



**The Abdus Salam
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Large-scale Structure - Lecture Note 4

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Recall the equations of motion in Newtonian case for dark matter (see previous)

$$\left\{ \begin{array}{l} \frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta) \vec{v}] = 0 \\ \frac{\partial \vec{v}}{\partial \tau} + H \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \vec{\nabla} \Phi \\ \nabla^2 \Phi = \frac{3}{2} H^2 \rho_m \delta \end{array} \right.$$

Since linear fluctuations grow, $\delta \sim \frac{a}{Hf} \sim a$, eventually the quadratic terms in these equations become important. This only happens at scales smaller than H^{-1} (after fluctuations entered H^{-1} and grew for a while), "recently" in the universe, i.e. for $z \leq 10$. So we are allowed to use Newtonian equations of motion.

- When $\delta \approx 1$ it is obvious the quadratic terms are of similar order. To derive non-linear solutions we will proceed perturbatively. We will work in Fourier space and go back to the 2-scalar fields description (in terms of δ and $\Theta = -\dot{\theta}/Hf$) developed in the linear case.

$$\Rightarrow \cancel{\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta) \vec{v}]} \quad \text{We will assume } \Omega_m = 1 \text{ for simplicity.}$$

$$\text{Then } f = \frac{d \ln D}{d \ln a} = 1 \quad \Theta = -\frac{\dot{\theta}}{H} \quad H' = -\frac{H^2}{2} \quad \bar{u} = -\frac{\vec{v}}{H}$$

$$\Rightarrow \frac{1}{H} \frac{\partial \delta}{\partial a} - \vec{\nabla} \cdot [(1+\delta) \vec{v}] = 0$$

$$\cancel{\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot [(1+\delta) \vec{v}]} \Rightarrow \frac{\partial \delta}{\partial a} - \Theta - \vec{\nabla} \cdot (\delta \vec{v}) = 0$$

Euler equation: taking divergence:

$$\frac{\partial \Theta}{\partial \tau} + H \Theta + \vec{\nabla} \cdot [\vec{\nabla} \cdot \vec{v}] = -\frac{3}{2} H^2 \delta$$

$$\frac{\partial \theta}{\partial t} = - \frac{\partial (\Theta H)}{\partial t} = -H \frac{\partial \Theta}{\partial t} + \frac{H^2}{2} \Theta$$

(2)

\Rightarrow dividing by H^2 :

$$-\frac{1}{H} \frac{\partial \Theta}{\partial t} + \frac{3}{2} \Theta + \frac{3}{2} \delta = -\bar{\nabla} \cdot [(\bar{u} \cdot \bar{\nabla}) \bar{u}]$$

So, we have: $t = \ln a$

$$\begin{cases} \partial_t \delta - \Theta = \bar{\nabla} \cdot (\delta \bar{u}) \\ \partial_t \Theta + \frac{1}{2} \Theta - \frac{3}{2} \delta = \bar{\nabla} \cdot [(\bar{u} \cdot \bar{\nabla}) \bar{u}] \end{cases} \quad (*)$$

the terms on right hand side were dropped in the linear analysis -

let's calculate things in Fourier space

$$\text{recall } \bar{u}(\vec{k}) = \frac{i \vec{k}}{k^2} \Theta(k)$$

(Sorry ~~about~~ $\Theta \rightarrow \Theta$ below)

$$\begin{aligned} & \boxed{\int \frac{-i \vec{k}_1 \cdot \vec{x}}{(2\pi)^3} d^3x \bar{\nabla} \cdot (\delta \bar{u})} = \int \frac{e^{-i \vec{k}_1 \cdot \vec{x}}}{(2\pi)^3} d^3x \bar{\nabla} \cdot \left[\int e^{i \vec{k}_1 \cdot \vec{x}} d^3k_1 \delta(k_1) \right] \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \int e^{i \vec{k}_2 \cdot \vec{x}} d^3k_2 \frac{-i \vec{k}_2}{k_2^2} \Theta(k_2) \\ & = \int \frac{e^{-i \vec{k}_1 \cdot \vec{x}}}{(2\pi)^3} e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{x}} d^3k_1 d^3k_2 \frac{(\vec{k}_1 \vec{k}_2)}{k_1 k_2} \frac{i \cancel{k}_1 \cdot \cancel{k}_2}{k_2^2} \delta(k_1) \Theta(k_2) \\ & = \int d^3k_1 d^3k_2 \delta_D(\vec{k}_1 - \vec{k}_1 - \vec{k}_2) \frac{\vec{k}_1 \cdot \vec{k}_2}{k_2^2} \delta(k_1) \Theta(k_2) \end{aligned}$$

a convolution, as expected,
Fourier transform of a product-

Doing same for the other non-linear term:

$$\begin{aligned} & \int \frac{-i \vec{k} \cdot \vec{x}}{(2\pi)^3} d^3x \bar{\nabla} \cdot [(\bar{u} \cdot \bar{\nabla}) \bar{u}] = \int \frac{-i \vec{k} \cdot \vec{x}}{(2\pi)^3} d^3x \bar{\nabla} \cdot \left[\int e^{i \vec{k} \cdot \vec{x}} d^3k \frac{-i \vec{k}}{k^2} \Theta(k) \cdot \bar{\nabla} \times \text{some term} \right] \\ & = \int \frac{e^{-i \vec{k} \cdot \vec{x}}}{(2\pi)^3} d^3x e^{i(\vec{k}_1 + \vec{k}_2) \cdot \vec{x}} d^3k_1 d^3k_2 i(\vec{k}_1 + \vec{k}_2) \cdot \left[-i \frac{\vec{k}_1 \cdot (\vec{k}_1 + \vec{k}_2)}{k_1^2} \right] \frac{-i \vec{k}_2}{k_2^2} \Theta(k_1) \Theta(k_2) \end{aligned}$$

Notice that integrand is symmetric due to $\Theta(k_1) \Theta(k_2)$, then we can write

$$\frac{\bar{k} \cdot \bar{k}_1}{k_1^2} \frac{\bar{k}_1 \cdot \bar{k}_2}{k_2^2} = \bar{k} \cdot \bar{k}_2 \quad \left(\frac{\bar{k}_1 \cdot \bar{k}_2}{k_1^2 k_2^2} \right) \quad (\bar{k} = \bar{k}_1 + \bar{k}_2 \text{ because of } \delta)$$

Symmetric in $k_1 \leftrightarrow k_2$

we symmetrize: $\bar{k} \cdot \bar{k}_2 \rightarrow \frac{1}{2} (\bar{k} \cdot \bar{k}_1 + \bar{k} \cdot \bar{k}_2) = \frac{1}{2} (k_1^2 + k_2^2 + i\bar{k}_1 \cdot \bar{k}_2) = \frac{k^2}{2}$

