But since time between collisions is smaller than Hubble time this effect can accumulate to suppress perturbations on scales much larger than λ_{γ} - It's a random walk problem, where photons are diffusing



← attempt to draw random walk

The effect will only operate until decoupling (afterwards photon travel freely, and baryons don't care about δ 's), afterwards time between collisions is longer than Hubble time (recall problem 5 again)

In comoving coordinates (the mean displacement is zero, calculate variance)

$$\Delta x^2 = N_{\text{coll}} \left(\frac{\lambda_{\gamma}}{a}\right)^2 \bar{\left(\frac{\Delta t}{a^2} \lambda_{\gamma}\right)^2}$$

$N_{\text{coll}} = \frac{c \Delta t}{\lambda_{\gamma}}$ (recall problem 5c)

$$\Rightarrow \Delta x^2 \approx \int_{t_{\text{dec}}}^{t_{\text{dec}}} \frac{dt}{a^2(t)} \lambda_{\gamma}^2(t) = \frac{3}{5} \frac{t_{\text{dec}}}{a^2(t_{\text{dec}})} \lambda_{\gamma}^2(t_{\text{dec}})$$

(it is dominated by $t=t_{\text{dec}}$, as density is too high at earlier times)

$$\Rightarrow \lambda_{\text{Silk}} = a(t_{\text{dec}}) \Delta x = \sqrt{\frac{3}{5} t_{\text{dec}} \lambda_{\gamma}^2(t_{\text{dec}})} \approx 3.5 \text{ Mpc} \sqrt{\frac{\Omega_M}{\Omega_B}} (\Omega_M h^2)^{-3/4}$$

Since baryons are tightly coupled to δ 's, all perturbations in $\lambda < \lambda_{\text{Silk}}$ are suppressed - The λ_{Silk} corresponds to a M_{Jes} :

$$M_{\text{Silk}} = 6.2 \times 10^{12} \left(\frac{\Omega_M}{\Omega_B}\right)^{3/2} (\Omega_M h^2)^{-5/4} M_\odot$$

So, this is close to the mass of clusters of galaxies. Recall, however, that $M_J^B > M_{\text{Silk}}$, so perturbations of this size could not grow anyway ~~before~~ before decoupling due to photon pressure. After decoupling, M_J^B drops to $\approx 10^5 M_\odot$, and $M_{\text{Silk}} \rightarrow \text{zero}$, so M_J^B is the relevant scale.

