The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

2354-21

Summer School on Cosmology

16 - 27 July 2012

The Cosmic Microwave Background - Notes

R. Durrer
Geneva University

→ slides 1–6

- Before recombination photons are in thermal equilibrium (via Compton scattering) with the electrons. → Planck distribution

$$f(p,t) = \frac{N_\gamma}{(2\pi)^3} \frac{1}{e^{ap/T} - 1}, \quad p = \sqrt{\gamma_{ij} p^i p^j}$$

- After recombination they are free and propagate along geodesics (geodesic spray). The distribution function satisfies Liouville's eqn.:

$$0 = \frac{df}{dt} = \dot{x}^\mu \partial_\mu f + \dot{p}^i \frac{\partial f}{\partial p^i}$$

$$\dot{x}^\mu = p^\mu, \quad \dot{p}^i = -\Gamma^i_{\mu\nu} p^\mu p^\nu,$$

$$0 = \left(p^\mu \partial_\mu - \Gamma^i_{\alpha\beta} p^\alpha p^\beta \frac{\partial}{\partial p^i}\right) f = L_{X_g} f.$$

If collisions are present this becomes the Boltzmann eqn.:

$$L_{X_g} f = C[f] \leftarrow \text{collision integral.}$$

In a homogeneous & isotropic universe f depends on p^i via $|p|$, $|p|^2 = \gamma_{ij} p^i p^j = (p^0)^2 = p^2$ so that

$$\frac{\partial f}{\partial p^i} = \frac{\partial p}{\partial p^i} \frac{\partial f}{\partial p} = \frac{p_i}{p} \frac{\partial f}{\partial p}$$

Furthermore, f depends on x^i only via

$$p = \sqrt{\gamma_{ij} p^i p^j}.$$

Therefore

$$p^i \partial_i f = \frac{1}{2} \frac{p^i \gamma_{lm,i} p^l p^m}{p} \frac{\partial f}{\partial p} = \frac{1}{2} \frac{p_j \gamma^{ij} \gamma_{lm,i} p^l p^m}{p} \frac{\partial f}{\partial p}$$

$$= \frac{1}{2} \gamma^{ij} \left(\gamma_{li,m} + \gamma_{mi,l} - \gamma_{lm,i} \right) \frac{p_j p^l p^m}{p} \frac{\partial f}{\partial p}$$

$$= \Gamma^j_{lm} \frac{p_j p^l p^m}{p} \frac{\partial f}{\partial p}$$

Together with the expression for $\frac{\partial f}{\partial p^i}$ this leads to

$$p^i \partial_i f - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = -2 \Gamma^i_{0j} \frac{p^j p_i p}{p} \frac{\partial f}{\partial p}$$

$$= -2 \frac{\dot a}{a} p^2 \frac{\partial f}{\partial p}$$

Hence the Liouville eqn. reduces to

$$\partial_t f - 2 \frac{\dot a}{a} p \frac{\partial f}{\partial p} = 0 \iff \cancel{f} \qquad (40)$$

$$f = f(a^2 p) = f(a\varepsilon) \qquad (41)$$

where $\varepsilon = a p^0 = ap$ is the physical photon energy (p^0 is the "comoving" photon energy).

⇒ In a Friedmann universe, the distribution function of free streaming massless particles is modified simply by redshifting the energy.

Hence in our case, where initially

$f = f(E/T)$ we can keep this form if we define $T = \frac{a_{dec}}{a} T_{dec}$,

hence the "temperature" keeps decaying like in the ~~mass~~ thermal equilibrium.

At present, the CMB spectrum is very close to a Planck spectrum with temperature

$$T_0 = 2.728K$$ (see graphic 6) (42).

This CMB radiation is very isotropic. It is a relic from the hot big bang and confirms the idea that the universe was once hot and very isotropic.

It was predicted by Gamow et al (1948) and observed first by Penzias & Wilson (1965), Nobel Price (1978).

<u>Dipole</u>

Our motion w.r.t the surface of last scattering induces a dipole in this radiation.

Indeed, setting $\underline{p} = -\varepsilon \underline{n}$, where ε is the photon energy and $\underline{n}$ its direction. An observer which moves with velocity $\underline{v}$ w.r. to this radiation receives it at energy

$$\varepsilon' = \gamma \varepsilon (1 - \underline{v} \cdot \underline{n}) \qquad (43)$$

where $\gamma = \frac{1}{\sqrt{1-(v)^2}}$ is the relativistic γ-factor.

To lowest order in $\underline{v}$, this induces a dipole anisotropy in the temperature,

$$\left(\frac{\Delta T}{T}\right)_{dipole} = |\underline{v}|$$

The observed dipole anisotropy is

$$\left(\frac{\Delta T}{T}\right)_{dipole} = 1.2 \times 10^{-3}$$

indicating that we move w.r.t the surface of last scattering with a speed of about 370 $\frac{km}{s}$.

More precisely $v = (371 \pm 0.5) \frac{km}{s}$ (68%)
(COBE)

On smaller scales, the temperature fluctuations are much smaller $\left(\frac{\Delta T}{T}\right) \sim$ a few $\times 10^{-5}$.