Then we have:

$$\int \frac{-i\bar{k} \cdot \bar{v}}{(2\pi)^3} \bar{\nabla} \cdot \left[(\bar{v} \cdot \bar{v}) \bar{u} \right] = \int d^3 k_1 d^3 k_2 \delta_D(\bar{k} - \bar{k}_1 - \bar{k}_2) \frac{1}{2} \frac{k^2 (k_1 \cdot k_2)}{k_1^2 k_2^2} \Theta(k_1) \Theta(k_2)$$

Sorry, again, but $\Theta(k)$ is actually $\Theta(k^2)$ -

Now we are ready to go back to (x) and Fourier transform:

$$\partial_t \Psi_a(t) + \Sigma_{ab} \Psi_b(k) = \int \gamma_{abc}(t, k_1, k_2) \Psi_b(k_1) \Psi_c(k_2) d^3 k_1 d^3 k_2$$

where, "as usual" $\Sigma_{ab} = \begin{bmatrix} 0 & -1 \\ -3k_2 & 1/2 \end{bmatrix}$ $\Psi_a(k) = \begin{pmatrix} \delta(k) \\ \Theta(k^2) \end{pmatrix}$

and $\gamma_{112} = \frac{\bar{k} \cdot \bar{k}_2}{k_2^2} \delta_D(\bar{k} - \bar{k}_1 - \bar{k}_2)$, $\gamma_{222} = \frac{1}{2} \frac{k^2 (k_1 \cdot k_2)}{k_1^2 k_2^2} \delta_D(\bar{k} - \bar{k}_1 - \bar{k}_2)$

and all other $\gamma_{abc} = 0$

We try solutions to the above non-linear equations by expanding in terms of linear growing modes to some power, this way we can generate the growing modes for non-linear case,

$$\Psi_a(k, t) = \sum_{n=1}^{\infty} a_{(n)}^{(n)} \Psi_a^{(n)}(k)$$

(notice we are writing separable solutions, $\Psi_a^{(n)}(k)$ does not depend on time)

(4)

$$\partial_t \Psi_a(t_1, t_2) = \partial_{t_1} \Psi_a(t_1, t_2) = \sum_n n a_{(T)}^n \Psi_a^{(n)}(t_2)$$

$$\Rightarrow \sum_{n=1}^{\infty} a^n \Psi_b^{(n)}(t_1) (n \delta_{ab} + \gamma_{ab}) = \int \gamma_{abc} \sum_n \sum_{m=1}^{n-1} a^n(t_1) \Psi_b^{(n-m)}(t_1) \Psi_c^{(m)}(t_2) d^3 k_1 d^3 k_2$$

Matching a^n on both sides we must have:

$$\underbrace{(n \delta_{ab} + \gamma_{ab})}_{\text{"}} \Psi_b^{(n)}(t_1) = \int \gamma_{abc} \sum_{m=1}^{n-1} \Psi_b^{(n-m)}(t_1) \Psi_c^{(m)}(t_2) d^3 k_1 d^3 k_2$$

$\gamma_{ab}^{-1}(n)$: this is an "old friend", its inverse is

$$\gamma_{ab}(n) = \frac{1}{(2n+3)(n+1)} \begin{bmatrix} 2n+1 & 2 \\ 3 & 2n \end{bmatrix}$$

Multiplying by this we get,

$$\boxed{\Psi_a^{(n)}(t_1) = \gamma_{ab}(n) \int \gamma_{bcd}(t_1, t_1, t_2) \sum_{m=1}^{n-1} \Psi_c^{(n-m)}(t_1) \Psi_d^{(m)}(t_2) d^3 k_1 d^3 k_2}$$

This is the general solution for the t_1 dependence of n th order solution, note that it is a function of the initial condition $\Psi_a^{(1)}(t_1)$ and the k -dependent "matrix" γ_{abc} - [Recall this assumes fastest growing mode, ω_a]

It is easy to write down the 2nd order solution ($n=2$), after some algebra (homework?) it is easy to show that,

$$\Psi_a^{(2)}(t_1) = \int \delta_D(t_1 - t_1 - t_2) d^3 k_1 d^3 k_2 \begin{bmatrix} F_2(t_1, t_2) \\ G_2(t_1, t_2) \end{bmatrix} \delta_L(t_1) \delta_L(t_2)$$

Where : $\Psi_a^{(1)}(t_1) = \delta_L(t_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (we are assume growing mode)
linear solutions

(5)

and the kernels F_2 and G_2 obey

$$\left\{ \begin{array}{l} F_2(t_1, t_2) = \frac{5}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left(\frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \\ G_2(t_1, t_2) = \frac{3}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{4}{7} \left(\frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \end{array} \right.$$

These two specify the 2nd order solution for the density and velocity divergence fields,

$$\begin{pmatrix} \delta^{(2)}(t_1, \tau) \\ \Theta^{(2)}(t_1, \tau) \end{pmatrix} = a_{(\tau)}^2 \Psi_a^{(2)}(t_1) \sim [a \Psi_a^{(1)}(\tau)]^2$$

and the full solution to 2nd order is,

$$\Psi_a(t_1, \tau) = \begin{pmatrix} \delta(t_1, \tau) \\ \Theta(t_1, \tau) \end{pmatrix} = a_{(\tau)} \Psi_a^{(1)}(t_1) + a_{(\tau)}^2 \Psi_a^{(2)}(t_1) \left\{ + \Theta [a \Psi_a^{(1)}]^3 \right\}$$

Before we turn to application of these results, note that these solutions are the fastest growing modes; in general non-linear solutions have more growing and decaying solutions, as we now derive -

OPTIONAL STUFF 

We go back to the equations of motion,

$$\partial_t \Psi_a(t_1, \tau) + \gamma_{ab} \Psi_b(t_1, \tau) = \int \gamma_{abc}(t_1, t_2, t_3) \Psi_b(t_2, \tau) \Psi_c(t_3, \tau) d^3 h_1 d^3 h_2$$

and again Laplace transform in the time variable $t = \ln a$ (as we did for linear solns)

$$\tilde{\Psi}_a(t_1, w) \equiv \int_0^\infty e^{-wt} \Psi_a(t_1, t) dt$$

$$\Rightarrow w \tilde{\Psi}_a(t_1, w) + \gamma_{ab} \tilde{\Psi}_b(t_1, w) = \Phi_a(t_1) + \int \gamma_{abc} d^3 h_1 d^3 h_2 \oint \frac{dw'}{2\pi i} \tilde{\Psi}_b(t_1, w') \tilde{\Psi}_c(t_2, w-w')$$

where we used that the Laplace transform of a product is the

convolution of Laplace transforms (Convolution theorem, like for Fourier transforms) (6)

i.e.

