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Large-scale Structure - Lecture Note 5

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XVI Large-scale Clustering Statistics : Bispectrum, Skewness and PDF

Last class we discussed solutions beyond linear perturbation theory (PT), now we look at what can be used for - the simplest application is the calculation of the development of Non-Gaussianity due to non-linearity in the equations of motion - Non-Gaussianity is characterized, to lowest order, by the appearance of a three-point function (which vanishes for a Gaussian field) or its Fourier transform, the bispectrum. An average of these quantities determines the skewness, which measures the asymmetry in evolution between underdense and overdense regions - let's consider this in more detail -

We write the PT solution for the density,

$$\delta(\vec{r}) = \delta_0(\vec{r}) + \sum_{n=2}^{\infty} \int [\delta_0] F_n(k_1, \dots, k_n) \delta_1(k_1) \dots \delta_n(k_n) = \sum_{n=1}^{\infty} \delta_n(\vec{r})$$

where we include the growth factor inside each $\delta_n(\vec{r})$, and we work in the approximation that the kernels are independent of time, i.e. we assume $f^2 = \text{const}$ at all times. Here $[\delta_0] \equiv \delta_0(t - \sum_{i=1}^3 \frac{k_i}{a_i})$ and a similar expression holds for the velocity divergence replacing F_n by G_n .

Now consider schematically the calculation of the third moment of the density

$$\begin{aligned} \langle \delta^3 \rangle &= \langle (\delta_1 + \delta_2 + \dots)^3 \rangle \\ &= \underbrace{\langle \delta_1^3 \rangle}_{O(\delta^3)} + \underbrace{3 \langle \delta_1^2 \delta_2 \rangle}_{O(\delta^4)} + \underbrace{3(\langle \delta_1 \delta_2^2 \rangle + \langle \delta_1^2 \delta_3 \rangle)}_{O(\delta^5)} + \underbrace{\langle \delta_2^3 \rangle + 6 \langle \delta_1 \delta_2 \delta_3 \rangle + 3 \langle \delta_1^2 \delta_4 \rangle}_{O(\delta^6)} \end{aligned}$$

where we have organized different terms by the number of powers in the linear density field - Now, if we assume Gaussian initial conditions, the expectation value of odd numbers of δ_1 's is zero, then

$$\langle \delta^3 \rangle = \underset{\text{GIC}}{3 \langle \delta_1^2 \delta_2 \rangle} + [\langle \delta_2^3 \rangle + 6 \langle \delta_1 \delta_2 \delta_3 \rangle + 3 \langle \delta_1^2 \delta_4 \rangle] + \dots$$

The first term scales as σ^4 where σ^2 is the linear variance of the density field, the second term scales as σ^6 , so it will be much smaller than the first at large scales where $\sigma^2 \ll 1$.

then, in the large scale limit we have

$$\langle \delta^3 \rangle = 3 \langle \delta_1^2 \delta_2 \rangle$$

Now, back to the details. We can calculate the three-point function in Fourier space or "bispectrum", by

$$\langle \delta(\vec{k}_1) \delta(\vec{k}_2) \delta(\vec{k}_3) \rangle = B_{123} \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$$

$$\Rightarrow \delta_D(\vec{k}_{123}) B_{123} = \langle \delta_1(\vec{k}_1) \delta_1(\vec{k}_2) \delta_2(\vec{k}_3) \rangle + \text{cyc.}$$

the 2 other cyclic permutations

let's calculate one term:

$$\langle \delta_1(\vec{k}_1) \delta_1(\vec{k}_2) \delta_2(\vec{k}_3) \rangle = \int \delta_D(\vec{k}_3 - \vec{q}_1 - \vec{q}_2) F_2(\vec{q}_1, \vec{q}_2) \underbrace{\langle \delta_1(\vec{k}_1) \delta_1(\vec{k}_2) \delta_1(\vec{q}_1) \delta_1(\vec{q}_2) \rangle}_{\delta_1's \text{ are Gaussian, so this is:}}$$

Now, the first term does not contribute because it sets $\vec{q}_1 + \vec{q}_2 = 0 \Rightarrow F_2 = 0$ (remember this is because $\langle \delta_2(\vec{q}) \rangle = 0$). Then we have

$$= \int \delta_D(\vec{k}_3 - \vec{q}_1 - \vec{q}_2) 2F_2(\vec{q}_1, \vec{q}_2) P(k_1) P(k_2) d^3 q_1 d^3 q_2 \delta_D(\vec{k}_1 + \vec{q}_1) \delta_D(\vec{k}_2 + \vec{q}_2)$$

where we introduce a factor of 2 since last two terms give a small contribution. Then we have

$$= \delta_D(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) 2F_2(k_1, k_2) P(k_1) P(k_2)$$