3 Cosmological perturbation theory

On small and intermediate scales the universe is not homogeneous and isotropic. The idea is that the observed structure grew out of small initial fluctuations (as they are generated during inflation).

Here I develop linear cosmological pert. theory, the main tool to determine anisotropies and polarisation of the CMB.

3.1 Metric perturbations

Be $\bar{g}_{\mu\nu}$ the Friedmann metric with scale factor $a(t)$. We define the perturbed metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \tag{44}$$

We parameterize the coefficients of $h_{\mu\nu}$ ~~where~~ by

$$h_{\mu\nu}\, dx^\mu dx^\nu = -2A\,dt^2 - 2B_i\,dt\,dx^i + 2H_{ij}\,dx^i dx^j$$

We decompose the vector B_i and the tensor H_{ij} (under spatial rotations) into their scalar (spin 0), vector (spin 1) and tensor (spin 2) degrees of freedom:

$$B_i = B_i^{(0)} + B_i^{(1)} \quad , \quad H_{ij} = H_{ij}^{(0)} + H_{ij}^{(1)} + H_{ij}^{(2)} \tag{45}$$

where $B_i^{(0)} = \nabla_i B$ and $\nabla^i B_i^{(1)} = 0$.

3.1 Gauge transformations

Of course, the background $\bar{g}_{\mu\nu}$ is not truly observable, only the full metric $g_{\mu\nu}$ is.

Furthermore, if we "define" some averaged metric $\bar{g}_{\mu\nu}$ and some averaged energy momentum tensor $\bar{T}_{\mu\nu}$, if $T_{\mu\nu}$ satisfies Einstein's eqn. for the metric $g_{\mu\nu}$, it is by no means clear that $\bar{g}_{\mu\nu}$ is the metric for which $\bar{T}_{\mu\nu}$ satisfies Einstein's eqn.

Einstein's eqn. are non-linear in the metric.

Furthermore, an averaging procedure requires the choice of a spatial 3-surface over which to average. This choice has two problems: it is not unique and it is non-local and in this sens not feasible in practice. The second problem is the averaging problem which I shall not address in this course. The first point is the issue of gauge freedom which we discuss now.

Linear perturbation theory is valid only if there exist averaging procedures leading to an average metric $\bar{g}_{\mu\nu}$ and an ~~ag~~ average energy momenten tensor $\bar{T}_{\mu\nu}$ such that

$$G_{\mu\nu}(\bar{g}_{\mu\nu}) = 8\pi G\, \bar{T}_{\mu\nu} \qquad (51)$$

and
$$|h_{\mu\nu}| = \frac{|g_{\mu\nu} - \bar{g}_{\mu\nu}|}{a^2} \sim \varepsilon \ll 1 \qquad (52)$$

$$\frac{|T_{\mu\nu} - \bar{T}_{\mu\nu}|}{|\bar{T}_{00}|} \sim \varepsilon \ll 1 \qquad (53)$$

(51) can easily be achieved by defining $\bar{T}_{\mu\nu}$ via Einstein's eqn. but linear perturbation theory is of course only meaningfull if also (52) and (53) are satisfied (at least on sufficiently large scales).

We call an averaging procedure 'admissible' if it satisfies (51) to (53).

From a given admissible averaging procedure we can find the others via infinitesimal diffeomorphisms: Let us consider a vector field X and its flow $\phi \equiv \phi_\varepsilon^X : x \mapsto \gamma_\varepsilon(x)$ where $\gamma_s(x)$ is the integral curve to X with starting point $x = \gamma_0(x)$. To first order in ε, a tensor field S changes under ϕ with the push-foreward

$$S \mapsto \phi_* S = S + \varepsilon L_X S + O(\varepsilon^2) \qquad (54)$$

Be therefore $S = \bar{S} + \varepsilon S_{(1)}$, where $\bar{S}$ is the value of S in the Friedmann background and $\varepsilon S_{(1)}$ the perturbation.

To first order in ε, $S_{(1)}$ changes under an infinitesimal change of the background (or change of coordinates) by

$$S_{(1)} \mapsto S_{(1)} + \mathcal{L}_X \bar{S}. \qquad (55)$$

A gauge-invariant perturbation variable is a variable which is invariant under all infinitesimal diffeomorphisms (called gauge transformations in this context), hence one for which

$$\mathcal{L}_X \bar{S} = 0 \quad \forall \text{ vector fields } X.$$

This fact is called the "Steward-Walker lemma" (Steward & Walker, 1974):
"A perturbation variable $S_{(1)}$ is gauge invariant if and only if its background component is constant."

<u>Examples:</u> . The Weyl tensor

. $h^{\mu\nu}\nabla_\nu \rho$, $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

. $h^{\mu\nu}(\nabla_\nu u_\mu + \nabla_\mu u_\nu)$

~~Let us now~~ To compute how metric perturbations change under gauge transformations:
~~For this~~ we set

$$X = T\partial_t + L^i \partial_i$$

and compute $\mathcal{L}_X \bar{g}$. (For details, see book)

One may choose the gauge transformation $kL = H_T$ and $kT = B - H_T$ so that both, B and H_T vanish. In the new <u>longitudinal</u> or <u>Newtonian</u> gauge scalar metric perturbations are of the form

$$h^{(S)}_{\mu\nu}\, dx^\mu dx^\nu = -2\Psi\, dt^2 - 2\Phi\, \gamma_{ij}\, dx^i dx^j . \quad (67)$$

Ψ and Φ are the so called Bardeen potentials.