$$\mathcal{L} \left[A(t) B(t) \right]_{(w)} = \oint \frac{dw'}{2\pi i} \tilde{A}(w') \tilde{B}(w-w')$$

$$\text{where } \tilde{A}(w) \equiv \mathcal{L}[A(t)]_{(w)} = \int_0^\infty e^{-wt} A(t) dt -$$

$$\text{Again, we can write } w \tilde{\psi}_a + \tau_{ab} \tilde{\psi}_b = (\omega \delta_{ab} + \tau_{ab}) \tilde{\psi}_b = \tau_{ab}^{-1}(w) \tilde{\psi}_b$$

Then, we multiply by inverse and have:

$$\tilde{\psi}_{a(\vec{t}_1, w)} = \tau_{ab}(w) \phi_b(\vec{t}_1) + \tau_{ab}(w) \int \delta_{bcd} d^3 t_1 d^3 t_2 \oint \frac{dw'}{2\pi i} \tilde{\psi}_c(\vec{t}_1, w) \tilde{\psi}_d(\vec{t}_2, w)$$

We then inverse transform back to time domain, recall that

$$A(t) = \oint_{C+i\infty}^{C-i\infty} \frac{dw}{2\pi i} e^{wt} \tilde{A}(w) \quad \left(\text{where } c \text{ is a constant to the right of any singularities of } A(w) \right)$$

$$\text{then, } g_{ab}(t) \equiv \oint_{C-i\infty}^{C+i\infty} \frac{dw}{2\pi i} e^{wt} \tau_{ab}(w) = e^t \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} - \bar{e}^{-3t/2} \frac{1}{5} \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$$

giving the linear propagator, with linear growing $e^t = \alpha(t)$ and decaying modes $\bar{e}^{-3t/2} = \alpha(t)^{-3/2}$. This takes care of the first term. The non-linear term involves,

$$\oint \frac{dw}{2\pi i} e^{wt} \tau_{ab}(w) \quad \oint \frac{dw'}{2\pi i} \underbrace{\psi_c(w')} \underbrace{\psi_d(w-w')} \quad \left(\begin{array}{l} \text{we only include here} \\ \text{(time-dependent) factors} \end{array} \right)$$

$$\stackrel{||}{\int} \bar{e}^{wt'} g_{ab}(t') dt' \quad \int \bar{e}^{ws} \psi_c(s) ds \quad \int \bar{e}^{-(w-w')s'} \psi_d(s') ds'$$

So, we have:

$$dw e^{w(t-t'-s)} \Rightarrow \text{sets } t = t' + s'$$

$$dw' e^{w'(s'-s)} \Rightarrow \text{sets } s = s'$$

Thus we have,

$$\int_0^t ds g_{ab}(t-s) \gamma_{bcd} \Psi_c(s) \Psi_d(s)$$

(7)
integral goes from 0 to t
because $g_{ab}(x)=0$ if $x < 0$
 g_{ab} is "retarded" propagator

So, the final solution is

$$\Psi_a(t, \vec{r}) = g_{ab}(t) \phi_b(\vec{r}) + \int_0^t ds g_{ab}(t-s) \gamma_{bcd}(t, t_s) \Psi_c(t_s, s) \Psi_d(t_s, s) d^3 k_1 d^3 k_2$$

(Notice that non-linear term vanishes at $t=0 \Rightarrow \Psi = \phi$)

This gives $\Psi_a(t, \vec{r})$ in terms of $\Psi_c(t_s, s)$ for time $s < t$ - Notice this is a full solution, so it includes all decaying and growing modes - For example, to 2nd order the full solution will be to use the linear solution for $\Psi_c(t_s, s)$ and $\Psi_d(t_s, s)$,

$$\Psi_a^{(2)}(t, \vec{r}) = \int_0^t ds g_{ab}(t-s) \gamma_{bcd}(t, t_s) [g_{ce}(s) \phi_e(t_s)] [g_{df}(s) \phi_f(t_s)] d^3 k_1 d^3 k_2$$

And so the full time dependence will come from

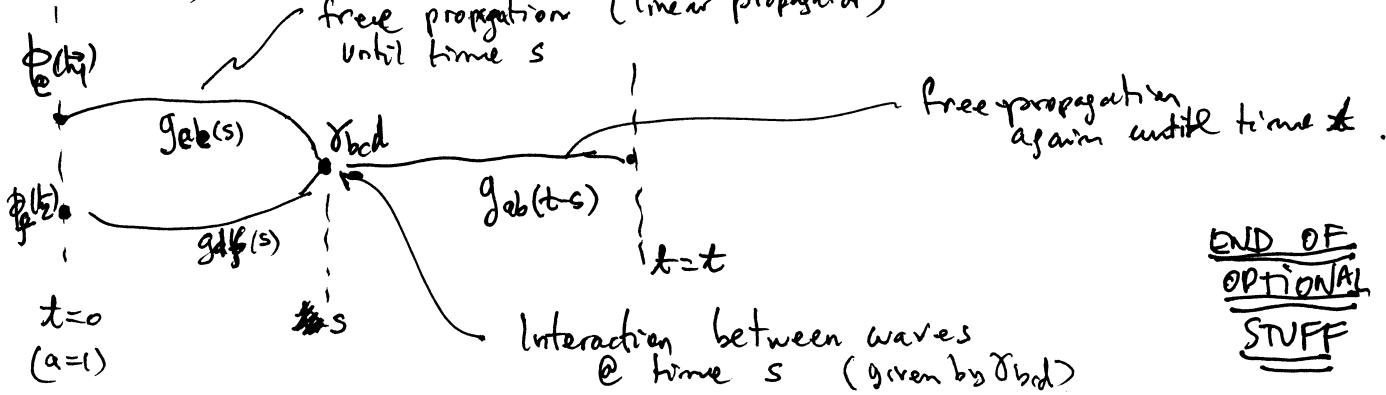
$$\int_0^t ds g_{ab}(t-s) g_{ce}(s) g_{df}(s)$$

One can check that you get terms going like $a^2, a, a^{-3/2}$

The term going as a^2 reduces to the one derived before.

The full solution above can be interpreted in terms of Feynman diagrams,

Initial conditions



Anyway, we won't go there ... (One can draw diagrams...etc) -

- We want to use the 2nd order solution derived above to show how non-linearities in the equations of motion generate non-Gaussianity from Gaussian initial perturbations (e.g. generated by inflation). -

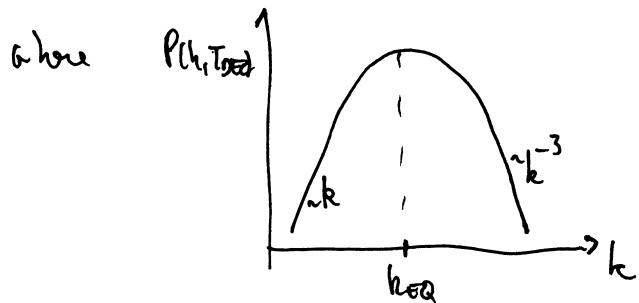
From previous class we know that after the shape of the power spectrum is set after MFT-NAD equality, i.e. when the shape of the transfer function is set [actually to do it right one has to wait until decoupling to include the baryon physics effects on DM, etc.]