Then, we have for the bispectrum,

$$\begin{aligned} B_{123} &= 2F_2(k_1, k_2) P(k_1) P(k_2) + 2F_2(k_1, k_3) P(k_1) P(k_3) + 2F_2(k_2, k_3) P(k_2) P(k_3) \\ &= 2F_2(k_1, k_2) P(k_1) P(k_2) + \text{cyc.} \end{aligned}$$

Note here that all time dependence in the bispectrum is through the power spectrum. Also note that the quadratic nonlinearities have induce a scaling

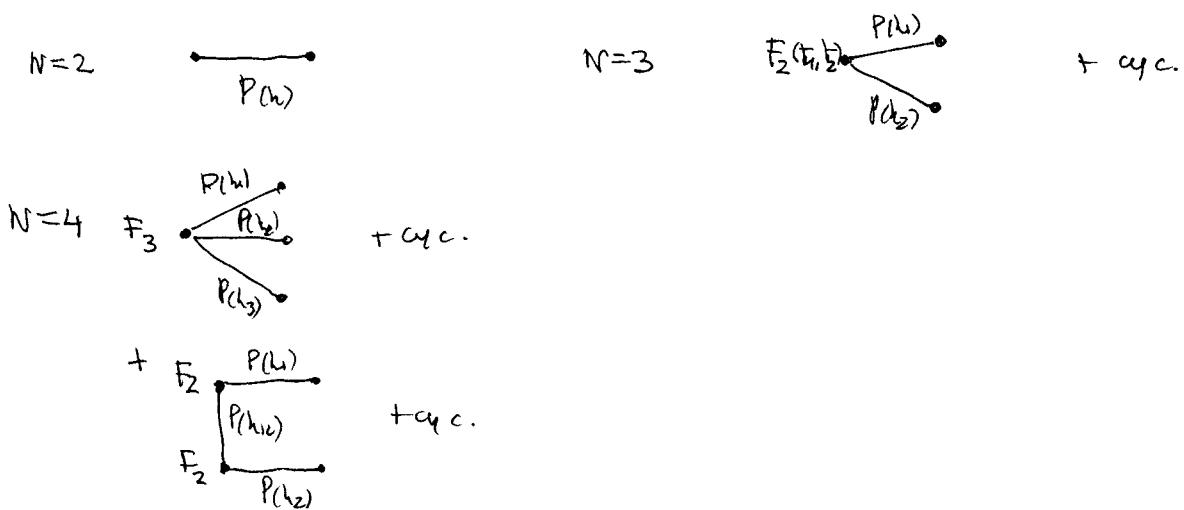
$$B \sim P^2$$

This in fact can be generalized to all higher-order correlation functions. For the N -point spectrum T_N we have

$$T_N \sim P^{N-1}$$

which corresponds to terms like $\langle \delta_{N-1} \underbrace{\delta_1 \dots \delta_s}_{N-1} \rangle_c \propto \langle \delta^N \rangle$

The "c" here means we take connected part, i.e. get rid of Gaussian contribution to even-order correlation functions. Such a structure of scalings is related to a tree with N points, which requires $N-1$ links to make it connected, e.g.



etc... Note that we defined vertices of different types, a vertex with n lines going out of it gets a factor of F_n (with $F_1 \equiv 1$). And each link carries a Power spectrum of the particular wavevector carried by that line. At each vertex "momentum conservation" is imposed, i.e. $k_1 + \dots + k_m = 0$ -

Note that the next-to leading terms of the same order organize themselves into 1-loop diagrams in this language,

$$\langle \delta_2^3 \rangle + 6 \langle \delta_1 \delta_2 \delta_3 \rangle + 3 \langle \delta_1^2 \delta_4 \rangle \stackrel{``="}{=} \text{Diagram } 1 + \text{Diagram } 2 + \text{Diagram } 3$$

where the diagrams are:

- Diagram 1: A triangle with three vertices and three edges, labeled $\langle \delta_2^3 \rangle$.
- Diagram 2: A triangle with three vertices and three edges, labeled $\langle \delta_1 \delta_2 \delta_3 \rangle$.
- Diagram 3: A triangle with three vertices and three edges, labeled $\langle \delta_4 \delta_1^3 \rangle$.

and similarly for the even spectrum,

$$\langle \delta^2 \rangle = \underbrace{\langle \delta_1^2 \rangle}_{\mathcal{O}(k^2)} + \underbrace{2 \langle \delta_1 \delta_2 \rangle}_{\mathcal{O}(k^3)} + \underbrace{[\langle \delta_2^2 \rangle + 2 \langle \delta_1 \delta_3 \rangle]}_{\mathcal{O}(k^4)} + \dots \quad (4)$$