In a generic gauge they are given by

$$\Psi = A - \frac{\mathcal{H}}{k}\sigma - \frac{1}{k}\dot{\sigma} \quad , \sigma = k^{-1}\dot{H}_T - B \quad (68)$$

$$\Phi = -H_L - \frac{1}{3}H_T + \frac{\mathcal{H}}{k}\sigma \quad (69)$$

Note that in a generic gauge, Ψ contains second derivatives of the metric fluctuations in $\ddot{\sigma}$!

Also, the choice $kL = H_T$ is <u>non local</u>; i.e. longitudinal gauge is not unproblematic. It can lead to counter intuitive results.

One can compute the Weyl tensor of the perturbed metric and finds to first order (in an arbitrary gauge)

$$E_{ij} = C^\mu{}_{i\nu j}\, u_\mu u^\nu = -C^0{}_{i0j} =$$

$$-\frac{1}{2}\left[(\Psi + \Phi)_{|ij} - \frac{1}{3}\Delta(\Phi + \Psi)\gamma_{ij} \right] \quad (70)$$

$$E_{ij}^{(1)} = \frac{1}{2}\dot{\sigma}^{(1)}_{(i|j)}$$

$$B_{ij}^{(1)} := \frac{1}{2}\varepsilon_{i\nu}{}^{\rho\sigma}C_{\rho\sigma j}{}^{\alpha}u^{\nu}u_{\alpha} = \varepsilon_{i\ell m}C^{0}_{j\ell m}$$

$$= \frac{1}{2}\varepsilon_{i\ell m}\left(\sigma^{(1)}_{\ell|jm} - \sigma^{(1)}_{m|j\ell} - \frac{k^2}{2}\gamma_{j\ell}\sigma^{(1)}_{m} + \frac{k^2}{2}\gamma_{jm}\sigma^{(1)}_{\ell}\right) \qquad (71)$$

For tensor perturbations one finds

$$E_{ij}^{(2)} = \frac{1}{2}(\partial_t^2 - k^2)H_{ij}^{(2)} \qquad (72)$$

$$B_{ij}^{(2)} = -\varepsilon_{i\ell m}(\dot{H}_{j\ell|m} - \dot{H}_{jm|\ell}).$$

The quantities related to the Weyl tensor are local and gauge invariant. For linear perturbation theory to apply, they have to be much smaller than the Ricci tensor which is of order $\mathcal{H}^2$. Note also that for scalar perturbations they are of order $k^2\Psi$ hence we need $\left|\frac{k^2\Psi}{\mathcal{H}^2}\right| \ll 1$ for linear perturbation theory to apply.

~~Let us mo~~

3.2. Perturbations of the energy momentum tensor

Let us now go on to define perturbations of the energy momentum tensor

$$T_{\nu}{}^{\mu} = \overline{T}_{\nu}{}^{\mu} + \Theta_{\nu}{}^{\mu} \qquad (73)$$

where $\bar{T}_\nu{}^\mu$ is the background e.m. tensor which satisfies Einstein's eqn. with the background metric, $\bar{g}_{\mu\nu}$ and $\Theta_\nu{}^\mu$ is a perturbation.

We define the energy density ρ and the energy flux (u^μ) as the timelike eigenvalue and eigenvector of $T_\nu{}^\mu$,

$$T_\nu{}^\mu u^\nu = -\rho u^\mu \quad , \quad u^2 = -1 .$$

$$\rho = \bar{\rho}(1+\delta) \quad , \quad u = u^0 \partial_t + u^i \partial_i \qquad (74)$$

$u^0 = \frac{1}{a}(1-A)$ due to the normalisation condition

$$u^i = \frac{1}{a} v^i = \frac{1}{a}\left(v Q^i + v^{(1)} Q^{(1)i}\right). \qquad (75)$$

Now $P_\nu^{(4)\mu} = u^\mu u_\nu + \delta_\nu{}^\mu$ is the projector onto the subspace of tangent space normal to u. We define the stress tensor

$$\tau^{\mu\nu} = P^{(4)\mu}{}_\alpha P^{(4)\nu}{}_\beta T^{\alpha\beta}$$

With this we can write

$$T^\mu{}_\nu = \rho u^\mu u_\nu + \tau^\mu{}_\nu .$$

In the unperturbed case we have $\tau^0{}_\mu = \tau^\mu{}_0 = 0$ and $\tau^i{}_j = \bar{P}\delta^i{}_j$, where $\bar{P}$ is the background pressure.

In the perturbed universe we find

$$\tau^0_{\ 0} = 0\ , \quad \tau^i_{\ 0} = -\bar{P} v^i , \quad \tau^0_{\ i} = \bar{P}(v_i - B_i)$$

$$\tau^i_{\ j} = \bar{P}\left[(1+\pi_L)\delta^i_{\ j} + \Pi^i_{\ j}\right]. \tag{76}$$

$$\Pi_{ij} = \Pi Q_{ij} + \Pi^{(1)} Q^{(1)}_{ij} + \Pi^{(2)} Q^{(2)}_{ij}$$

is the traceless part of $\tau^\mu_{\ \nu}$, and it is gauge-invariant by itself. This is the anisotropic stress.