So, after decoupling we have linear evolution of dark matter (DM) perturbations (we work on scales smaller than H^{-1}),

$$\delta(t, \tau) \propto \frac{a(\tau)}{a_{\text{dec}}} \delta(t^*) \quad \text{where } \delta(t^*) = \delta(t^*, T_{\text{dec}})$$

And thus the power spectrum $P(k)$ evolves as

$$\langle \delta(t, \tau) \delta(t', \tau') \rangle = \delta(t+t') P(k, \tau) = \left(\frac{a}{a_{\text{dec}}} \right)^2 P(k, T_{\text{dec}}) \delta_D(t+t')$$



is the power spectrum @ decoupling
(which includes effects of transfer function)

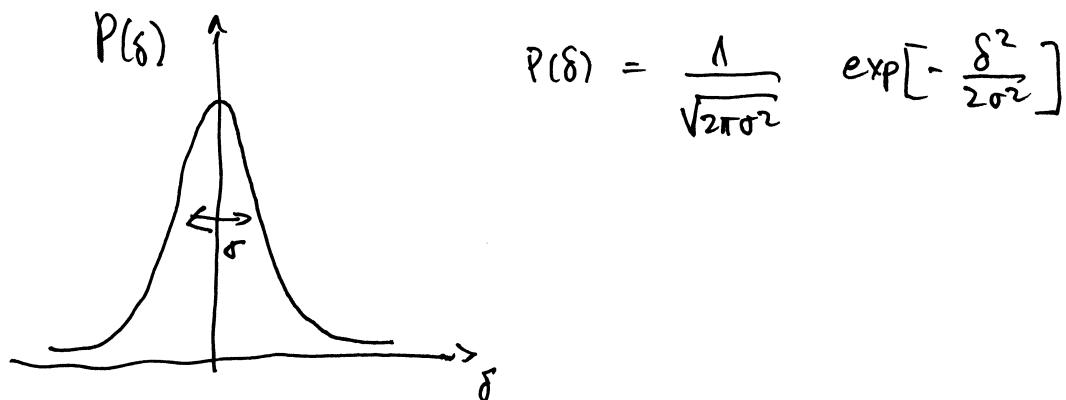
Since evolution is linear until decoupling (and actually until much later, ≈ 10)
If the ~~initial~~ ~~entire~~ primordial fluctuations are Gaussian (e.g. generated by simplest inflationary models), they remain Gaussian:

$$\langle \delta_{(t)}^n \rangle_c = a_{(t)}^n \quad \langle \delta_0^n \rangle_c = 0 \quad \text{for } n > 2 \quad \text{if } \delta_0 \text{ is Gaussian.}$$

So, linear time evolution does not change Gaussianity -

A simple consideration of the physics of gravity will tell us that Gaussianity cannot be preserved.

Recall that the fact that $\delta(x)$ [or $\delta(\vec{x})$] is Gaussian means that its probability distribution is Gaussian so



where σ represents the width of the Gaussian, related to the size of fluctuations,

$$\sigma^2 = \int_{\text{Variance}} d^3k P(k) = \int_{\text{Power Spectrum}} 4\pi k^2 P(k) dk = \int \frac{dk}{k} \Delta(k)$$

Recall that the density is defined as $\rho = \bar{\rho}(t) [1 + \delta(x(t))]$

Positivity of density $\rho(x(t))$ means that $\delta \geq -1$ at all $x(t)$.

In the early universe fluctuations are very small, i.e. at ~~deeper~~ $\sigma \approx 10^{-5}$. So the probability of having $\delta < -1$ is tremendously suppressed $P \propto \exp\left[-\frac{1}{10^{-10}}\right] \sim e^{-10^{10}}$, so a Gaussian manifold will not yield negative densities. As fluctuations grow and σ becomes close to unity, obviously it cannot stay Gaussian, this will yield non-negligible probability that density is negative, which doesn't make sense physically.

- Also, more importantly, the physics of gravitational clustering tells us that there should be asymmetry between δ and $-\delta$.

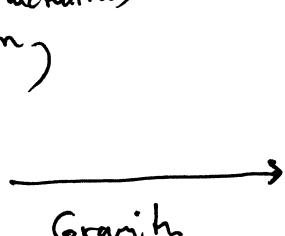
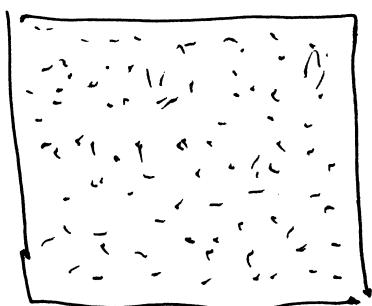
Suppose we start with some Gaussian initial condition

As higher densities attract more matter, the density increases (like a in linear case). Also, underdense regions where $\delta < 0$, become more evacuated, again because $\delta(t) \propto a\delta_0$. So becomes more negative as time goes on.

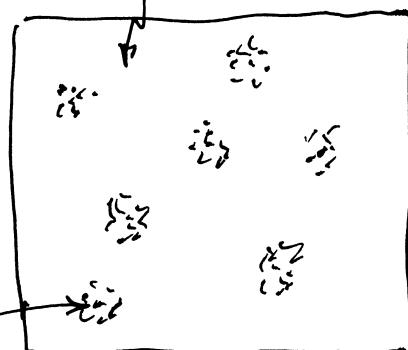
But since $\rho > 0$, and ρ can grow as time goes on in dense regions, the symmetry between δ and $-\delta$ will be lost.