For Gaussian initial conditions the second term vanishes and one has,

$$P(k) = \text{---} + [\text{---} + \text{---}] + \dots$$

$\langle \delta_1^2 \rangle \quad \langle \delta_2^2 \rangle \quad \langle \delta_1 \delta_3 \rangle$

the first being a tree diagram, the second contributions 1-loop diagrams - they correspond to the non-linear corrections to the power spectrum - We will discuss this next class -

OK, let's go back to the bispectrum and interpret physically what we get - Since $B \sim k^2$ it is convenient to define the reduced bispectrum

$$Q_{123} = \frac{B_{123}}{P_1 P_2 + P_2 P_3 + P_3 P_1}$$

i.e., we divide by the symmetrized combinations of k^2 . This definition removes the main dependencies on B_{123} , in fact:

- i) Q_{123} is indep of time (since linear growth factor cancels)
- ii) " " " " " cosmological parameters (" ")
- iii) " " " " overall scale, since F_2 depends on ratios of k 's - for a power-law power spectrum, $P(k) \sim k^n$.

The first two properties remain true for a CDM spectrum, the last one does not because the spectral index depends on scales in CDM - so the bottom line is that Q_{123} only depends on the shape of the triangle and the spectral index at the range of scales involved in the k 's that correspond to the triangle sides -

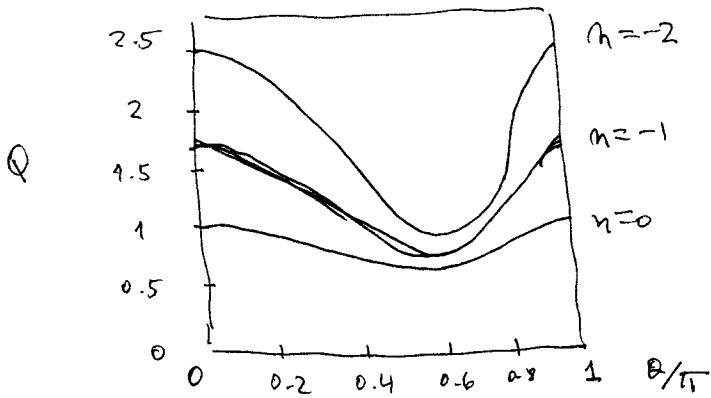
One case that is particularly simple is that of an equilateral triangle, $k_1 \cdot k_2 = -1/2$ and $|k_i| = k$, thus

$$Q_{\text{eq}} = \frac{4}{7}$$

(5)

this is independent of the power spectrum.

For other triangles the dependence is more complicated, but to give an example consider $k_2 = 2k_1$ and plot things as a function of θ where $\hat{k}_1 \cdot \hat{k}_2 = \cos \theta$,



Q is maximal for colinear triangles and minimum for triangles close to equilateral. This reflects the dependence on the tidal field and on the transformation from a given mass element to Eulerian space. In summary it reflects the shapes of structures generated by gravitational instability, in fact in order to talk about shapes you need at least 3 points, so in this sense the bispectrum is the lowest order statistic sensitive to shapes of structures generated by gravity. Note that for a more negative spectral index the bispectrum is more anisotropic, this is what we expect by just looking at the distribution of matter -

An averaged version of the bispectrum is to compute the skewness - let's first calculate the skewness at a given point in space - we have

$$\xi_3 \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2}$$

i.e. the same definition as for Q_{123} - Similarly, for higher-order moments one defines

$$S_N \equiv \frac{\langle \delta^N \rangle_c}{\langle \delta^2 \rangle^{N-1}}$$

again, to scale out the dependence on time, scale and cosmological parameters in the large-scale limit -

$$\langle \delta_{(x)}^3 \rangle = 3 \langle \delta_2(x) \delta_1^2(x) \rangle$$

$$= 3 \left\langle \int e^{i k_3 \cdot \bar{x}} d^3 k_3 \delta_2(k_3) \int e^{i k_1 \cdot \bar{x}} \delta_1(k_1) x^3 k_1 \int e^{i k_2 \cdot \bar{x}} \delta_1(k_2) x^3 k_2 \right\rangle$$

$$= \star \int e^{i(k_3 + k_1 + k_2) \cdot \bar{x}} B(k_1, k_2, k_3) \delta_D(k_1 + k_2 + k_3) d^3 k_1 d^3 k_2 d^3 k_3$$

$$= \int B(k_1, k_2) d^3 k_1 d^3 k_2 = 6 \int F_2(k_1, k_2) P(k_1) P(k_2) d^3 k_1 d^3 k_2$$

Now, we write as last class

$$F_2(k_1, k_2) = \frac{v_2}{2} + k_1 \cdot k_2 \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left[\left(\frac{k_1 \cdot k_2}{k_1 + k_2} \right)^2 - \frac{1}{3} \right]$$

$$v_2 \approx \frac{34}{21}$$

$$\Rightarrow \langle \delta^3(x) \rangle = 3 v_2 \left[\int P(k) d^3 k \right]^2 = 3 v_2 \star \langle \delta^2(x) \rangle^2$$

$$\Rightarrow \xi_3 = 3 v_2 = \frac{34}{7}$$

We see here that, i) only spherical dynamics enters

ii) does not depend on power spectrum or cosmological parameters -

iii) by translation invariance does not depend on \bar{x} .