The density and pressure perturbations transform under gauge transformations as

$$\delta \longrightarrow \delta - 3(1+w)\mathcal{H}T \tag{77}$$

$$\pi_L \longrightarrow \pi_L - \frac{3c_s^2}{w}(1+w)\mathcal{H}T$$

$$v \longrightarrow v - \dot{L}\ , \quad v^{(1)} \longrightarrow v^{(1)} - \dot{L}^{(1)}) \tag{78}$$

The quantity $\Gamma = \pi_L - \frac{c_s^2}{w}\delta$ is gauge invariant

On can show that it is proportional to the divergence of the entropy flux. For adiabatic perturbations, $\Gamma = 0$.

To obtain gauge invariant perturbations for the density contrast or the velocity, one has to com- ~~by~~ bine δ and v with metric perturbations:

Gauge invariant variables for the density and the velocity are

$$V = v - \frac{1}{k}\dot{H}_T = v_e$$

$$D_s = \delta + 3(1+w)\frac{\mathcal{H}}{k}\sigma = \delta_e$$

$$D_g = \delta + 3(1+w)\mathcal{R} = D_s - 3(1+w)\Phi = \delta_{flat}$$

$$D = \delta + 3(1+w)(k^{-2}\dot{H}_T - k^{-1}B) = D_s + 3(1+w)\frac{\mathcal{H}}{k}V = \delta_{com}$$

23.7.12

→ slide 7

Using e.g. Ψ, ϕ and V, D, Γ, Π we can now express the perturbed Einstein & conservation eqns. in these variables.

→ see slides

spatial curvature in comoving gauge $(H_L + \frac{1}{3}H_T)$

$$-\mathcal{R}_{co} = k^{-1}\mathcal{H}V + \Phi = \frac{2}{3(1+w)}[\Psi + \mathcal{H}^{-1}\dot{\phi}] + \Phi = \zeta$$

$$\frac{1}{\mathcal{H}}\dot{\zeta} = \frac{w}{1+w}\Gamma - \frac{2}{3(1+w)}\frac{k^2}{\mathcal{H}^2}\Psi \Rightarrow$$

$\dot{\zeta} \simeq 0$ for on large scales, $k^2/\mathcal{H}^2 \ll 1$ adiabatic perturbations $(\Gamma = 0)$

→ slides 8 & 9

3.4 Simple examples

case $K = 0$, $c_s^2 = w = \text{const.}$

$\Rightarrow a(t) \propto t^{\frac{2}{1+3w}}$, $\mathcal{H} = \frac{2}{1+3w}\frac{1}{t}$, $q = \frac{2}{1+3w}$

In addition, we set $\Pi = \Gamma = 0$, hence $\phi = \Psi$.

⇒ The Bardeen eqn. reduces to

$$\ddot{\Psi} + 6\frac{1+w}{(1+3w)t}\dot{\Psi} + wk^2\Psi = 0 . \qquad (93)$$

Setting $q = \frac{2}{1+3w}$, hence $a \propto t^q$, $\mathcal{H} = \frac{q}{t}$

we find that the generic solution of (93) is simply a Bessel function

$$\Psi = \frac{1}{a}\left[A j_q(\sqrt{w}\,kt) + B y_q(\sqrt{w}\,kt)\right] \qquad (94)$$

Using that $j_q(x) \propto x^q$, $x \ll 1$ and $y_q \propto x^{-q-1}$, $x \ll 1$ we find that the B-mode is decaying and the A-mode is constant for small $x = \sqrt{w}\,kt$,

$$\Psi = \tilde{A} + \frac{\tilde{B}}{a(kt)^{q+1}} \quad \text{for} \quad \sqrt{w}\,kt \ll 1$$

On subhorizon scales, the solution oscillates so that, considering only the growing mode, we have

$$\Psi = \begin{cases} \text{constant}, & \sqrt{w}\,kt \ll 1 \\ \frac{A}{a\sqrt{w}\,kt}\sin\left(\sqrt{w}\,kt - \frac{q}{2}\pi\right) & \text{for } \sqrt{w}\,kt \gg 1 . \end{cases} \qquad (95)$$

$-\frac{\sqrt{3}A}{akt}\cos\left(\frac{kt}{\sqrt{3}}\right)$, $q = 1$, $w = 1/3$, radiation

Considering now a radiation dominated universe which starts out with an initial spectrum of the form

$$\langle|\Psi|^2\rangle k^3\Big|_{in} = A_s\left(\frac{k}{\mathcal{H}_0}\right)^{n_s - 1}$$

we obtain today (neglecting the Λ-dominated phase)

$$\langle|\Psi|^2\rangle k^3 = A_S\,(k/H_0)^{n_s-1}\cdot\begin{cases}1\,, & kt_{eq}\ll 1\\ (kt_{eq})^{-4}\cos^2\left(\frac{kt_{eq}}{\sqrt{3}}\right), & kt_{eq}\gg 1\end{cases} \qquad (96)$$

where t_{eq} is the time of equal matter and radiation.