In a pictorial way,

early state (close to uniform, only small Gaussian fluctuations away from random)

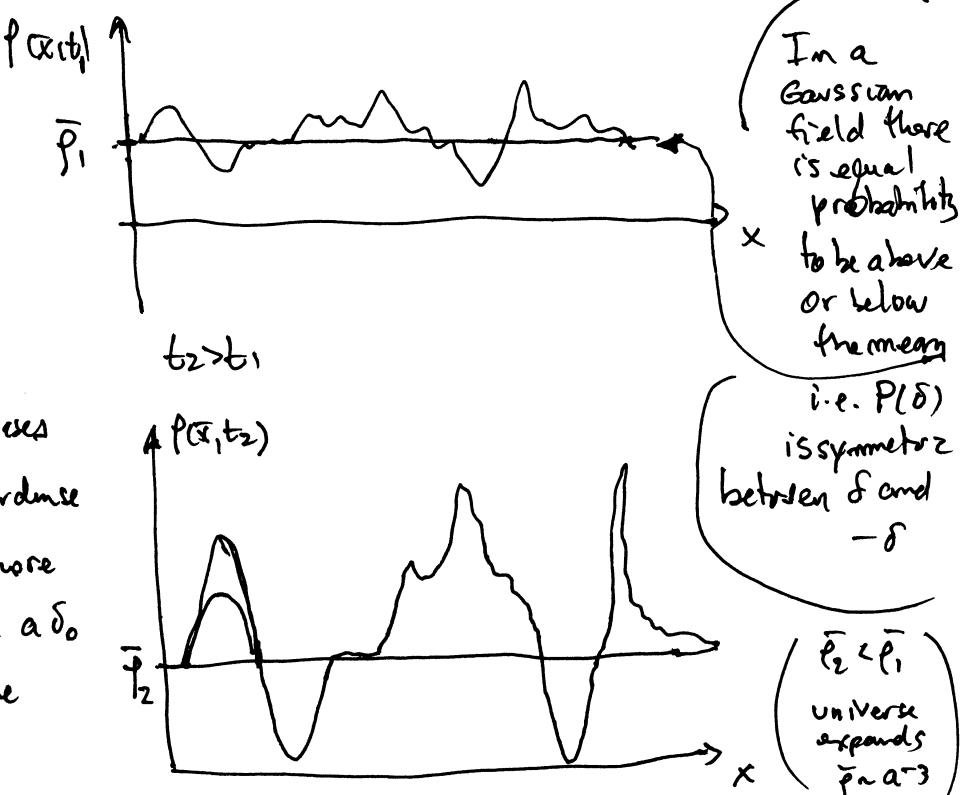


regions that initially have $\delta < 0$ become void as time goes on



regions that initially had $\delta > 0$ grow and grow creating clusters of small size.

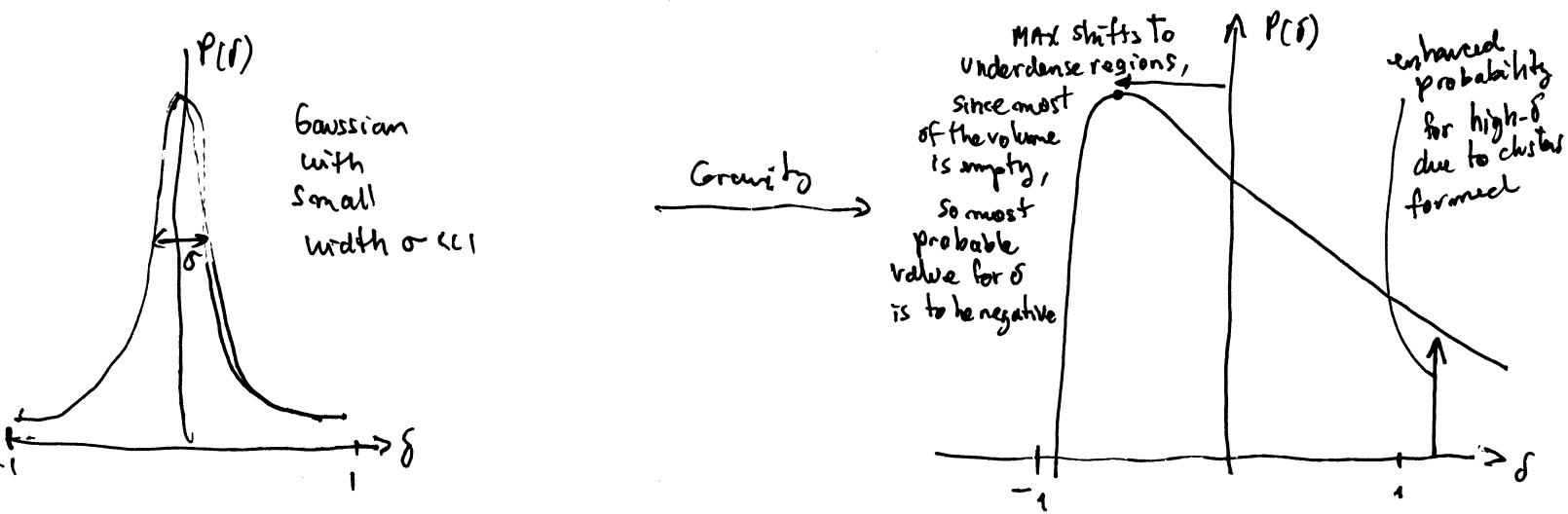
Note that although initially the size of regions with $\delta > 0$ and $\delta < 0$ is equal, as time goes on the $\delta > 0$ occupy a lot smaller volume than regions with $\delta < 0$, most of the universe is "empty"!!



In a Gaussian field there is equal probability to be above or below the mean i.e. $P(\delta)$ is symmetric between δ and $-\delta$

This means that the PDF (probability distribution function) (11)

develops a significant asymmetry,



Note the asymmetry between δ and $-\delta$.
Also σ now is larger (since in linear PT $\sigma \propto a$).

We can understand the deviations from Gaussianity that develops as non-linear dynamics becomes important by expanding the PDF about a Gaussian and its derivatives:

Let $P(\delta) d\delta = \phi(v) dv$ where $v \equiv \frac{\delta}{\sigma}$ i.e. normalized variable so it has unit variance

$$\text{For a Gaussian } P_G(\delta) d\delta = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\delta^2/2\sigma^2} d\delta = \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv = \phi_G(v) dv$$

We expand about a Gaussian,

$$\phi(v) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \frac{d^n \phi_G(v)}{dv^n}$$

Want to calculate c_n 's.

Obviously, they depend on the higher-order connected moments, which are zero for a Gaussian but non-zero for $P(\delta)$ or $\phi(v)$

Recall that derivatives of a Gaussian generate the Hermite polynomials,

$$\frac{d^n}{dv^n} e^{-v^2/2} = (-1)^n H_n(v) e^{-v^2/2}$$

where,

where $n=0$ is Gaussian PDF, next terms are corrections. Expansion will converge if c_n are increasingly small as n increases.

$$H_0 = 1 \quad H_1 = v \quad H_2 = v^2 - 1 \quad H_3 = v^3 - 3v \quad H_4 = v^4 - 6v^2 + 3 \quad \dots$$