Now, in practice one does not observe density field at "a point", but rather smoothed over some scale R . The most typical filter one smooths with is a top-hat filter, which in real space corresponds to counting things inside a sphere of radius R :

$$W_{TH}(x) = \Theta(|x| - R) / \frac{4\pi}{3} R^3$$

$$\text{where } \Theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

We can then write

$$\delta_R(\bar{x}) = \int \delta(y) W_{TH}(\bar{x} - y) d^3 y$$

or, in Fourier space,

$$\delta_R(k) = \delta(k) W_{TH}(kR)$$

where the Fourier transform is given by

$$W_{TH}(x) = \frac{3}{x^3} (\sin x - x \cos x) = 3 \sqrt{\frac{\pi}{2}} \frac{J_{3/2}(x)}{x^{3/2}}$$

$$\Rightarrow \langle \delta_R^2(x) \rangle = \int P(k) W_{TH}^2(kR) d^3k = \sigma_8^2(R)$$

This is used to specify the normalization of the power spectrum, e.g. at 8 Mpc/h it is called σ_8 ,

$$\sigma_8^2 = \int P(k) W_{TH}^2(k8 \text{ Mpc}/h) d^3k$$

For the third moment we have,

$$\begin{aligned} \langle \delta_R^3(x) \rangle &= \int B(k_1 + k_2 + k_3) \delta_D(k_{123}) W(k_1 R) W(k_2 R) W(k_3 R) d^3k_1 d^3k_2 d^3k_3 \\ &= 6 \int F_2(k_1, k_2) P(k_1) P(k_2) W(k_1 R) W(k_2 R) W(|k_1 + k_2| R) d^3k_1 d^3k_2 \end{aligned}$$

Now we see that the integration over angles is a bit more complicated because of the angular dependence inside $W(|k_1 + k_2| R)$ - But fortunately, this can be taken care of by the summation theorem for Bessel function which imply:

$$\int \frac{d\Omega_{12}}{4\pi} W(|k_1 + k_2|) [1 - (k_1 \cdot k_2)^2] = \frac{2}{3} W(k_1) W(k_2)$$

Note this is the same as doing $W \rightarrow W(k_1) W(k_2)$ and doing angle average.

Also,

$$\int \frac{d\Omega_{12}}{4\pi} W(|k_1 + k_2|) \left(1 + k_1 \cdot k_2 \frac{k_2}{k_1} \right) = W(k_1) [W(k_2) + \frac{1}{3} k_2 W'(k_2)]$$

Then we have

$$\begin{aligned} &\int F_2(k_1, k_2) W(k_1 R) W(k_2 R) W(|k_1 + k_2| R) \frac{d\Omega_{12}}{4\pi} \\ &= \underbrace{-\frac{2}{7} \times \frac{2}{3} W_1 W_2}_{\frac{17}{21} W_1 W_2} + W_1 W_2 + \underbrace{\frac{1}{6} (W_1 k_2 R W_2' + W_2 k_1 R W_1')}_{\text{contribution from "kinetic" L to E term}} \end{aligned}$$

writing that

$$\frac{d\sigma^2}{d\ln R} = 2 \int P(h) W(hR) W'(hR) kR d^3k$$

We have:

$$\begin{aligned}\langle \delta_R^3 \rangle &= 3v_2 \langle \delta_R^2 \rangle^2 + \frac{1}{3} \cancel{\langle \delta_R^2 \rangle} 6 \int P(h) W(hR) kR W'(hR) d^3k \\ &= 3v_2 \langle \delta_R^2 \rangle^2 + \langle \delta_R^2 \rangle^2 \frac{d \ln \sigma^2}{d \ln R}\end{aligned}$$

$$\Rightarrow S_3 = \frac{34}{7} + \frac{d \ln \sigma^2}{d \ln R}$$

We see that smoothing has introduced a correction that depends on spectral index at the smoothing scale.

For a power-law spectrum $P(h) \sim h^{-n}$

$$\sigma^2(R) \sim R^{-(n+3)} \Rightarrow \frac{d \ln \sigma^2}{d \ln R} = -(n+3)$$

$$\Rightarrow S_3 = \frac{34}{7} - (n+3)$$

By doing a similar calculation for all higher-order cumulants (that is, knowing V_n from spherical collapse and incorporating smoothing through L \rightarrow E mapping) one can sum up the cumulant generating function and compute the PDF. Show results.