(see fig. for $n_S = 1$) , → slide 10

Dust: For dust, $w=0$ the above considerations cannot be applied strictly., ~~since~~ There $q=2$ and the Bardeen eqn. reduces to

$$\ddot\Psi + \frac{6}{t}\dot\Psi = 0 \quad \text{with solution}$$

$$\Psi = A + \frac{B}{(kt)^5}\,. \quad \Rightarrow \Psi(t_0)\simeq\Psi(t_{eq}),\quad D = \frac{2}{3}\frac{k^2}{\mathcal{H}^2}\Psi$$

Neglecting the decaying mode one finds that the gravitational potential remains constant in the matter era.

From momentum conservation one obtains

$$\dot V + \mathcal{H}V = k\Psi$$

$$V' + \frac{2}{x}V = \Psi\,, \quad x = kt,\ \ ' = \frac{d}{dx}$$

$$\Rightarrow V = V_0 x \quad \text{with } 3V_0 = A$$

$$V = \frac{A}{3}x$$

For the density we obtain from energy conservation

$$\dot{D}_g = -kV, \quad D_g' = -V$$

$$\Rightarrow D_g = -\frac{A}{6}x^2 + C$$

From the constraint eqn. we find $C = -5A$ so that

$$D_g = -A\left(5 + \frac{x^2}{6}\right). \tag{97}$$

On small scales, $x \gg 1$ density fluctuations grow like $x^2 \propto a$ and velocity fluctuations grow like $x \propto \sqrt{a}$. On large scales, $x \ll 1$, perturbations remain constant.

One can also determine V and D_g for radiation, $w = c_s^2 = 1/3$, $q = 1$. On small scales $x = c_s kt \gg 1$ one finds

$$D_g = 2A\cos(x), \quad \Psi = -\frac{A\cos(x)}{x^2} \tag{98}$$

$$V = \frac{\sqrt{3}A}{2}\sin(x)$$

3.5 Lightlike geodesics, CMB anisotropies

After decoupling, $t > t_{dec}$, photons move along geodesics from the last scattering surface to us. We consider an unperturbed trajectory of the form $(x^\mu(t)) = (t, \underline{x}_0 - \underline{n}(t_0 - t))$ of a photon moving in direction $\underline{n}$ and arriving at time t_0 at our telescope, positionned at $\underline{x}_0$.

For $\underline{n}^2 = 1$, $\underline{n} = const.$ this is a lightlike geodesic in a Friedmann universe with vanishing spatial curvature, $K = 0$.

Our metric is of the form

$$d\tilde{s}^2 = a^2 ds^2 \quad \text{where}$$

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$

We make use of the fact that lightlike geodesics are conformally invariant. Only the affine parameter depends on the conformal factor a.

$$n = \frac{dx}{d\lambda}, \quad \tilde{n} = \frac{d\tilde{x}}{d\tilde{\lambda}} \quad \tilde{n}^2 = n^2 = 0, \quad n^0 = 1, \ \underline{n}^2 = 1$$

We have seen that photon momenta are redshifted so that their components behave like

$$\tilde{n}^i \propto \frac{1}{a^2}, \ \sum_i \tilde{n}^i \tilde{n}_i \propto \frac{1}{a^2}. \quad \text{Hence we have}$$

to choose $d\tilde{\lambda} = a^2 d\lambda$.

We now proceed to compute the perturbed geodesic for the metric ds^2;

Keeping this in mind, we can consider a geodesic w.r.t the metric

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

$$n = (n^0, \underline{n}) \quad , \quad n^0 = 1 + \delta n^0$$

To first order in δn^μ we obtain

$$\frac{d\,\delta n^\mu}{d\lambda} = -\delta\Gamma^\mu_{\alpha\beta}\, n^\alpha n^\beta$$

$$\delta\Gamma^0_{\alpha\beta} = -\frac{1}{2}\left(h_{\alpha 0,\beta} + h_{\beta 0,\alpha} - \dot{h}_{\alpha\beta}\right) \quad \text{so that}$$

$$\frac{d\delta n^0}{d\lambda} = \underbrace{h_{\alpha 0,\beta}\, n^\alpha n^\beta}_{\frac{d}{d\lambda} h_{\alpha 0} n^\alpha} - \frac{1}{2}\dot{h}_{\alpha\beta}\, n^\alpha n^\beta$$

$$\frac{\delta n^0}{d\lambda}\Big|_s^0 = \left[h_{00} + h_{0j} n^j\right]_s^0 - \frac{1}{2}\int_s^0 \dot{h}_{\mu\nu}\, n^\mu n^\nu\, d\lambda$$

The energy of a photon with 4-momentum $\tilde{p}^\mu$ seen by an observer with 4-velocity $\tilde{u}^\mu$ is

$$E = -(\hat{u} \,\tilde{\cdot}\, \hat{p})$$

where the $\tilde{\cdot}$ indicates that the scalar product has to be taken w.r.t. the expanding metric

Hence the ratio (1+redshift)$^{-1}$ $\frac{E_0}{E_s} = \frac{(\tilde{n} \tilde{\cdot} \tilde{u})_0}{(\tilde{n} \tilde{\cdot} \tilde{u})_s}$

$= \frac{a_s (n \cdot u)_0}{a_0 (n \cdot u)_s}$ Here we have used that

$\tilde{n} = \frac{1}{a^2} n$, $\tilde{u} = a^{-1} u = a^{-1}((1-A)\partial_t + v^i \partial_i)$

and $(\tilde{\cdot}) = a^2 (\,\cdot\,)$.