Hermite polynomials are orthogonal with Gaussian weights,

$$\int_{-\infty}^{\infty} H_n(v) H_m(v) \phi_G(v) dv = \delta_{nm} m!$$

This allows us to calculate c_n . We write,

$$\phi(v) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (-1)^n H_n(v) \phi_G(v)$$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(v) H_m(v) dv = \sum_{n=0}^{\infty} \frac{c_n}{n!} (-1)^n \underbrace{\int_{-\infty}^{\infty} H_n(v) H_m(v) \phi_G(v) dv}_{m! \delta_{nm}}$$

$$\Rightarrow c_m = (-1)^m \int \phi(v) H_m(v) dv = (-1)^m \langle H_m(v) \rangle$$

$$c_0 = \langle 1 \rangle = 1$$

$$c_1 = -\langle v \rangle = -\frac{1}{\sigma} \langle \delta \rangle = 0$$

$$c_2 = \langle v^2 - 1 \rangle = \frac{1}{\sigma^2} \langle \delta^2 \rangle - \langle 1 \rangle = 1 - 1 = 0$$

$$c_3 = -\langle v^3 - 3v \rangle = -\frac{1}{\sigma^3} \langle \delta^3 \rangle + \frac{3}{\sigma} \underbrace{\langle \delta \rangle}_{=1} = -\frac{\langle \delta^3 \rangle}{\sigma^3}$$

$$c_4 = \langle v^4 - 6v^2 + 3 \rangle = \frac{1}{\sigma^4} \langle \delta^4 \rangle - \frac{6}{\sigma^2} \underbrace{\langle \delta^2 \rangle}_{=1} + 3$$

$$\text{Now } \langle \delta^4 \rangle = \langle \delta^4 \rangle_c + 3 \langle \delta^2 \rangle^2 = \langle \delta^4 \rangle_c + 3\sigma^4$$

$$\Rightarrow c_4 = \frac{\langle \delta^4 \rangle_c}{\sigma^4} + \cancel{\beta} - \cancel{6} + \cancel{\beta}$$

$$\text{Similarly, } c_5 = -\frac{\langle \delta^5 \rangle}{\sigma^5}$$

$$c_6 = \frac{\langle \delta^6 \rangle_c}{\sigma^6} + 10 \left[\frac{\langle \delta^3 \rangle_c}{\sigma^3} \right]^2$$

Remember, for n odd $\langle \delta^n \rangle = \langle \delta^n \rangle_c$ since $\langle \delta \rangle = 0$.

We define the dimensionless parameters,

$$s_n = \frac{\langle \delta^n \rangle_c}{\sigma^n} \quad n \geq 2$$

for a Gaussian $s_2 = 1$ $s_n = 0$ for $n > 2$. Clearly, these parameters determine how non-Gaussian is the distribution, since

$$\phi(v) = \phi_G(v) \left[1 + \frac{s_3}{3!} H_3(v) + \frac{s_4}{4!} H_4(v) - \frac{s_5}{5!} H_5(v) + \frac{s_6 + 10s_3^2}{6!} H_6(v) + \dots \right]$$

if all s_n are small $s_n \ll 1 \Rightarrow \phi(v) \approx \phi_G(v)$ to a good approximation

As we shall see in a minute, gravity generates the following hierarchy,

$$\langle \delta^n \rangle_c \sim \langle \delta^2 \rangle^{n-1} \quad \text{to leading order in PT, when fluctuations are small}$$

$\Rightarrow s_n = \frac{\langle \delta^n \rangle_c}{\langle \delta^2 \rangle^{n-1}}$ are numbers that can be calculated from the equations of motion (we will calculate s_3 below)

$$\Rightarrow s_n = \frac{\langle \delta^n \rangle_c}{\sigma^n} = \frac{\langle \delta^n \rangle_c}{\langle \delta^2 \rangle^{n/2}} = \frac{\langle \delta^n \rangle_c}{\langle \delta^2 \rangle^{n-1}} \sigma^{n-2} = s_n \sigma^{n-2} \quad n \geq 2$$

i.e. $s_3 = S_3 \sigma$; $s_4 = S_4 \sigma^2$ and so on

This for $\sigma \ll 1$, ~~the~~ the s_n 's are increasingly smaller and the expansion above converges. In order to properly talk about expansion we have to collect all terms of the same power in σ , which is the small parameter, then:

$$P(\delta) = P_G(\delta) \left[1 + \frac{S_3 \sigma}{3!} H_3(v) + \sigma^2 \left[\frac{S_4}{4!} H_4(v) + \frac{10}{6!} S_3^2 H_6(v) \right] + O(\sigma^3) \right]$$

This is the so-called "Edgeworth Expansion" of the PDF.

We see then when fluctuations are small $\sigma^2 \ll 1$, the PDF approaches Gaussianity. The 1st order deviation is given by S_3 , known as the "skewness", the 2nd order deviation involves S_4 , known as the "kurtosis".

Note that to first order in σ , the maximum of the PDF is at

$$\frac{dP}{d\delta} = 0 \approx P_G \frac{-\delta}{\sigma^2} \left[1 + \frac{S_2}{6} \sigma H_3 + \dots \right] + P_G \left[\frac{S_2 \sigma}{6} \left(\frac{3\delta^2}{\sigma^3} - \frac{3}{\sigma} \right) \right]$$

Since the maximum is close to $\delta=0$ (in the limit $\sigma \rightarrow 0$), we can throw away terms δ^2 or higher:

$$0 = -\frac{\delta}{\sigma^2} - \frac{S_3}{2} \Rightarrow \boxed{\delta_{\max} = -\frac{S_3}{2} \sigma^2}$$

So the shift to the left of the maximum of the PDF is related directly to the skewness.

Now we are ready to calculate S_3 induced by gravity:

$$\langle \delta^3 \rangle = \langle (\delta_1 + \delta_2 + \dots)^3 \rangle$$

where δ_1 is the linear solution, δ_2 is the 2nd order solution, etc.

$$\Rightarrow \langle \delta^3 \rangle = \langle \delta_1^3 \rangle + 3 \langle \delta_1^2 \delta_2 \rangle + \dots$$

If primordial fluctuations are Gaussian the linear contribution vanishes,

$$\langle \delta_1^3 \rangle = \langle \delta_1^3 \rangle_{\text{PT}} \langle \delta_0^3 \rangle = 0 \quad \text{because } \delta_0(x) \text{ is Gaussian}$$

Then, leading order contribution comes from 2nd order PT,

$$\langle \delta^3 \rangle = 3 \langle \delta_1^2(x) \delta_2(y) \rangle$$

Since solution for 2nd order is easiest written in Fourier space,
we use Fourier space:

$$\langle \delta_{(x)}^3 \rangle = 3 \int e^{ik_1 \cdot \bar{x}} e^{ik_2 \cdot \bar{x}} e^{ik_3 \cdot \bar{x}} \langle \delta_L(k_1, z) \delta_L(k_2, z) \delta_L(k_3, z) \rangle d^3 k_1 d^3 k_2 d^3 k_3$$

and for $n_{\text{m}}=1$, we have:

$$\begin{cases} \delta_L(k_i, z) = a(z) \delta_L(k_i) \\ \delta_L(k_j, z) = a^2(z) \int \delta_D(k_i - \vec{q}_1 - \vec{q}_2) F_2(\vec{q}_1, \vec{q}_2) d^3 q_1 d^3 q_2 \delta_L(\vec{q}_1) \delta_L(\vec{q}_2) \end{cases}$$

$$\Rightarrow \langle \delta_{(x)}^3 \rangle = 3 a^4(z) \int e^{i(k_1 + k_2 + k_3) \cdot \bar{x}} d^3 k_1 d^3 k_2 d^3 k_3 \delta_D(k_3 - \vec{q}_1 - \vec{q}_2) d^3 q_1 d^3 q_2 \times \langle \delta_L(k_1) \delta_L(k_2) \delta_L(\vec{q}_1) \delta_L(\vec{q}_2) \rangle F_2(\vec{q}_1, \vec{q}_2)$$

Since $\delta_L(k)$ is a Gaussian random field, we can easily calculate its 4-pt correlator, as the 3 possible pairings which give power spectra

$$\begin{aligned} \langle \delta_L(k_1) \delta_L(k_2) \delta_L(\vec{q}_1) \delta_L(\vec{q}_2) \rangle &= \langle \delta_L(k_1) \delta_L(k_2) \rangle \langle \delta_L(\vec{q}_1) \delta_L(\vec{q}_2) \rangle + \langle \delta_L(k_1) \delta_L(\vec{q}_1) \rangle \langle \delta_L(k_2) \delta_L(\vec{q}_2) \rangle \\ &\quad + \langle \delta_L(k_1) \delta_L(\vec{q}_2) \rangle \langle \delta_L(k_2) \delta_L(\vec{q}_1) \rangle \end{aligned}$$

Now, the first term will not contribute, since it corresponds to taking the mean value ~~$\langle \delta_L(k_1, z) \rangle$~~ $\langle \delta_L(k_1, z) \rangle$, which is zero (see homework #5..)

The other 2 terms contribute the same, they are equal by $\vec{q}_1 \leftrightarrow \vec{q}_2$,

so:

$$\begin{aligned} \langle \delta_{(x)}^3 \rangle &= 6 a^4(z) \int e^{i(k_1 + k_2 + k_3) \cdot \bar{x}} d^3 k_1 d^3 k_2 d^3 k_3 \delta_D(k_3 - \vec{q}_1 - \vec{q}_2) d^3 q_1 d^3 q_2 \\ &\quad P(q_1) \delta_D(k_1 + \vec{q}_1) P(q_2) \delta_D(k_2 + \vec{q}_2) F_2(\vec{q}_1, \vec{q}_2) \end{aligned}$$

$$\Rightarrow \langle \delta^3 \rangle = 6 a^4 \int F_2(\vec{q}_1, \vec{q}_2) P(q_1) P(q_2) d^3 q_1 d^3 q_2$$

Obviously, the \vec{x} dependence had to go away by translation invariance. Now, the only tricky integration is the angular part over wavevectors \vec{q}_1, \vec{q}_2 :

$$\langle \delta^3 \rangle = 6\alpha t \int P(q_1) q_1^2 dq_1 \int P(q_2) q_2^2 dq_2 \underbrace{\int F_2(\vec{q}_1, \vec{q}_2) d\Omega_1 d\Omega_2}_{F_2}$$

Since $F_2(\vec{q}_1, \vec{q}_2)$ depends only on scalar product of $\vec{q}_1 \cdot \vec{q}_2$, we can take \vec{q}_1 in, say, \hat{z} direction and $\vec{q}_1 \cdot \vec{q}_2 = q_1 q_2 x$ where x is cosine, then:

$$\begin{aligned} F_2 &= \int_{-1}^1 (4\pi)^2 \frac{d\Omega}{2} \left[\frac{5}{7} + \pi \left(\frac{h_1}{h_2} + \frac{h_2}{h_1} \right) + \frac{2}{7} x^2 \right] \\ &= (4\pi)^2 \left[\frac{5}{7} + \frac{2}{7} \frac{1}{3} \right] = (4\pi)^2 \frac{17}{21} \end{aligned}$$

So \bar{F}_2 corresponds to taking the monopole (spherical average) of the F_2 kernel.

$$\Rightarrow \langle \delta^3 \rangle = 6 \cancel{\alpha t} \frac{17}{21} \left[\alpha^2 \int P(k) 4\pi k^2 dk \right]^2 = \frac{34}{7} \sigma^4 = \frac{34}{7} \langle \delta^2 \rangle^2$$

$$\Rightarrow S_3 = \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2} = \frac{34}{7}$$

Note this is a number, independent of the normalization and shape of the power spectrum! [We shall see later how this changes when we consider smoothing].

A similar calculation can be done for the kurtosis S_4 ,

$$S_4 = \frac{\langle \delta^4 \rangle_c}{\langle \delta^2 \rangle^3} = \frac{60712}{1323} \quad \left[\begin{array}{l} \text{This calculation involves 3rd-order PT} \\ \text{in general for } S_p \text{ need } (p+1)\text{th order PT} \end{array} \right]$$

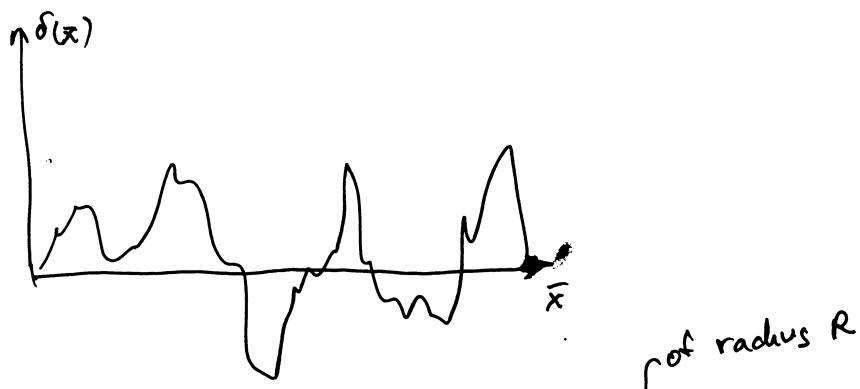
and similarly for higher-order S_p . Notice the scaling induced by gravity

$$\langle \delta^n \rangle_c \sim \langle \delta^2 \rangle^{n-1}$$

This is so because the non-linear terms in the equations of motion are quadratic. This is why we defined S_n the way we did, to cancel the time and scale dependence.

In practice we cannot measure $\langle \delta_{(x)}^n \rangle$, with $\delta(x)$ at a single point.