Finally, we have to take into acount that 0 and s correspond to fixed temperatures, not to fixed scale factors

$T_0 = \bar{T}_0 + \delta T_0$, $T_s = T_{dec} = \bar{T}_s + \delta T_s$ and

$$\frac{a_s}{a_0} = \frac{\bar{T}_0}{\bar{T}_s} = \frac{T_0}{T_{dec}}\left(1 - \frac{\delta T_0}{T_0} + \frac{\delta T_s}{T_s}\right) = \frac{T_0}{T_{dec}}\left(1 - \frac{1}{4}\delta_r \Big|_i^f\right)$$

Putting it all together yields (in longitudinal gauge scalar perts.)

$$\frac{E_0}{E_s} = \frac{T_0}{T_{dec}}\left[1 - \left(\frac{1}{4}\delta_r + \Psi - \underline{V}^{(b)} \cdot \underline{n}\right)\Big|_s^0 + \int_s^0 (\dot{\Psi} + \dot{\Phi})\, d\lambda\right] \qquad (99)$$

Evaluating this result for photons coming from different directions we obtain the temperature difference

$$\frac{\Delta T}{T} = \frac{\Delta T(\underline{n}_1) - \Delta T(\underline{n}_2)}{\overline{T}} = \left[\frac{E_0}{E_s}(n_1) - \frac{E_0}{E_s}(n_2)\right]\frac{T_{dec}}{T_0}$$

Neglecting the dipole $\underline{V}^{(0)} \cdot n \,|^0$ due to our proper motion w.r.t. the CMB, we find

$$\frac{\Delta T}{T}(\underline{n}) = \left(\frac{1}{4} D_s^{(r)} + V_{|i}^{(b)} n^i + \Psi\right)(t_{dec}, \underline{x}_{dec}) + \int_{t_{dec}}^{t_0} \partial_t (\Psi + \Phi)(t, \underline{x}(t))\, dt \qquad (100)$$

where $\underline{x}(t) = \underline{x}_0 + \underline{n}(t - t_0)$.

In a universe filled with dust & radiation and with adiabatic perturbations we have on large scales

$$D_s^{(r)} = \frac{4}{3} D_s^{(m)} \simeq -4 \frac{\mathcal{H}}{k} V = \frac{-4\mathcal{H}^2 \Psi}{\underbrace{4\pi G a^2 \rho}_{3/2\, \mathcal{H}^2}} = -\frac{8}{3}\Psi + \mathcal{O}\left(\left(\frac{k}{\mathcal{H}}\right)^2\right)\Psi$$

so that

$$\left(\frac{\Delta T}{T}\right)_{OSW} = \frac{1}{3}\Psi(t_{dec}, \underline{x}_{dec}) \qquad (101)$$

$$\left(\frac{\Delta T}{T}\right)_{ISW} = \int_{t_{dec}}^{t_0} (\dot{\Psi} + \dot{\Phi})(t, \underline{x}(t))\, dt \qquad (102)$$

On intermediate scales the dominant term comes from the combination $\left(\frac{1}{4}D_s^{(r)} + V_{1i}^{(b)} n^i\right)(t_{dec}, \underline{x}_{dec})$ and gives raise to the <u>acoustic peaks</u> which reflect the acoustic oscillations in $D^{(r)}$ prior to recombination:

$$D_g^{(r)} = 4\Psi_0\left(\frac{1}{3}\cos(c_s k t) - 1\right)$$ adiabatic perturbations.

in the matter dominated regime,

$$D_s^{(r)} = D_g + 4\Phi_0 \Rightarrow \frac{1}{4}D_s^{(r)} + \Phi_0 = \frac{1}{3}\cos(c_s k t).$$

For tensor perturbations, only the ISW term survives,

$$\left(\frac{\Delta T}{T}\right)^T(\underline{n}) = -\int_S^0 \dot{H}_{ij}(t, \underline{x}(t))\, n^i n^j\, dt. \qquad (103)$$

2.6 Power spectra

In a given (e.g. inflationary) model, one can usually not compute the value of a perturbation at a given position, e.g. $\Psi(\underline{x}, t)$, but only expectation values, or correlations like

$$\langle \Psi(\underline{x}, t)\Psi(\underline{y}, t)\rangle = \xi_\Psi(\underline{x}, \underline{y}, t)$$

We shall assume that the stochastic process which generates the perturbations is statistically homogeneous and isotropic; i.e. it has no preferred directions or positions in space. Then $\xi_\psi(\underline{x},\underline{y},t) = \xi_\psi(|\underline{x}-\underline{y}|,t)$.