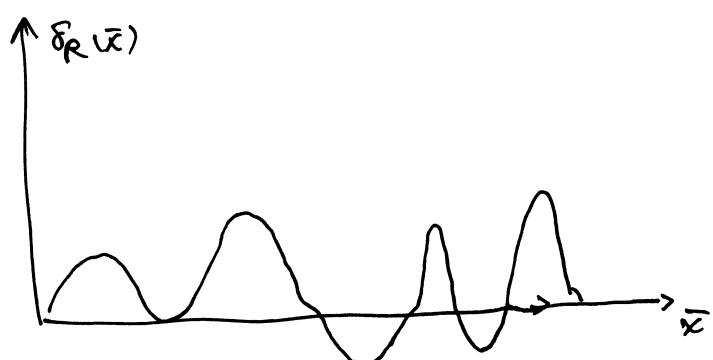
This is so because what we see is galaxies, which are discrete in nature. What they sample is a discrete version of the continuous field - To measure density (galaxy) fields we use smoothing. Think of a density field,



If at each point \bar{x} we put a top-hat sphere of radius R so that $W_R(\bar{x}-\bar{x}') = \begin{cases} 1 & |\bar{x}-\bar{x}'| \leq R \\ 0 & |\bar{x}-\bar{x}'| > R \end{cases}$
we are smoothing the field,

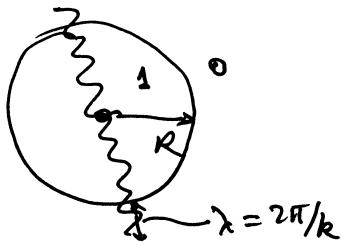
$$\delta_R(\bar{x}) = \int \delta(\bar{x}') W_R(\bar{x}-\bar{x}') d^3x'$$

So at each point \bar{x} all points $|\bar{x}-\bar{x}'| \leq R$ contribute - This creates the smoothed field,



where all high-frequency waves with $kR > 1$ have been washed-out by the smoothing operation.

This is so because if I count everything inside a sphere of radius R ,



all waves for which $kR \gg 1$ oscillate a lot inside and cancel out. Because a convolution in Fourier space becomes multiplication we have that the smoothed field Fourier coefficients obey,

$$\delta_R(\vec{k}) = \delta(\vec{k}) W(kR)$$

where for a top-hat filter $W(x) = \frac{3}{x^3} (\sin x - x \cos x)$

This implies that the smoothed field power spectrum is

$$P_R(k) = P(k) W^2(kR) \quad \begin{pmatrix} \text{(note that power is cut off for} \\ kR \gg 1 \end{pmatrix}$$

and the variance smoothed at scale R is:

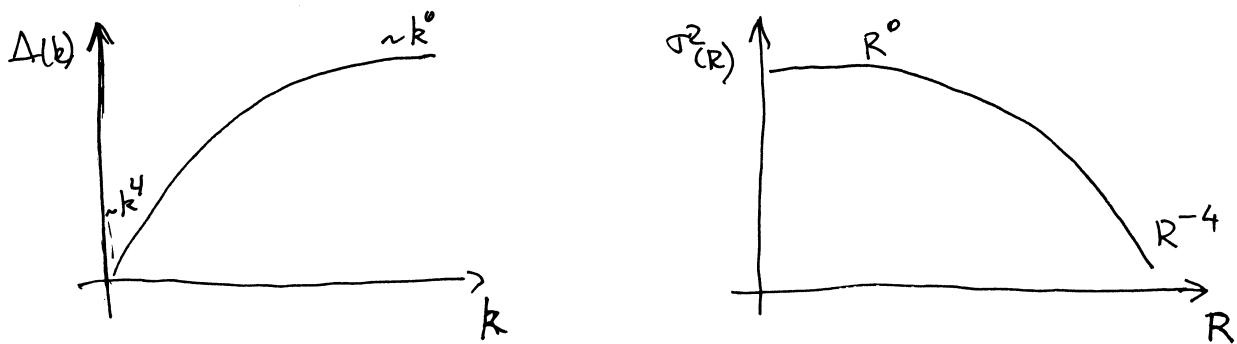
$$\sigma_{(R)}^2 = \int d^3k P_R(k) = \int d^3k P(k) W^2(kR)$$

It is customary to talk about the normalization of fluctuations at $R=8 \text{ Mpc}/h$,

$$\sigma_{(R=8 \text{ Mpc}/h)}^2 \equiv \sigma_8^2 = \int d^3k P(k) W^2(k8)$$

This gives a measure of the size of fluctuations, today ($t=0$) $\sigma_8 \approx 1$ (it depends on galaxy type if we talk about σ_8 for galaxies)

Since $P(k)$ has a definite shape $\sigma_{(R)}^2$ will inherit the shape of $\Delta(k) = \eta \pi h^3 P(k)$ with " $k \rightarrow \frac{1}{R}$ ":



In general if $P(k) \sim k^n$, $\Delta(k) \sim k^{n+3}$, $\sigma^2(R) \sim R^{(n+3)}$

We can now go back to the calculation of S_3 and include smoothing. All we need is to change $\delta(x)$ by $\delta_R(\vec{x})$, which in Fourier space means just changing $\delta(\vec{k})$ by $\delta(\vec{q}) W(kR)$, then

$$\langle \delta_R^3(\vec{x}) \rangle = 6\alpha^4 \int F_2(\vec{q}_1, \vec{q}_2) P(q_1) P(q_2) W(q_1 R) W(|\vec{q}_1 + \vec{q}_2| R) d^3 q_1 d^3 q_2$$

Note the $W(|\vec{q}_1 + \vec{q}_2| R)$ term appears because of $\delta_R(\vec{k}_3)$ and $\vec{k}_3 = \vec{q}_1 + \vec{q}_2$ - There are 3 W 's because there are 3 δ_R 's.

The angular integration now becomes more tricky; we won't do the math here but the final result is easy to write down [comes from summation theorem of Bessel Functions],

$$\langle \delta_R^3(\vec{x}) \rangle = \langle \delta_R^2 \rangle^2 \left[\frac{34}{7} + \frac{d \ln \sigma^2(R)}{d \ln R} \right]$$

Essentially angular integration gives $W(\vec{q}_1 + \vec{q}_2) \rightarrow W(\vec{q}_1) W(\vec{q}_2)$, so the total result is again proportional to $\sigma_{(R)}^4$ - The coefficient is now changed a bit by smoothing,

$$S_3(R) = \frac{34}{7} + \frac{d \ln \sigma^2(R)}{d \ln R}$$

For a power-law $P(k) \sim k^n \Rightarrow S_3 = \frac{34}{7} - (n+3)$ - For a CDM type spectrum,

$$S_3(r) = \frac{34}{7} - (n_{\text{eff}}(r) + 3)$$

where the effective spectral index depends on scale. Similar result holds
for $S_4(r)$, etc -