Furthermore, for its Fourier transforms we find

$$\Psi(\underline{k},t) = \int e^{i\underline{k}x}\,\psi(\underline{x},t)\,d^3x$$

$$\langle\psi(\underline{k},t)\,\psi^*(k',t)\rangle = \int e^{i(\underline{k}x-\underline{k}'y)}\,\langle\psi(\underline{x},t)\,\psi(\underline{y},t)\rangle\, d^3x\, d^3y$$

$$= \int d^3r\, e^{i\underline{k}\,\underline{r}}\, e^{i(\underline{k}-\underline{k}')y}\,\xi_\psi(r,t)\,d^3r\,d^3y$$

$$= \delta^3(k-k')\,(2\pi)^3\,P_\psi(k)\,, \quad \text{where} \quad (104)$$

$\underline{r} = \underline{x}-\underline{y}$

$P_\psi(k) := \int e^{i\underline{k}\,\underline{r}}\,\xi_\psi(r)\,d^3r$ and we have used

$$\int e^{i(\underline{k}-\underline{k}')y}\,d^3y = (2\pi)^3\,\delta^3(\underline{k}-\underline{k}').$$

2.7 The CMB power spectrum

The CMB temperature fluctuations are functions on the sphere. At our position $\underline{x}_0$, today, t_0

$$\frac{\Delta T}{T}(\underline{x}_0, t_0, \underline{n}) = \sum_{\ell,m} a_{\ell m}(\underline{x}_0, t_0)\, Y_{\ell m}(\underline{n}) \qquad (105)$$

To require that $\frac{\Delta T}{T}(\underline{n})$ be statistically isotropic is equivalent to $\langle \frac{\Delta T}{T}(\underline{n}) \frac{\Delta T}{T}(\underline{n}')\rangle$ to be a function of the angle θ, or

$$\mu = \cos\theta = \underline{n}\cdot\underline{n}'.$$

This requires that $\langle a_{\ell m} \cdot a^*_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell$. (106)

If only diagonal terms are non-vanishing,

$$\langle \frac{\Delta T}{T}(n)\cdot\frac{\Delta T}{T}(n')\rangle = \sum_{\ell, m \ell' m'} \langle a_{\ell m} a^*_{\ell' m'}\rangle Y_{\ell m}(\underline{n}) Y^*_{\ell' m'}(\underline{n}')$$

$$= \sum_{\ell} C_\ell \underbrace{\sum_m Y_{\ell m}(n) Y^*_{\ell m}(n')}_{\frac{2\ell+1}{4\pi} P_\ell(\underline{n}\cdot\underline{n}')} = \frac{1}{4\pi}\sum_\ell (2\ell+1) C_\ell P_\ell(\mu) \qquad (107)$$

Because of its importance, let us compute the C_ℓ's from the OSW in detail.

For this we assume that inflation provides a nearly scale invariant spectrum of scalar fluctuations,

$$k^3 P_\psi(\underline{k},t) = A\cdot(kt_0)^{n_s-1}\,, \quad n_s \simeq 1\,,\quad kt_{eq} < 1$$

We want to compute the C_ℓ's from

$$\left(\frac{\Delta T}{T}\right)^{OSW}(\underline{x}_0,t_0,\underline{n}) = \frac{1}{3}\Psi(\underline{x}_{dec},t_{dec}) \qquad \underline{x}_{dec} = \underline{x}_0 + \underline{n}(t_0-t_{dec})$$

$$\left(\frac{\Delta T}{T}\right)^{OSW}(\underline{k},t_0,\underline{n}) = \frac{1}{3}\Psi(\underline{k},t_{dec})e^{+i\underline{k}\underline{n}(t_0-t_{dec})}$$

We now use that

$$e^{i\underline{k}\underline{n}(t_0-t_{dec})} = \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell j_\ell(k(t_0-t_{dec}))P_\ell(\hat{\underline{k}}\cdot\underline{n})$$

$$\Rightarrow \left\langle\left(\frac{\Delta T}{T}\right)^{OSW}(\underline{n})\left(\frac{\Delta T}{T}\right)^{OSW}(\underline{n}')\right\rangle = \frac{1/9}{(2\pi)^6}\int d^3k\,d^3k'\,\langle\Psi(\underline{k})\Psi^*(\underline{k}')\rangle\cdot$$

$$\sum_{\ell,\ell'=0}^{\infty}(2\ell+1)(2\ell'+1)i^{\ell-\ell'}j_\ell(k(t_0-t_{dec}))j_{\ell'}(k'(t_0-t_{dec}))P_\ell(\hat{\underline{k}}\underline{n})P_{\ell'}(\hat{\underline{k}}'\cdot\underline{n}')$$

Now $\langle\Psi(\underline{k})\Psi^*(\underline{k}')\rangle = (2\pi)^3P_\psi(k)\delta(\underline{k}-\underline{k}')$

and $\int d\Omega_{\hat{k}} \, \underbrace{P_\ell(\hat{k}\cdot\underline{n}) P_{\ell'}(\hat{k}\underline{n}')}$

$$\frac{(4\pi)^2}{(2\ell+1)(2\ell'+1)} \sum_{m m'} Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\underline{n}) Y^*_{\ell' m'}(\hat{k}) Y_{\ell' m'}(\underline{n}')$$

$$= \frac{(4\pi)^2}{(2\ell+1)^2} \underbrace{\sum_m Y^*_{\ell m}(\underline{n}) Y_{\ell m}(\underline{n}')}_{\frac{2\ell+1}{4\pi} P_\ell(\underline{n}\cdot\underline{n}')}$$

$=$

$\Rightarrow$

$$\left\langle \left(\frac{\Delta T}{T}(n) \frac{\Delta T}{T}(n')\right)^{OSW} \right\rangle = \cancel{\frac{1}{9}\frac{2}{\pi}} \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(n\cdot n')$$

with

$$C_\ell = \frac{1}{9}\frac{2}{\pi}\int_0^\infty \frac{dk}{k} \underbrace{k^3 P_\psi(k)}_{A(kt_0)^{n-1}} j_\ell^2(k(t_0-t_{dec}))$$

$$\cong \frac{A}{9}\cdot\frac{2}{\pi}\int \frac{dx}{x} x^{n-1} j_\ell^2(x) \quad , \; x = kt_0$$

$$= \frac{A}{9} \frac{\Gamma(3-n)\Gamma(\ell+\frac{n}{2}-\frac{1}{2})}{2^{3-n}\Gamma^2(2-\frac{n}{2})\Gamma(\ell+\frac{5}{2}-\frac{n}{2})} \, , \; -3<n<3 \qquad (108)$$

$$= \frac{A}{9\pi}\cdot\frac{1}{\ell(\ell+1)} \quad \text{for } n=1$$

The observed CMB anisotropies require

$$A \simeq 0.9 \times 10^{-8}, \quad n \cong 0.96.$$

(see graphic)

→ slide 11

4. The Boltzmann eqn. for the CMB

In our treatment of CMB anisotropies so far, we have assumed that photons are a tightly coupled fluid until they reach the temperature T_{dec} and at T_{dec} they decouple and free-stream into our antennas.

In reality of course this decoupling process is not abrupt, but gradual and this influences CMB anisotropies in several ways:

1) Silk damping: When the photon/baryon/electron fluid is no longer perfect but has a finite collision time, photons can stream from high density to low density regions which damps fluctuations (diffusion damping). This is especially important on small scales, but already reduces the amplitude of the first peak by more than 20%!

2) Projection effects: The fact that the last scattering surface actually has a finite thickness means that

by integrating through it, small scale fluctuations are averaged over. This leads to damping on similar scales as Silk damping.

3) Polarization: The only scattering process relevant prior to decoupling is non-relativistic Thomson scattering. But this depends on polarisation: photons which are polarized in the scattering plane have their cross section reduced by a factor $\cos^2\hat{\theta}$ scattering angle.
Therefore, if the incoming radiation has a non-vanishing quadrupole, the outgoing radiation is polarized (see graphic).

3.

4.1 Liouville's equation

An alternative to the fluid description is the discription via a distribution function f defined on the 7-dimensional mass bundle

$$P_m\mathcal{M} := \{(x,p) \in T\mathcal{M} \,/\, \underbrace{g_{\mu\nu}(x)\tilde{p}^\mu\tilde{p}^\nu}_{p^2} = -m^2\}$$

An invariant measure on this space is given by

$$|\det g|\, d^4x\, d^4\tilde{p} \cdot \delta(p^2+m^2)$$

$$= \frac{|\det g|}{2|p_0|}\, d^3\tilde{p}\, d^4x \quad , \quad f = \bar{f}(ap_{phys}) + F(x^\mu, \tilde{p}^i)$$

$$ap_{phys} \equiv v$$

$$\tilde{p}^\mu \partial_\mu F\big|_p - \mathcal{H}\tilde{p}^0 p \frac{\partial F}{\partial p} = \frac{1}{a} v \frac{d\bar{f}}{dv}\left[-p^0\dot{\Phi} + \frac{(\tilde{p}^0)^2}{\tilde{p}^2} p^k \partial_k \Psi\right]$$

Setting $\tilde{p}^i = \frac{p}{a} n^i = \frac{v}{a^2} n^i$, $\underline{n}^2 = 1$ and

$$q = a^2 \tilde{p}^0 = a p^0 = \sqrt{v^2 + a^2 m^2}$$

as well as
$$\mathcal{F} = F + \Phi v \frac{d\bar{f}}{dv} \qquad (110)$$

we obtain

$$q\partial_0 \mathcal{F} + v n^i \partial_i \mathcal{F} = n^i \partial_i [q^2 \Psi + v^2 \Phi] \frac{d\bar{f}}{dv}. \qquad (111)$$

For massless particles $q = v$, this simply reduces to

$$(\partial_0 + n^i \partial_i)\mathcal{F} = n^i \partial_i (\Phi + \Psi) \cdot v \frac{d\bar{f}}{dv}. \qquad (112)$$

Defining the energy integrated distribution function by

$$\mathcal{M} = \frac{1}{4}\left(\frac{4\pi}{a^4 \bar{\rho}} \int v^3 \mathcal{F} dv\right) \quad \text{with} \qquad (113)$$

$$4\pi \int v^4 \frac{d\bar{f}}{dv} dv = -4\left(4\pi \int v^3 \bar{f} dv\right) = -4a^4 \bar{\rho}$$

$$(\partial_0 + n^i \partial_i)\mathcal{M} = -n^i \partial_i (\Phi + \Psi). \qquad (114)$$

$\rightarrow$ slides